The Bare-Bones, Minimal, Abbreviated and All-Around Shortest Discussion of Tensors Needed to Understand Curvature on a Semi-Riemannian Manifold

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February 12, 2017

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Tensor Notation

Since they appear in every calculation and *almost* every page in books on General Relativity, we need to become comfortable with mathematical entities called tensors and the standard operations on them. These things looks like:

$$X^i \quad \text{or} \quad \Lambda_\alpha \quad \text{or} \quad g_{\alpha\,\beta} \quad \text{or} \quad R^\alpha{}_{\beta\,\gamma\,\delta}$$

$$\text{or} \qquad R^\alpha{}_{\beta\,\alpha\,\delta} \to R_{\beta\,\delta}$$

$$\text{or} \quad g^{\alpha\,\beta}\, R_{\beta\,\delta} \to R^\alpha{}_\delta$$

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What does "handling" mean?

Except in carefully arranged cases these calculations are impossible to perform in a timely fashion by a human.

So "handling" definitely does not mean taking a general tensor and writing down all the coordinates in a basis.

"Handling" means knowing what is going on in simple cases and understanding the basic properties of tensors.

. . . and being aware of that which is **not** written, understood by all as background information: part of the "set-up".

Let's begin . . .

$$X^i \quad \text{or} \quad X^\alpha$$

Notice that the index is "high" as if it were an exponent. It is not. These are coordinates of a vector in a specific ordered basis of some agreed-upon underlying vector space V. These are called contravectors, or simply vectors.

The basis is not mentioned, so these are really real-valued *functions* whose domain is the set of all ordered bases of the vector space V: a different list of coordinates for each basis.

Following the standard convention of Linear Algebra, these correspond to the entries of a column (that is, $n \times 1$) matrix. You tell me the basis, I tell you the matrix.

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$$\Lambda_\alpha$$

Notice here that the index is "low". These are coordinates in a specific ordered basis of a member of the dual of our underlying vector space. These are called covectors and the dual vector space is denoted V^*.

Recall that members of the dual are linear functionals: linear maps from the underlying vector space V to the real numbers.

Following the standard convention of Linear Algebra, these correspond to the entries of a row (that is, $1 \times n$) matrix.

Evaluating a functional on a vector corresponds to left matrix multiplication by the functional representative of the vector representative. $1 \times n$ times $n \times 1$ is 1×1, a real number.

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The underlying vector space we have in mind will have dimension 4 and any subscripts or superscripts you see will generally be assumed to run over four possible values, independently of the values assigned to different subscripts or superscripts.

So $R^\alpha{}_{\beta\gamma\delta}$ will refer to $4^4 = 256$ different numbers.

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Usually Mathematicians would follow something like X^i with "$i = 1, \ldots, 4$" to indicate the range of index values. . .

. . . but one typical Physics convention, which we will occasionally use, is to let a lower case Greek index such as α denote an integer $0, 1, 2$ or 3 while a lower case Latin index such as i denotes one of $1, 2$ or 3.

The reason for this comes from the intended meaning of the coordinates. X^0 will often correspond to something like time, while the last three coordinates have more to do with space, and it shortens some formulae if there is a notation to reflect this.

THE SUMMATION CONVENTION

If you see two of these objects ***juxtaposed*** with ***different indices*** such as

$$X^\alpha \Lambda_\beta$$

the intent is that these numbers are to be multiplied. In our case this represents 16 different numbers.

$X^\gamma \Lambda_\delta$ is the same 16 numbers: new unused index symbols can be substituted without changing the meaning of the expression.

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If you see juxtaposed terms with ***repeated indices, exactly once "high" and exactly once "low"*** such as

$$X^\alpha \Lambda_\alpha$$

the intent is that these numbers are to be summed (a sum of products) over the range of the repeated index values.

The sum shown here is the result of causing the covector Λ to act on the vector X and is *physical.* The result cannot depend on choice of basis, and that is what we mean when we call *anything* "invariant."

For instance if Λ is a (constant-nothing dynamic here!) electric field in ordinary three dimensional space and X is a displacement vector the number ΛX is the electric potential change experienced during the displacement.

$Volts * m^{-1} * m = Volts$. The number $\Lambda_i X^i$ must be invariant.

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If there are unrepeated indices as in

$$X^\alpha g_{\alpha\,\beta}$$

the intent is to indicate four different sums, one for each value of β.

$$X^\alpha g_{\alpha\,\alpha} \quad \text{and} \quad X^\alpha X^\alpha$$

are undefined. This restriction will help you avoid or *recognize* mistakes. The usual summation with "Σ" must be used for sums of products that look like this.

