CHAPTER 4

THE CURVATURE OF HIGHER DIMENSIONAL MANIFOLDS

A. AN INAUGURAL LECTURE

On June 10, 1854 the faculty of Göttingen University heard a lecture entitled *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses which lie at the Foundations of Geometry). This lecture was delivered by Georg Friedrich Bernhard Riemann, who had been born just a year before Gauss' paper of 1827. Although the lecture was not published until 1866, the ideas contained within it proved to be the most influential in the entire history of differential geometry. To be sure, mathematicians had not neglected the study of surfaces in the meantime; in fact, Gauss' work had inspired a tremendous amount of work along these lines. But the results obtained in those years can all be proved with much greater ease after we have followed the long series of developments initiated by the turning point in differential geometry which Riemann's lecture provided.

A short account of the life and character of Riemann can be found in the biography by Dedekind* which is included in Riemann's collected works (published by Dover). His interest in many fields of mathematical physics, together with a demand for perfection in all he did, delayed until 1851 the submission of his doctoral dissertation *Grundlagen für eine allegemeine Theorie der Functionen einer veränderlichen complexen Grösse* (Foundations for a general theory of functions of a complex variable). Gauss' official report to the Philosophical Faculty of the University of Göttingen stated "The dissertation submitted by Herr Riemann offers convincing evidence of the author's thorough and penetrating investigations in those parts of the subject treated in the dissertation, of a creative, active truly mathematical mind, and of a gloriously fertile originality."

Riemann was now qualified to seek the position of Privatdocent (a lecturer who received no salary, but was merely forwarded fees paid by those students

* Even for those who can only plod through German, this is preferable to the account in E. T. Bell's *Men of Mathematics*, which is hardly more than a translation of Dedekind, written in a racy style and interlarded with supercilious remarks of questionable taste.

who elected to attend his lectures). To attain this position he first had to submit an "inaugural paper" (Habilitationsschrift). Again there were delays, and it was not until the end of 1853 that Riemann submitted the Habilitationsschrift, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series). Now Riemann still had to give a probationary inaugural lecture on a topic chosen by the faculty, from a list of three proposed by the candidate. The first two topics which Riemann submitted were ones on which he had already worked, and he had every reason to expect that one of these two would be picked; for the third topic he chose the foundations of geometry. Contrary to all traditions, Gauss passed over the first two, and picked instead the third, in which he had been interested for years. At this time Riemann was also investigating the connection between electricity, magnetism, light, and gravitation, in addition to acting as an assistant in a seminar on mathematical physics. The strain of carrying out another major investigation, aggravated perhaps by the hardships of poverty, brought on a temporary breakdown. However, Riemann soon recovered, disposed of some other work which had to be completed, and then finished his inaugural lecture in about seven more weeks.

Riemann hoped to make his lecture intelligible even to those members of the faculty who knew little mathematics. Consequently, hardly any formulas appear and the analytic investigations are completely suppressed. Although Dedekind describes the lecture as a masterpiece of exposition, it is questionable how many of the faculty comprehended it. In making the following translation,* I was aided by the fact that I already had some idea what the mathematical results were supposed to be. The uninitiated reader will probably experience a great deal of difficulty merely understanding what Riemann is trying to say (the proofs of Riemann's assertions are spread out over the next several chapters). We can be sure, however, that one member of the faculty appreciated Riemann's work. Dedekind tells us that Gauss sat at the lecture "which surpassed all his expectations, in the greatest astonishment, and on the way back from the faculty meeting he spoke to Wilhelm Weber, with the greatest appreciation, and with an excitement rare for him, about the depth of the ideas presented by Riemann".

*The original is contained, of course, in Riemann's collected works. Two English translations are readily available, one in Volume 2 of Smith's *Source Book in Mathematics* (Dover), and one in Clifford's *Mathematical Papers* (Chelsea).

On the Hypotheses which lie at The Foundations of Geometry

Plan of the Investigation.

As is well known, geometry presupposes the concept of space, as well as assuming the basic principles for constructions in space. It gives only nominal definitions of these things, while their essential specifications appear in the form of axioms. The relationship between these presuppositions is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or *a priori* whether any connection between them is even possible.

From Euclid to Legendre, the most famous of the modern reformers of geometry, this darkness has been dispelled neither by the mathematicians nor by the philosophers who have concerned themselves with it. This is undoubtedly because the general concept of multiply extended quantities, which includes spatial quantities, remains completely unexplored. I have therefore first set myself the task of constructing the concept of a multiply extended quantity from general notions of quantity. It will be shown that a multiply extended quantity is susceptible of various metric relations, so that Space constitutes only a special case of a triply extended quantity. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties which distinguish Space from other conceivable triply extended quantities can only be deduced from experience. Thus arises the problem of seeking out the simplest data from which the metric relations of Space can be determined, a problem which by its very nature is not completely determined, for there may be several systems of simple data which suffice to determine the metric relations of Space; for the present purposes, the most important system is that laid down as a foundation of geometry by Euclid. These data are — like all data — not logically necessary, but only of empirical certainty, they are hypotheses; one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small.

I. Concept of an n fold extended quantity.

In proceeding to attempt the solution of the first of these problems, the development of the concept of multiply extended quantity, I feel particularly entitled to request an indulgent hearing, as I am little practiced in these tasks of a philosophical nature where the difficulties lie more in the concepts than in the construction, and because I could not make use of any previous studies, except for some very brief hints on the subject which Privy Councilor Gauss has given in his second memoir on Biquadratic Residues, in the Gottingen Gelehrte Anzeige and in the Gottingen Jubilee-book, and some philosophical researches of Herbart.

1.

Notions of quantity are possible only when there already exists a general concept which admits particular instances. These instances form either a continuous or a discrete manifold, depending on whether or not a continuous transition of instances can be found between any two of them; individual instances are called points in the first case and elements of the manifold in the second. Concepts whose particular instances form a discrete manifold are so numerous that some concept can always be found, at least in the more highly developed languages, under which any given collection of things can be comprehended (and consequently, in the study of discrete quantities, mathematicians could unhesitatingly proceed from the principle that given objects are to be regarded as all of one kind). On the other hand, opportunities for creating concepts whose instances form a continuous manifold occur so seldom in everyday life that color and the position of sensible objects are perhaps the only simple concepts whose instances form a multiply extended manifold. More frequent opportunities for creating and developing these concepts first occur in higher mathematics.

Particular portions of a manifold, distinguished by a mark or by a boundary, are called quanta. Their quantitative comparison is effected in the case of discrete quantities by counting, in the case of continuous quantities by measurement. Measuring involves the superposition of the quantities to be compared; it therefore requires a means of transporting one quantity to be used as a standard for the others. Otherwise, one can compare two quantities only when one is a part of the other, and then only as to "more" or "less", not as to "how much". The investigations which can be carried out in this case form a general division of the science of quantity, independent of measurement, where quantities are regarded, not as existing independent of position and not as expressible in terms of a unit, but as regions in a manifold. Such investigations have become a necessity for several parts of mathematics, e.g., for the treatment of many-valued analytic functions, and the dearth of such studies is one of the principal reasons why the celebrated theorem of Abel and the contributions of Lagrange, Pfaff and Jacobi to the general theory of differential equations have remained unfruitful for so long. From this portion of the science of extended quantity, a portion which proceeds without any further assumptions, it suffices for the present purposes to emphasize two points, which will make clear the essential characteristic of an $n$fold extension. The first of these concerns the generation of the concept of a multiply extended manifold, the second involves reducing position fixing in a given manifold to numerical determinations.

2.

In a concept whose instances form a continuous manifold, if one passes from one instance to another in a well-determined way, the instances through which one has passed form a simply extended manifold, whose essential characteristic is, that from any point in it a continuous movement is possible in only two directions, forwards and backwards. If one now imagines that this manifold passes to another, completely different one, and once again in a well-determined way, that is, so that every point passes to a well-determined point of the other, then the instances form, similarly, a doubly extended manifold. In a similar way, one obtains a triply extended manifold when one imagines that a doubly extended one passes in a well-determined way to a completely different one, and it is easy to see how one can continue this construction. If one considers the process as one in which the objects vary, instead of regarding the concept as fixed, then this construction can be characterized as a synthesis of a variability of $n + 1$ dimensions from a variability of n dimensions and a variability of one dimension.

3.

I will now show, conversely, how one can break up a variability, whose boundary is given, into a variability of one dimension and a variability of lower dimension. One considers a piece of a manifold of one dimension — with a fixed origin, so that points of it may be compared with one another — varying so that for every point of the given manifold it has a definite value, continuously changing with this point. In other words, we take within the given manifold a continuous function of position, which, moreover, is not constant on any part of the manifold. Every system of points where the function has a constant value then forms a continuous manifold of fewer dimensions than the given one. These manifolds pass continuously from one to another as the function changes; one can therefore assume that they all emanate from one of them, and generally speaking this will occur in such a way that every point of the first passes to a definite point of any other; the exceptional cases, whose investigation is important, need not be considered here. In this way, the determination of position in the given manifold is reduced to a numerical determination and to the determination of position in a manifold of fewer dimensions. It is now easy to show that this manifold has $n-1$ dimensions, if the given manifold is an $n$fold extension. By an $n$time repetition of this process, the determination of position in an $n$fold extended manifold is reduced to n numerical determinations, and therefore the determination of position in a given manifold is reduced, whenever this is possible, to a finite number of numerical determinations. There are, however, also manifolds in which the fixing of position requires not a finite number, but either an infinite sequence or a continuous manifold of numerical measurements. Such manifolds form, e.g., the possibilities for a function in a given region, the possible shapes of a solid figure, etc.

II. Metric relations of which a manifold of n dimensions is susceptible, on the assumption that lines have a length independent of their configuration, so that every line can be measured by every other.

Now that the concept of an $n$fold extended manifold has been constructed, and its essential characteristic has been found in the fact that position fixing in the manifold can be reduced to n numerical determinations, there follows, as the second of the problems proposed above, an investigation of the metric relations of which such a manifold is susceptible, and of the conditions which suffice to determine them. These metric relations can be investigated only in abstract terms, and their interdependence exhibited only through formulas. Under certain assumptions, however, one can resolve them into relations which are individually capable of geometric representation, and in this way it becomes possible to express the results of calculation geometrically. Thus, although an abstract investigation with formulas certainly cannot be avoided, the results can be presented in geometric garb. The foundations of both parts of the question are contained in the celebrated treatise of Privy Councilor Gauss on curved surfaces.

1.

Measurement requires an independence of quantity from position, which can occur in more than one way. The hypothesis which first presents itself, and which I shall develop here, is just this, that the length of lines is independent of their configuration, so that every line can be measured by every other. If position-fixing is reduced to numerical determinations, so that the position of a point in the given $n$fold extended manifold is expressed by n varying quantities x_1, x_2, x_3, and so forth up to x_n, then specifying a line amounts to giving the quantities x as functions of one variable. The problem then is, to set up a mathematical expression for the length of a line, for which purpose the quantities x must be thought of as expressible in units. I will treat this problem only under certain restrictions, and I first limit myself to lines in which the ratios of the quantities dx — the increments in the quantities x — vary continuously; one can then regard the lines as broken up into elements within which the ratios of the quantities dx may be considered to be constant, and the problem then reduces to setting up a general expression for the line element ds at every point, an expression which will involve the quantities x and the quantities dx. I assume, secondly, that the length of the line element remains unchanged, up to first order, when all the points of this line element suffer the same infinitesimal displacement, whereby I simply mean that if all the quantities dx increase in the same ratio, the line element changes by the same ratio. Under these assumptions, the line element can be an arbitrary homogeneous function of the first degree in the quantities dx which remains the same when all the quantities dx change sign, and in which the arbitrary constants are functions of the quantities x. To find the simplest cases, I first seek an expression for the $(n - 1)$fold extended manifolds which are everywhere equidistant from the origin of the line element, i.e., I seek a continuous function of position which distinguishes them from one another. This must either decrease or increase in all directions from the origin; I will assume that it increases in all directions and therefore has a minimum at the origin. Then if its first and second differential quotients are finite, the first order differential must vanish and the second order differential cannot be negative; I assume that it is always positive. This differential expression of the second order remains constant if ds remains constant and increases quadratically when the quantities dx, and thus also ds, all increase in the same ratio; it is therefore = constant. ds^2 and consequently ds = the square root of an everywhere positive homogeneous function of the second degree in the quantities dx, in which the coefficients are continuous functions of the quantities x. In Space, if one expresses the location of a point by rectilinear coordinates, then $ds = \sqrt{\Sigma(dx)^2}$; Space is therefore included in this simplest case. The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essentially different principles, but it would be rather time consuming and throw proportionally little new light on the study of Space, especially since the results cannot be expressed geometrically; I consequently restrict myself to those manifolds where the line element can be expressed by the square root of a differential expression of the second degree. One can transform such an expression into another similar one by substituting for the n independent variables, functions of n new independent variables.

