MANIFOLD NOTES PART II
(DRAFT)

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Contents

In Part II we discuss issues raised by differentiating the entities created in Part I, which "live at points" and correspond to linearization and first derivatives. Higher derivatives are embodied in, for instance, exterior derivative, the Lie derivatives and curvature.

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1. The Exterior Derivative

There are several different kinds of differentiation-related operations in common use when discussing tensor fields. We will first define the exterior derivative on each space of r-forms $\Lambda^r(\mathcal{M})$ and, by application at each grade, the corresponding Grassmann algebra $\mathcal{G}(\mathcal{M})$.

We already have a start on the exterior derivative. We have defined the differential df of a real-valued smooth function f on $\mathcal{M}$, and that is the exterior derivative on these functions, the number fields $\Lambda^0(\mathcal{M})$.

Pick coordinate system x around one point $p \in \mathcal{M}$. We will suppress mention of p to avoid clutter, but the tensors we produce will be restrictions to U_x of members of $\Lambda^r(\mathcal{M}_p)$, and therefore members of $\Lambda^r((U_x)_p)$, for various r.

For $f \in \Lambda^0(U_x) = \mathcal{F}^\infty(U_x)$ define

$$d_x f \;=\; df \;=\; D_i(f \circ x^{-1})\, dx^i \;=\; \frac{\partial f}{\partial x^i}\, dx^i \;\in\; \Lambda^1(U_x)\,.$$

In particular, $d_x x^i = dx^i$ for each i.

In the higher grades, if $\omega = \omega_{i_1,\dots,i_r}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \in \Lambda^r(\mathcal{M})$ (increasing indices) we define

$$\begin{aligned} d_x\omega \;&=\; (\, d\omega_{i_1,\dots,i_r}(x)\,) \wedge \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r}\right) \\ &=\; \left(\frac{\partial\, \omega_{i_1,\dots,i_r}(x)}{\partial x^j}\, dx^j\right) \wedge \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r}\right) \\ &=\; \frac{\partial\, \omega_{i_1,\dots,i_r}(x)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \end{aligned}$$

where the indices $i_1, \dots, i_r$ are increasing but j is arbitrary.

Since the coefficients $\omega_{i_1,\dots,i_r}(x)$ are C^∞ on U_x so too will be the coefficients $\frac{\partial\, \omega_{i_1,\dots,i_r}(x)}{\partial x^j}$ of $d_x\omega$ on U_x. The sum of wedge product of tensors is a tensor so we have succeeded in creating a member of $\Lambda^{r+1}(U_x)$ from $\omega \in \Lambda^r(U_x)$.

We don't know, however, if this procedure has tensor character. When you define a multilinear function in a basis you can force it to be a tensor by defining it in other bases to have the appropriately modified coefficients.

We would like to show that if $y\colon U_y \to R_y$ then on $U_y \cap U_x$ the tensor $d_x\omega$ agrees with the tensor $d_y\omega$ calculated as

$$d_y\omega = \frac{\partial\, \omega_{i_1,\dots,i_r}(y)}{\partial y^j}\, dy^j \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_r}.$$

We accumulate a few interesting properties of d_x (which are also possessed by d_y) that will characterize d_x in useful ways and help with this.

First, it is obvious that in any grade, $d_x(\omega + \tau) = d_x\omega + d_x\tau$.

Further, if c is the constant function on $\mathcal{M}$ then $d_x(c\,\omega) = c\, d_x\omega$.

Third, we examine $d_x(\omega \wedge \tau)$ when both ω and τ are single-term tensors

$$\omega = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \quad \text{and} \quad \tau = g\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

In this case

$$\begin{aligned} d_x(\omega \wedge \tau) &= d_x \left(fg\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \\ &= \left(\frac{\partial f}{\partial x^j}\, g + \frac{\partial g}{\partial x^j}\, f \right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \\ &= \left(\frac{\partial f}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) \wedge \left(g\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \\ &\qquad + (-1)^r \left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) \wedge \left(\frac{\partial g}{\partial x^j}\, dx^j \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \\ &= (d_x\omega) \wedge \tau + (-1)^r\, \omega \wedge d_x\tau. \end{aligned}$$

The same result holds for any r-form ω and k-form τ (not just single-term tensors) by the linearity properties of wedge product and d_x.