They virtually never occur.

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The Encyclopedia Brittanica article on Tensors refers to Tensor Notation as an invention of "almost magical efficiency."

Used properly the notation itself will guide you toward the proper calculation and make most common mistakes obvious.

The rule:

Once high and once low in a product indicates summation.

A Few Words About What is Missing

We will do calculations using convenient coordinates, but one must keep straight what the coordinates represent.

Numerical coordinates are (practically) without physical meaning unless an ordered basis is specified!!! The basis "attaches" the numbers to reality.

If X is a vector in the underlying vector space V and $\mathcal{A} = \{ a_0, a_1, a_2, a_3 \}$ is an ordered basis of V then

$$X = X^\alpha a_\alpha$$

for certain real numbers X^α.

These numbers obviously depend on $\mathcal{A}$ so in the interests of precision if not clarity we may write $X^\alpha(\mathcal{A})$ to remind us of the basis-dependence of these four numbers.

Define the standard ordered basis $\mathcal{E}$ of $\mathbb{R}^4$ to have αth basis member e_α which is (and will always be) the unit column vector with the number one α down from the top.

The coordinate vector for X in the basis is

$$[X]_{\mathcal{A}} = X^\alpha(\mathcal{A}) e_\alpha$$

which is the column vector in $\mathbb{R}^4$ we mentioned before.

It is convenient to think of $[X]_{\mathcal{A}}$ as a complete description of X in "language" $\mathcal{A}$.

The actual vector is $X = X^\alpha(\mathcal{A}) a_\alpha$.

Its representative in $\mathbb{R}^4$ ***in this basis*** is

$$[X]_{\mathcal{A}} = X^\alpha(\mathcal{A}) e_\alpha.$$

Suppose $\mathcal{A}^* = \{ a^0, a^1, a^2, a^3 \}$ is the ordered basis of V^* dual to $\mathcal{A}$, which is **defined** by the condition

$$a^\alpha a_\beta = \delta^\alpha_\beta$$

where δ^α_β is the "Kronecker delta" symbol.

The functional a^α "picks off" the coefficient on a_α when you represent a vector such as X in terms of basis $\mathcal{A}$.

For example $\mathcal{E}^*$ is (identified with) the ordered basis containing the row matrices e^α, which have one α spaces from the left and zeroes elsewhere.

$$e^\alpha X^\beta e_\beta = X^\alpha.$$

Our description in what follows requires us to decide in advance which underlying vector space is to provide the vectors and which will provide the covectors. Both choices are legitimate—the dual of the dual can be identified with the original space—but we must *choose.*

The only effect of a different choice would be to switch which indices are "high" and which are "low."

If Λ is a covector, a member of the dual vector space, then there are constants $\Lambda_\alpha(\mathcal{A})$ for which

$$\Lambda = \Lambda_\alpha(\mathcal{A})\boldsymbol{a}^{\boldsymbol{\alpha}} = \Lambda(\boldsymbol{a}_{\boldsymbol{\alpha}})\boldsymbol{a}^{\boldsymbol{\alpha}}.$$

We then define the row matrix

$$[\Lambda]_{\mathcal{A}} = \Lambda_\alpha(\mathcal{A})\boldsymbol{e}^{\boldsymbol{\alpha}} = \left(\Lambda(\boldsymbol{a}_0)\ \Lambda(\boldsymbol{a}_1)\ \Lambda(\boldsymbol{a}_2)\ \Lambda(\boldsymbol{a}_3) \right)$$

which is the matrix that represents Λ when we start with basis $\mathcal{A}$ on our original space of vectors.

We think of $[\Lambda]_{\mathcal{A}}$ as a complete description of covector Λ when members of the original vector space are understood to be described using "language" $\mathcal{A}$.

Changing Bases

Suppose we change bases from basis $\mathcal{A}$ to basis $\mathcal{B}$ on our original vector space. We need to know how this affects coordinates of vectors and covectors.

Define the matrix of transition from basis $\mathcal{A}$ to basis $\mathcal{B}$ to be the square matrix

$$P_{\mathcal{B}\leftarrow\mathcal{A}} = \Big([\mathbf{a}_1]_{\mathcal{B}}\ \ldots [\mathbf{a}_n]_{\mathcal{B}} \Big).$$

A calculation now shows that for any vector X

$$[X]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}\,[X]_{\mathcal{A}}\,.$$

In other words, left multiplication by the matrix $P_{\mathcal{B}\leftarrow\mathcal{A}}$ is the translator from language $\mathcal{A}$ to language $\mathcal{B}$.