However, one cannot transform any expression into any other in this way; for the expression contains $n\frac{n+1}{2}$ coefficients which are arbitrary functions of the independent variables; by the introduction of new variables one can satisfy only n conditions, and can therefore make only n of the coefficients equal to given quantities. There remain $n\frac{n-1}{2}$ others, already completely determined by the nature of the manifold to be represented, and consequently $n\frac{n-1}{2}$ functions of position are required to determine its metric relations. Manifolds, like the Plane and Space, in which the line element can be brought into the form $\sqrt{\Sigma dx^2}$ thus constitute only a special case of the manifolds to be investigated here; they clearly deserve a special name, and consequently, these manifolds, in which the square of the lines element can be expressed as the sum of the squares of complete differentials, I propose to call flat. In order to survey the essential differences of the manifolds representable in the assumed form, it is necessary to eliminate the features depending on the mode of presentation, which is accomplished by choosing the variable quantities according to a definite principle.

2.

For this purpose, one constructs the system of shortest lines emanating from a given point; the position of an arbitrary point can then be determined by the initial direction of the shortest line in which it lies, and its distance, in this line, from the initial point. It can therefore be expressed by the ratios of the quantities dx^0, i.e., the quantities dx at the origin of this shortest line, and by the length s of this line. In place of the dx^0 one now introduces linear expressions $d\alpha$ formed from them in such a way that the initial value of the square of the line element will be equal to the sum of the squares of these expressions, so that the independent variables are: the quantity s and the ratio of the quantities $d\alpha$. Finally, in place of the $d\alpha$ choose quantities $x_1, x_2, \ldots, x_n$ proportional to them, but such that the sum of their squares equals s^2. If one introduces these quantities, then for infinitely small values of x the square of the line element $= \Sigma dx^2$, but the next order term in its expansion equals a homogeneous expression of the second degree in the $n\frac{n-1}{2}$ quantities $(x_1\, dx_2 - x_2\, dx_1)$, $(x_1\, dx_3 - x_3\, dx_1)$, $\ldots$, and is consequently an infinitely small quantity of the fourth order, so that one obtains a finite quantity if one divides it by the square of the infinitely small triangle at whose vertices the variables have the values $(0, 0, 0, \ldots)$, $(x_1, x_2, x_3 \ldots)$, $(dx_1, dx_2, dx_3, \ldots)$. This quantity remains the same as long as the quantities x and dx are contained in the same binary linear forms, or as long as the two shortest lines from the initial point to x and from the initial point to dx remain in the same surface element, and therefore depends only on the position and direction of that element. It obviously = zero if the manifold in question is flat, i.e., if the square of the line element is reducible to Σdx^2, and can therefore be regarded as the measure of deviation from flatness in this surface direction at this point. When multiplied by $-\frac{3}{4}$ it becomes equal to the quantity which Privy Councilor Gauss has called the curvature of a surface. Previously, $n\frac{n-1}{2}$ functions of position were found necessary in order to determine the metric relations of an n fold extended manifold representable in the assumed form; hence if the curvature is given in $n\frac{n-1}{2}$ surface directions at every point, then the metric relations of the manifold may be determined, provided only that no identical relations can be found between these values, and indeed in general this does not occur. The metric relations of these manifolds, in which the line element can be represented as the square root of a differential expression of the second degree, can thus be expressed in a way completely independent of the choice of the varying quantities. A similar path to the same goal could also be taken in those manifolds in which the line element is expressed in a less simple way, e.g., by the fourth root of a differential expression of the fourth degree. The line element in this more general case would not be reducible to the square root of a quadratic sum of differential expressions, and therefore in the expression for the square of the line element the deviation from flatness would be an infinitely small quantity of the second dimension, whereas for the other manifolds it was an infinitely small quantity of the fourth dimension. This peculiarity of the latter manifolds therefore might well be called plainness in the smallest parts. For present purposes, however, the most important peculiarity of these manifolds, on whose account alone they have been examined here, is this, that the metric relations of the doubly extended ones can be represented geometrically by surfaces and those of the multiply extended ones can be reduced to those of the surfaces contained within them, which still requires a brief discussion.

3.

In the conception of surfaces, the inner metric relations, which involve only the lengths of paths within them, are always bound up with the way the surfaces are situated with respect to points outside them. We may, however, abstract from external relations by considering deformations which leave the lengths of lines within the surfaces unaltered, i.e., by considering arbitrary bendings — without stretching — of such surfaces, and by regarding all surfaces obtained from one another in this way as equivalent. Thus, for example, arbitrary cylindrical or conical surfaces count as equivalent to a plane, since they can be formed from a plane by mere bending, under which the inner metric relations remain the same; and all theorems about the plane — hence all of planimetry

— retain their validity. On the other hand, they count as essentially different from the sphere, which cannot be transformed into the plane without stretching. According to the previous investigations, the inner metric relations at every point of a doubly extended quantity, if its line element can be expressed as the square root of a differential expression of the second degree, which is the case with surfaces, is characterized by the curvature. For surfaces, this quantity can be given a visual interpretation as the product of the two curvatures of the surface at this point, or by the fact that its product with an infinitely small triangle formed from shortest lines is, in proportion to the radius, half the excess of the sum of its angles over two right angles. The first definition would presuppose the theorem that the product of the two radii of curvatures is unaltered by mere bendings of a surface, the second, that at each point the excess over two right angles of the sum of the angles of any infinitely small triangle is proportional to its area. To give a tangible meaning to the curvature of an n fold extended manifold at a given point, and in a given surface direction through it, we first mention that a shortest line emanating from a point is completely determined if its initial direction is given. Consequently we obtain a certain surface if we prolong all the initial directions from the given point which lie in the given surface element, into shortest lines; and this surface has a definite curvature at the given point, which is equal to the curvature of the n fold extended manifold at the given point, in the given surface direction.

4.

Before applying these results to Space, it is still necessary to make some general considerations about flat manifolds, i.e., about manifolds in which the square of the line element can be represented as the sum of squares of complete differentials.

In a flat n fold extended manifold the curvature in every direction, at every point, is zero; but according to the preceding investigation, in order to determine the metric relations it suffices to know that at each point the curvature is zero in $n\frac{n-1}{2}$ independent surface-directions. The manifolds whose curvature is everywhere $= 0$ can be considered as a special case of those manifolds whose curvature is everywhere constant. The common character of those manifolds whose curvature is constant may be expressed as follows: figures can be moved in them without stretching. For obviously figures could not be freely shifted and rotated in them if the curvature were not the same in all directions, at all points. On the other hand, the metric properties of the manifold are completely determined by the curvature; they are therefore exactly the same in all the directions around any one point as in the directions around any other, and thus the same constructions can be effected starting from either; consequently, in the manifolds with constant curvature figures may be given any arbitrary position. The metric relations of these manifolds depend only on the value of the curvature, and it may be mentioned, as regards the analytic presentation, that if one denotes this value by α, then the expression for the line element can be put in the form

$$\frac{1}{1+\frac{\alpha}{4}\Sigma x^2}\sqrt{\Sigma dx^2}$$

5.

The consideration of *surfaces* with constant curvature may serve for a geometric illustration. It is easy to see that the surfaces whose curvature is positive can always be rolled onto a sphere whose radius is the reciprocal of the curvature; but in order to survey the multiplicity of these surfaces, let one of them be given the shape of a sphere, and the others the shape of surfaces of rotation which touch it along the equator. The surfaces with greater curvature than the sphere will then touch the sphere from inside and take a form like the portion of the surface of a ring, which is situated away from the axis; they could be rolled upon zones of spheres with smaller radii, but would go round more than once. Surfaces with smaller positive curvature are obtained from spheres of larger radii by cutting out a portion bounded by two great semi-circles, and bringing together the cut-lines. The surface of curvature zero will be a cylinder standing on the equator; the surfaces with negative curvature will touch this cylinder from outside and be formed like the part of the surface of a ring which is situated near the axis. If one regards these surfaces as possible positions for pieces of surface moving in them, as Space is for bodies, then pieces of surface can be moved in all these surfaces without stretching. The surfaces with positive curvature can always be so formed that pieces of surface can even be moved arbitrarily without bending, namely as spherical surfaces, but those with negative curvature cannot. Aside from this independence of position for surface pieces, in surfaces with zero curvature there is also an independence of position for directions, which does not hold in the other surfaces.

III. Applications to Space.

1.

Following these investigations into the determination of the metric relations of an n fold extended quantity, the conditions may be given which are sufficient and necessary for determining the metric relations of Space, if we assume beforehand the independence of lines from configuration and the possibility of expressing the line element as the square root of a second order differential expression, and thus flatness in the smallest parts.

First, these conditions may be expressed by saying that the curvature at every point equals zero in three surface directions, and thus the metric relations of Space are implied if the sum of the angles of a triangle always equals two right angles.

But secondly, if one assumes with Euclid not only the existence of lines independently of configuration, but also of bodies, then it follows that the curvature is everywhere constant, and the angle sum in all triangles is determined if it is known in one.

In the third place, finally, instead of assuming the length of lines to be independent of place and direction, one might assume that their length and direction is independent of place. According to this conception, changes or differences in position are complex quantities expressible in three independent units.

2.

In the course of the previous considerations, the relations of extension or regionality were first distinguished from the metric relations, and it was found that different metric relations were conceivable along with the same relations of extension; then systems of simple metric specifications were sought by means of which the metric relations of Space are completely determined, and from which all theorems about it are a necessary consequence. It remains now to discuss the question how, to what degree, and to what extent these assumptions are borne out by experience. In this connection there is an essential difference between mere relations of extension and metric relations, in that among the first, where the possible cases form a discrete manifold, the declarations of experience are to be sure never completely certain, but they are not inexact, while for the second, where the possible cases form a continuous manifold, every determination from experience always remains inexact — be the probability ever so great that it is nearly exact. This circumstance becomes important when these empirical determinations are extended beyond the limits of observation into the immeasurably large and the immeasurably small; for the latter may obviously become ever more inexact beyond the boundary of observation, but not so the former.

When constructions in Space are extended into the immeasurably large, unboundedness is to be distinguished from infinitude; one belongs to relations of extension, the other to metric relations. That Space is an unbounded triply extended manifold is an assumption which is employed for every apprehension of the external world, by which at every moment the domain of actual perception is supplemented, and by which the possible locations of a sought for object are constructed; and in these applications it is continually confirmed. The unboundedness of space consequently has a greater empirical certainty than any experience of the external world. But its infinitude does not in any way follow from this; quite to the contrary, Space would necessarily be finite if one assumed independence of bodies from position, and thus ascribed to it a constant curvature, as long as this curvature had ever so small a positive value. If one prolonged the initial directions lying in a surface direction into shortest lines, one would obtain an unbounded surface with constant positive curvature, and thus a surface which in a flat triply extended manifold would take the form of a sphere, and consequently be finite.

3.

Questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small. Upon the exactness with which we pursue phenomena into the infinitely small, does our knowledge of their causal connections essentially depend. The progress of recent centuries in understanding the mechanisms of Nature depends almost entirely on the exactness of construction which has become possible through the invention of the analysis of the infinite and through the simple principles discovered by Archimedes, Galileo, and Newton, which modern physics makes use of. By contrast, in the natural sciences where the simple principles for such constructions are still lacking, to discover causal connections one pursues phenomenon into the spatially small, just so far as the microscope permits. Questions about the metric relations of Space in the immeasurably small are thus not idle ones.

If one assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes can be neglected. But if such an independence of bodies from position does not exist, then one cannot draw conclusions about metric relations in the infinitely small from those in the large; at every point the curvature can have arbitrary values in three directions, provided only that the total curvature of every measurable portion of Space is not perceptibly different from zero. Still more complicated relations can occur if the line element cannot be represented, as was presupposed, by the square root of a differential expression of the second degree. Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of Space. In connection with this question, which may indeed still be ranked as part of the study of Space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.

An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by unduly restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices.

This leads us away into the domain of another science, the realm of physics, into which the nature of the present occasion does not allow us to enter.

B. WHAT DID RIEMANN SAY?

Upon a first reading, Riemann's lecture may appear to have almost no mathematical content. But this is only because the analytic investigations, which occur in Part II, have been drastically condensed, while Part I explains, in general philosophical terms, important mathematical concepts which succeeding generations of investigators were eventually able to express with mathematical precision; finally, Part III of the lecture deals with applications of the mathematical discoveries to questions in physics, a process which is perhaps not yet complete.

In this commentary on Riemann's lecture, we will follow closely the order of Riemann's exposition, referring often to the various sections (1, 2, etc.) within each part (I, II, III). It should not be expected that all details will be cleared up, even in the remaining portions of this chapter, for the complete consideration of Riemann's ideas will occupy several of the succeeding chapters. Consequently, the remaining parts of Chapter 4 may be the hardest reading encountered in either of the two volumes of these notes. Nevertheless, we hope that in the end a clear view of all these ideas will be obtained.

In the "Plan of the Investigation", Riemann begins by accounting for the confusion over the status of non-Euclidean geometry, which at this time was still not completely accepted. In 1829, Lobachevsky and Bolyai had independently constructed a system of geometry which began by assuming that through a point not on a line there was more than one line parallel to it (as opposed to the assumption that there is only one parallel line, which is equivalent to Euclid's Fifth Postulate); but it was still supposed by some that contradictions in this system would eventually be found.

