# Some Aspects of Rigid Body Motion

## Rigid Body Motion with a Point Fixed

A solid body $\mathcal{B}$ moves in space so that one point in the body (taken to be the origin) remains fixed.

Consider an initial position of $\mathcal{B}$ and a final position of $\mathcal{B}$: there is a mapping $f$ that takes each point in $\mathcal{B}$ from its initial to its final position.

**Theorem** (Euler, 1775) The mapping $f$ is accomplished by a rotation about some axis through the origin.

To rephrase: There is an orthogonal matrix $A$ with determinant 1 such that $f(x) = A\,x$ for all $x$.

### Kinematics of Rigid Body Motion

A rigid body begins moving at $t = 0$, still assuming the origin is left fixed by the motion.

The initial position of the body we call the reference configuration. Let $\mathbf{r}(t)$ be the position at time $t$ of a particular point in the body.

Initially, this point is located at $\mathbf{r}(0)$ and at time $t > 0$ it is at $\mathbf{r}(t)$. According to Euler, there is a rotation matrix $A(t)$ such that

$$\mathbf{r}(t) = A(t)\,\mathbf{r}(0). \qquad (1)$$

We have

$$A(0) = I \qquad (I \text{ is the } 3\times 3 \text{ identity matrix}).$$

Differentiate (1) to get

$$\begin{aligned} \mathbf{r}'(t) &= A'(t)\,\mathbf{r}(0) \\ &= A'(t)\,A^{-1}(t)\,\mathbf{r}(t) \\ &= A'(t)\,A^{T}(t)\,\mathbf{r}(t). \end{aligned}$$

We define

$$B(t) = A'(t)\,A^{T}(t),$$

and conclude that the position vector $\mathbf{r}(t)$ of a particle in the rigid body satisfies the system of differential equations

$$\mathbf{r}'(t) = B(t)\,\mathbf{r}(t).$$

Recall: matrix $P$ is antisymmetric if $P^T = -P$.

**Lemma:** For each $t$, $B(t)$ is an antisymmetric matrix ( a matrix $P$ is antisymmetric if $P^T = -P$).

**Proof**: We know that $A(t)$ is orthogonal for each $t$. Therefore we have $A(t)\,A^T(t) = I$ for all $t$. Differentiating this equation gives

$$A'(t)\,A^T(t) + A(t)\left(A^T\right)'(t) = 0.$$

Therefore,

$$B(t) = -A(t)\left(A^T\right)'(t).$$

Then (omitting the $t$ to simplify notation)

$$\begin{aligned} B^T &= \left(-A\left(A^T\right)'\right)^T \\ &= -\left(A(A')^T\right)^T \\ &= -\left(A'\,A^T\right) \\ &= -B. \end{aligned}$$

Since $B(t)$ is antisymmetric, we have functions $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$, such that

$$B(t) = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}.$$

If the position vector of a particle in the rigid body is

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

and if we define the vector $\boldsymbol{\omega}(t)$ by

$$\boldsymbol{\omega}(t) = \omega_1(t)\,\mathbf{i} + \omega_2(t)\,\mathbf{j} + \omega_3(t)\,\mathbf{k},$$

then we have

$$\begin{aligned} \mathbf{r}'(t) &= B(t)\,\mathbf{r}(t) \\ &= \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \\ &= \begin{pmatrix} \omega_2(t)\,z(t) - \omega_3(t)\,y(t) \\ \omega_3(t)\,x(t) - \omega_1(t)\,z(t) \\ \omega_1(t)\,y(t) - \omega_2(t)\,x(t) \end{pmatrix} \\ &= \boldsymbol{\omega}(t) \times \mathbf{r}(t). \end{aligned}$$

This is a <u>fundamental</u> <u>result</u>: If a particle moves in space so that its position vector at time $t$ is obtained from the position vector at time 0 by a rotation matrix, that is

$$\mathbf{r}(t) = A(t)\,\mathbf{r}(0),$$

then there is a <u>vector</u> <u>function</u> of time $\boldsymbol{\omega}(t)$, called the <u>angular</u> <u>velocity</u> <u>vector</u>, such that the particle's velocity vector $\mathbf{r}'(t)$ satisfies the equation

$$\mathbf{r}'(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t).$$

The vector $\boldsymbol{\omega}(t)$ determines an <u>instantaneous</u> <u>axis</u> <u>of</u> <u>rotation</u> : those vectors that are at rest at time $t$ are precisely those whose position at time $t$ lies along the line through the origin determined by $\boldsymbol{\omega}(t)$.

## Dynamical Equations for Systems of Particles

View our rigid body as consisting of masses $m_1, m_2, \ldots, m_N$ with position vectors $\mathbf{r}_1(t), \ldots, \mathbf{r}_N(t)$. Let the total force on $m_i$ be

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{i\,j}$$

where $\mathbf{F}_i^{\text{ext}}$ is an external force, exerted from outside the system, and $\mathbf{F}_{i\,j}$ is an internal force exerted on mass $m_i$ due to the presence of $m_j$. We define the total external force

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}}$$

and we assume

$$\mathbf{F}^{\text{ext}} = \mathbf{0}.$$

The angular momentum of the $i$-th particle about the origin is

$$\mathbf{L}_i = m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i{}'(t).$$

Define the quantities:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i \qquad \text{(total angular momentum)}$$

$$\mathbf{N} = \sum_{i=1}^{N} \mathbf{r}_i(t) \times \mathbf{F}_i^{\text{ext}} \quad \text{(total torque)}$$

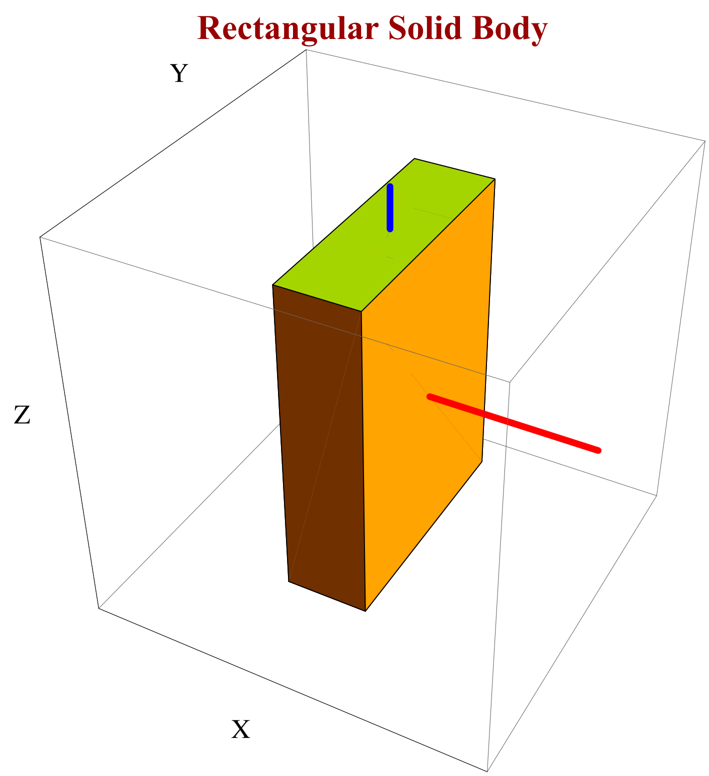
We can then derive the dynamical law

$$\frac{d\,\mathbf{L}}{d\,t} = \mathbf{N}.$$

## The Inertia Tensor of a Rigid Body

The rigid body that we will consider is the rectangular solid shown below.