And it follows easily that $P_{\mathcal{A}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{A}})^{-1}$.

The situation is very similar for covectors. If Λ is a covector and X *any* vector the number ΛX cannot depend on basis. Therefore

$$\Lambda\, X = [\Lambda]_{\mathcal{A}}\,[X]_{\mathcal{A}} = [\Lambda]_{\mathcal{B}}\,[X]_{\mathcal{B}} = [\Lambda]_{\mathcal{B}}\,P_{\mathcal{B}\leftarrow\mathcal{A}}\,[X]_{\mathcal{A}}\,.$$

It now follows that $[\Lambda]_{\mathcal{B}} = [\Lambda]_{\mathcal{A}}\,P_{\mathcal{A}\leftarrow\mathcal{B}}$.

In other words, right multiplication by the matrix $P_{\mathcal{A}\leftarrow\mathcal{B}}$ is the translator from language $\mathcal{A}$ to language $\mathcal{B}$ for coordinates of covectors.

An internal narrative might go as follows. $[\Lambda]_{\mathcal{B}}$ is the row matrix that calculates ΛX using the description of X in the $\mathcal{B}$ language. Reading from right to left, this number can be calculated by translating the coordinate vector $[X]_{\mathcal{B}}$ to the $\mathcal{A}$ language, and then using $[\Lambda]_{\mathcal{A}}$, which understands $\mathcal{A}$-speak.

The Simplest Tensors

We need a concise way of referring to the entries of the matrix of transition $P_{\mathcal{B}\leftarrow\mathcal{A}}$.

We will refer row α column β entry of $P_{\mathcal{B}\leftarrow\mathcal{A}}$ by $\frac{d\mathcal{B}^\alpha}{d\mathcal{A}^\beta}$.

Since $P_{\mathcal{A}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{A}}$ is the identity matrix we have $\frac{d\mathcal{A}^\alpha}{d\mathcal{B}^\gamma}\frac{d\mathcal{B}^\gamma}{d\mathcal{A}^\beta} = \delta^\alpha_\beta$.

And $X^\alpha(\mathcal{B}) = \frac{d\mathcal{B}^\alpha}{d\mathcal{A}^\beta} X^\beta(\mathcal{A})$.
$\left(\text{new vector coordinates} = \frac{d\,\text{new coordinates}}{d\,\text{old coordinates}}\ \text{old vector coordinates}\right)$

And $\Lambda_\alpha(\mathcal{B}) = \Lambda_\gamma(\mathcal{A})\frac{d\mathcal{A}^\gamma}{d\mathcal{B}^\alpha}$.

So

$$\Lambda X = \Lambda_\alpha(\mathcal{B})X^\alpha(\mathcal{B}) = \Lambda_\gamma(\mathcal{A})\frac{d\mathcal{A}^\gamma}{d\mathcal{B}^\alpha}\frac{d\mathcal{B}^\alpha}{d\mathcal{A}^\beta}X^\beta(\mathcal{A})$$
$$= \Lambda_\gamma(\mathcal{A})\,\delta^\gamma_\beta\,X^\beta(\mathcal{A}) = \Lambda_\gamma(\mathcal{A})\,X^\gamma(\mathcal{A}).$$

Example in $\mathbb{R}^2$

Suppose given vector v with coordinate vector $\begin{pmatrix}1\\2\end{pmatrix} = [v]_\mathcal{A}$.

Suppose we have a second basis both $\mathcal{B}$ with each vector one-fifth as long as the corresponding vector from $\mathcal{A}$. So $5\,\boldsymbol{b_1} = \boldsymbol{a_1}$ and $5\,\boldsymbol{b_2} = \boldsymbol{a_2}$.

Since the basis vectors are shorter, coordinates in the new basis will all be 5 times bigger.

$$\begin{pmatrix}5\\10\end{pmatrix} = [v]_\mathcal{B} = \frac{d\,\text{new coordinates}}{d\,\text{old coordinates}}[v]_\mathcal{A} = \begin{pmatrix}5&0\\0&5\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}.$$

Coordinate Changes for Tensors of Higher Order

So how do more complicated tensors, such as $g_{\alpha\,\beta}$ and $R^\alpha{}_{\beta\,\gamma\,\delta}$ change coordinates?