Riemann attributes the difficulties encountered in the study of non-Euclidean geometry to the fact that geometers had never separated what we would call the topological properties of space from its metric properties; in the axiomatic development of geometry, even the notion of space itself is undefined, and its properties are developed through the axioms.

Riemann proposes to distinguish the metric properties from the topological properties, and promises that we will discover how different metric structures can be put on the triply extended quantity which constitutes Space, so that one cannot possibly expect to deduce the parallel postulate of Euclid from topological considerations alone. This implies that experimental data must be used to determine what metric properties Space actually has, and raises the question which data we should seek, and what we can expect to say about the regions of Space too distant, or too small, to be investigated experimentally.

In Part I, "Concept of an n fold extended quantity", Riemann is clearly trying to define a manifold.

It is impossible to tell from this lecture, intended for non-mathematicians, how far Riemann had advanced toward the precise solution of this problem, and whether he had any way of expressing concretely the notion of a metric or topological space, which is essentially prerequisite to the definition of a manifold. However, it is quite obvious that the notion was thoroughly clear in his own mind and that he recognized that manifolds were characterized by the fact that they are locally like n-dimensional Euclidean space. It is also clear that he understood the importance of infinite dimensional spaces, such as the set of all real-valued functions on a space (it is interesting that quite recently some of these infinite dimensional spaces have been given the structure of "infinite dimensional manifolds", and differential geometric methods have been applied to them with great success).

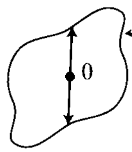
Part II contains nearly all the mathematical results, and the discussion of this Part will take up most of the present chapter.

The difficulties in Part II begin right with the title, "Metric relations of which a manifold of n dimensions is susceptible, on the assumption that lines have a length independent of their configuration, so that every line can be measured by every other". To understand what Riemann means, it is necessary to recall the process by which lengths are assigned to curves in the plane or 3-dimensional space of analytic geometry. In this case, we begin with the notion of distance between pairs of points, which amounts to saying that we first assign a length to *straight* lines; the length of other lines is then defined as the least upper bound of inscribed curves made up of straight lines, a process which can be reduced to integration. In this method of assigning lengths to curves, it may be said that all curves are measured by means of straight lines.

By contrast, Riemann proposes to consider a uniform method of assigning lengths to all curves in a manifold, a method which does not depend on first distinguishing a particular class of curves. This is to be done by measuring the lengths of tangent vectors, so that the lengths of curves can be defined by an integral (a restriction to C^1 curves is first indicated). Riemann assumes that this "length" function f is continuous on each tangent space and also positive homogeneous—the length $f(\lambda v)$ of λv is $|\lambda|$ times the length $f(v)$ of v.

Now there are many kinds of positive homogeneous functions on a finite dimensional vector space; any subset of the vector space which is symmetric with respect to the origin, and intersects each ray through 0 just once, can be used as the set of vectors of length 1. Riemann notes that the partial derivative of f^2 (with respect to some basis of the tangent space M_p) must vanish at

$0 \in M_p$, and that the matrix of second order partial derivatives is positive semi-



definite. He then assumes, as the simplest possibility, that it is actually positive definite. This means that f can be expressed as $\sqrt{\sum g_{ij}(p)\, dx^i \cdot dx^j}$ for certain numbers $g_{ij}(p)$. An assignment to each tangent space M_p of such a norm, or more precisely the inner product from which it comes, is, of course, what we now call a Riemannian metric on the manifold M.

Riemann points out that it is merely to save time, and to allow geometric descriptions of the results, that he restricts his attention to the special case. Certain more general cases, though not the most general of all, were investigated by Finsler in his thesis (1918), and are now known as Finsler metrics; it seems clear, however, that Riemann must have already known the basic facts about these more general metrics (some information on Finsler metrics is given in the Addendum).

Having restricted his attention to "Riemannian manifolds", Riemann now asks the crucial question: when does the introduction of a new coordinate system change the metric $\sum g_{ij}\, dy^i \otimes dy^j$ into some given metric $\sum a_{ij}\, dx^i \otimes dx^j$; in other words, when are two Riemannian manifolds locally isometric? Riemann here presents one of his famous "counting arguments", which enabled him to guess results that in some cases were not rigorously proved until a hundred years later. Riemann argues that the expression $\sum g_{ij}\, dx^i \otimes dx^j$ contains $n\frac{n+1}{2}$ functions (not n^2, for $g_{ij} = g_{ji}$) while a new coordinate system involves only n functions, so that we can change only n of the g_{ij}, leaving $n\frac{n-1}{2}$ other functions which depend on the metric; consequently, Riemann argues, there should be some set of $n\frac{n-1}{2}$ functions which will determine the metric completely.

In section 2 of Part II, Riemann indicates how such functions are to be found. We are going to apply a standard technique for the study of differentiable functions—we examine the Taylor polynomials approximating the functions g_{ij}. If x is a coordinate system on M, with $x(p) = 0$, and the Riemannian metric is given by $\langle\ ,\ \rangle = \sum g_{ij}\, dx^i \otimes dx^j$, then for the Taylor expansion of the function $g_{ij} \circ x^{-1} \colon \mathbb{R}^n \to \mathbb{R}$ we have

$$g_{ij} \circ x^{-1}(t) = (g_{ij} \circ x^{-1})(0) + \sum_{k=1}^{n} D_k(g_{ij} \circ x^{-1})(0)t^k + \frac{1}{2} \sum_{k,l=1}^{n} D_{k,l}(g_{ij} \circ x^{-1})(0)t^k t^l + o(|t|^2).$$

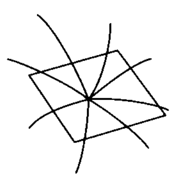
Hence on M we have

$$(*) \qquad g_{ij} = g_{ij}(p) + \sum_{k=1}^{n} \frac{\partial g_{ij}}{\partial x^k}(p)x^k + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(p)x^k x^l + o(|x|^2),$$

where $o(|x|^2)$ denotes a function f on M such that

$$\lim_{q \to p} \frac{f(q)}{|x(q)|^2} = 0.$$

However, and this is the important device Riemann introduces, we will select a very special coordinate system around each point $p \in M$. We choose an orthonormal basis $X_1, \dots, X_n \in M_p$, and define a coordinate system $\chi \colon M_p \to \mathbb{R}^n$ on M_p by $\chi\left(\sum a^i X_i\right) = (a^1, \dots, a^n)$. Then we let x be the coordinate system $\chi \circ \exp^{-1}$. (This coordinate system is introduced at the very beginning



of section 2, but it takes a little work to decipher Riemann's description of it.)

The coordinate system x is not uniquely determined, for it depends on the choice of the orthonormal basis $X_1, \dots, X_n \in M_p$; but any two differ by an element of $O(n)$, so it will not be hard to take into account the way any of our results depend on this choice. These coordinate systems are called **Riemannian normal coordinates** at p. Notice that since $\exp_* \colon (M_p)_0 \to M_p$ is the identity (upon identifying $(M_p)_0$ with M_p), we have

$$\left.\frac{\partial}{\partial x^i}\right|_p = \exp_* X_i = X_i \in M_p.$$

We can quickly give some information about the first two terms in the expansion $(*)$ of g_{ij}:

1. PROPOSITION. In a Riemannian normal coordinate system x at p we have

$$g_{ij}(p) = \delta_{ij}$$
$$\frac{\partial g_{ij}}{\partial x^k}(p) = 0.$$

PROOF. The first set of equations is clear, for

$$g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right\rangle = \langle X_i, X_j \rangle = \delta_{ij}.$$

To prove the second set of equations, we recall the equations for a geodesic γ:

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t)) \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

In Riemannian normal coordinates the geodesics through p are just $\exp \circ c$, where c is a straight line in M_p. This means that for all n-tuples $(\xi^1, \ldots, \xi^n)$, the geodesics through p are the curves γ with $\gamma^k(t) = \xi^k t$. Hence

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))\xi^i\xi^j = 0 \qquad \text{for the geodesic} \quad \gamma^k(t) = \xi^k t.$$

In particular, since $p = \gamma(0)$ is on all these geodesics, we have

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(p)\xi^i\xi^j = 0 \qquad \text{for } all \text{ } n\text{-tuples } (\xi^1, \ldots, \xi^n).$$

This shows that all $\Gamma_{ij}^k(p)$ are 0: choosing all $\xi^\alpha = 0$ except $\xi^i = 1$ gives $\Gamma_{ii}^k = 0$; then choosing all $\xi^\alpha = 0$ except $\xi^i = \xi^j = 1$ gives

$$0 = \Gamma_{ij}^k(p) + \Gamma_{ji}^k(p) + \Gamma_{ii}^k(p) + \Gamma_{jj}^k(p) = 2\Gamma_{ij}^k(p).$$

It follows that

$$[ij, k] = \sum_{\alpha=1}^{n} g_{\alpha l}\Gamma_{ij}^l = 0 \quad \text{at } p.$$

Making use of equation (∗) on pg. I.331, we have finally,

$$\frac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i] = 0 \quad \text{at } p. ❖$$

In view of Proposition 1, we can now use (*) to expand the squared norm $\| \ \|^2$ as

$$\| \ \|^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

$$= \sum_{i=1}^{n} dx^i dx^i + \frac{1}{2} \sum_{i,j;k,l} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(p) x^k x^l dx^i dx^j + o(|x|^2).$$

[This is an equation for tangent vectors near p, and $o(|x|^2)$ now denotes a function f on tangent vectors; in order to have

$$\lim_{q \to p} \frac{f(v_q)}{|x|^2} = 0,$$

we must restrict v_q to be of some bounded length.] Riemann's main assertion involves the term

$$\frac{1}{2} \sum_{i,j;k,l} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(p) x^k x^l dx^i dx^j = \sum_{i,j;k,l} c_{ij,kl} x^k x^l dx^i dx^j, \quad \text{say.}$$

Riemann asserts that there are numbers $C_{ij,kl}$ such that we can write

$$\sum_{i,j;k,l} c_{ij,kl} x^k x^l dx^i dx^j = \sum_{i,j;k,l} C_{ij,kl} (x^k dx^i - x^i dx^k) \cdot (x^l dx^j - x^j dx^l).$$

This assertion immediately suggests three questions—Why did Riemann suspect this was true? How did he prove it? What is its significance?

We will begin by giving a partial answer to the third of these questions. Notice that the equation in question is supposed to hold for all tangent vectors v at all points q in a neighborhood of p. Consequently, the numbers $dx^i(v)$ [and $x^i(q)$] can take on all [sufficiently small] values. The coordinate system x and the Riemannian metric $\langle \ , \ \rangle$ are used to obtain the n^4 numbers $c_{ij,kl} = \frac{1}{2} \partial^2 g_{ij} / \partial x^k \partial x^l (p)$; but beyond this, the above equation has nothing to do with the manifold at all. If we define a quadratic polynomial Q in $2n$ variables by

$$Q(X, Y) = Q(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l,$$

then Riemann is asserting that this quadratic polynomial can be written as

$$Q(X, Y) = \sum_{i,j;k,l} C_{ij,kl} (X_i Y_k - X_k Y_i) \cdot (X_j Y_l - X_l Y_j).$$

To obtain the geometric consequences of this fact, we observe what it says when we select two vectors $v_p, w_p \in M_p$ and let $X_i = dx^i(v_p)$ and $Y_i = dx^i(w_p)$; denoting $Q(X,Y)$ by $Q(v_p, w_p)$, we have

$$Q(v_p, w_p) = \sum_{i,j;k,l} c_{ij,kl}\, dx^i(v_p)\, dx^j(v_p) \cdot dx^k(w_p)\, dx^l(w_p)$$

$$= \sum_{i,j;k,l} C_{ij,kl}[(dx^i \wedge dx^k)(v_p, w_p)] \cdot [(dx^j \wedge dx^l)(v_p, w_p)].$$

[We can also write

$$Q = \sum_{i,j;k,l} c_{ij,kl}\, dx^i dx^j \otimes dx^k dx^l = \sum_{i,j;k,l} C_{ij,kl}(dx^i \wedge dx^k) \cdot (dx^j \wedge dx^l),$$

a little more simply.] Now suppose $v'_p, w'_p \in M_p$ span the same subspace as v_p, w_p, so that we can write

$$\begin{aligned} v'_p &= a_{11}v_p + a_{21}w_p \\ w'_p &= a_{12}v_p + a_{22}w_p \end{aligned} \qquad \det(a_{ij}) \neq 0.$$

The right side of the above equation for $Q(v_p, w_p)$ shows that

$$Q(v'_p, w'_p) = [\det(a_{ij})]^2 \cdot Q(v_p, w_p),$$

since each $dx^\alpha \wedge dx^\beta$ is multiplied by the factor $\det(a_{ij})$. If we use $\|v_p, w_p\|$ to denote the area of the parallelogram spanned by v_p and w_p, then we also have

$$\|v'_p, w'_p\|^2 = [\det(a_{ij})]^2 \cdot \|v_p, w_p\|^2.$$

Consequently,

$$\frac{Q(v'_p, w'_p)}{\|v'_p, w'_p\|^2} = \frac{Q(v_p, w_p)}{\|v_p, w_p\|^2}.$$

We therefore have a way of assigning a number to every 2-dimensional subspace of the tangent space at p. (Riemann sticks to the original quadratic function of the x^i and dx^i, which puts him in the position of having to divide by the squared area of a very strange triangle, with one vertex at x^i, and one at dx^i.)