These three properties, possessed by d_x, are the defining characteristics of a **$\wedge$-antiderivation on $\mathcal{G}\,(U_x)$**.

Finally, if $\omega = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ consists of a single term then

$$\begin{aligned} d_x d_x \omega &= d_x \frac{\partial f}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \\ &= \frac{\partial^2 f}{\partial x^j x^k}\, dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \end{aligned}$$

where the indices $i_1, \ldots, i_r$ are increasing but j and k are arbitrary. Requiring j and k to be unequal (dropping only terms that are zero with this restriction), the last becomes a sum of terms of the form

$$\sum_{j<k} \left(\frac{\partial^2 f}{\partial x^j x^k} \; - \; \frac{\partial^2 f}{\partial x^k x^j} \right) dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Under our differentiability conditions, the mixed partials are equal and we conclude that $d_x d_x \omega = 0$. Since any r-form is the sum of r-forms like this single-term ω, the additivity property mentioned above implies that $d_x d_x \omega = 0$ for any r-form and any r.

We now come to the main result.

Suppose d is any $\wedge$-antiderivation on $\mathcal{G}\,(U_x)$ that agrees with d_x on $\Lambda^0\,(U_x)$ and for which $d\left(\Lambda^j\,(U_x)\right) \subset \Lambda^{j+1}\,(U_x)$ for each j. If $d \circ d = 0$ then $d = d_x$. In other words, the exterior derivative can be calculated by the same formula (as given above in basis x) using any coordinates.

It follows that $d_x\omega = d_y\omega$ at points on the manifold in $U_x \cap U_y$.

So the procedure given locally by d_x serves to define a $\wedge$-antiderivation d on all of $\mathcal{G}\,(\mathcal{M})$ that agrees with the original definition of d on $\Lambda^0\,(\mathcal{M})$, satisfies $d\left(\Lambda^j\,(\mathcal{M})\right) \subset \Lambda^{j+1}\,(\mathcal{M})$ for all j, and for which $d \circ d = 0$.

And d is the only $\wedge$-antiderivation on $\mathcal{G}\,(\mathcal{M})$ with these properties.

The proof has several steps. First, we have the result we seek (by assumption) when d and d_x are applied to functions. In particular, $dx^i = d_x x^i$ for each i.

Suppose we have shown $d\left(dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \right) = 0$ for all increasing sequences $i_1, \dots, i_{r-1}$ where $r > 1$.

So if $i_1, \dots, i_r$ is an increasing sequence

$$
\begin{aligned}
&d\left(dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) \\
&\quad = d\left(dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \right) \wedge dx^{i_r} + (-1)^{r-1} \left(dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \right) \wedge d(dx^{i_r}) \\
&\quad = 0 + 0.
\end{aligned}
$$

So by induction on r we have $d\left(dx^{i_1} \wedge \cdots \wedge dx^{r} \right) = 0$ for all increasing sequences $i_1, \dots, i_r$ and any $r > 0$. As a special case, this result holds if $d = d_x$.

Now we find that if f is any differentiable function on $\mathcal{M}$ and for all increasing sequences $i_1, \dots, i_r$ and any $r > 0$

$$
\begin{aligned}
&d\left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) - d_x\left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) \\
&\quad = (df) \wedge \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) - (d_x f) \wedge \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) \\
&\qquad + f d\left(dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) - f d_x \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right).
\end{aligned}
$$

The first two of the last four terms are equal and the last two are both zero. So

$$
d\left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) = d_x\left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right).
$$

Since any r-form is the sum of r-forms of the type in this last calculation, $d = d_x$ on each $\Lambda^r(\mathcal{M})$ and so, therefore, on $\mathcal{G}(\mathcal{M})$.