Just like the simpler coordinates do!

Specifically, each "high" index will correspond to a $P_{\mathcal{B}\leftarrow\mathcal{A}} = \frac{d\mathcal{B}^\alpha}{d\mathcal{A}^\beta}$ factor, and each "low" index induces a $P_{\mathcal{A}\leftarrow\mathcal{B}} = \frac{d\mathcal{A}^\alpha}{d\mathcal{B}^\beta}$ factor.

$$g_{\alpha\,\beta}(\mathcal{B}) = g_{\gamma\,\delta}(\mathcal{A})\frac{d\mathcal{A}^\gamma}{d\mathcal{B}^\alpha}\frac{d\mathcal{A}^\delta}{d\mathcal{B}^\beta}$$

$$R^\alpha{}_{\beta\,\gamma\,\delta}(\mathcal{B})$$
$$= R^\tau{}_{\zeta\,\sigma\,\mu}(\mathcal{A})\frac{d\mathcal{B}^\alpha}{d\mathcal{A}^\tau}\frac{d\mathcal{A}^\zeta}{d\mathcal{B}^\beta}\frac{d\mathcal{A}^\sigma}{d\mathcal{B}^\gamma}\frac{d\mathcal{A}^\mu}{d\mathcal{B}^\delta}$$

Tensors of Higher Order

Now let's get back to $g_{\alpha\,\beta}$ and $R^{\alpha}{}_{\beta\,\gamma\,\delta}$ in a **fixed** basis $\mathcal{A}$.

In this basis these numbers are the coefficients from

$$g_{\alpha\,\beta}\, \boldsymbol{a}^{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}}$$

and

$$R^{\alpha}{}_{\beta\,\gamma\,\delta}\, \boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\gamma}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}}.$$

(Note that basis dependence is missing from the coefficients. It is "understood" unless you are changing coordinates.)

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But what do $\boldsymbol{a}^{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}}$ and $\boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\gamma}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}}$ mean?

They are real-valued functions from products of V and V^* in an order specified by the left-to-right position of the index in the coefficients (each has its own column there) and the consequent order in the tensor product of basis vectors.

Specifically, $\boldsymbol{a}^{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \colon V \times V \to \mathbb{R}$ is given by

$$\boldsymbol{a}^{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}}(X, Y) = \boldsymbol{a}^{\boldsymbol{\alpha}}(X)\, \boldsymbol{a}^{\boldsymbol{\beta}}(Y) = X^{\alpha} Y^{\beta}$$

where X^{α} and Y^{β} are coordinates of X and Y in the $\mathcal{A}$ basis.

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And $\boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\gamma}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}} \colon V^* \times V \times V \times V \to \mathbb{R}$ is given by

$$\begin{aligned} &\boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\gamma}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}}(\Lambda, X, Y, Z) \\ &\quad = \boldsymbol{a}_{\boldsymbol{\alpha}}(\Lambda)\, \boldsymbol{a}^{\boldsymbol{\beta}}(X)\, \boldsymbol{a}^{\boldsymbol{\gamma}}(Y)\, \boldsymbol{a}^{\boldsymbol{\delta}}(Z) = \Lambda_{\alpha} X^{\beta} Y^{\gamma} Z^{\delta}. \end{aligned}$$

These functions are linear in each "slot" separately: fix a vector or covector in each "slot" but one and the result is linear.

So any linear combination of tensors (with the same domain) is linear in each slot.

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Alternating and Symmetric Tensors

A tensor is called covariant if all its indices are low. It is called contravariant if all indices are high.

A covariant or contravariant tensor is called symmetric if switching any two index values in a given basis leaves the coefficient value unchanged. If this is true in one basis, it is true in *any* basis.

A covariant or contravariant tensor is called alternating if switching any two index values in a given basis introduces a minus sign in the coefficient value, but otherwise leaves the coefficient unchanged. If this is true in one basis, it is true in *any* basis.

Suppose θ is a covariant tensor with L indices—these are called L-forms— and $\mathcal{A}$ is a basis.