Out[56]= Rectangular Solid




A rigid body is a system of particles satisfying the additional condition that for each mass we have

$$\mathbf{r}'(t) = \omega(t) \times \mathbf{r}(t).$$

The total angular momentum is then

$$\mathbf{L} = \sum_{i=1}^{N} m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i'(t) = \sum_{i=1}^{N} m_i\, \mathbf{r}_i(t) \times (\omega(t) \times \mathbf{r}_i(t)).$$

Given any list of masses $m_1, m_2, \ldots, m_N$ and any set of $N$ vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$, we can define a linear transformation

$$\mathbf{I} : \mathbb{R}^3 \to \mathbb{R}^3$$

by

$$\mathbf{I}(\mathbf{v}) = \sum_{i=1}^{N} m_i\, \mathbf{r}_i \times (\mathbf{v} \times \mathbf{r}_i).$$

This is the inertia tensor for the given masses and position vectors. If the position vectors are changing with time, so will this inertia tensor be a function of time. Therefore, for a moving rigid body, we have the inertia tensor

$$\mathbf{I}_t(\mathbf{v}) = \sum_{i=1}^{N} m_i\, \mathbf{r}_i(t) \times (\mathbf{v} \times \mathbf{r}_i(t)),$$

and the following relationship holds:

$$\mathbf{L}(t) = \mathbf{I}_t(\omega(t))$$

or, as it is often written,

$$\mathbf{L} = \mathbf{I} \cdot \omega.$$

So the total angular momentum of a rigid body is expressible in terms of the inertia tensor and the angular velocity vector.

The kinetic energy $E$ of the body is similarly expressible:

$$E = \frac{1}{2}\,\omega(t) \cdot \mathbf{I}_t(\omega(t)).$$

Because:

$$\begin{aligned}
\frac{1}{2}\,\omega(t) \cdot \mathbf{I}_t(\omega(t)) &= \frac{1}{2}\,\omega(t) \cdot \sum_{i=1}^{N} m_i\, \mathbf{r}_i(t) \times (\omega(t) \times \mathbf{r}_i(t)) \\
&= \frac{1}{2}\,\omega(t) \cdot \sum_{i=1}^{N} m_i\, \mathbf{r}_i(t) \times \mathbf{r}_i{}'(t) \\
&= \frac{1}{2} \sum_{i=1}^{N} m_i\, \omega(t) \cdot \mathbf{r}_i(t) \times \mathbf{r}_i{}'(t) \\
&= \frac{1}{2} \sum_{i=1}^{N} m_i\, \omega(t) \times \mathbf{r}_i(t) \cdot \mathbf{r}_i{}'(t) \\
&= \frac{1}{2} \sum_{i=1}^{N} m_i\, \mathbf{r}_i{}'(t) \cdot \mathbf{r}_i{}'(t) \\
&= E.
\end{aligned}$$

# Torque-Free Motion with One Point Fixed

If a rigid-body moves so that one point is fixed and there is <u>no</u> <u>external</u> <u>torque</u>, then

$$\frac{d\mathbf{L}}{dt} = \mathbf{0},$$

so that

$$\mathbf{L} = \mathbf{L}_0 = \text{constant} \quad \text{(Conservation of angular momentum)}$$

Then

$$\frac{d}{dt}\,(\mathbf{I} \cdot \omega) = \mathbf{0}.$$

# Equations of Motion in Body-Coordinates

A body-coordinate system is a right-handed Cartesian coordinate system that moves with the body.

Let $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ be the unit vectors for a body-coordinate system.

Components of the inertia tensor are constant when expressed in body-coordinates and the matrix is symmetric and positive definite.

Choose another set of body-axes so that $(I_{i\,j})$ is diagonal

$$(I_{i\,j}) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (I_1 > I_2 > I_3 > 0).$$

In terms of components in body coordinates, we have

$$\begin{aligned}\mathbf{0} = \frac{d}{dt}(\mathbf{I}_t \cdot \boldsymbol{\omega}(t)) &= \frac{d}{dt}\left(\sum_{i=1}^{3} I_i\ \omega_i(t)\,\mathbf{u}_i(t)\right) \\ &= \sum_{i=1}^{3} I_i\ \omega_i{}'(t)\,\mathbf{u}_i(t) + \sum_{i=1}^{3} I_i\ \omega_i(t)\,\mathbf{u}_i{}'(t) \\ &= \sum_{i=1}^{3} I_i\ \omega_i{}'(t)\,\mathbf{u}_i(t) + \sum_{i=1}^{3} I_i\ \omega_i(t)\,\boldsymbol{\omega}(t)\times\mathbf{u}_i(t).\end{aligned}$$

Now the second sum can be written as

$$\begin{aligned}\boldsymbol{\omega}(t)\times\sum_{i=1}^{3} I_i\ \omega_i(t)\,\mathbf{u}_i(t) &= (\omega_1(t)\,\mathbf{u}_1(t) + \omega_2(t)\,\mathbf{u}_2(t) + \omega_3(t)\,\mathbf{u}_3(t))\times(I_1\,\omega_1(t)\,\mathbf{u}_1(t) + I_2\,\omega_2(t)\,\mathbf{u}_2(t) + I_3\,\omega_3(t)\,\mathbf{u}_3(t)) \\ &= (I_3 - I_2)\,\omega_2\,\omega_3\,\mathbf{u}_1(t) + (I_1 - I_3)\,\omega_1\,\omega_3\,\mathbf{u}_2(t) + (I_2 - I_1)\,\omega_1\,\omega_2\,\mathbf{u}_3(t).\end{aligned}$$

Setting to zero each component of the above equation, we get Euler’s equations of torque-free rigid-body motion:

$$\begin{aligned} &I_1\,\omega_1{}' + (I_3 - I_2)\,\omega_2\,\omega_3 = 0 \\ &I_2\,\omega_2{}' + (I_1 - I_3)\,\omega_1\,\omega_3 = 0 \qquad (\textbf{Euler's equations}) \\ &I_3\,\omega_3{}' + (I_2 - I_1)\,\omega_1\,\omega_2 = 0.\end{aligned}$$

Euler’s equations comprise a system of three ODEs for the three unknowns $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$.

