Geometry, Reality and the World of Sensation

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Introduction

Over the last few millennia, since the invention of the modern idea of axioms and logical consequence as used by the Greeks, we have had an evolving relationship between the

mathematical subject of geometry and

the universe that contains us and

our sensations that provide us certain types of information about things external to individual humans,

in time.

Geometry

Ancient Geometry was the study of undefined entities using a particular short list of assumed facts about them and subsequent knowledge derived by rules of inference: Logic.

Philosophers use the words ontology and epistemology to talk about the objects of study and the means by which we acquire knowledge of these objects.

The Universe

This is the "meta" world.

In Mathematics it is this world that gives meaning to the axioms.

We already understood that $2 + 3 = 5$ before we invented "the ring of integers."

A line of symbols assembled according to rules without referents is meaningless, even if the rules are carefully adhered to and the symbols form a "valid statement."

Sensations

In Physics the "meta" world consists of the actual objects in the physical world, though over the centuries the presumed nature of these objects has changed.

Sensations provide us certain types of information about things external to individual humans, in time.

Patterns of sensations can be used to form the actual objects of study in Physics, or they can be used to acquire knowledge of these objects.

Plato and Socrates

According to Socrates (470-399 BCE) as reported in the dialogues of Plato (425-348 BCE) physical objects and physical events are "distorted shadows" of their ideal or perfect Forms.

There are no Circles in the world, nor is there Love or Justice. Socrates is said to think that perfect Justice exists and his own trial was a corrupt instance.

There is an ideal "Red" and all red things in the world are instantiations of it.

Aristotle (384-322 BCE) codified and amplified our understanding of logical consequence (among many other things, of course).

He claimed that Plato took from the Pythagoreans (Pythagorus of Samos lived roughly 570-495 BCE) the idea that mathematics and, generally speaking, abstract thinking can be taken as a secure basis for conclusions about Archetypal Entities in areas beyond mathematics.

The idea here is that the process of extracting Axioms from observations about corrupt instances of lines and circles in the world and thereby creating Theorems about objects in the Perfect Ideal world to which the visible versions refer offered the cleanest possible examples of how to obtain knowledge about Perfect Ideal Things more generally, such as Virtue, Beauty, the Good and so on.

$$\begin{array}{c}\text{Corrupt}\\\text{Lines}\end{array} \Longrightarrow \begin{array}{c}\text{Axioms and}\\\text{Logic}\end{array} \Longrightarrow \begin{array}{c}\text{Theorems about}\\\text{Lines}\end{array} \Longrightarrow \begin{array}{c}\text{Facts About}\\\text{Ideal Lines}\end{array}$$

So to the Greeks, there is something innate in humans that let's us see aspects of lines and circles and other objects which exist in an ideal world: perfect, unchanging, out of time.

We have a mysterious natural affinity for these Forms, but cannot see them directly.

Mathematics is to be the conduit, our window into the ideal world.

EUCLID

Euclid of Alexandria (323-283 BCE) is reported to have organized the geometrical Axioms and Theorems into "The Elements", which was the heart and gold-standard of mathematical thought in Europe, the Eastern Empire and the Islamic World for over 2000 years, known and used by everyone who studied mathematics.

It WAS mathematics, unquestionably the most widely read book during all that time. (The Bible and The Koran were not generally available to common folk.)

Objects of Study: the Forms of Circle, Line, Point etc.

Means of Acquiring Knowledge: Mathematics/Logic

The First Four of Euclid's Postulates:

1. A straight line segment is determined by any two distinct points.
2. Any straight line segment can be extended indefinitely.
3. Given any straight line segment, a circle exists having the segment as radius and one endpoint as center.
4. All right angles are congruent.

The Fifth

5. If two straight line segments intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

This postulate in its contra-positive form, is known as the parallel postulate.

Mathematicians were quite sure that the Forms must have the first four qualities. The fifth was widely assumed to be true . . . but many had an intuition that the fifth was not an independent axiom and could be proved from the other four.

Many incorrect "proofs" of this appeared by famous mathematicians.