$\mathcal{P}_L$ is the group of all permutations on the set $\{1, \ldots, L\}$.

$$\begin{aligned} Alt(\theta) &= \frac{1}{L!} \sum_{Q \in \mathcal{P}_L} sgn(Q)\, \theta_{i_1,\ldots,i_L}\, \boldsymbol{a}^{\boldsymbol{i}_{Q(1)}} \otimes \cdots \otimes \boldsymbol{a}^{\boldsymbol{i}_{Q(L)}} \\ &= \frac{1}{L!} \sum_{Q \in \mathcal{P}_L} sgn(Q)\, \theta_{i_{Q(1)},\ldots,i_{Q(L)}}\, \boldsymbol{a}^{\boldsymbol{i}_1} \otimes \cdots \otimes \boldsymbol{a}^{\boldsymbol{i}_L}. \end{aligned}$$

(Sum on an index pair i_m and $i_{Q(q)}$ whenever $m = Q(q)$.)

$$\begin{aligned} Sym(\theta) &= \frac{1}{L!} \sum_{Q \in \mathcal{P}_L} \theta_{i_1,\ldots,i_L}\, \boldsymbol{a}^{\boldsymbol{i}_{Q(1)}} \otimes \cdots \otimes \boldsymbol{a}^{\boldsymbol{i}_{Q(L)}} \\ &= \frac{1}{L!} \sum_{Q \in \mathcal{P}_L} \theta_{i_{Q(1)},\ldots,i_{Q(L)}}\, \boldsymbol{a}^{\boldsymbol{i}_1} \otimes \cdots \otimes \boldsymbol{a}^{\boldsymbol{i}_L}. \end{aligned}$$

(Sum on an index pair i_m and $i_{Q(q)}$ whenever $m = Q(q)$.)

$Alt(\theta) = \theta$ if and only if θ is alternating, while $Sym(\theta) = \theta$ if and only if θ is symmetric.

EXAMPLE

Suppose $\theta = \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{2}}$ and $\sigma = \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{3}}$.

$$Alt(\theta) = \frac{1}{2}(\, \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{2}} - \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{1}}\,)$$

Three-index tensors generate six terms for each original term:

$$\begin{aligned} Alt(\sigma) = \frac{1}{6}(\, &\boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{3}} - \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{3}} \\ &- \boldsymbol{a}^{\boldsymbol{3}} \otimes \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{1}} + \boldsymbol{a}^{\boldsymbol{2}} \otimes \boldsymbol{a}^{\boldsymbol{3}} \otimes \boldsymbol{a}^{\boldsymbol{1}} \\ &- \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{3}} \otimes \boldsymbol{a}^{\boldsymbol{2}} + \boldsymbol{a}^{\boldsymbol{3}} \otimes \boldsymbol{a}^{\boldsymbol{1}} \otimes \boldsymbol{a}^{\boldsymbol{2}}\,) \end{aligned}$$

CONTRACTION, OR THE TRACE OPERATION

The contraction or trace operation is a means of reducing the rank of a tensor by two indices, one contravariant and one covariant index. *The result of doing this also has tensor character.*

For instance

$$R^{\alpha}{}_{\beta\,\gamma\,\delta}\, \boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\gamma}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}} : V^* \times V \times V \times V \to \mathbb{R}$$

can be converted to

$$R_{\beta\,\delta}\, \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}^{\boldsymbol{\delta}} : V \times V \to \mathbb{R}$$

where $R_{\beta\,\delta}$ is the sum $R^{\alpha}{}_{\beta\,\alpha\,\delta}$.

This example will be important to us. If the original tensor is the Riemann curvature tensor then this contraction is the Ricci curvature tensor. Einstein's field equations are expressed using the Ricci curvature.

Raising and Lowering Indices

Indices are raised or lowered by specifying a metric tensor by analogy with the way functionals on Euclidean space are represented by vectors acting via the dot product, or how bras and kets from QM can be transformed into each other using the inner product on the state space.

A metric in this context is a symmetric nondegenerate bilinear form defined, for us, on a 4 dimensional vector space. In basis $\mathcal{A}$ a metric g can be represented as

$$g \;=\; g_{\alpha\,\beta}\, \boldsymbol{a^{\alpha}} \otimes \boldsymbol{a^{\beta}}.$$

Symmetry of $g_{\alpha\,\beta}\, \boldsymbol{a^{\alpha}} \otimes \boldsymbol{a^{\beta}}$ means here that $g_{\alpha\,\beta} = g_{\beta\,\alpha}$ and nondegeneracy means that, as a matrix, $(\,g_{\alpha\,\beta}\,)$ is invertible.

The inverse matrix $(\,g_{\alpha\,\beta}\,)^{-1}$ is also symmetric, and the entries of the inverse matrix are denoted $g^{\alpha\,\beta}$.