It is easy to see that if we pick a different Riemannian normal coordinate system at p, then the resulting function on the 2-dimensional subspaces of M_p will be the same, for $Q(v_p, w_p) = Q(dx^1(v_p), \ldots, dx^n(v_p), dx^1(w_p), \ldots, dx^n(w_p))$ will change by $(\det B)^2$, where $B \in \mathrm{O}(n)$, so that $\det B = \pm 1$. We will examine later the significance of this new function on 2-dimensional subspaces of the tangent space. For the present we take up the other questions—Why did Riemann think it was true, and how did he prove it?

Of course, an answer to the first question is not only doomed to be mere conjecture, but is always foolhardy to put forth, for there is no accounting for genius. The best suggestion I can offer is that the dependence of $Q(v_p, w_p)$ on the span of v_p and w_p alone is certainly an attractive one, and as we shall see later, in one special case which Riemann may have investigated first, the result appears in a rather natural way. It is also impossible to say for sure how Riemann proved the result, for his own investigations were never published. I have used the remarks by H. Weber in Riemann's collected works (pp. 405–409), as well as the commentary given by Herman Weyl in a special edition of Riemann's lecture. There are two parts to the proof, a purely algebraic one about quadratic functions, which determines what relations the numbers $c_{ij,kl}$ ought to satisfy, and an analytic one which establishes these relations.

For the algebraic part, we will be considering a quadratic function Q of $2n$ variables

$$Q(X,Y) = Q(X_1,\dots,X_n,Y_1,\dots,Y_n) = \sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l.$$

Note that for our Q we have

$$c_{ij,kl} = c_{ji,kl} = c_{ij,lk},$$

(using $g_{ij} = g_{ji}$, and $\partial^2/\partial x^k \partial x^l = \partial^2/\partial x^l \partial x^k$). If $A = (a_{ij})$ is a 2×2 matrix, we will use $A(X,Y)$ to denote the $2n$-tuple

$$\begin{aligned} A(X,Y) &= (a_{11}X + a_{21}Y,\ a_{12}X + a_{22}Y) \\ &= (a_{11}X_1 + a_{21}Y_1,\ \dots,\ a_{11}X_n + a_{21}Y_n,\ a_{12}X_1 + a_{22}Y_1,\ \dots,\ a_{12}X_n + a_{22}Y_n). \end{aligned}$$

2. PROPOSITION. Let Q be a quadratic function of $2n$ variables,

$$Q(X,Y) = \sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l,$$

where

$$c_{ij,kl} = c_{ji,kl} = c_{ij,lk}. \tag{1}$$

Then

$$Q(A(X,Y)) = (\det A)^2 Q(X,Y)$$

for all 2×2 matrices A if and only if:

$$c_{ij,kl} = c_{kl,ij} \tag{2}$$

$$c_{li,jk} + c_{lj,ki} + c_{lk,ij} = 0. \tag{3}$$

PROOF. First of all, the equation $Q(A(X,Y)) = (\det A)^2 Q(X,Y)$ clearly holds for all 2×2 matrices A if and only if it holds for the non-singular ones, since both sides of the equation are continuous functions of A, and the non-singular matrices are dense.

Now it is well known that all non-singular 2×2 matrices can be written as a product of the matrices

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

[This comes from the fact that any non-singular matrix can be obtained from the identity matrix by a sequence of elementary row operations, and every row operation may be accomplished by multiplying by one of the above matrices.] So our condition holds for all A if and only if it holds for the above matrices. We can disregard the last matrix, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For the matrix $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, the condition $Q(A(X,Y)) = (\det A)^2 Q(X,Y)$ becomes simply

$$Q(aX,Y) = a^2 Q(X,Y),$$

which is automatically true. The same result holds for the second matrix on our list, so all the conditions finally come down to

(a) $$Q(Y,X) = Q(X,Y) \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b) $$Q(X+Y,Y) = Q(X,Y) \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Now equation (a) becomes

$$\sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l = \sum_{i,j;k,l} c_{ij,kl} Y_i Y_j X_k X_l.$$

Since this must be a polynomial identity, we obtain (2) immediately, by looking at the coefficient of $X_i X_j Y_k Y_l$ on both sides.

Equation (b) becomes

$$\sum_{i,j;k,l} c_{ij,kl} (X_i + Y_i)(X_j + Y_j) Y_k Y_l = \sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l,$$

or

$$\sum_{i,j;k,l} c_{ij,kl}[X_i Y_j Y_k Y_l + X_j Y_i Y_k Y_l + Y_i Y_j Y_k Y_l] = 0.$$

Letting $X = 0$, we obtain

(bl)
$$\sum_{i,j;k,l} c_{ij,kl} Y_i Y_j Y_k Y_l = 0$$

and then, in consequence,

(b2)
$$\sum_{i,j;k,l} c_{ij,kl}[X_i Y_j Y_k Y_l + X_j Y_i Y_k Y_l] = 0.$$

On the other hand, (b2) implies (bl), so (b) is equivalent to (b2) alone. Finally, since $c_{ij,kl} = c_{ji,kl}$, equation (b2) is equivalent to

(b3)
$$\sum_{i,j;k,l} c_{ij,kl} X_i Y_j Y_k Y_l = 0.$$

Looking at the coefficient of a particular $X_i Y_j Y_k Y_l$ we obtain

$$c_{ij,kl} + c_{ij,lk} + c_{ik,jl} + c_{ik,lj} + c_{il,jk} + c_{il,kj} = 0.$$

Using the symmetry with respect to the last two indices, this is equivalent to equation (3). ❖

3. PROPOSITION. A quadratic function

$$Q(X,Y) = \sum_{i,j;k,l} c_{ij,kl} X_i X_j Y_k Y_l$$

with

(1)
$$c_{ij,kl} = c_{ji,kl} = c_{ij,kl}$$

satisfies the two equivalent conditions of Proposition 2 if and only if it can be written as

$$Q(X,Y) = \sum_{i,j;k,l} C_{ij,kl}(X_i Y_k - X_k Y_i) \cdot (X_j Y_l - X_l Y_j).$$

PROOF. If Q can be written this way, then we will clearly have $Q(A(X,Y)) = (\det A)^2 Q(X,Y)$ for all A. Conversely, suppose this holds for all A, so that we also have

$$c_{ij,kl} = c_{kl,ij} \tag{2}$$

$$c_{li,jk} + c_{lj,ki} + c_{lk,ij} = 0. \tag{3}$$

We begin by writing four equivalent expressions for Q:

$$\begin{aligned} Q(X,Y) &= \sum c_{ij,kl} X_i X_j Y_k Y_l \\ &= \sum c_{jk,il} X_j X_k Y_i Y_l \\ &= \sum c_{il,jk} X_i X_l Y_j Y_k \\ &= \sum c_{kl,ij} X_k X_l Y_i Y_j . \end{aligned}$$

Now, by (3) we have

$$c_{jk,il} = -c_{ji,lk} - c_{jl,ki},$$

so

$$\begin{aligned} \sum_{i,j,k,l} c_{jk,il} X_j X_k Y_i Y_l &= - \sum_{i,j,k,l} c_{ji,lk} X_j X_k Y_i Y_l - \sum_{i,j,k,l} c_{jl,ki} X_j X_k Y_i Y_l \\ &= - \sum_{i,j,k,l} c_{ji,lk} X_j X_k Y_i Y_l - \sum_{i,j,k,l} c_{kl,ji} X_j X_k Y_i Y_l \\ &\qquad \text{(interchanging } j \text{ and } k \text{ in the second sum)} \\ &= -2 \sum_{i,j,k,l} c_{ij,kl} X_j X_k Y_i Y_l, \qquad \text{using (1) and (2).} \end{aligned}$$

If we apply a similar process to the third expression for Q, use (2) on the fourth, and leave the first unaltered, we obtain

$$\begin{aligned} Q(X,Y) &= \sum c_{ij,kl} X_i X_j Y_k Y_l \\ \tfrac{1}{2} Q(X,Y) &= - \sum c_{ij,kl} X_j X_k Y_i Y_l \\ \tfrac{1}{2} Q(X,Y) &= - \sum c_{ij,kl} X_i X_l Y_j Y_k \\ Q(X,Y) &= \sum c_{ij,kl} X_k X_l Y_i Y_j . \end{aligned}$$

Adding, we obtain the desired result,

$$3Q(X,Y) = \sum_{i,j;k,l} c_{ij,kl} (X_i Y_k - X_k Y_i) \cdot (X_j Y_l - X_l Y_j). \quad ❖$$

We now proceed with the hardest part of the investigation, a hairy calculation indeed.

4. PROPOSITION. In a Riemannian normal coordinate system x at p, the numbers

$$c_{ij,kl} = \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(p)$$

satisfy

$$c_{ij,kl} = c_{kl,ij}$$
$$c_{li,jk} + c_{lj,ki} + c_{lk,ij} = 0.$$

PROOF. We begin with an equation derived in the proof of Proposition 1. For the geodesic $\gamma^k(t) = \xi^k t$ we have

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))\xi^i\xi^j = 0;$$

multiplying by t^2, we have

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t))x^i(\gamma(t))x^j(\gamma(t)) = 0.$$

Since these geodesics go through all points in a neighborhood of p, we have the following relation between the functions Γ_{ij}^k and x^i:

$$\sum_{i,j=1}^{n} \Gamma_{ij}^k x^i x^j = 0. \tag{1}$$

Since the tangent vector to the geodesic $\gamma^k(t) = \xi^k t$ has constant length, we also obtain

$$\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = \sum_{i=1}^{n}(\xi^i)^2,$$

which leads, in the same way, to the equation

$$\sum_{i,j=1}^{n} g_{ij}x^i x^j = \sum_{i=1}^{n}(x^i)^2. \tag{2}$$

Now equation (1) leads to

$$\sum_{i,j=1}^{n} [ij,k]x^i x^j = 0,$$

i.e., to

$$\sum_{i,j=1}^{n} \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0.$$

Interchanging the indices i and j in the second term, we can write

(3)
$$\sum_{i,j=1}^{n} \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0.$$

Our penultimate goal is to break this equation up into two sums, each of which is individually 0; the conditions on the c's, which are our ultimate goal, will then follow fairly easily. To achieve this, our antepenultimate goal is to prove that $x^\beta = \sum_\alpha g_{\beta\alpha} x^\alpha$; these equations are at least reasonable, for they imply (2). To prove these relations, we first introduce the functions $\bar{x}^\beta$ defined by

$$\bar{x}^\beta = \sum_{\alpha=1}^{n} g_{\beta\alpha} x^\alpha.$$

Note that

$$\frac{\partial \bar{x}^\beta}{\partial x^\gamma} = \sum_{\alpha=1}^{n} \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} x^\alpha + g_{\beta\gamma}.$$

Substituting in (3), we obtain

$$\begin{aligned}
0 &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} \frac{\partial g_{ik}}{\partial x^j} x^i\right) x^j - \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial g_{ij}}{\partial x^k} x^j\right) x^i \\
&= \sum_{j=1}^{n}\left(\frac{\partial \bar{x}^k}{\partial x^j} - g_{kj}\right) x^j - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial \bar{x}^i}{\partial x^k} - g_{ik}\right) x^i \\
&= \sum_{j=1}^{n} \frac{\partial \bar{x}^k}{\partial x^j} x^j - \bar{x}^k - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} x^i - \bar{x}^k\right) \\
&= \sum_{j=1}^{n} \frac{\partial \bar{x}^k}{\partial x^j} x^j - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial \bar{x}^i}{\partial x^k} x^i + \bar{x}^k\right) \\
&= \sum_{j=1}^{n} \frac{\partial \bar{x}^k}{\partial x^j} x^j - \frac{1}{2} \cdot \frac{\partial\left(\sum_{i=1}^{n} x^i \bar{x}^i\right)}{\partial x^k}.
\end{aligned}$$

Now by (2) and the definition of $\bar{x}^i$, we have

$$\sum_{i=1}^{n} x^i \bar{x}^i = \sum_{i=1}^{n} (x^i)^2,$$

so we obtain

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \frac{\partial \bar{x}^k}{\partial x^j} x^j - x^k \\ &= \sum_{j=1}^{n} \frac{\partial(\bar{x}^k - x^k)}{\partial x^j} x^j. \end{aligned}$$

This equation shows that along any geodesic $\gamma(t) = \xi^i t$ we have

$$\frac{d[\bar{x}^k - x^k](\gamma(t))}{dt} = 0,$$

so that $\bar{x}^k - x^k$ is constant along the geodesic. Since $g_{ij}(p) = \delta_{ij}$, we clearly have $\bar{x}^k(p) = x^k(p)$. Moreover, these geodesics pass through all points in a neighborhood of p. Thus $\bar{x}^k = x^k$ in a neighborhood of p, so that we finally obtain the desired equations

