Finally, we examine another way to produce the associated matrix from axis vector and angle by taking another look at

$$q\,\vec{v}\,q^* = \cos(2\,\theta)\,\vec{v} + (1 - \cos(2\,\theta))\,(\mathbf{U}\cdot\vec{v})\,\mathbf{U} + \sin(2\,\theta)\,\mathbf{U}\times\vec{v}.$$

If generic vector $\vec{v}$ in $\mathbb{R}^3$ has coordinates (v_1, v_2, v_3) and axis vector $\mathbf{U}$ has coordinates (U_1, U_2, U_3), both written as columns, then a direct calculation shows that the above equation for the rotated vector $\mathbf{M}\,\vec{v}$ equals

$$\mathbf{M}\,\vec{v} = \left(\ \cos(2\,\theta)\,\mathbf{I} + (1 - \cos(2\,\theta))\,\mathbf{U}\mathbf{U}^T + \sin(2\,\theta)\,\mathbf{U}^\times\ \right)\vec{v}$$

where $\mathbf{I}$ is the 3×3 identity matrix, $\mathbf{U}\mathbf{U}^T$ is a symmetric 3×3 matrix and $\mathbf{U}^\times$ is the 3×3 matrix that implements cross product with $\mathbf{U}$ (on the left):

$$\mathbf{U}\mathbf{U}^T = \begin{pmatrix} (U_1)^2 & U_1U_2 & U_1U_3 \\ U_2U_1 & (U_2)^2 & U_2U_3 \\ U_3U_1 & U_3U_2 & (U_3)^2 \end{pmatrix} \qquad \mathbf{U}^\times = \begin{pmatrix} 0 & -U_3 & U_2 \\ U_3 & 0 & -U_1 \\ -U_2 & U_1 & 0 \end{pmatrix}.$$

This provides a direct and somewhat faster way (without the choices involved in finding the matrix of transition $\mathbf{P}$) to create a matrix for the rotation around normalized axis $\mathbf{U}$ by angle $2\,\theta$. It is called **Rodrigues' formula.**

More importantly, if you actually have a rotation matrix $\mathbf{M}$ in hand, Rodrigues' formula shows that

$$\mathbf{M} - \mathbf{M}^T = 2\,\sin(2\,\theta)\,\mathbf{U}^\times = 2\,\sin(2\,\theta)\begin{pmatrix} 0 & -U_3 & U_2 \\ U_3 & 0 & -U_1 \\ -U_2 & U_1 & 0 \end{pmatrix}$$

so by normalizing you can determine the coordinates U_1, U_2 and U_3 of the unit rotation axis and the rotation angle $2\,\theta$.