Complex Analysis

The Way It Ought To Be*

Larry Curnutt, March 2024

*Complex analysis is often described as more elegant and perfect than real analysis, which is often viewed as messy and chaotic. More on this later, when we discuss analytic functions

Contents*

*Larry S. – Notice that I did not include "Contents" in the table of contents!

$$imaginary : i = \sqrt{-1} \qquad i^2 = -1$$

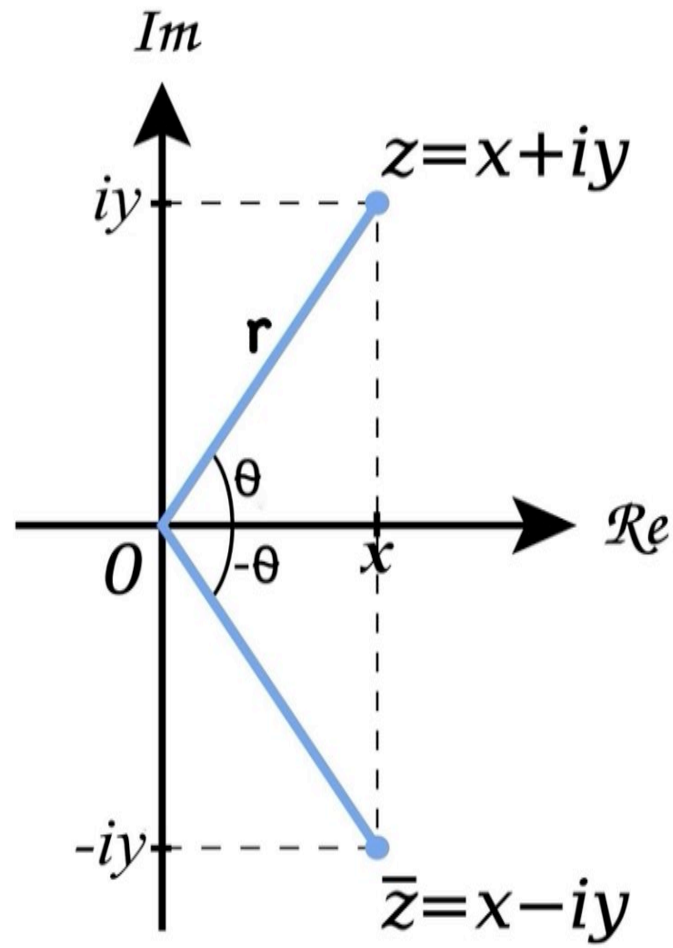
$$complex : z = x + iy$$

$$polar : z = r(cos\theta + isin\theta)$$

$$congugate : \bar{z} = x - iy$$

$$modulus : |z| = \sqrt{x^2 + y^2}$$

$$argument : \theta = tan^{-1}(y/x)$$



$\mathbb{C}$ denotes the set of all complex numbers, or geometrically the complex plane. Unless specified otherwise, functions $(f, g, h, etc.)$ have domains and ranges contained in $\mathbb{C}$.

Complex Arithmetic

$$z = 3 + 4i \qquad w = 1 - 3i$$

$$\bar{z} = 3 - 4i \qquad \overline{w} = 1 + 3i$$

- $z + w = (3 + 1) + (4 - 3)i = 4 + i$

- $z \cdot w = (3 + 4i)(1 - 3i) = 3 - 9i + 4i - 12i^2 = 15 - 5i$

- $\dfrac{z}{w} = \dfrac{(3 + 4i) \cdot (1 + 3i)}{(1 - 3i) \cdot (1 + 3i)} = \dfrac{-9 + 13i}{10} = -0.9 + 1.3i$

- $|z \cdot w| = \sqrt{15^2 + 5^2} = 5 \cdot \sqrt{10} = |z| \cdot |w|$

- $|z\bar{z}| = |z|^2$

Very Little History

(a lesson in name-dropping)

• **Complex analysis, the theory of functions of a complex variable, is one of the classical branches of mathematics; it uses algebra, geometry, limits, derivatives, integrals, topology, and more.**

• Square roots of negative numbers have been around (though neither understood nor trusted) since the pre-algebra days of the Greeks and even the Babylonians.

• In the late 1500s **Nicholas Tartaglia**, **Hieronymus Cardano** and others ran smack dab into square roots of negatives in their attempts to solve cubic equations.

• In the 1600s such notables as **Rene Descartes**, **Leonard Euler** and **Carl Gauss** established vocabulary, notation, and properties of imaginary and complex numbers, and began the investigation of complex functions.

• Around 1800 **Caspar Wesell** and **Ami Argand** "invented" the complex plane. A little later **William Rowan Hamilton** formalized the algebra of complex numbers (and generalized to quaternions).

• During the 19th century **Augustin-Louis Cauchy**, **Karl Weierstrass**, and **Georg Friedrich Bernard Riemann** developed the heart of what is considered today to be complex analysis: analytic functions, complex integrals, power series, complex manifolds.