(4) $$\sum_{\alpha=1}^{n} g_{k\alpha} x^\alpha = x^k.$$

Now we differentiate (4) to obtain

$$\sum_{\alpha=1}^{n} \frac{\partial g_{k\alpha}}{\partial x^l} x^\alpha + g_{kl} = \delta_{kl};$$

multiplying by x^l and summing, we obtain

$$\sum_{\alpha,l=1}^{n} \frac{\partial g_{k\alpha}}{\partial x^l} x^\alpha x^l = \sum_{l=1}^{n} -g_{kl} x^l + \delta_{kl} x^l,$$

which, together with (4) gives

$$\sum_{\alpha,l=1}^{n} \frac{\partial g_{k\alpha}}{\partial x^l} x^\alpha x^l = -x^k + x^k = 0,$$

and we have thus obtained the first part of our penultimate goal,

$$\sum_{i,j=1}^{n} \frac{\partial g_{ik}}{\partial x^j} x^i x^j = 0. \tag{5}$$

Together with (3), it implies the other part,

$$\sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k} x^i x^j = 0. \tag{6}$$

We now obtain the desired equations as follows. Along the geodesic $\gamma^k(t) = \xi^k t$ we have, by (6),

$$\sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k}(\gamma(t))\xi^i\xi^j t^2 = 0. \tag{7}$$

This implies that

$$\sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k}(\gamma(t))\xi^i\xi^j = 0 \tag{8}$$

for $t \neq 0$, and hence even for $t = 0$, by continuity. Differentiating (7) with respect to t gives

$$\begin{aligned} 0 &= \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k}(\gamma(t))\xi^i\xi^j \cdot 2t + \sum_{i,j,l=1}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(\gamma(t))\xi^l\xi^i\xi^j t^2 \\ &= \sum_{i,j,l=1}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(\gamma(t))\xi^l\xi^i\xi^j t^2, \qquad \text{by (8);} \end{aligned}$$

consequently,

$$0 = \sum_{i,j,l=1}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(\gamma(t))\xi^i\xi^j\xi^l$$

for all $t \neq 0$, and hence also for $t = 0$. Setting $t = 0$, we obtain

$$\sum_{i,j,l=1}^{n} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(p)\xi^i\xi^j\xi^l = 0.$$

This equation holds for *all* n-tuples $\xi^1, \dots, \xi^n$. From this we easily derive

(A) $$c_{ij,kl} + c_{il,jk} + c_{jl,ik} = 0.$$

Applying the same process to (5), we obtain

(B) $$c_{ki,jl} + c_{kj,li} + c_{kl,ij} = 0.$$

In (B) we interchange k and l, to obtain

$$c_{li,jk} + c_{lj,ki} + c_{kl,ij} = 0.$$

Comparing this equation with (A), we obtain the first of the desired relations,

$$c_{ij,kl} = c_{kl,ij}.$$

Moreover, using this relation with either (A) or (B), we obtain the second of the desired relations,

$$c_{li,jk} + c_{lj,ki} + c_{lk,ij} = 0.$$

And thus we are done! ❖

When we put all these results together we see that the quadratic function

$$Q(v_p, w_p) = \frac{1}{2} \sum_{i,j;k,l} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} dx^i(v_p)\, dx^j(v_p)\, dx^k(w_p)\, dx^l(w_p)$$

can be written

$$\boxed{Q(v_p, w_p) = \frac{1}{3} \sum_{i,j;k,l} c_{ij,kl} (dx^i \wedge dx^k) \cdot (dx^j \wedge dx^l)(v_p, w_p).}$$

We thus see that the quadratic function Q, obtained from the Taylor expansion of $\| \ \|^2$ in Riemannian normal coordinates, has special properties which allow us to define, for any 2-dimensional subspace $W \subset M_p$, a number

$$Q(W) = \frac{Q(v_p, w_p)}{\|v_p, w_p\|^2} \qquad v_p, w_p \text{ any basis for } W.$$

The work of the last four Propositions, which establishes this fact, is completely suppressed in Riemann's account, where the final result is merely stated, at the beginning of section 2 of Part II. Riemann then makes some remarkable claims. First, Riemann interprets Q for a surface:

(1) If M is 2-dimensional and $W = M_p$, then $-3Q(W)$ is just the Gaussian curvature $K(p)$ given by Theorem 3-7; we thus have an intrinsic definition of K, obtained by picking a special class of coordinate systems determined by the metric. (Riemann needs the factor $-3/4$ because he divides $Q(v_p, w_p)$ by the square of the area of the *triangle* spanned by v_p and w_p.)

At the end of section 2, Riemann interprets Q for an n-manifold:

(2) If M is n-dimensional, $W \subset M_p$ is a 2-dimensional subspace, and $\mathcal{O} \subset W$ is a neighborhood of $0 \in W$ on which exp is a diffeomorphism, then $-3Q(W)$ is the Gaussian curvature at p of the surface $\exp(\mathcal{O})$, with the metric it inherits as a submanifold of M.

But the most important claim is made in section 2. In an n-dimensional vector space there are $n\frac{n-1}{2}$ "independent" 2-dimensional subspaces: if $v_1, \dots, v_n$ is a basis, we can choose the subspaces spanned by v_i and v_j, for $i < j$. Riemann claims that the metric $\langle\ ,\ \rangle$ is determined if $Q(W)$ is known for $n\frac{n-1}{2}$ independent 2-dimensional subspaces $W \subset M_q$ at each point q, for example, if Q is known for the subspaces spanned by each $\partial/\partial x^i|_q$ and $\partial/\partial x^j|_q$ $(i < j)$.

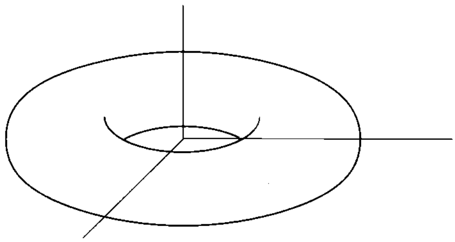
A very special case of this general claim is the following, which we will henceforth call the *Test Case*:

(3) If M is n-dimensional and $Q = 0$ for $n\frac{n-1}{2}$ independent 2-dimensional subspaces of each M_q, then M is **flat**, that is, M is locally isometric to $\mathbb{R}^n$ with its usual inner product.

In connection with the Test Case, it should be pointed out that a local isometry with $\mathbb{R}^n$ is the best we can hope for, since there are Riemannian manifolds which are not homeomorphic to $\mathbb{R}^n$, but which are locally isometric to $\mathbb{R}^n$, and hence have $Q = 0$ everywhere. The simplest example of such a manifold, the "flat torus", is constructed as follows. The torus T can be obtained from $\mathbb{R}^2$ by identifying (x, y) with (x', y') if and only if

$$y' - y,\ x' - x \in \mathbb{Z}$$

(compare pg. I.372). The map $\pi : \mathbb{R}^2 \to T$, defined by taking (x, y) to its equivalence class, is locally a diffeomorphism, and there is clearly a unique metric $\langle\ ,\ \rangle$ on T such that $\pi^*\langle\ ,\ \rangle$ is the usual Riemannian metric on $\mathbb{R}^2$; consequently, $(T, \langle\ ,\ \rangle)$ is locally isometric to $\mathbb{R}^2$ with its usual Riemannian metric. Notice that the usual torus in $\mathbb{R}^3$, with the induced Riemannian metric, is *not* flat; it has positive Gaussian curvature on the part furthest from the axis,



and negative Gaussian curvature on the part nearest the axis. However, if we consider $S^1 \subset \mathbb{R}^2$, then it is easy to see that $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$, with the induced Riemannian metric, is flat.

One other remark should probably be made about the Test Case. At first sight, the Test Case might seem to be little more than a theorem about functions whose second partial derivatives are everywhere zero. However, it is actually quite different from this simple sort of result, since the value of Q at different points is defined in terms of different coordinate systems, each chosen specifically for one point.

Now our aim in the rest of this chapter is to prove assertions (1), (2), and (3). (The general claim that Q determines the metric will be considered later, as will the information given in sections 4 and 5 of Part II.) However, we will defer the proofs of assertions (1), (2), and (3) to another section of this chapter, not only in order to provide ourselves with a brief respite, but also to allow Riemann to add one or two more brilliant ideas.

C. A PRIZE ESSAY

The second edition of Riemann's collected works includes an unpublished paper, in Latin, which was submitted to the Paris Academy in 1861, to compete for a prize on a question involving heat conduction. In 1868, ten years after it had been offered, the prize was finally withdrawn. Because the way of obtaining the results of his essay were not fully explained, the prize was not awarded to Riemann, whose health prevented the more detailed handling of the subject which he had intended.

An extract from this paper is given below.* It should not be very hard to read, but the significance of the equations obtained there is only suggested by Riemann's final remarks; in the next part of this chapter we will have a great deal more to add. In the translation I have made some minor changes of notation.

An Extract From Riemann's Paper of 1861

Second Part

On the transformation of the expression $\sum\limits_{i,j} g_{ij}\, dy^i\, dy^j$
into the given form $\sum\limits_{i,j} a_{ij}\, dx^i\, dx^j$.

When the inquiry of the third Academy is restricted to homogeneous bodies, in which the resulting conductivities are constants, we develop the first condition that the expression $\sum\limits_{i,j} g_{ij}\, dy^i\, dy^j$, in which the y^i are functions of the x^i, can be transformed into the form $\sum\limits_{i,j} a_{ij}\, dx^i\, dx^j$ with given constant coefficients a_{ij}.

The expression $\sum\limits_{i,j} a_{ij}\, dx^i\, dx^j$, if it is, as we shall suppose, a positive form in the dx^i, can always be put in the simplified form $\sum\limits_{i} (dx^i)^2$. Thus if $\sum\limits_{i,j} g_{ij}\, dy^i\, dy^j$ can be transformed into the form $\sum\limits_{i,j} a_{ij}\, dx^i\, dx^j$, it can likewise be reduced to the form $\sum\limits_{i} (dx^i)^2$ and vice versa. We therefore ask whether it can be put in the form $\sum\limits_{i} (dx^i)^2$.

* Certain omissions, indicated by " . . . ", are considered in Addendum 2 to Chapter 6.

Let $G = \det(g_{ij})$ and let γ_{ij} be the cofactor; in this way $\sum_i g_{ij}\gamma_{ij} = G$ and $\sum_i g_{ij}\gamma_{ik} = 0$ if $j \gtrless k$.

If $\sum_{i,j} g_{ij}\, dy^i\, dy^j = \sum_i (dx^i)^2$ for arbitrary values of the dx^i, substituting $d + \delta$ for d leads also to $\sum_{i,j} g_{ij}\, dy^i\, \delta y^j = \sum_i dx^i\, \delta x^i$ for arbitrary values of the dx^i and δx^i.

Consequently, if the dy^i are expressed in terms of the dx^i and the δx^i in terms of the δy^i, it follows that

$$\frac{\partial x^\beta}{\partial y^\alpha} = \sum_i g_{\alpha i} \frac{\partial y^i}{\partial x^\beta} \tag{1}$$

and consequently

$$\frac{\partial y^i}{\partial x^\beta} = \sum_\alpha \frac{\gamma_{\alpha i}}{G} \frac{\partial x^\beta}{\partial y^\alpha}. \tag{2}$$

Thus we further deduce, seeing that

$$\sum_\alpha \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} = 1 \qquad \text{and} \qquad \sum_\alpha \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^j} = 0 \quad \text{if} \quad i \gtrless j,$$

$$\text{(3)} \quad \sum_\alpha \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\alpha}{\partial y^j} = g_{ij}, \qquad \text{(4)} \quad \sum_\alpha \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\alpha} = \frac{\gamma_{ij}}{G}$$

and differentiating formula (3),

$$\sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} \frac{\partial x^\alpha}{\partial y^j} + \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} \frac{\partial x^\alpha}{\partial y^i} = \frac{\partial g_{ij}}{\partial y^k}.$$

Now from these expressions for

$$\frac{\partial g_{ij}}{\partial y^k}, \quad \frac{\partial g_{ik}}{\partial y^j}, \quad \frac{\partial g_{jk}}{\partial y^i}$$

we can write

$$2 \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} \frac{\partial x^\alpha}{\partial y^i} = \frac{\partial g_{ij}}{\partial y^k} + \frac{\partial g_{ik}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^i} \tag{5}$$

and if these quantities are designated by p_{ijk}, then

$$2\frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} = \sum_i \frac{\partial y^i}{\partial x^\alpha} p_{ijk}. \tag{6}$$

Differentiating the quantities p_{ijk} again yields

$$\frac{\partial p_{ijk}}{\partial y^l} - \frac{\partial p_{ijl}}{\partial y^k} = 2\sum_\nu \frac{\partial^2 x^\nu}{\partial y^j \partial y^k}\frac{\partial^2 x^\nu}{\partial y^i \partial y^l} - 2\sum_\nu \frac{\partial^2 x^\nu}{\partial y^j \partial y^l}\frac{\partial^2 x^\nu}{\partial y^i \partial y^k},$$

whence finally, substituting the values found in (6) and (4),

$$\frac{\partial^2 g_{ik}}{\partial y^j \partial y^l} + \frac{\partial^2 g_{jl}}{\partial y^i \partial y^k} - \frac{\partial^2 g_{il}}{\partial y^j \partial y^k} - \frac{\partial^2 g_{jk}}{\partial y^i \partial y^l} + \frac{1}{2}\sum_{\alpha,\beta} (p_{\alpha jl} p_{\beta ik} - p_{\alpha il} p_{\beta jk})\frac{\gamma_{\alpha\beta}}{G} = 0. \tag{I}$$

The functions g_{ij} must necessarily satisfy these equations whenever $\sum_{i,j} g_{ij}\, dy^i\, dy^j$ can be transformed into the form $\sum_i (dx^i)^2$: we denote the left side of this equation by

$$(ij,kl).$$

... Given an acquaintance with the traditional methods, it is demonstrated without difficulty that these ... conditions when they are satisfied, suffice ....