Define the **exterior derivative $\boldsymbol{d\colon \mathcal{G}(U_x) \to \mathcal{G}(U_x)}$** at each grade by

$$
\boxed{d\omega = d_x\omega = \frac{\partial\, \omega_{i_1,\dots,i_r}(x)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}}
$$

as calculated in any coordinates, where j is an arbitrary index and the summation indices $i_1, \dots, i_r$ are increasing.

As noted above, this local procedure serves to define d on all of $\mathcal{G}(\mathcal{M})$.

As a final point, tensor operations such as contraction or tensor product, can be made to apply to tensor fields by implementing them point-by-point. To calculate them at a point in the resulting field you do not need to know the values of the fields nearby, only at the point itself. Though construction of the tangent map H_* and the pullback H^* does involve the values of H itself at nearby points, once these maps are created they are applied to field values point-by-point. Knowledge of field values away from the point are not needed.

On the other hand, the exterior derivative (and the Lie derivative we define using it in the next sections) are operations on tensor *fields*. Without knowledge of how field values change in a (possibly tiny) neighborhood of a point you cannot calculate the exterior derivative and the actual *value* of a tensor field is not involved at all. The Lie derivative which follows will involve both field values *and* local changes, and is also, inherently, a "field" construction in this sense.

2. The Lie Derivative Applied to Number and Vector Fields

The **Lie derivative $\boldsymbol{\mathcal{L}_X(f)}$** of a smooth number field f with respect to a smooth vector field X on $\mathcal{M}$ is just $X(df)$. This is a new smooth number field on $\mathcal{M}$.

To calculate it at point p pick differentiable curve c in X_p with $c(\alpha) = p$. Then $\mathcal{L}_X(f)(p) = (f \circ c)'(\alpha)$. This is the rate of change of f when moving through p at the pace and in the direction determined by the curves, such as c, in X_p.

Since each X_p is a point derivation we have $\mathcal{L}_X(fg) = f\mathcal{L}_X(g) + g\mathcal{L}_X(f)$.

$\mathcal{L}_X(f)$ is often called the **directional derivative of f with respect to X**.

When this vocabulary is used, the symbol $\boldsymbol{\nabla_X f}$ may be found in reference to it.

It is also possible to create a **Lie derivative on fields of tangent vectors** producing, as a result, another vector field. We turn our attention to this new process.

Most typically, you will see a bracket notation, the **Lie bracket**, employed below.

$$\boldsymbol{\mathcal{L}_X(Y) = [X, Y] = XY - YX} \in \mathcal{T}_0^1(\mathcal{M}) \quad \text{for } X,\, Y \in \mathcal{T}_0^1(\mathcal{M}).$$

This operation, also called the **Lie derivative with respect to X**, looks like it should mean something. But what?

Thinking of tangent vectors as acting on (and determined by their action upon) differentiable functions (number fields) this is intended to mean

$$\mathcal{L}_X(Y)(f) = [X, Y](f) = X(Y(f)) - Y(X(f)),$$

the composition of the actions of the vectors involved applied to generic smooth function f, one after the other.

It is clear that for *constant* r and vector fields Y and Z

$$\mathcal{L}_X(Y + r\,Z)(f) = \mathcal{L}_X(Y)(f) + r\,\mathcal{L}_X(Z)(f).$$

The tangent vectors X and Y correspond to derivations on number fields, and this implies (after a calculation) that for twice continuously differentiable number fields f and g, and evaluated at each point in the manifold,

$$\mathcal{L}_X(Y)(fg) = f\,\mathcal{L}_X(Y)(g) + g\,\mathcal{L}_X(Y)(f).$$

So $\mathcal{L}_X(Y)$ is also a derivation for each Y at each point in the manifold. Hence there is a unique vector field corresponding to this action: that is, $\mathcal{L}_X(Y)$ evaluated at point p is in $\mathcal{M}_p$ for each $p \in \mathcal{M}$. It remains to verify that the vector field defined by this process is smooth.