So $g^{\alpha\,\gamma}\, g_{\gamma\,\beta} = \delta^{\alpha}_{\beta}$.

$$g^{\star} \;=\; g^{\alpha\,\beta}\, \boldsymbol{a_{\alpha}} \otimes \boldsymbol{a_{\beta}}$$

is also a tensor, said to be conjugate or dual to the metric tensor. It is an inner product on V^*.

Indices are raised or lowered using these two tensors.

Suppose given covector and vector

$$\Lambda_{\alpha}\, \boldsymbol{a^{\alpha}} \qquad \text{and} \qquad X^{\alpha}\, \boldsymbol{a_{\alpha}}.$$

The “raised index version” of Λ has coordinates

$$\Lambda^{\alpha} = \Lambda_{\beta}\, g^{\alpha\,\beta}$$

which corresponds to the vector $\Lambda^{\alpha}\, \boldsymbol{a_{\alpha}}$.

Similarly the “lowered index version” of X has coordinates

$$X_{\alpha} = X^{\beta}\, g_{\alpha\,\beta}$$

corresponding to the covector $X_{\alpha}\, \boldsymbol{a^{\alpha}}$.

These procedures are inverse to each other: lowering followed by raising has no net effect.

Let’s look at a 4 dimensional vector space V and suppose we have a metric tensor g given in basis $\mathcal{A}$ by

$$g = -\,\boldsymbol{a^0} \otimes \boldsymbol{a^0} + \boldsymbol{a^1} \otimes \boldsymbol{a^1} + \boldsymbol{a^2} \otimes \boldsymbol{a^2} + \boldsymbol{a^3} \otimes \boldsymbol{a^3}.$$

This is a very atypical basis since twelve of the sixteen possible $g_{\alpha\,\beta}$ values are 0. Thought of as the entries in a matrix, the nonzero coefficients are ± 1 and on the diagonal.

If X and Y are vectors, then using coordinates in this basis

$$g(X, Y) = -X^0 Y^0 + X^1 Y^1 + X^2 Y^2 + X^3 Y^3.$$

The conjugate metric tensor has coordinates which are numerically the same in this unusual basis: i.e. $g^{\alpha\beta} = g_{\alpha\beta}$. But the tensor to which these coordinates refer is

$$g^\star = -\boldsymbol{a_0} \otimes \boldsymbol{a_0} + \boldsymbol{a_1} \otimes \boldsymbol{a_1} + \boldsymbol{a_2} \otimes \boldsymbol{a_2} + \boldsymbol{a_3} \otimes \boldsymbol{a_3}.$$

If Λ and Ψ are covectors, then using coordinates in this basis

$$g^*(\Lambda, \Psi) = -\Lambda_0\Psi_0 + \Lambda_1\Psi_1 + \Lambda_2\Psi_2 + \Lambda_3\Psi_3.$$

Let's lower the index of vector X in this basis.

$X_\alpha = g_{\alpha\beta} X^\beta$ means vector

$$X = X^0\boldsymbol{a_0} + X^1\boldsymbol{a_1} + X^2\boldsymbol{a_2} + X^3\boldsymbol{a_3}$$

has been converted into covector

$$\begin{aligned} X_\alpha \boldsymbol{a^\alpha} &= X_0\boldsymbol{a^0} + X_1\boldsymbol{a^1} + X_2\boldsymbol{a^2} + X_3\boldsymbol{a^3} \\ &= -X^0\boldsymbol{a^0} + X^1\boldsymbol{a^1} + X^2\boldsymbol{a^2} + X^3\boldsymbol{a^3}. \end{aligned}$$

Raising the index on a covector is handled similarly.

$\Lambda^\alpha = g^{\alpha\beta}\Lambda_\beta$ means covector

$$\Lambda = \Lambda_0\boldsymbol{a^0} + \Lambda_1\boldsymbol{a^1} + \Lambda_2\boldsymbol{a^2} + \Lambda_3\boldsymbol{a^3}$$

has been converted into vector

$$\begin{aligned} \Lambda^\alpha \boldsymbol{a_\alpha} &= \Lambda^0\boldsymbol{a_0} + \Lambda^1\boldsymbol{a_1} + \Lambda^2\boldsymbol{a_2} + \Lambda^3\boldsymbol{a_3} \\ &= -\Lambda_0\boldsymbol{a_0} + \Lambda_1\boldsymbol{a_1} + \Lambda_2\boldsymbol{a_2} + \Lambda_3\boldsymbol{a_3}. \end{aligned}$$