D. THE BIRTH OF THE RIEMANN CURVATURE TENSOR

All the developments in this part of the chapter have their origin in the following question, which Riemann considers in the paper of Part C: When is a Riemannian manifold $(M, \langle\ ,\ \rangle)$ *flat* (locally isometric to $\mathbb{R}^n$ with its usual Riemannian metric)? In other words, when is there a coordinate system $x^1, \ldots, x^n$ on M for which

$$(1) \qquad \langle\ ,\ \rangle = \sum_{i=1}^{n} dx^i \otimes dx^i\,?$$

We are going to seek an answer to this question in as straightforward a manner as possible; the quadratic function Q will not be used at all, but at the end it will make a surprise appearance.

We begin by choosing an arbitrary coordinate system y, in terms of which the metric $\langle\ ,\ \rangle$ can be written

$$(2) \qquad \langle\ ,\ \rangle = \sum_{i,j=1}^{n} g_{ij}\, dy^i \otimes dy^j;$$

and we then seek conditions on the g_{ij} in order for (1) to hold for some coordinate system x. Since this is a purely local question, we can assume that $y^1, \ldots, y^n$ is just the standard coordinate system on $\mathbb{R}^n$.

If we express the dx^i in terms of the dy^j, and equate the coefficients of $dy^i \otimes dy^j$ in (2) with the resulting coefficients in (1), we find that the coordinate system $x^1, \ldots, x^n$ has the desired property if and only if

$$(3) \qquad \sum_{\alpha} \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\alpha}{\partial y^j} = g_{ij}.$$

From equation (3) we can immediately derive another, for we obtain

$$\begin{aligned}
\sum_{j,\beta} g_{ij}\frac{\partial y^j}{\partial x^\alpha}\frac{\partial y^k}{\partial x^\beta} &= \sum_{j,\beta,\alpha} \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\alpha}{\partial y^j}\frac{\partial y^j}{\partial x^\beta}\frac{\partial y^k}{\partial x^\beta} \\
&= \sum_{\alpha,\beta} \frac{\partial x^\alpha}{\partial y^i}\delta^\alpha_\beta\frac{\partial y^k}{\partial x^\beta} \\
&= \sum_{\alpha} \frac{\partial x^\alpha}{\partial y^i}\frac{\partial y^k}{\partial x^\alpha} = \delta^k_i,
\end{aligned}$$

which shows that if the coordinate system $x^1, \dots, x^n$ has the desired property, then

$$\sum_{\beta} \frac{\partial y^i}{\partial x^\beta} \frac{\partial y^j}{\partial x^\beta} = g^{ij}. \tag{4}$$

Conversely, (4) implies (3). These results are just the equations (3) and (4) that Riemann obtains. Notice that Riemann begins with the square of the norm $\|\ \|^2 = \sum g_{ij}\, dy^i \cdot dy^j$, and then uses polarization to obtain the inner product, which he writes as $\sum g_{ij}\, dy^i\, \delta y^j$. Riemann also treats the two coordinate systems x and y on an equal footing throughout, so that his derivations of (3) and (4) are somewhat different. From (4) we obtain

$$\begin{aligned}\sum_{i,j} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} &= \sum_{\beta,i,j} \frac{\partial y^i}{\partial x^\beta} \frac{\partial y^j}{\partial x^\beta} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \\ &= \sum_{\beta} \delta^\mu_\beta \delta^\nu_\beta,\end{aligned}$$

and thus the coordinate system $x^1, \dots, x^n$ has the desired property if and only if

$$\sum_{i,j} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} = \delta_{\mu\nu}. \tag{4$'$}$$

This equation, which we will find more useful than (4), can be derived directly from (3) in the following way. If $A = (a_{ij}) = (\partial x^i / \partial y^j)$, and $G = (g_{ij})$, then (3) says that

$$A^{\mathbf{t}} \cdot A = G,$$

where $A^{\mathbf{t}}$ is the transpose of A; this is equivalent to

$$G^{-1} = A^{-1} \cdot (A^{\mathbf{t}})^{-1},$$

and hence to

$$AG^{-1}A^{\mathbf{t}} = I,$$

which is just (4′). In particular, this shows immediately that (4′) is equivalent to (3).

Now equation (3) is a partial differential equation for the functions x^α. In Chapter I.6 we developed a general theory for partial differential equations, but we notice at once that (3) is not an equation of the type to which our theory applies. Our first task will thus be to obtain from (3) an equation that we do know how to handle. The situation is very much like, and may profitably be compared to, that which occurs in Problem I.7-19, where the analysis of a certain set of partial differential equations is reduced to Theorem I.6-1, together with the Poincaré Lemma. To treat equation (3), we begin by differentiating (about all we can do), to obtain

$$\sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^k}\frac{\partial x^\alpha}{\partial y^j} + \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\frac{\partial x^\alpha}{\partial y^i} = \frac{\partial g_{ij}}{\partial y^k}.$$

By writing down this equation for

$$\frac{\partial g_{ij}}{\partial y^k}, \quad \frac{\partial g_{ik}}{\partial y^j}, \quad \frac{\partial g_{jk}}{\partial y^i},$$

and combining, we obtain an equation equivalent to Riemann's,

$$(5) \qquad \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\frac{\partial x^\alpha}{\partial y^i} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial y^k} + \frac{\partial g_{ik}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^i}\right) = [jk,i].$$

Thus, the symbols $[jk,i]$, which came up naturally in the calculus of variations, also come up naturally in this different context. After Riemann's Habilitations lecture was published, in 1866, several mathematicians independently derived his results or considered related questions. Christoffel, in particular, introduced these combinations of the partial derivatives of the g_{ij}'s, and the symbols $[ij,k]$ and Γ_{ij}^k are called the **Christoffel symbols** of the **first** and **second kinds**, respectively (Christoffel actually used $\left[{ij \atop k}\right]$ and $\left\{{ij \atop k}\right\}$, which do not accommodate themselves to the summation convention). In the next chapter we will see one important use which Christoffel made of these symbols.

At this point we will depart slightly from Riemann's treatment, in order to obtain equations to which Theorem I.6-1 directly applies. From (5) we obtain

$$\begin{aligned}\sum_{i,\gamma} g^{i\gamma}\frac{\partial x^\lambda}{\partial y^\gamma}[jk,i] &= \sum_{\alpha,i,\gamma} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\lambda}{\partial y^\gamma} g^{i\gamma} \\ &= \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\left(\sum_{i,\gamma}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\lambda}{\partial y^\gamma} g^{i\gamma}\right) \\ &= \sum_\alpha \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\delta_{\alpha\lambda} \qquad \text{by } (4'),\end{aligned}$$

so we obtain, finally,

$$\frac{\partial^2 x^\lambda}{\partial y^j \partial y^k} = \sum_{\gamma=1}^{n} \Gamma_{jk}^{\gamma} \frac{\partial x^\lambda}{\partial y^\gamma} \tag{6}$$

(which is easily seen to be equivalent to Riemann's equation (6)); we will also write this equation as

$$\frac{\partial\left(\dfrac{\partial x^\lambda}{\partial y^j}\right)}{\partial y^k} = \sum_{\gamma=1}^{n} \Gamma_{jk}^{\gamma} \frac{\partial x^\lambda}{\partial y^\gamma}.$$

Notice that the index λ plays no special role here; all functions x^λ satisfy the *same* equation. Thus, for each λ the n-tuple of functions

$$\alpha = \left(\frac{\partial x^\lambda}{\partial y^1}, \dots, \frac{\partial x^\lambda}{\partial y^n}\right) \qquad \alpha\colon \mathbb{R}^n \to \mathbb{R}^n$$

satisfies the set of partial differential equations

$$\frac{\partial \alpha}{\partial y^k}(y) = f_k(y, \alpha(y)), \tag{$*$}$$

where $f_k\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f_k^j(y,z) = \sum_{\gamma=1}^{n} \Gamma_{jk}^{\gamma}(y) \cdot z^\gamma.$$

Since this is true for every λ, the equation $(*)$ has n solutions whose initial values at some point, 0 say, are linearly independent. Since constant linear combinations of solutions of $(*)$ are also solutions, it follows that $(*)$ has solutions with arbitrary initial conditions at 0. From Theorem I.6-1 we thus obtain necessary integrability conditions,

$$\frac{\partial f_k}{\partial y^l} - \frac{\partial f_l}{\partial y^k} + \sum_{\mu=1}^{n} \frac{\partial f_k}{\partial z^\mu} f_l^\mu - \sum_{\mu=1}^{n} \frac{\partial f_l}{\partial z^\mu} f_k^\mu = 0.$$

In our case, looking at the j^{th} components of these equations, we obtain

$$\sum_{\gamma=1}^{n} \frac{\partial \Gamma_{jk}^{\gamma}}{\partial y^l} z^\gamma - \sum_{\gamma=1}^{n} \frac{\partial \Gamma_{jl}^{\gamma}}{\partial y^k} z^\gamma + \sum_{\mu=1}^{n} \Gamma_{jk}^{\mu} \sum_{\gamma=1}^{n} \Gamma_{\mu l}^{\gamma} z^\gamma - \sum_{\mu=1}^{n} \Gamma_{jl}^{\mu} \sum_{\gamma=1}^{n} \Gamma_{\mu k}^{\gamma} z^\gamma = 0.$$

Since these relations must hold for all $z = (z^1, \dots, z^n)$, we obtain

$$(**) \qquad 0 = R^{\gamma}{}_{jlk} \overset{\text{def}}{=} \frac{\partial \Gamma^{\gamma}_{kj}}{\partial y^l} - \frac{\partial \Gamma^{\gamma}_{lj}}{\partial y^k} + \sum_{\mu=1}^{n} (\Gamma^{\mu}_{kj}\Gamma^{\gamma}_{l\mu} - \Gamma^{\mu}_{lj}\Gamma^{\gamma}_{k\mu})$$

as necessary conditions that $\sum g_{ij}\, dy^i \otimes dy^j = \sum dx^i \otimes dx^i$ for some coordinate system $x = (x^1, \dots, x^n)$. Notice that the set of equations $R^{\gamma}{}_{jlk} = 0$ is equivalent to the set of equations

$$R_{ijlk} \overset{\text{def}}{=} \sum_{\gamma=1}^{n} g_{i\gamma} R^{\gamma}{}_{jlk} = 0.$$

The quantities R_{ijlk} can be expressed in another way, after a little calculation. Note first that

$$\begin{aligned} \sum_{\gamma=1}^{n} g_{i\gamma} \frac{\partial \Gamma^{\gamma}_{jk}}{\partial y^l} &= \frac{\partial}{\partial y^l}\left(\sum_{\gamma=1}^{n} g_{i\gamma}\Gamma^{\gamma}_{jk}\right) - \sum_{\gamma=1}^{n} \Gamma^{\gamma}_{jk} \frac{\partial g_{i\gamma}}{\partial y^l} \\ &= \frac{\partial [jk,i]}{\partial y^l} - \sum_{\gamma=1}^{n} \Gamma^{\gamma}_{jk}\big([il,\gamma] + [\gamma l, i]\big). \end{aligned}$$

Substituting into (**), and remembering the definition of $[ij,k]$, we obtain

$$(***) \qquad \begin{aligned} R_{ijlk} = \frac{1}{2}&\left(\frac{\partial^2 g_{ik}}{\partial y^j \partial y^l} + \frac{\partial^2 g_{jl}}{\partial y^i \partial y^k} - \frac{\partial^2 g_{il}}{\partial y^j \partial y^k} - \frac{\partial^2 g_{jk}}{\partial y^i \partial y^l}\right) \\ &+ \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}\big([jl,\alpha]\cdot[ik,\beta] - [il,\alpha]\cdot[jk,\beta]\big). \end{aligned}$$

The condition $R_{ijlk} = 0$ is just the condition (I) which Riemann obtains (note that Riemann's p_{ijk} equals $2[jk,i]$)—the quantity which we have denoted by R_{ijlk} is what Riemann denotes by $2(ij,kl)$; the factor of 2 is not particularly significant, nor is the interchange of l and k, for it is easily seen that $R_{ijlk} = -R_{ijkl}$.