These functions are components of the angular velocity vector $\boldsymbol{\omega}(t)$ with respect to the body-fixed axes.

# Equations of Motion in Space Coordinates

To describe the motion in space, we need to know the rotation matrix $A(t)$ that rotates the body from its reference position to its position at time $t$. This can be done by solving the system of differential equations consisting of the three Euler equations and the equations

$$\mathbf{u}_i'(t) = \omega(t) \times \mathbf{u}_i(t) \qquad i = 1, 2, 3$$

If we use initial-conditions specifying that $\mathbf{u}_i(0) = \mathbf{e}_i$ (where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors of a spatially-fixed reference coordinate system), then we have

$$A(t) = [\mathbf{u}_1(t)\, \mathbf{u}_2(t)\, \mathbf{u}_3(t)].$$

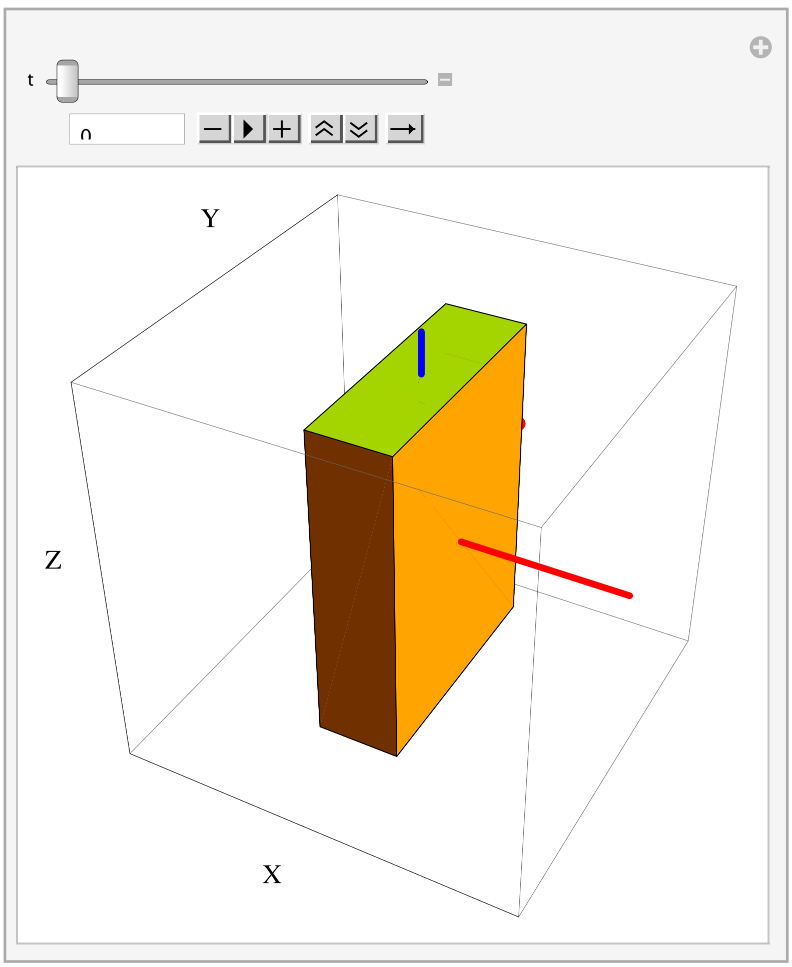
# Stationary Rotations

Eigenvectors of $(I_{i\,j})$ give solutions constant in body-coordinates. So an eigenvector for the eigenvalue $I_1$ is a constant solution

$$(\omega_1(t),\ \omega_2(t), \omega_3(t)) = (C_1, 0, 0).$$

- **A Stationary Rotation in Space**

Out[57]= A Stationary Rotation



# The Energy and Momentum Ellipsoids

Any solution of the Euler equations satisfies conservation of kinetic energy and angular momentum. That is, the quantities

$$I_1\,\omega_1{}^2 + I_2\,\omega_2{}^2 + I_3\,\omega_3{}^2$$

$$I_1{}^2\,\omega_1{}^2 + I_2{}^2\,\omega_2{}^2 + I_3{}^2\,\omega_3{}^2$$

are constant along any solution.

Pick an initial point satisfying

$$\mathcal{E}_T : I_1\,x^2 + I_2\,y^2 + I_3\,z^2 = 2\,E_0 \qquad (\text{KE } E_0)$$

Compute the momentum quantity for the chosen point and then our solution must lie in the intersection of the ellipsoid $\mathcal{E}_T$with the momentum ellipsoid

$$\mathcal{E}_M : I_1{}^2\,\omega_1{}^2 + I_2{}^2\,\omega_2{}^2 + I_3{}^2\,\omega_3{}^2 = L_0{}^2$$

- **Energy and Momentum Ellipsoids**

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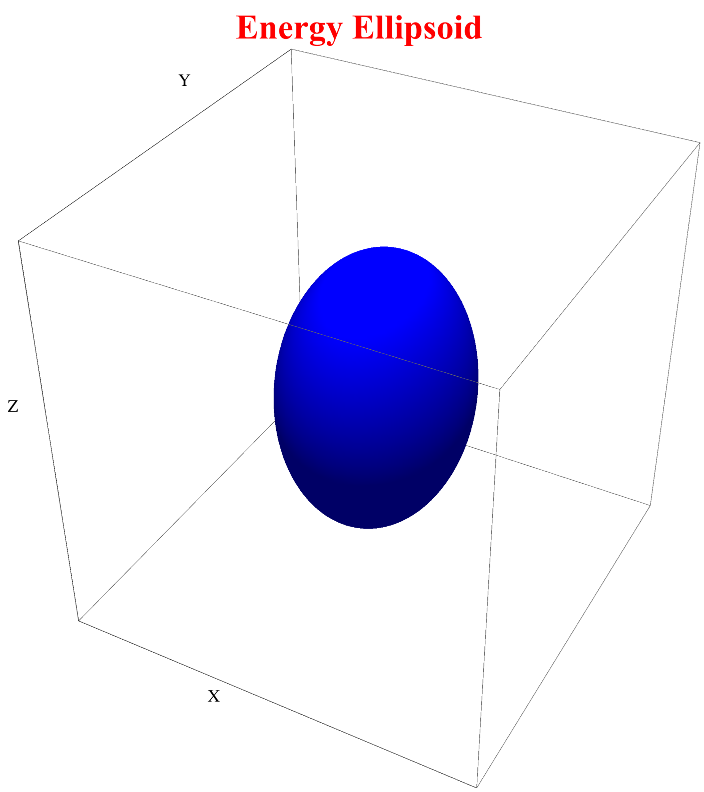