We can apply this procedure to a more general tensor in any basis and the process becomes automatic with practice. Let's consider our friend the Riemann curvature tensor.

$$R^\alpha{}_{\beta\gamma\delta}\, \boldsymbol{a_\alpha} \otimes \boldsymbol{a^\beta} \otimes \boldsymbol{a^\gamma} \otimes \boldsymbol{a^\delta} : V^* \times V \times V \times V \to \mathbb{R}$$

We can lower the first index or raise the third index to produce

$$R_{\alpha\beta\gamma\delta}\, \boldsymbol{a^\alpha} \otimes \boldsymbol{a^\beta} \otimes \boldsymbol{a^\gamma} \otimes \boldsymbol{a^\delta} : V \times V \times V \times V \to \mathbb{R}$$

$$\text{and } \; R^\alpha{}_\beta{}^\gamma{}_\delta\, \boldsymbol{a_\alpha} \otimes \boldsymbol{a^\beta} \otimes \boldsymbol{a_\gamma} \otimes \boldsymbol{a^\delta} : V^* \times V \times V^* \times V \to \mathbb{R}$$

where $\quad R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^\mu{}_{\beta\gamma\delta} \quad$ and $\quad R^\alpha{}_\beta{}^\gamma{}_\delta = g^{\gamma\mu} R^\alpha{}_{\beta\mu\delta}$.

In the very special basis discussed earlier this gives

$$R_{0\beta\gamma\delta} = -R^0{}_{\beta\gamma\delta} \quad \text{and} \quad R^0{}_\beta{}^\gamma{}_\delta = -R^0{}_{\beta\mu\delta}$$

$$\text{and } \; R_{i\beta\gamma\delta} = R^i{}_{\beta\gamma\delta} \quad \text{and} \quad R^i{}_\beta{}^\gamma{}_\delta = R^i{}_{\beta\mu\delta}.$$

Tensor Product

Tensors of higher rank can be created from tensors of lower rank by tensor product. We illustrate this by example.

Suppose given tensors

$$R = R^{\alpha}{}_{\beta}\, \boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \colon V^* \times V \to \mathbb{R} \quad \text{and} \quad X = X^{\alpha}\, \boldsymbol{a}_{\boldsymbol{\alpha}} \colon V^* \to \mathbb{R}.$$

The tensor $T = R \otimes X$ is defined to be

$$T^{\alpha}{}_{\beta}{}^{\gamma}\, \boldsymbol{a}_{\boldsymbol{\alpha}} \otimes \boldsymbol{a}^{\boldsymbol{\beta}} \otimes \boldsymbol{a}_{\boldsymbol{\gamma}} \colon V^* \times V \times V^* \to \mathbb{R}$$

with coordinates calculated as $T^{\alpha}{}_{\beta}{}^{\gamma} = R^{\alpha}{}_{\beta} X^{\gamma}$.

It is important to note that even though the numbers $R^{\alpha}{}_{\beta} X^{\gamma}$ and $X^{\gamma} R^{\alpha}{}_{\beta}$ are equal, the tensors $R \otimes X$ and $X \otimes R$ *are not.*

For instance, the domain of $R \otimes X$ is $V^* \times V \times V^*$ while the domain of $X \otimes R$ is $V^* \times V^* \times V$.

But even if the domains *are* equal, which will be the case for example if both tensors are covariant, these tensor products need not be equal.

Tensor product is not commutative.

Wedge Product

Define **wedge product** between s-forms and t-forms by

$$\sigma \wedge \tau = \frac{(s+t)!}{s!\, t!} Alt(\sigma \otimes \tau).$$

From the standpoint of calculation this can be impossibly arduous in view of the fact that $\mathcal{P}_L$ has $L!$ members. However, it is not completely unmanageable when L is 4 or less, as it usually is.

Example

Suppose now that $\mathcal{M}$ is a 3-manifold and $\sigma = 4\,\boldsymbol{a}^1 + 8\,\boldsymbol{a}^2$ and $\tau = 6\,\boldsymbol{a}^1 + 5\,\boldsymbol{a}^3$.