Caliph Harun al-Rashid (786-809) created the House of Wisdom in Baghdad. He collected scholars from around the world to translate all of the world's classical knowledge into Arabic. (Including "Arabic numbers," invented in the Indian subcontinent around 500 CE.)

Galen of Pergamon (129-200 CE) and some other Romans emphasized both theory and observation/experiment, and this heritage became common knowledge among Arabic readers.

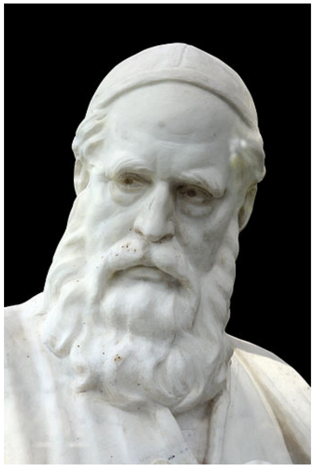
But it is fair to say that a modern concept of Science was created by thinkers from the Islamic world.

Persian al-Khwārizmī (780-850) is known as the inventor of algebra, building on the work started by Diophantus of Alexandria (circa 210-290).

The Persian al-Biruni (973-1048) contributed to development of experimental science, combined static and dynamic ideas into mechanics and hydrodynamics.

Egyptian Ibn al-Haytham (born in Basra, lived 965-1040) is usually regarded as the “father” of the modern notion of scientific progress: emphasized experimental data, reproducability. (optics)

Persian Omar Khayyám (1034-1131) assembled many of the facts of arithmetic (ratios) and also algebra in more-or-less modern form, and explored its relation to geometry. He spent a long time re-working Euclid.



Khayyám (1034-1131)

A interesting quote from Khayyám:

“I have seen many books which have objected to [Euclid’s Fifth Postulate], among the earlier ones Heron and Autolycus, and the later ones Al-Khazen, Al-Sheni, Al-Neyrizi, etc.

None has given a proof.

Then I have seen the book of Ibn Haytham, God bless his soul, called the *Solution of Doubt*, in Chapter One. This postulate among other things was accepted without proof.”

More from Khayyám:

“There are many other things foreign to this field [from this book] such as:

If a straight line segment moves so that it remains perpendicular to a given line, and one end of it remains on the given line, then the other end of it draws a parallel.

There are many things wrong here. How could a proof be based on this idea? How could geometry and motion be connected?”

It seems that Khayyám was onto something here.

Because if a whole line segment is moving does that imply that the two ends travel the same distance in the same time.?

To which observer?

We have arrived at issues of simultaneity and distance that are at the heart of Relativity Theory and which were bothering Khayyám (and others) *before the Eleventh Century.*

Khayyám did not have physical intuition, I presume, about this.

He arrived at his point by examining a mathematical model and finding "holes" in the arguments, unproven assumptions proposed as part of an argument related to the fifth postulate.

And was led thereby to the central issue in Relativity Theory.

In Western Europe the Catholic Church found the intellectual ferment from the East and North Africa to be potentially de-stabilizing and suppressed it where possible.

But the increasing wealth and broader education of the Renaissance (1300-1600), and exposure through trade and travel to the Islamic and Eastern world, overwhelmed the ability of the Church to restrict the spread of Eastern knowledge.

The European Dark Ages (they were never dark in the Islamic world and Asia) came, unevenly and over centuries, to an end and the tools needed for a Newton to play his role were assembled.

But to tie up loose ends in our axiomatic-geometric story we briefly leap forward a few centuries . . .

In 1829 Nikolai Ivanovich Lobachevsky and (1831) János Bolyai described geometries in which the parallel postulate was false.

Gauss claimed (and who could doubt him?) that he had been proving results about these geometries for 35 years.

Eugenio Beltrami in 1868 finally proved that these other geometries (hyperbolic and elliptic) were actually consistent establishing the independence of the fifth from the first four axioms, settling a 2000 year-old question.

So that is the end of that thread.

Greek mathematics was all about proving things about Forms: they were only marginally interested in things in our "corporeal" world which, to them, was not real but a result of defects in our being and nature.

The world of the senses was not real to them.

So let's move on to The Scientists.