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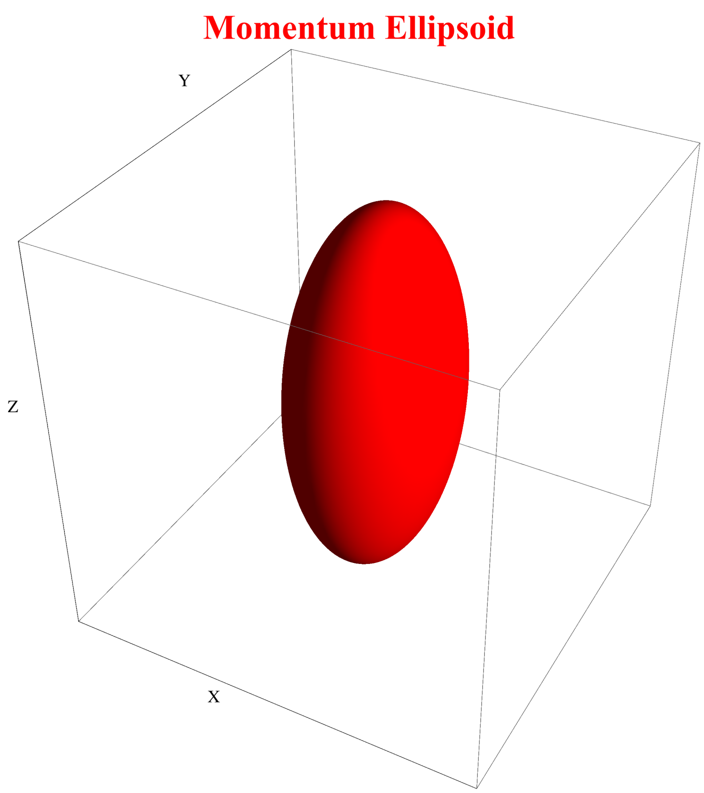


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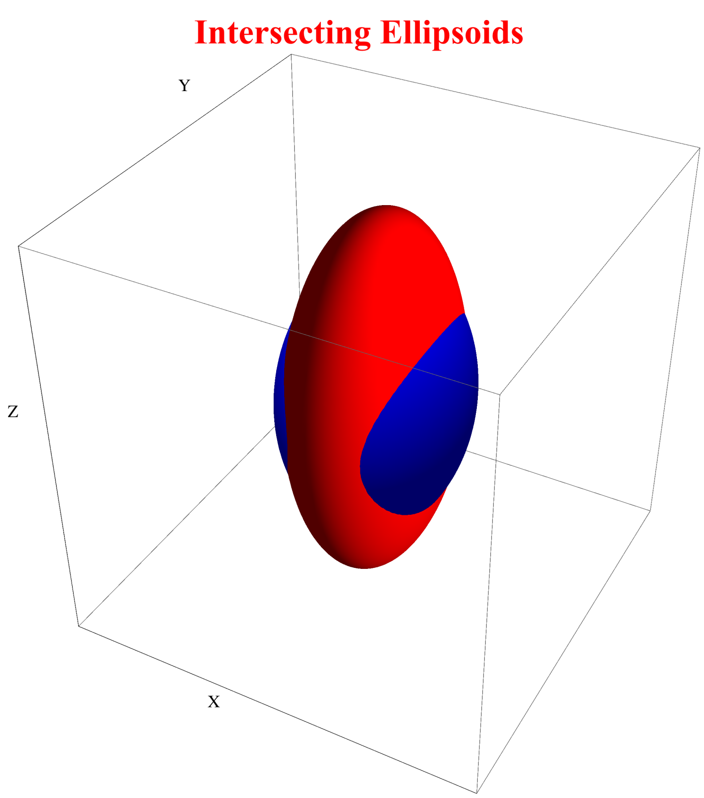
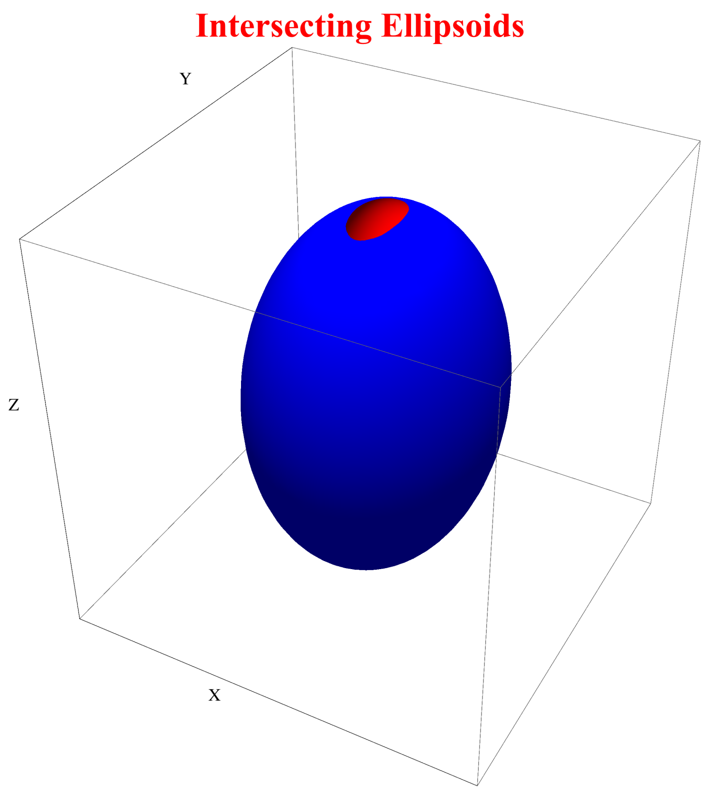