If x is a coordinate system on $\mathcal{M}$ then at points p in U_x we have a representation $X_p = [x, p, v]$ and $Y_p = [x, p, w]$ so evaluated at p

$$\boldsymbol{\mathcal{L}_X(Y)} = v^i \frac{\partial}{\partial x^i}\, w^j \frac{\partial}{\partial x^j} \; - \; w^i \frac{\partial}{\partial x^i}\, v^j \frac{\partial}{\partial x^j}$$

where the coefficients v^i and w^i depend, of course, on x and by assumption vary from point to point in U_x in a smooth way.

Calculating directly, when applied to a number field f and using these coordinates and evaluated at a point in U_x, the action of $\mathcal{L}_X(Y)$ is given by

$$\begin{aligned}\mathcal{L}_X(Y)(f) &= v^i\frac{\partial}{\partial x^i}\left(w^j\frac{\partial}{\partial x^j}f\right) - w^i\frac{\partial}{\partial x^i}\left(v^j\frac{\partial}{\partial x^j}f\right)\\ &= v^i\frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} + v^iw^j\frac{\partial^2 f}{\partial x^i\partial x^j}\\ &\qquad\qquad - w^i\frac{\partial v^j}{\partial x^i}\frac{\partial f}{\partial x^j} - w^iv^j\frac{\partial^2 f}{\partial x^i\partial x^j}.\end{aligned}$$

In the last two lines above the second and fourth terms (they are, each, sums) cancel because of the equality of mixed partial derivatives. The first and third terms become

$$\mathcal{L}_X(Y)(f) = \sum_{j=1}^n\left(v^i\frac{\partial w^j}{\partial x^i} - w^i\frac{\partial v^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}(f).$$

The coefficients are obviously differentiable and so at any point in U_x

$$\boxed{\mathcal{L}_X(Y) = [X,Y] = \sum_{j=1}^n\left(v^i\frac{\partial w^j}{\partial x^i} - w^i\frac{\partial v^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \in \mathcal{T}_0^1(U_x).}$$

Since this can be carried out for each chart in an atlas we have $\mathcal{L}_X(Y) \in \mathcal{T}_0^1(\mathcal{M})$.

If only one of X_p or Y_p is 0 there is no reason to suppose[1] $\mathcal{L}_X(Y)_p = 0$. The point here is that knowing X_p and Y_p does not tell you much about $\mathcal{L}_X(Y)_p$. You need to know also the *rates of change* of X and Y at p to determine $\mathcal{L}_X(Y)_p$.

Let's remind ourselves of the structure of the argument given above. First we showed that this Lie derivative was a point derivation on $\mathcal{F}^\infty(\mathcal{M})$ at each point so there is a tangent vector at each point that implements the Lie derivative at that point. At that point we had not, however, shown that these tangent vectors fit together in a smooth way. For that we went to coordinates at a point, found a formula for the Lie derivative output on the corresponding coordinate neighborhood, and observed that the coefficient functions were smooth in that neighborhood. And since they are smooth in a neighborhood of every point they are smooth on $\mathcal{M}$.

It is interesting to note[2] using the same ideas that the operator XY on $\mathcal{F}^\infty(\mathcal{M})$ is *not* a vector field. Its values when applied to a smooth function depend only on the "local" values of the function so it can be calculated using derivatives in coordinates. But these calculations do not reduce to the action of a tangent vector.

Generally, $XY = YX$ when and only when $[X,Y] = 0$ from which we deduce that these vector fields commute when and only when, within the domain of *any* coordinate system x, we have

$$v^i\frac{\partial w^j}{\partial x^i} = w^i\frac{\partial v^j}{\partial x^i} \quad \text{for} \quad j = 1,\dots,n.$$

[1]Though if *both* are 0 at p then $\mathcal{L}_X(Y)_p = 0$.

[2]For smooth functions f and g calculate $XY(fg) - fXY(g) - gXY(f)$. Show that this is not always 0 unless the value X_p of the vector field X is the zero tangent vector whenever Y_p is nonzero. So in this case XY is not a point derivation everywhere on $\mathcal{M}$.

The vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ corresponding to the coordinate gridcurves themselves do commute as members of $\mathcal{T}_0^1(U_x)$, and we phrase a converse to this observation later.