Setting The Stage

al-Khwārizmī's algebra was widely understood by a group of Western European scholars, who propagated there the work of the Islamic Proto-Scientists of the previous centuries. They even added *a little* to the discussion . . .

—Fibonacci (1170-1250) *"Arabic" numerals to Western Europe*
—Robert Grosseteste (1175-1253)
—Roger Bacon (1219-1292)
— Albertus Magnus (1200-1280)
—Duns Scotus (1266-1308)
—Theodoric of Freiberg (1250-1310)
— William of Ockham (1285-1350)
—Jean Buridan (1300-1358)

da Vinci to Newton

—Leonardo da Vinci (1452-1519)
—Nicolaus Copernicus (1473-1543)
—Gerolamo Cardano (1501-1576)
—Francis Bacon (1561-1626)
—Galileo Galilei (1564-1642)
—Johannes Kepler (1571-1630)

René Descartes (1596-1650) invented the concept of coordinate system as we know it, the Cartesian coordinate system, "algebraizing" geometry into analytic geometry.

—Christiaan Huygens (1629-1695)
—Robert Hook (1635-1703)

After all this stage-setting and technique-polishing Isaac Newton (1642-1726) created his version of Calculus and, subsequently, what is now known as Classical Mechanics.

He was a key participant in a paradigm shift.

The Scientific Revolution

In this new paradigm, the World of the Senses *is* the world.

Reality is not an out-of-space-and-time universe of Forms.

Reality is nothing more than what you can sense and measure.

The interesting things in this world, the things that catch the eye, are things that change. Scientific Theories propose a cause of this change, and if correct that thing becomes predictable.

If it is predictable it can be used.

Many Scientific Theories (most theories in Physics) are encapsulated in Differential Equations, which propose a cause of changes in things that can be measured in terms of changes in *other* things that can be measured.

The Scientific Revolution proposed that names be given to these real things, things that can be measured, and knowledge of these things is obtained by

1) creating proposals for differential equations which these things must satisfy and

2) verifying through observation that the predictions are correct.

A new ontology. A new epistemology.

Unobservable things are not found anywhere!

One thing that was temporarily lost was *classical mathematical rigor*, though these defects have been repaired over the centuries.

Newton and Leibnitz and the Bernoulli's (and . . .) did not care. These folks just went about answering question after question that had been in the air for thousands of years.

Bishop Berkeley's (1685-1753) snarky critique, including a reference to "the ghosts of departed quantities" regarding infinitesimals which could be divided by other infinitesimals to get measurable things (if done *just right*) couldn't, in the end, stand up to this success.

And this new framework has been fantastically successful.

It is not perfect, not the only "way of knowing", nor does it claim to be.

It does not apply to everything humans want to know.

Nor does it tell any scientist *how* to invent these Scientific Theories, which sensory patterns to name and which differential equations these named entities should satisfy.

What it *does* do is just one simple but *completely new*, revolutionary little thing.

This method tells us when we are wrong, and provides a social imperative requiring the most ardent believer in a theory to give it up when it is shown to be wrong.

The reasoning behind the calculations in Newtonian Mechanics involves, implicitly, the ideas of vectors and coordinate space—though Newton himself in his published work often used laborious calculations involving intricate geometrical constructions.

Real progress was made, big jumps in understanding occurred, whenever notation and linearization techniques improved.

Leibnitz: Determinants 1693

Cramer: Cramer's Rule 1750

Gauss: Gaussian Elimination, FFT (Ceres) 1809

Caspar Wessel: (1745-1818) was the first to use complex numbers ("On the Analytical Representation of Direction" 1799) as tools to handle rotations.

Hamilton: Quaternions 1843

Graßmann: 1844 Grassmann Algebra

Sylvester: matrix 1848

Cayley: 1853, Matrix Multiplication and Inverses

Hüseyin Tevfik Pasha wrote the book (1882) "Linear Algebra".

Giuseppe Peano (1888) put together the main ideas of Vector Spaces in a palatable and (mostly) modern form.

So during this time Reality was assumed *to be* the world of our senses, possibly augmented with microscopes or galvanometers and so on, but the normal human interpretation of the world around was not in doubt.