# Solution Initially near the Long Axis

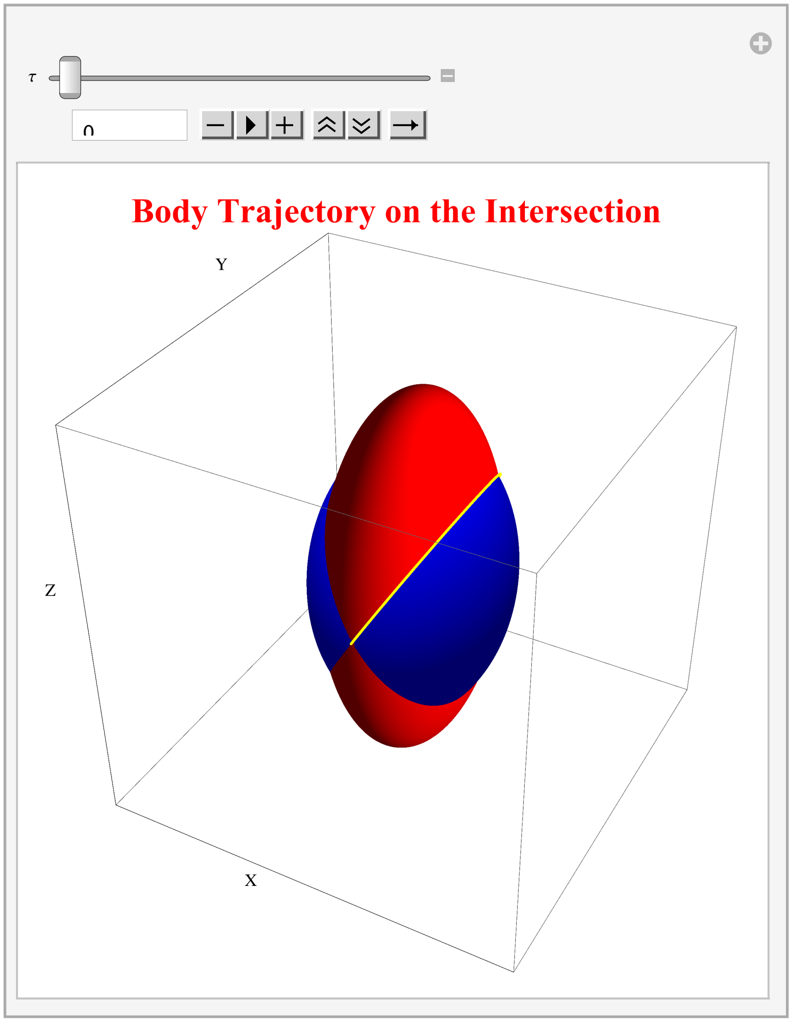
### ■ Energy and Momentum Ellipsoids

Out[75]= Intersecting Ellipsoids



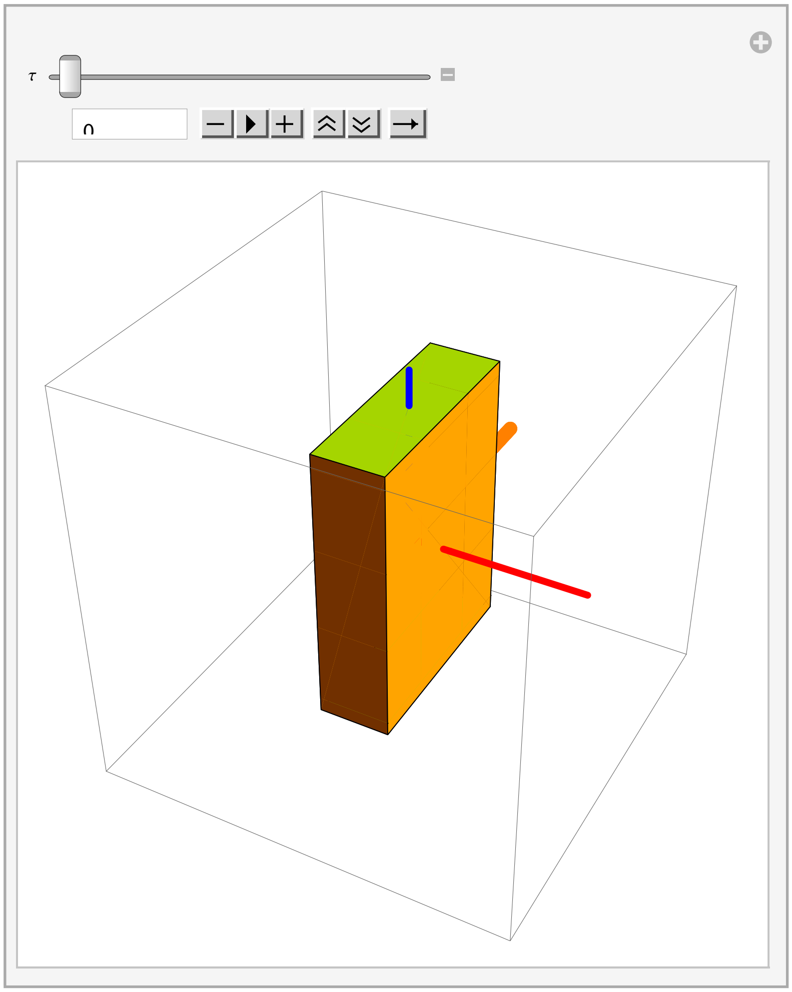

### ■ Motion in the Body Reference Frame

Out[76]= Motion in the Body Frame




- **Motion in the Spatial Reference Frame**

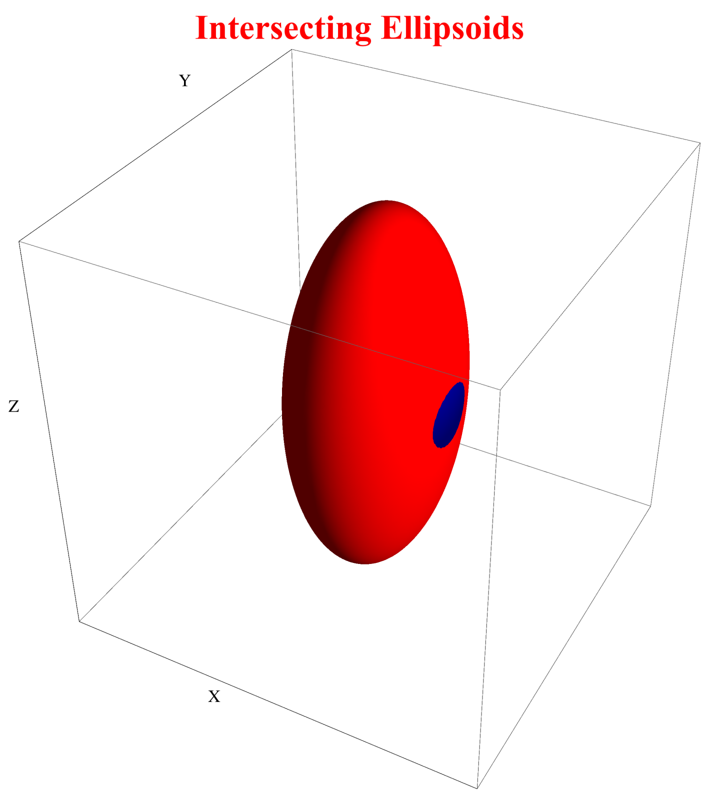
Out[77]= Motion in Space




# Solution Initially near the Short Axis

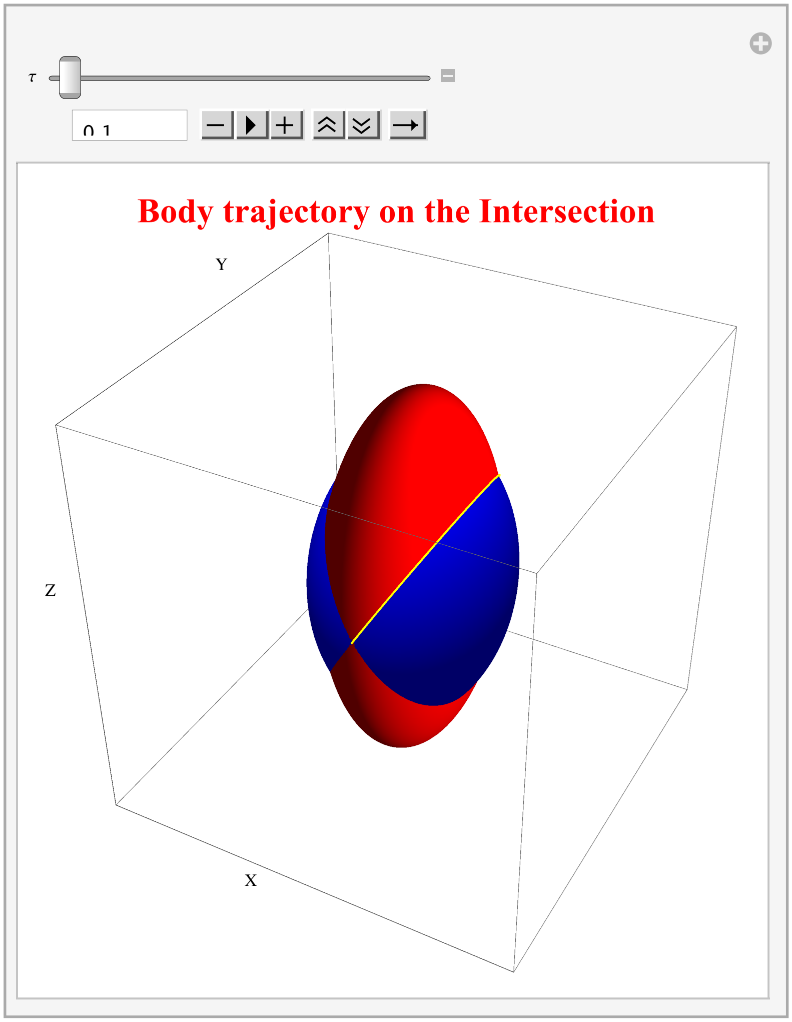
- **Energy and Momentum Ellipsoids**

Out[78]= Intersecting Ellipsoids