The notation $R^i{}_{jkl}$ has been picked in anticipation of the following result.

5. PROPOSITION. On a Riemannian manifold $(M, \langle\ ,\ \rangle)$ there is a tensor of type $\binom{3}{1}$ whose components in any coordinate system y are

$$R^i{}_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial y^k} - \frac{\partial \Gamma^i_{jk}}{\partial y^l} + \sum_{\mu=1}^{n} (\Gamma^{\mu}_{jl}\Gamma^i_{\mu k} - \Gamma^{\mu}_{jk}\Gamma^i_{\mu l})$$

(where $\langle\ ,\ \rangle = \sum g_{ij}\, dy^i \otimes dy^j$, and the Christoffel symbols Γ are defined as usual).

PROOF. We just compute that the components transform "correctly"!! In other words, if ${R'}^{i}{}_{jlk}$ are defined by the same formula, with respect to the coordinate system y', we show that

$$ {R'}^{\alpha}{}_{\beta\gamma\delta} = \sum_{i,j,k,l} R^{i}{}_{jkl} \frac{\partial y^j}{\partial y'^\beta} \frac{\partial y^k}{\partial y'^\gamma} \frac{\partial y^l}{\partial y'^\delta} \frac{\partial y'^\alpha}{\partial y^i}. $$

To do this, all one needs is the result from Problem I.9-22,

$$ {\Gamma'}^{\gamma}_{\alpha\beta} = \sum_{i,j,k} \Gamma^{k}_{ij} \frac{\partial y^i}{\partial y'^\alpha} \frac{\partial y^j}{\partial y'^\beta} \frac{\partial y'^\gamma}{\partial y^k} + \sum_{\mu=1}^{n} \frac{\partial^2 y^\mu}{\partial y'^\alpha \partial y'^\beta} \frac{\partial y'^\gamma}{\partial y^\mu}, $$

and plenty of perseverance.

SLIGHTLY MORE MOTIVATED PROOF. Begin with the equation

$$ \sum_{i,j} g_{ij}\, dy^i \otimes dy^j = \langle\ ,\ \rangle = \sum_{i,j} {g'}_{ij}\, dy'^i \otimes dy'^j, $$

and repeat the whole sequence of computations which we performed in the special case that ${g'}_{ij} = \delta_{ij}$. The result will be the desired transformation law. (The integrability conditions (**) then follow as a necessary condition for the existence of a coordinate system y' with ${g'}_{ij} = \delta_{ij}$, for in such a coordinate system we clearly have ${R'}^{i}{}_{jkl} = 0$, which in turn implies that all $R^{i}{}_{jkl} = 0$.) ❖

We have thus stumbled onto a new tensor, the **Riemann curvature tensor,** which in the coordinate system y equals

$$ \sum_{i,j,k,l} R^{i}{}_{jkl}\, dy^j \otimes dy^k \otimes dy^l \otimes \frac{\partial}{\partial y^i}. $$

Eventually we hope to have a useful invariant definition of this tensor; this will involve an enormous amount of exploration. For the time being, we simply accept the classical definition, which arises naturally as an integrability condition, and explain how it is connected with curvature. In the process we will obtain an invariant, but extraordinarily clumsy, definition of the curvature tensor.

It will be convenient to introduce a bit of modern terminology, and denote by R the tensor with components $R^{i}{}_{jkl}$. Since this tensor is of type $\binom{3}{1}$ it may be regarded as a function taking three vectors to another vector. The value of R on $X, Y, Z \in M_p$ will be denoted by

$$ R(Y, Z)X \in M_p, $$

and hence we have

$$R\left(\left.\frac{\partial}{\partial y^k}\right|_p, \left.\frac{\partial}{\partial y^l}\right|_p\right)\left.\frac{\partial}{\partial y^j}\right|_p = \sum_{i=1}^n R^i{}_{jkl}(p)\cdot\left.\frac{\partial}{\partial y^i}\right|_p$$

(the reason for choosing the notation $R(Y,Z)X$ comes out in Proposition 6).

The numbers $R_{ijkl} = \sum_\gamma g_{i\gamma}R^\gamma{}_{jkl}$ are also the components of a tensor, of type $\binom{4}{0}$, but it is unnecessary to perform any calculations to verify this. Clearly

$$R_{ijkl}(p) = \left\langle R\left(\left.\frac{\partial}{\partial y^k}\right|_p, \left.\frac{\partial}{\partial y^l}\right|_p\right)\left.\frac{\partial}{\partial y^j}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_p\right\rangle,$$

so the tensor in question is just the multilinear map

$$(X,Y,Z,W)\mapsto\langle R(Z,W)Y,X\rangle.$$

This function of four tangent vectors is closely connected with the quadratic function introduced in Part B of this chapter:

6. PROPOSITION. Let x be a Riemannian normal coordinate system at p, and Q the quadratic function on $M_p\times M_p$ defined by

$$Q(X,Y) = \sum_{i,j;k,l} c_{ij,kl}\,dx^i(X)\,dx^j(X)\,dx^k(Y)\,dx^l(Y),$$

where

$$c_{ij,kl} = \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}.$$

Then

$$Q(X,Y) = -\frac{1}{3}\langle R(X,Y)Y,X\rangle.$$

PROOF. We have seen that

$$\begin{aligned}
3Q(X,Y) &= \sum_{i,j,k,l} c_{ij,kl}(dx^i\wedge dx^k)\cdot(dx^j\wedge dx^l)(X,Y)\\
&= \sum_{i,j,k,l} c_{ij,kl}\,dx^i(X)\,dx^j(X)\,dx^k(Y)\,dx^l(Y)\\
&\quad+\sum_{i,j,k,l} c_{ij,kl}\,dx^k(X)\,dx^l(X)\,dx^i(Y)\,dx^j(Y)\\
&\quad-\sum_{i,j,k,l} c_{ij,kl}\,dx^j(X)\,dx^k(X)\,dx^i(Y)\,dx^l(Y)\\
&\quad-\sum_{i,j,k,l} c_{ij,kl}\,dx^i(X)\,dx^l(X)\,dx^j(Y)\,dx^k(Y).
\end{aligned}$$

By switching indices we can rewrite this as

$$
\begin{aligned}
3Q(X,Y) = &\sum_{i,j,k,l} c_{ik,jl}\, dx^i(X)\, dx^j(Y)\, dx^k(X)\, dx^l(Y) && \text{[interchange } j \text{ and } k\text{]} \\
&+ \sum_{i,j,k,l} c_{jl,ik} \qquad \text{''} && \text{[interchange } i \text{ and } l\text{]} \\
&- \sum_{i,j,k,l} c_{il,jk} \qquad \text{''} && \text{[change } i \text{ to } l;\ j \text{ to } i;\ l \text{ to } j\text{]} \\
&- \sum_{i,j,k,l} c_{jk,il} \qquad \text{''} && \text{[change } i \text{ to } k;\ k \text{ to } l;\ l \text{ to } i\text{]}
\end{aligned}
$$

$$
= \sum_{i,j,k,l} (c_{ik,jl} + c_{jl,ik} - c_{il,jk} - c_{jk,il})\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l(X,Y,X,Y).
$$

Now in Riemannian normal coordinates, the Christoffel symbols $[ij,k]$ are all 0 at p, since all $\partial g_{ij}/\partial x^k$ are 0 at p. Referring to equation (***) we thus have

$$
\begin{aligned}
3Q(X,Y) &= \sum_{i,j,k,l} R_{ijlk}(p)\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l(X,Y,X,Y) \\
&= -\sum_{i,j,k,l} R_{ijkl}(p)\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l(X,Y,X,Y) \\
&= -\langle R(X,Y)Y,X\rangle. \quad ❖
\end{aligned}
$$

We are now ready to verify some of Riemann's claims.

7. PROPOSITION. Let $(M, \langle\ ,\ \rangle)$ be a 2-dimensional Riemannian manifold, and let $X, Y \in M_p$ be linearly independent. Let $\|X,Y\|$ denote the area of the parallelogram spanned by X and Y. Then

$$
K(p) = \frac{\langle R(X,Y)Y,X\rangle}{\|X,Y\|^2} \qquad [= \langle R(X,Y)Y,X\rangle \text{ if } X \text{ and } Y \text{ are orthornomal}]
$$

is the same as the Gaussian curvature at p defined by the formula in Theorem 3-7 (in particular, this proves that the formula in Theorem 3-7 is indeed independent of the coordinate system).

FIRST PROOF. Let (x, y) be a coordinate system on a neighborhood of p. It obviously suffices to verify the theorem when $X = \partial/\partial x|_p$ and $Y = \partial/\partial y|_p$, since by Proposition 6, and the results of Part B, the numerator is multiplied by the same factor as the denominator when we change to any other pair of vectors. In this case,

$$\begin{aligned}\langle R(X,Y)Y,X\rangle &= \left\langle R\left(\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p\right)\left.\frac{\partial}{\partial y}\right|_p, \left.\frac{\partial}{\partial x}\right|_p\right\rangle \\ &= R_{1212}(p).\end{aligned}$$

If we write

$$\langle\ ,\ \rangle = E\,dx\otimes dx + F\,dx\otimes dy + F\,dy\otimes dx + G\,dy\otimes dy,$$

so that

$$\begin{aligned} g_{11} &= E \\ g_{12} = g_{21} &= F \\ g_{22} &= G, \end{aligned}$$

then (by the formula on pg. I.308)

$$\|X,Y\|^2 = EG - F^2,$$

so we must prove that

$$4R_{1212}(EG-F^2) = 4(EG-F^2)^2K,$$

where the right side is given by the formula in Theorem 3-7. This is a fairly straightforward calculation from (∗∗∗) on page 188. The first term in (∗∗∗) corresponds to the last in the formula for $4(EG-F^2)^2K$, and the second corresponds to the first three in the latter formula. In carrying out the calculation, note that

$$\begin{aligned} g^{11} &= \frac{G}{EG-F^2} \\ g^{12} = g^{21} &= \frac{-F}{EG-F^2} \\ g^{22} &= \frac{E}{EG-F^2}; \end{aligned}$$

the denominators cancel out the unwanted factor in $4R_{1212}(EG-F^2)$.

SECOND PROOF (OUTLINE). Let (r, φ) be the coordinate system around p which is introduced on page 136. We know that in this coordinate system

$$\langle\ ,\ \rangle = dr \otimes dr + G\, d\varphi \otimes d\varphi$$

for some function G, and (see page 145) that

$$K(p) = -\frac{\partial^3 \sqrt{G}}{\partial r^3}(p).$$

Introduce a Riemannian normal coordinate system x^1, x^2 by the equations

$$x^1 = r\cos\varphi, \quad x^2 = r\sin\varphi.$$

We can then calculate the g_{ij} in terms of G, and use these results to show that the quantity

$$Q\left(\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p\right) = 2c_{11,22}(p)$$

is equal to $-K(p)/3$. The result then follows from Proposition 6. ❖

8. PROPOSITION. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, and let W be a 2-dimensional subspace of M_p, spanned by $X, Y \in M_p$. Let $\mathcal{O} \subset W$ be a neighborhood of $0 \in M_p$ on which exp is a diffeomorphism, let $i\colon \exp(\mathcal{O}) \to M$ be the inclusion, and let $\bar{R}$ be the Riemann curvature tensor for $\exp(\mathcal{O})$ with the induced Riemannian metric $i^*\langle\ ,\ \rangle$. Then

$$\langle \bar{R}(X,Y)Y, X\rangle = \langle R(X,Y)Y, X\rangle.$$

Consequently,

$$\frac{\langle R(X,Y)Y, X\rangle}{\|X, Y\|^2}$$

is the Gaussian curvature at p of the surface $\exp(\mathcal{O})$.

FIRST PROOF. It obviously suffices to prove the theorem when X and Y are orthonormal. Choose a Riemannian normal coordinate system at p with $X = \partial/\partial x^1|_p$, $Y = \partial/\partial x^2|_p$; then x^1, x^2 is a coordinate system on $\exp(\mathcal{O})$. Now we are trying to prove that $\bar{R}_{1212}(p) = R_{1212}(p)$. But in (***), the terms involving Christoffel symbols vanish at p. The theorem is now obvious, since the functions $\bar{g}_{ij}$ $(i, j = 1, 2)$ defining the metric $i^*\langle\ ,\ \rangle$ are just the corresponding g_{ij} restricted to $\exp(\mathcal{O})$, and they have the same mixed partial derivatives with respect to x^1 and x^2.

SECOND PROOF. It is even more obvious that the quadratic form $\bar{Q}$ associated with $(\exp(\mathcal{O}), i^*\langle\ ,\ \rangle)$ is the restriction to W of the quadratic form Q on M_p, for they are the second non-zero terms in the Taylor expansion of the same metric. ❖

9. COROLLARY. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, let $X, Y \in M_p$ span a 2-dimensional subspace W of M_p, and let $\mathcal{O} \subset W$ be a neighborhood of 0 on which exp is a diffeomorphism. If Q is the quadratic form on M defined previously, then

$$\frac{-3Q(X,Y)}{\|X,Y\|^2} = \frac{\langle R(X,Y)Y, X\rangle}{\|X,Y\|^2} = K,$$

where K is the Gaussian curvature at p of the surface $\exp(\mathcal{O})$.