Real things were measurable things.

The world is to be taken literally through our senses.

The Triumph of Geometry

Leonhard Euler (1707-1783) studied curvature on surfaces.

Gauss (1777-1855) published in 1827 "Disquisitiones Generales Circa Superficies Curvas" and set the stage for physical applications of these more general geometries, a study that resulted from his contract to improve the the geodetic description of Hanover.

This little book contains Gauss's "Theorema Egregium" which can be translated from the Latin roughly as "Totally Awesome Theorem" and states in modern language "The Gaussian curvature of a surface is invariant under local isometry."

Theorema Egregium

From Spivak, A Comprehensive Introduction to Differential Geometry Volume II:

> *If $f, g \colon \mathcal{M} \to \mathbb{R}^3$ are two embeddings (or even immersions) such that $f^*\langle \cdot, \cdot \rangle = g^*\langle \cdot, \cdot \rangle$, then the Gaussian curvature of $f(\mathcal{M})$ at $f(p)$ is the same as the Gaussian curvature of $g(\mathcal{M})$ at $g(p)$.*

Quote from Gauss:

> *Thus the formula of the preceding article leads itself to the remarkable Theorem. If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

Gauss's student Bernhard Riemann (1826-1866) gave a series of talks, prompted by Gauss and in support of Riemann's Habilitationsschrift, on the foundations of geometry. Riemann developed his theory of higher dimensions and in 1854 gave his famous lecture at Göttingen "On the Hypotheses Which Underlie Geometry".

Gauss, in the audience, was said to be astonished by the depth of the ideas presented but the results were not generally understood before Felix Klein (1849-1925) republished the results and explained why Riemann's results were so important.

Riemann

Although only living to age 40 the incredible richness and creativity of his accomplishments in pure mathematics and mathematical physics cannot be overstated.

In addition to his eponymous integral he made fundamental contributions to Complex variables, Riemann surfaces, methods of series solutions to differential equations still in use, special functions, number theory, algebraic geometry and the beginning of manifold theory in higher dimensions.

This last led to the Klein's Erlangen program and eventually to Einstein's General Relativity.

Riemann

From Riemann's Inaugeral Lecture in support of his Habilitationsshrift.

> *. . . in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in the continuous one it must come from something else. Therefore, either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.*

Erlangen Program

Klein's 1872 Erlangen Program proposal, classifying geometries by their underlying symmetry groups, was a tour de force, combining many different areas of mathematics with physical application a primary goal.

He had many students who developed these ideas, including Carl Runge, Max Planck, Luigi Bianchi, and Gregorio Ricci-Curbastro (1853-1925).

Ricci-Curbastro was the inventor of Tensor Calculus and the Ricci tensor together with Levi-Civita in 1904.

Tullio Levi-Civita (1873-1929) Technical tools: the Levi-Civita connection

His Swedish colleague Sophus Lie (1842-1899) told Klein about the concept of group and they worked both together and independently on this project.

Hermann Minkowski (1864-1909) was at Göttingen too.

Slightly before Ricci-Curbastro and Levi-Civita, the German mathematician and physicist Elwin Bruno Christoffel (1829-1900) was also studying surfaces. His Christoffel symbols describe a metric connection.

The metric connection is a special type of function defined on vector fields on surfaces or other manifolds endowed with a metric tensor, allowing distances to be measured on that surface.

Many additional concepts follow: parallel transport, covariant derivatives, geodesics, etc.

These concepts can be directly tied to the "shape" of the manifold itself— that shape is determined by how the tangent manifold is related to the cotangent manifold, and this is determined by the metric tensor.

Also in the late 1800's, French Mathematician Jules Henri Poincaré (1854-1912) was working to emphasize the importance of paying attention to the invariance of laws of physics under different transformations. This was along the same lines as the Ehrlangen program at Göttingen.