## ■ Motion in the Body Reference Frame

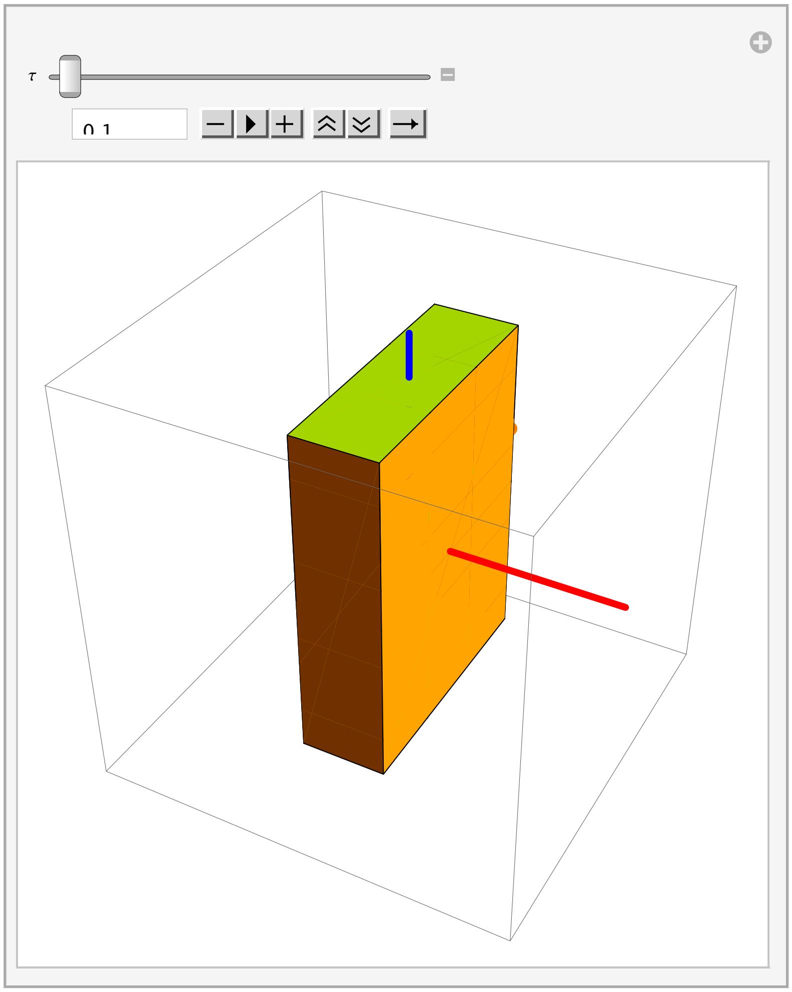
Out[79]= Motion in the Body Frame




In[80]:=

- **Motion in the Spatial Reference Frame**

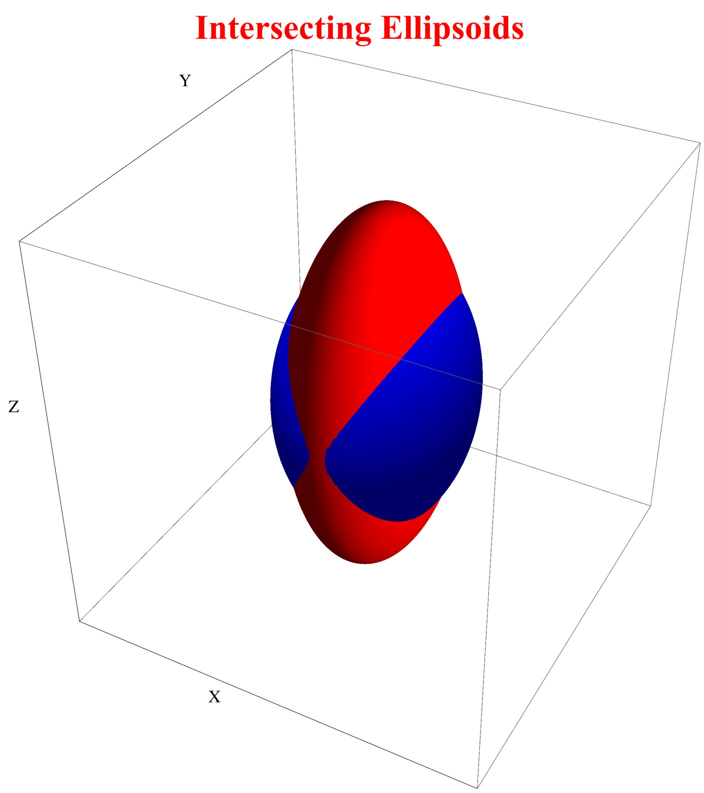
Out[81]= Motion in Space




## Solution Starting Near the Intersecting Axis

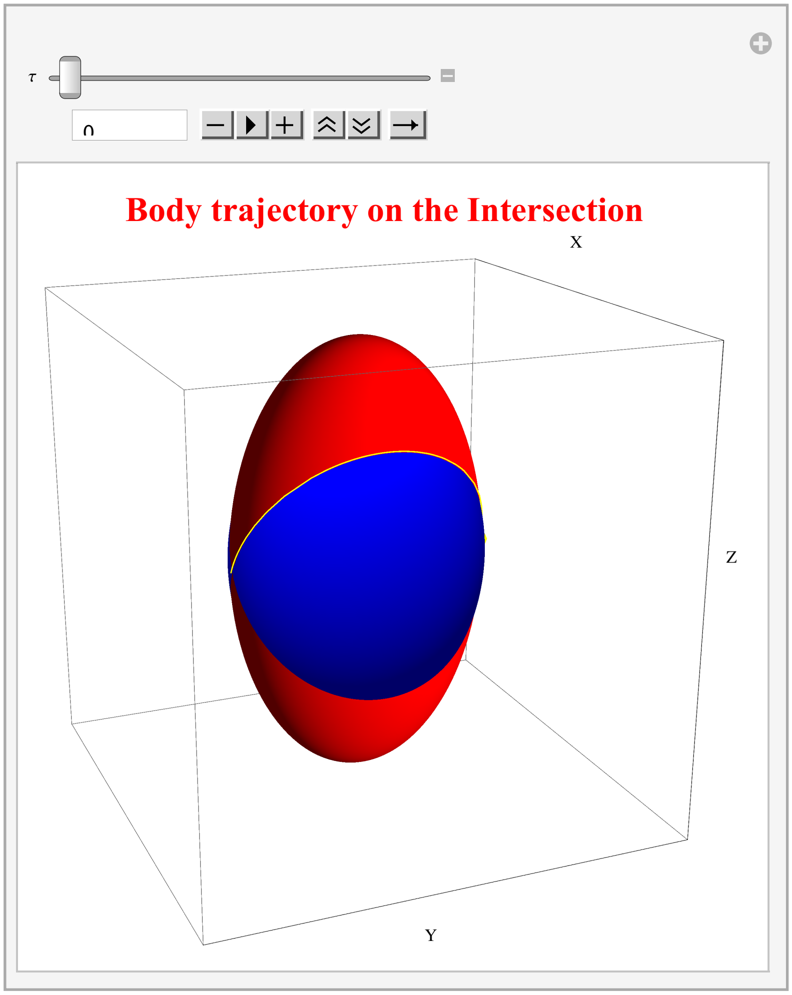
### ■ Energy and Momentum Ellipsoids

Out[82]= Intersecting Ellipsoids



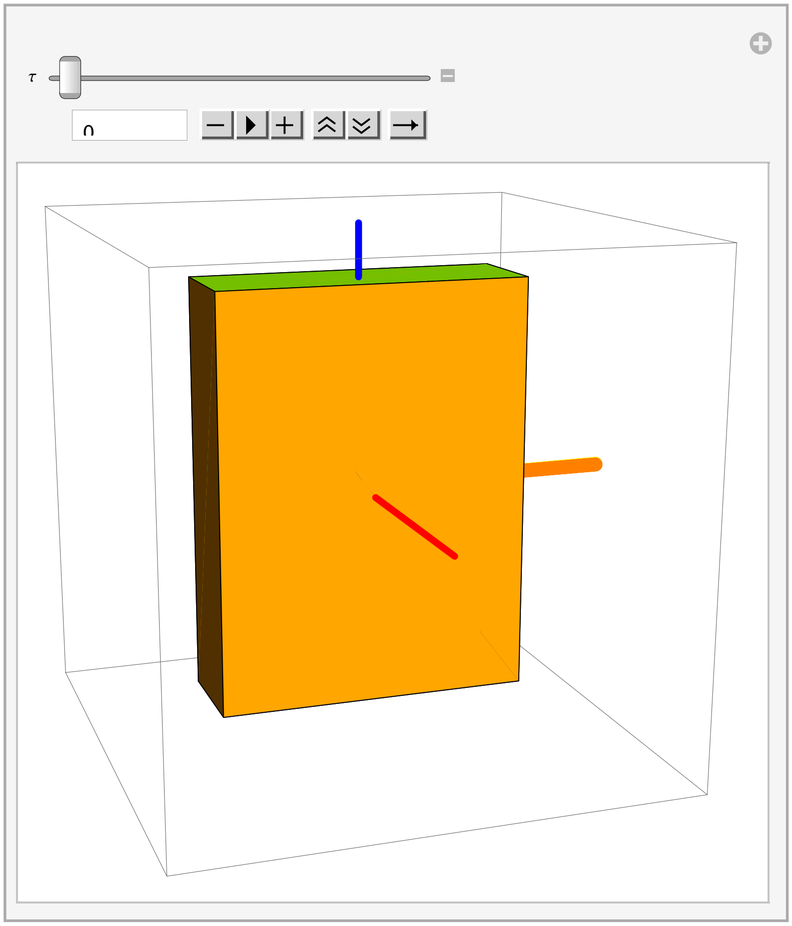

- **Motion in the Body Reference Frame**

Out[83]= Motion in the Body Frame