$\mathcal{P}_2$ contains two permutations only: the identity and the permutation that switches 1 and 2.

$$\begin{aligned}
\gamma = \sigma \wedge \tau &= \frac{(2)!}{1!\,1!} Alt(\sigma \otimes \tau) = 2\, Alt(\sigma \otimes \tau) \\
&= 2\, Alt(\, 24\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^1 + 20\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^3 + 48\,\boldsymbol{a}^2 \otimes \boldsymbol{a}^1 + 40\,\boldsymbol{a}^2 \otimes \boldsymbol{a}^3\,) \\
&= 24\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^1 + 20\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^3 + 48\,\boldsymbol{a}^2 \otimes \boldsymbol{a}^1 + 40\,\boldsymbol{a}^2 \otimes \boldsymbol{a}^3 \\
&\quad - (24\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^1 + 20\,\boldsymbol{a}^3 \otimes \boldsymbol{a}^1 + 48\,\boldsymbol{a}^1 \otimes \boldsymbol{a}^2 + 40\,\boldsymbol{a}^3 \otimes \boldsymbol{a}^2) \\
&= 20\,(\boldsymbol{a}^1 \otimes \boldsymbol{a}^3 - \boldsymbol{a}^3 \otimes \boldsymbol{a}^1) - 48\,(\boldsymbol{a}^1 \otimes \boldsymbol{a}^2 - \boldsymbol{a}^2 \otimes \boldsymbol{a}^1) \\
&\qquad + 40\,(\boldsymbol{a}^2 \otimes \boldsymbol{a}^3 - \boldsymbol{a}^3 \otimes \boldsymbol{a}^2) \\
&= 20\,\boldsymbol{a}^1 \wedge \boldsymbol{a}^3 - 48\,\boldsymbol{a}^1 \wedge \boldsymbol{a}^2 + 40\,\boldsymbol{a}^2 \wedge \boldsymbol{a}^3.
\end{aligned}$$

Back to Manifolds

On a manifold at point p we have made several equivalent definitions of the tangent space $\mathcal{M}_p$ and its dual $\mathcal{M}_p^*$.

Members of $\mathcal{M}_p$ are equivalence classes of differentiable curves through the point p at time 0 all of which have the *same derivative* at time 0. If $c\colon (a,b) \to \mathcal{M}$ is one such curve, $[c]_p$ is the class and we call these classes tangent vectors. We defined vector operations making $\mathcal{M}_p$ a vector space.

Members of $\mathcal{M}_p^*$ are equivalence classes of differentiable real valued functions defined on a neighborhood of p all of which have the *same derivative* at p. If $f\colon \mathcal{U} \to \mathbb{R}$ is one such function, df_p is the class and we call these classes cotangent vectors. We defined vector operations making $\mathcal{M}_p^*$ a vector space.

$\mathcal{M}_p$ and $\mathcal{M}_p^*$ are the vector spaces V and V^* which we will be using for our tensors, and we took pains to show that these vector spaces were defined independently of any particular coordinate system.

We then created specific bases for these vector spaces using coordinate maps such as $x\colon U_x \to \mathbb{R}^4$ and $y\colon U_y \to \mathbb{R}^4$ around p.

The dx_p^i form a basis for $\mathcal{M}_p^*$ and the tangent vectors $\frac{\partial}{\partial x^i}\Big|_p$ form a basis for $\mathcal{M}_p$.

$$df_p = \frac{\partial f}{\partial x^i}(p)\, dx_p^i \quad \text{and} \quad [c]_p = \frac{d(x^i \circ c)}{dt}(0)\, \frac{\partial}{\partial x^i}\bigg|_p .$$

df_p and $[c]_p$ act on each other by

$$df_p\,[c]_p = \frac{\partial f}{\partial x^i}(p)\frac{d(x^i \circ c)}{dt}(0) = (f \circ c)'(0).$$

A most convenient representation for these vectors and covectors is as equivalence classes

$$[x,\sigma]_p \in \mathcal{M}_p^* \text{ with } \sigma \in \mathbb{R}^{4*} \quad \text{and} \quad [x,v]_p \in \mathcal{M}_p \text{ with } v \in \mathbb{R}^4.$$

Representatives act on each other by $[x,\sigma]_p[x,v]_p = \sigma\, v$.

And coordinate changes—changing representatives—can be accomplished by multiplying by the appropriate Jacobian matrices.

$$\sigma\, v = \sigma \left(\frac{\partial x^i}{\partial y^j}(p)\right) \left(\frac{\partial y^i}{\partial x^j}(p)\right) v$$

Letting $\tau = \sigma \left(\frac{\partial x^i}{\partial y^j}(p)\right)$ and $w = \left(\frac{\partial y^i}{\partial x^j}(p)\right) v$ we have

$$[y,\tau]_p = [x,\sigma]_p \quad \text{and} \quad [y,w]_p = [x,v]_p.$$