Other facts involving the Lie bracket are also straightforward to show. For instance for $f, g \in \mathcal{F}^\infty(\mathcal{M})$ we have the formula

$$\mathcal{L}_{fX}(gY) = [fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

The special case of $\mathcal{L}_X(gY) = g\mathcal{L}_X(Y) + (Xg)Y$ tells us how the Lie derivative differs from an $\mathcal{F}^\infty(\mathcal{M})$-module homomorphism.

The **Jacobi identity** holds for the bracket:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Temporarily writing $[X, Y]$ as $X * Y$ and using $X * Y = -Y * X$ the Jacobi identity tells us that

$$(X * Y) * Z = X * (Y * Z) + Y * (Z * X)$$

so this operation is *not* associative on generic vector fields. Also,

$$\mathcal{L}_Y(X * Z) = X * \mathcal{L}_Y(Z) + Z * \mathcal{L}_Y(X).$$

In other words, the Lie derivative with respect to a fixed vector field is a derivation on the (non-associative) algebra of vector fields with the $*$ operation.

3. The Lie Derivative on the Grassmann Algebra

Before we generalize this notion to forms, we first introduce interior product on our manifold to help us with notation.

The interior product is a family of functions on $\mathcal{G}(\mathcal{M})$ indicated by the notation $\boldsymbol{X}\lrcorner$ for various tensor fields $X \in \mathcal{T}^1(\mathcal{M})$.

We calculate the interior product on 1-term forms, extending to all of $\mathcal{G}(\mathcal{M})$ by linearity.

If f has grade 0 (i.e. a smooth number field) we define $X\lrcorner(f) = 0$.

When $r > 0$ it is calculated on $\theta \in \Lambda^r(\mathcal{M})$ as $(X\lrcorner\theta)(p) = X_p\lrcorner\theta_p$ for each $p \in \mathcal{M}$, pushing the definition to a local calculation, defined next.

$X_p\lrcorner\theta_p$ is defined to be evaluation of the first index "slot" of θ_p at X_p. In terms of tensor operations, this is the tensor product $X_p \otimes \theta_p$ followed by contraction of X_p against the first index of θ_p.

This operation on $\mathcal{G}(\mathcal{M})$ has a number of properties.

If $\theta \in \Lambda^r(\mathcal{M})$ then $X\lrcorner\theta \in \Lambda^{r-1}(\mathcal{M})$. Thus, we have a grade-reducing map

$$X\lrcorner\colon \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M}).$$

Further, $X\lrcorner(Y\lrcorner\theta) = -Y\lrcorner(X\lrcorner\theta)$ so $X\lrcorner(X\lrcorner\theta) = 0$.

And, finally, $X\lrcorner$ is a $\wedge$-antiderivation on $\mathcal{G}(\mathcal{M})$:

$$X\lrcorner(\theta \wedge \tau) = (X\lrcorner\theta) \wedge \tau + (-1)^r\, \theta \wedge (X\lrcorner\tau)$$

Getting back to Lie derivatives, define

$$\mathcal{L}_X \colon \Lambda^r(\mathcal{M}) \to \Lambda^r(\mathcal{M}) \quad \text{by} \quad \mathcal{L}_X(\theta) = X \lrcorner\, d\theta + d(X \lrcorner\, \theta)$$

for any r-form θ, and to any multi-form field by application at each grade. You will note that this agrees with the earlier definition on functions: for smooth function θ the second term is zero and the first term is $X(d\theta)$.

The Lie derivative on the Grassmann algebra has numerous useful properties, of which we will explore few. We do note that the Lie derivative is a $\wedge$-derivation on the Grassmann algebra, not a $\wedge$-antiderivation. By this we mean that if θ and τ are two r-forms and c is a constant function and μ is a k-form then

$$\mathcal{L}_X(c\,\theta + \tau) = c\mathcal{L}_X(\theta) + \mathcal{L}_X(\tau) \quad \text{and} \quad \mathcal{L}_X(\theta \wedge \mu) = \mathcal{L}_X(\theta) \wedge \mu + \theta \wedge \mathcal{L}_X(\mu).$$

The first equation is obvious, while the second follows by expanding the wedge products and application of the antiderivation sign pattern for d and $X \lrcorner$.