The quantity $\langle R(X,Y)Y,X\rangle/\|X,Y\|^2$ appearing in Corollary 9 is called the **sectional curvature** $K(W)$ of W. It would seem that the function $(X,Y) \mapsto \langle R(X,Y)Y,X\rangle$ contains only a small portion of the total information contained in the curvature tensor, but Propositions 10 and 12, which follow, show that R satisfies certain identities which allow it to be determined in terms of the metric $\langle\ ,\ \rangle$ and the quadratic function Q which it determines.

10. PROPOSITION. The curvature tensor satisfies the following identities:

(1) $R(X,Y)Z = -R(Y,X)Z$, hence

$$\langle R(X,Y)Z, W\rangle = -\langle R(Y,X)Z, W\rangle.$$

(2) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$.

(3) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, hence

$$\langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle = 0.$$

(4) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

PROOF. In a coordinate system x, these relations are equivalent to

(1) $\quad R^i{}_{jkl} = -R^i{}_{jlk} \quad$ or $\quad R_{ijkl} = -R_{ijlk}$

(2) $\quad R_{ijkl} = -R_{jikl}$

(3) $\quad R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0 \quad$ or $\quad R_{ijkl} + R_{iklj} + R_{iljk} = 0$

(4) $\quad R_{klij} = R_{ijkl}$.

These are immediate from (∗∗) and (∗∗∗). ❖

Notice that $\langle R(X,Y)Z, W\rangle$ is skew-symmetric in both (X,Y) and (Z,W), which again shows that $\langle R(X,Y)Y,X\rangle$ changes by $\det(a_{ij})^2$ when X and Y are replaced by $a_{11}X + a_{21}Y, a_{21}X + a_{22}Y$. For later use, we insert a result which shows that the fourth property of R is a formal consequence of the others.

11. PROPOSITION. Let V be a vector space and $\mathcal{R} : V \times V \times V \times V \to \mathbb{R}$ a multilinear map satisfying

(1) $$\mathcal{R}(X, Y, Z, W) = \mathcal{R}(Y, X, Z, W)$$

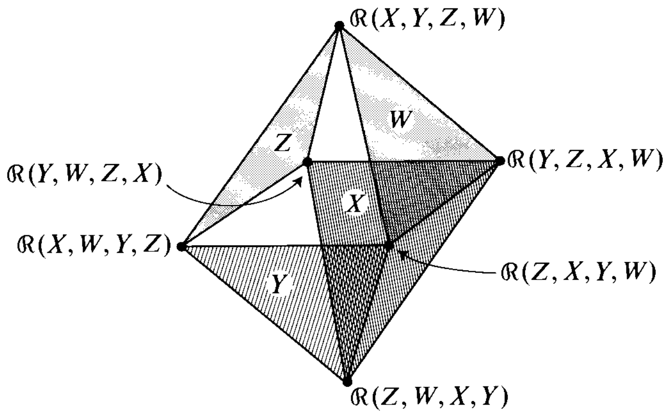
(2) $$\mathcal{R}(X, Y, Z, W) = \mathcal{R}(X, Y, W, Z)$$

(3) $$\mathcal{R}(X, Y, Z, W) + \mathcal{R}(Y, Z, X, W) + \mathcal{R}(Z, X, Y, W) = 0.$$

Then $\mathcal{R}$ also satsifies

(4) $$\mathcal{R}(X, Y, Z, W) = \mathcal{R}(Z, W, X, Y).$$

PROOF. The proof is a tricky manipulation, cleverly systematized by the following diagram from Milnor's *Morse Theory*.



Equation (3) shows that the sum of the numbers at the vertices of triangle W is zero. The sums of the vertices of triangles X, Y, and Z are also seen to be zero, using (1) and (2). Adding these identities for the top two triangles, and subtracting the identities for the bottom ones, we see that twice the top vertex minus twice the bottom vertex is zero. ❖

12. PROPOSITION. Let V be a vector space and $\mathcal{R}_i : V \times V \times V \times V \to \mathbb{R}$ two multilinear maps satisfying (1)–(4) of Proposition 11. Suppose $\mathcal{R}_1(X, Y, X, Y) = \mathcal{R}_2(X, Y, X, Y)$ for all $X, Y \in V$. Then $\mathcal{R}_1 = \mathcal{R}_2$.

PROOF. It clearly suffices to prove that a multilinear $\mathcal{R}$ satisfying (1)–(4) is 0 if $\mathcal{R}(X, Y, X, Y) = 0$ for all $X, Y \in V$. Now we have

$$\begin{aligned} 0 &= \mathcal{R}(X, Y + W, X, Y + W) \\ &= \mathcal{R}(X, Y, X, Y) + \mathcal{R}(X, Y, X, W) + \mathcal{R}(X, W, X, Y) + \mathcal{R}(X, W, X, W) \\ &= \mathcal{R}(X, Y, X, W) + \mathcal{R}(X, W, X, Y) \\ &= 2\mathcal{R}(X, Y, X, W). \end{aligned}$$

Using (1) and (2), we easily see that $\mathcal{R}$ is alternating, and hence skew-symmetric. Consequently, (3) gives

$$3\mathcal{R}(X, Y, Z, W) = 0. ❖$$

Propositions 10 and 12 tell us that the curvature tensor R is completely determined by the values of $\langle R(X, Y)Y, X\rangle$, and hence by the quadratic function Q. [This means that in a sense we can frame a coordinate-free definition of the curvature tensor, but it would certainly be an awkward one. Moreover, given a multilinear map $\mathcal{R}\colon V \times V \times V \times V \to \mathbb{R}$, satisfying (1)–(4), it is a fairly difficult exercise to work out a formula for $\mathcal{R}$ in terms of the quantities $\mathcal{R}(X, Y, X, Y)$.] In terms of a coordinate system, we see that the tensor $\mathcal{R}(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ is determined by the components R_{ijij}, of which there are $n\frac{n-1}{2}$ with $i < j$. According to Riemann, these $n\frac{n-1}{2}$ functions must determine the metric completely; in other words, the tensor $\mathcal{R}$ must determine the metric.*

Recall that we have selected one special case of this assertion as our Test Case, which can now be restated as follows: If $R = 0$, then the manifold is flat. We are ready to present the first, and longest, of our proofs of the Test Case. It is separated into three Steps, and all our subsequent proofs, no matter how elegant and brief, essentially contain these same three Steps.

Recall that for a coordinate system $y^1, \dots, y^n$ we have the formula (pg. I.331)

$$(*) \qquad \frac{\partial g_{ij}}{\partial y^k} = [ik, j] + [jk, i],$$

which is equivalent to the definition of the Christoffel symbols, as well as the formula (pg. I.331)

$$(**) \qquad \frac{\partial g^{ij}}{\partial y^k} = -\sum_{l=1}^{n} (g^{il}\Gamma_{lk}^{j} + g^{lj}\Gamma_{lk}^{i}),$$

which can be derived from it.

---

*This is not really the same as saying that R determines the metric, since we can't determine $R(X, Y)Z$ from $\mathcal{R}(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ unless the metric is already known! In fact, any numbers R_{ijkl} satisfying the identities of Proposition 6 can be realized as the components, at a point, of R for some metric (the R_{ijkl} determine the second derivatives of the metric at the point).

13. THEOREM (THE TEST CASE; FIRST VERSION). Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional Riemannian manifold for which the curvature tensor R is 0. Then M is locally isometric to $\mathbb{R}^n$ with its usual Riemannian metric.

PROOF. This is a purely local question, so we assume that M is $\mathbb{R}^n$, with the standard coordinate system $y^1, \dots, y^n$, and the Riemannian metric

$$\langle\ ,\ \rangle = \sum_{i,j=1}^{n} g_{ij}\, dy^i \otimes dy^j.$$

Step 1. We claim that there are functions $(h_1, \dots, h_n)$, with any desired initial conditions $(h_1(0), \dots, h_n(0))$, satisfying the equations

$$(*) \qquad \frac{\partial h_j}{\partial y^k} = \sum_{\gamma=1}^{n} \Gamma_{jk}^{\gamma} h_\gamma.$$

The reason for this is, of course, that the relations $R^\gamma{}_{jlk} = 0$, which express the vanishing of R, are just the integrability conditions for $(*)$, as we have already seen.

In particular, for $\alpha = 1, \dots, n$ we can choose such a set $(h^{(\alpha)}{}_1, \dots, h^{(\alpha)}{}_n)$ satisfying the initial condition

$$(h^{(\alpha)}{}_1(0), \dots, h^{(\alpha)}{}_n(0))_0 = X_\alpha,$$

where $X_1, \dots, X_n \in \mathbb{R}^n{}_0$ is orthonormal with respect to $\langle\ ,\ \rangle_0$.

Step 2. We claim that if $(h_1, \dots, h_n)$ satisfies $(*)$, then $h = dx$ for some function x, i.e., $h_j = \partial x/\partial y^j$. In terms of the form

$$\eta = h_1\, dy^1 + \cdots + h_n\, dy^n,$$

we are just saying that η is exact. We know (Corollary I. 7-15) that this is true if and only if

$$\frac{\partial h_j}{\partial y^k} = \frac{\partial h_k}{\partial y^j}.$$

Glancing at $(*)$, we see that this is indeed true, since $\Gamma_{jk}^{\gamma} = \Gamma_{kj}^{\gamma}$.

Now choose functions x^α with $h^{(\alpha)}{}_j = \partial x^\alpha/\partial y^j$. Then the functions x^α satisfy

$$(\dagger) \qquad \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} = \sum_{\gamma=1}^{n} \Gamma_{jk}^{\gamma} \frac{\partial x^\alpha}{\partial y^\gamma}$$

[these are the equations (6), obtained earlier, page 187] and

$$\left(\frac{\partial x^\alpha}{\partial y^j}(0)\right) = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix};$$

this matrix is non-singular, so $x^1, \ldots, x^n$ is a coordinate system in a neighborhood of 0.

Step 3. We claim that x is the desired coordinate system, i.e., that

$$(\ddagger) \qquad \delta_{\mu\upsilon} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j} \qquad \text{[equation (4'), page 185].}$$

We know that this equation holds at 0, by the choice of the initial conditions $\partial x^\alpha / \partial y^j (0)$. So it suffices to show that the right side of (‡) has all partial derivatives $\partial/\partial y^k$ equal to 0. But

$$\begin{aligned}
\frac{\partial}{\partial y^k}\left(\sum_{i,j} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j}\right) &= \sum_{i,j} \frac{\partial g^{ij}}{\partial y^k} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j} \\
&\quad + \sum_{i,j} g^{ij} \frac{\partial^2 x^\mu}{\partial y^i \partial y^k} \frac{\partial x^\upsilon}{\partial y^j} + \sum_{i,j} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial^2 x^\upsilon}{\partial y^j \partial y^k} \\
&= \sum_{i,j} \frac{\partial g^{ij}}{\partial y^k} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j} + \sum_{i,j} g^{ij} \sum_\gamma \Gamma_{ik}^\gamma \frac{\partial x^\mu}{\partial y^\gamma} \frac{\partial x^\upsilon}{\partial y^j} \\
&\quad + \sum_{i,j} g^{ij} \sum_\gamma \Gamma_{jk}^\gamma \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^\gamma} \qquad \text{by } (\dagger).
\end{aligned}$$

Switching some indices, we thus have

$$\begin{aligned}
\frac{\partial}{\partial y^k}\left(\sum_{i,j} g^{ij} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j}\right) &= \sum_{i,j} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\upsilon}{\partial y^j} \left(\frac{\partial g^{ij}}{\partial y^k} + \sum_\gamma g^{\gamma j} \Gamma_{\gamma k}^i + \sum_\gamma g^{i\gamma} \Gamma_{\gamma k}^j\right) \\
&= 0 \qquad \text{by } (**).
\end{aligned}$$

This completes the proof of the theorem. ❖

As a brief review of the proof, we note that

Step 1 uses the *integrability conditions*, $R = 0$, to obtain certain forms $\sum_i h^{(\alpha)}{}_i \, dy^i$, with any desired initial conditions;

Step 2 uses *symmetry of the Christoffel symbols*, $\Gamma^k_{ij} = \Gamma^k_{ji}$, to prove that $\sum_i h^{(\alpha)}{}_i\, dy^i = dx^\alpha$ for some x^α;

Step 3 uses the *definition of the Christoffel symbols* $[ij, k]$ to prove that the vectors $\partial/\partial x^\alpha$ are orthonormal.

Despite its length, the proof is essentially a straightforward application of the integrability conditions for partial differential equations. As Riemann says, at the end of the section in Part C, "Given an acquaintance with the traditional methods, it is demonstrated without difficulty that these ... conditions, when they are satisfied, suffice."

We have thus proved one special case of Riemann's assertion that the curvature determines the metric. We will not return to the more general assertion until Chapter 7, for our immediate task will be to begin systematizing all the results which have been uncovered so far.