In his work Poincaré explicitly talked about the "principle of relative motion" (1900) and named it the *Principle of Relativity* in 1904, according to which

> *...no physical experiment can discriminate between a state of uniform motion and a state of rest.*

In 1905 Poincaré wrote to the Physicist Hendrik Lorentz (1853-1928) about Lorentz's paper of 1904, which Poincaré described as a "paper of supreme importance" and in which he discussed what is now called Lorentz transformations—which preserve the space-time interval between events and stabilize the origin—as distinguished from the Galilean transformations which are generated by time and space translations and spatial uniform motion and a spatial rotation.

The Galilean transformations form a 10-dimensional Lie group and are the set of Euclidean isometries.

Poincaré Group: All the Lorentz transformations plus translations. These also form a 10-dimensional Lie group.

The Poincaré Group is exactly the set of Minkowski space isometries.

Let's take stock.

The evidence that things were seriously amiss in current physical explanations was apparent from the Michelson-Morley experiment of 1887.

The clues that geometry should be taken seriously were embedded in Maxwell's Electrodynamics.

The mathematical techniques and the imperative to "geometrize" were in place from the Erlangen program and Minkowski and Poincaré.

All is ready, from the Mathematical side, for the 1905 paper on Special Relativity by Albert Einstein.

As astounding as that paper was, it didn't fix all the problems.

It *did* provide strong evidence that it was pointing in the right direction.

But . . . it couldn't deal properly with acceleration or gravitation.

The paper on General Relativity of 1916 answered those questions. It took a very smart man 11 years to "get there" because the tools of Differential Geometry required for the solution were definitely not in a form that were easily adapted to his purpose . . .

. . . and because of the bizarre, almost unimaginable, unbelievable, nature of the objects in this new world.

Apparently (from notes after visit to Prague) in 1912 Einstein realized for the first time that spacetime coordinates need not determine the distances between spacetime points.

Einstein's quote from his "Autobiographical Notes" :

> *Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.*

And then there was Amalie (Emmy) Noether (1882-1935).

She was a protégé of David Hilbert and Felix Klein at Góttingen and was described by many (including Einstein) as the greatest female mathematician who ever lived.

She had numerous students who revered her, in spite of her eccentricities, and is widely acknowledged as the "mother" of modern abstract algebra.

Her unfortunate early death at age 53 deprived the world of a great genius.

The Noether Theorems (1915) on the relationship between conserved quantities and symmetry of physical law were regarded by Einstein as *beyond profound*.

These explained, for instance, the conserved quantities in Newtonian Mechanics related to the Galilean Group and those of Special Relativity related to the Poincaré Group.

Noether's brilliant work, finally, showed how *any* physical theory is related to geometry: Quantum Mechanics, Newtonian Mechanics, Relativity are merely instances.

Her work applies universally.

Einstein's big contribution was to decide to take geometry seriously and see where that leads—that the (Minkowski) geometrization of physics was not just a calculation trick but an expression of SOMETHING real.

Somebody, *finally*, took the issues at the heart of Khayyám's warnings at face value.

And Noether wrapped it all up by demonstrating the inevitability of the geometrization of physical law.

What can we do about our limitations?

So what is the lesson for the physical scientist here?

A partial return to Forms—coupled with experiment.

Trust the models: follow them where they lead as a partial substitute for natural intuition. Then, of course, test them.

Project onto a Hilbert subspace after a measurement?

We got to this idea for very good reasons. But the model leads us to the unimaginable.

Entangled states = no matter of fact regarding the "true state" until a state is measured?

Preposterous.

What can we do about our limitations?

Moving clocks run slow, yardsticks shrink in the direction of their motion?

Ridiculous. But true.

We have reached the point now where must pay—at least some—homage to the "pale shadow on the wall of our cave" interpretation of reality from the ancient Greeks.

Intuition and instinct based on sensory patterns and brain evolution may no longer be sufficient.

But we humans ought to be able to push this enterprise *just a little farther*, even so.

Theseus Comments on His Creator

The Poet's Eye in a Fine Frenzy Rolling

Doth Glance From Heaven to Earth, From Earth to Heaven,

And as Imagination Bodies Forth the Forms of Things Unknown

The Poet's Pen Turns Them to Shapes, and Gives to Airy Nothing

A Local Habitation and a Name.

William Shakespeare,
A Midsummer Night's Dream