It is also obvious, but useful, that

$$\mathcal{L}_{X+cY}(\theta) = \mathcal{L}_Y(\theta) + c\mathcal{L}_X(\theta) \quad \text{for constant } c \text{ and vector fields } X \text{ and } Y.$$

Next: $\mathcal{L}_{gX}(\theta), \mathcal{L}_X(g\theta), \mathcal{L}_X(Yf), \mathcal{L}_X(Y df)$

4. Lie Derivatives Via the Local Flow

For smooth vector field X, the various versions of Lie derivative $\mathcal{L}_X$ applied to functions, vector fields and forms, are not defined solely "at points" in the sense that to determine them at a point you must have knowledge of the functions, fields and forms on some (any) neighborhood around that point. But this kind of information, which might be described as "local," suffices.

We will concentrate here on a single point $p \in \mathcal{M}$ so that X determines a local flow $T \colon (-2r, 2r) \times \mathcal{U} \to \mathcal{V}$ for open neighborhood $\mathcal{U}$ of p.

Let's define $\mathcal{U}_t = T_t(\mathcal{U})$ and suppose $\mathcal{V}$ is $\bigcup_{t \in (-2r,2r)} \mathcal{U}_t$.

Each T_t is a diffeomorphism onto its image $\mathcal{U}_t$.

$T_{t+s} = T_t \circ T_s$ for all $s, t \in (-r, r)$ and so $T_0(q) = q$ for all $q \in \mathcal{U}$.

We will suppose $x \colon U_x \to R_x$ is a coordinate system around p and $\mathcal{V} \subset U_x$.

X_q is tangent to c_p, defined by $c_p(t) = T_t(p)$, whenever q is in the range of c_p.

In particular $X_q = [x, q, (x \circ c_q)'(0)]$ for each $q \in \mathcal{U}$.

By Theorem 19.4 of Part I we can specify x so that $X_q = [x, q, e_1]$ everywhere in $\mathcal{U}$: that is, all integral curves follow the coordinate gridcurves so $c_q = G^{e_1}_{x,q}$.

If $f \in \mathcal{F}^\infty\mathcal{M}$ we have defined $\mathcal{L}_X(f)_p$ to be $(f \circ c_p)'(0)$.

This ordinary derivative is

$$\mathcal{L}_X(f)_p = X_p(f) = \lim_{\varepsilon\to 0}\frac{f(c_p(\varepsilon)) - f(p)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{f(T_\varepsilon(p)) - f(p)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\frac{(T_{-\varepsilon})_{*p}(f) - f(p)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{(T_\varepsilon)^*_p(f) - f(p)}{\varepsilon}.$$

We have defined the Lie derivative on functions in terms of pullbacks or tangent maps using the diffeomorphisms from the local flow.

We can do the same thing for Lie derivatives on vector fields and one-forms.

Suppose $\omega \in \mathcal{T}_1(\mathcal{M})$. Near p we can represent ω as $\sigma_i\, dx^i$ for smooth σ_i.

$$\text{Define } \; A_\epsilon(\omega) = \frac{(T_\varepsilon)^*_p(\omega) - \omega_p}{\varepsilon}.$$

We would like to show that $\lim_{\varepsilon\to 0} A_\varepsilon(\omega)$ converges, and the limit is

$$\mathcal{L}_X(\omega)_p = (\, X \lrcorner\, d\omega\,)_p + (\, d(X \lrcorner\, \omega)\,)_p,$$

but first we have to be clear about what $(T_\varepsilon)^*_p(\omega)$ actually is.

It pulls back the action of ω at $c_p(\varepsilon)$ to p.

So if g is a real-valued function in $\omega_{c_p(\varepsilon)}$ then $g \circ T_\varepsilon$ is in $(T_\varepsilon)^*_p(\omega)$.

That is, $(T_\varepsilon)^*_p(\omega) = d(g \circ T_\varepsilon)_p$.

We have proven that

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