### ■ Motion in the Spatial Reference Frame

Out[84]= Motion in Space

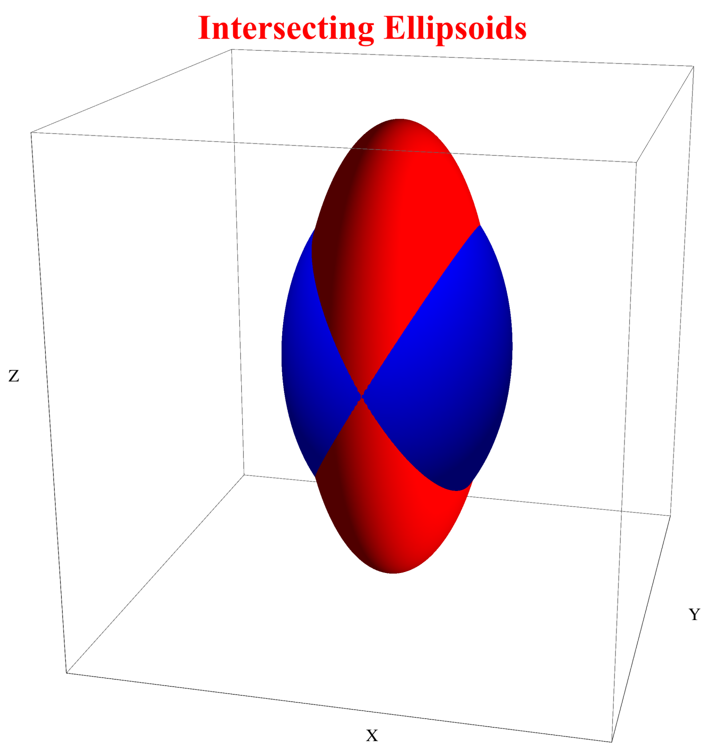



# A Heteroclinic Solution

If we choose the initial value $\omega(0)$ on the intermediate body axis, then the intersection of the ellipsoids will contain that point. There are then four solutions of the Euler equations whose trajectories join $\omega(0)$ to $-\omega(0)$. If we choose $\omega(0)$ to be on one of these intersection curves and very near the intermediate axis we get a solution the approaches one critical point as $t \to -\infty$ and the one on the opposite side of the ellipsoids as $t \to \infty$. Such a solution is called heteroclinic.

- **Energy and Momentum Ellipsoids**

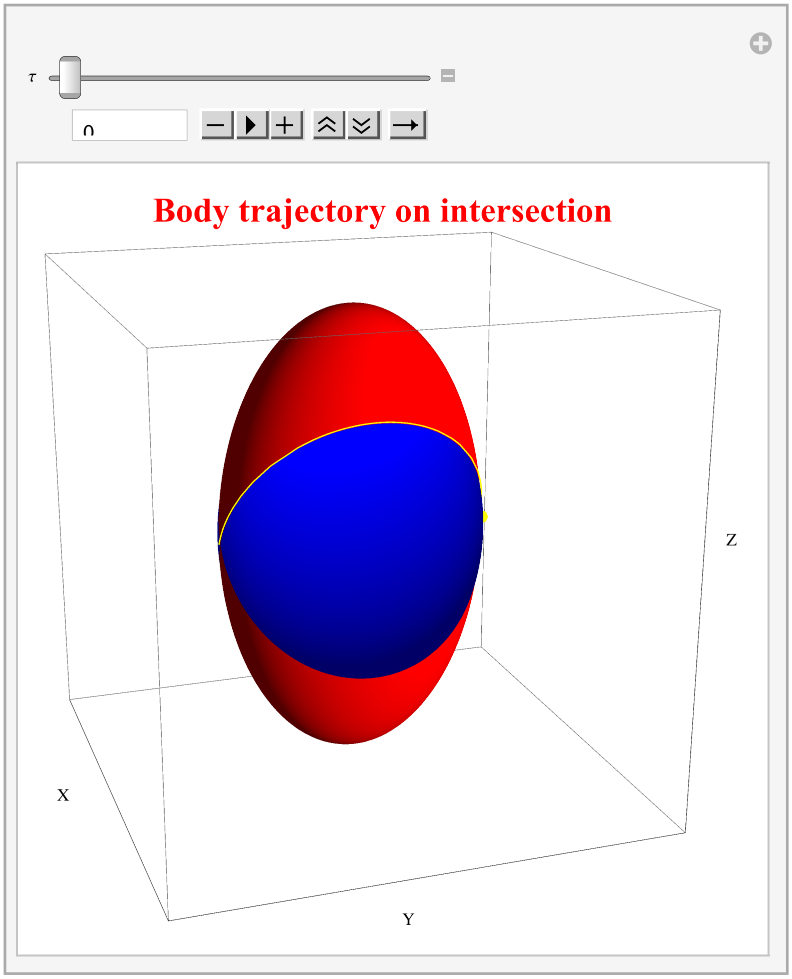
Out[85]= Intersecting Ellipsoids




- **Motion in the Body Reference Frame**

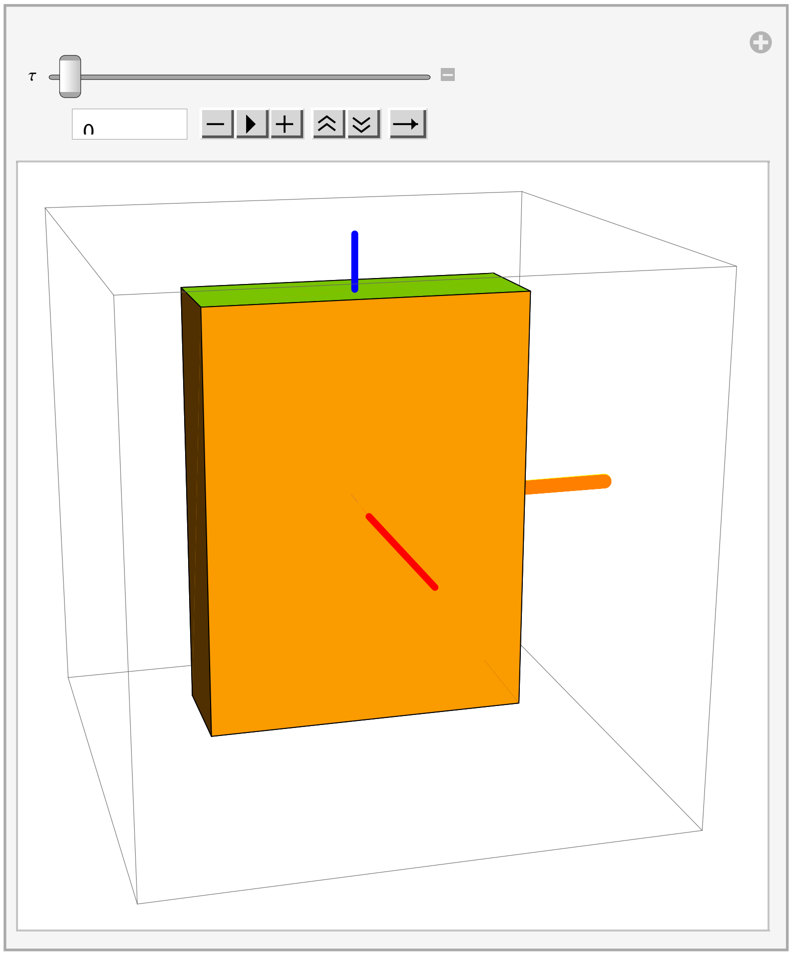
Take $\omega_0$ on the intersection; Then solution approaches stationary rotation about the intermediate axis.

Out[86]= Motion in the Body Frame




### ■ Motion in the Spatial Reference Frame

Out[87]= Motion in Space


τ
0