• **Cauchy is called "the father of of complex function theory."**

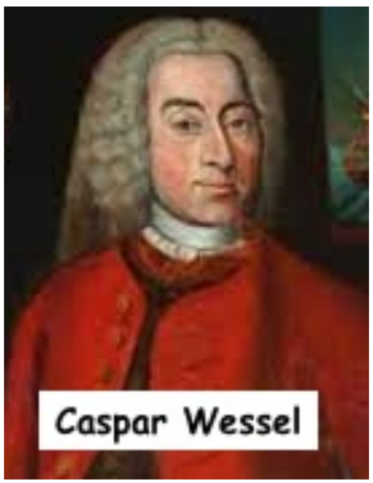

Nicolavs Tartaglia


Hieronymus Cardano


Rene Descartes

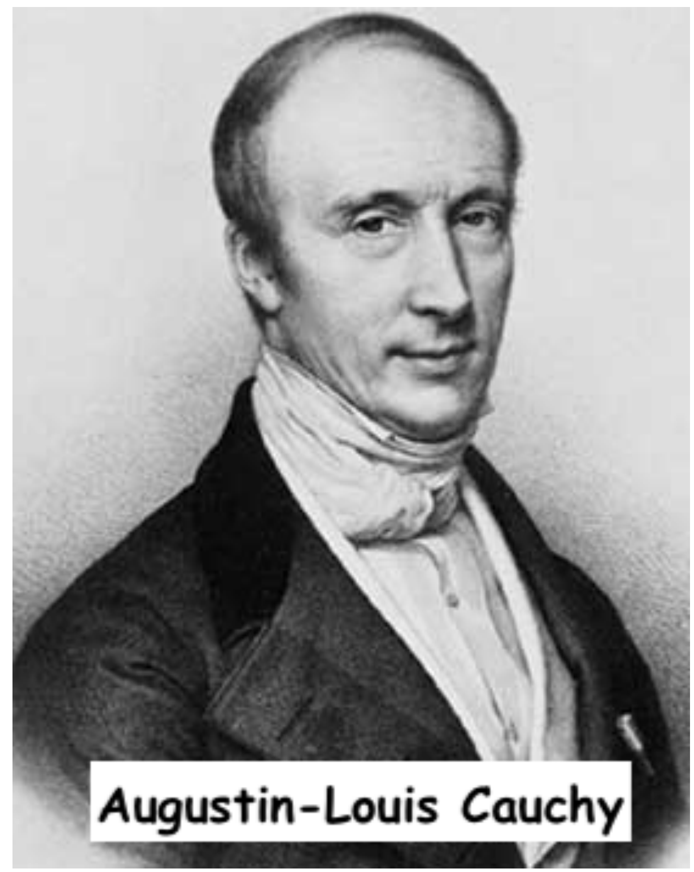

Leonard Euler

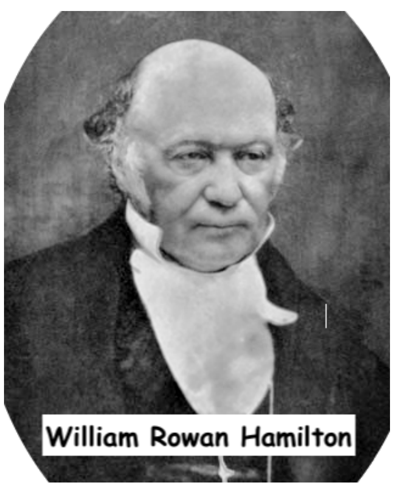

Carl Friedrich Gauss


Caspar Wessel


Ami Argand

A Gallery of Players


Augustin-Louis Cauchy


William Rowan Hamilton


Karl Weierstrass

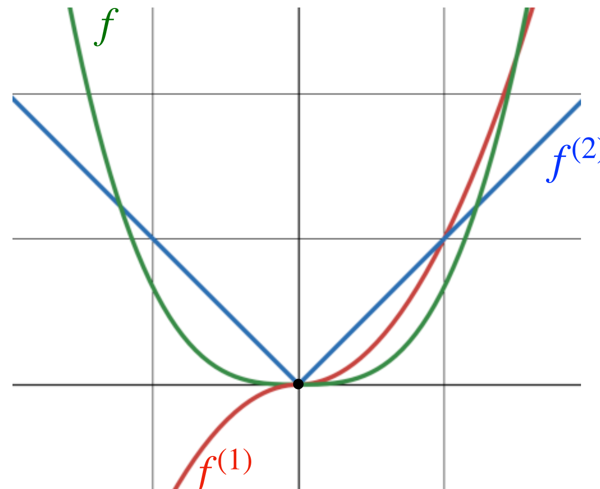

Georg Friedrich Bernard Riemann

Complex Derivatives

DEF. $f(z)$ is <u>analytic</u> if $f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ exists.

(holomorphic)

This definition is much more restrictive than the identical-looking one from (real) calculus. In calculus, for a limit to exist there are only two directions to consider: from the left and from the right. In complex analysis, for a limit to exist, limits from **all directions** around the compass must agree. This makes an incredible difference in the subjects. Here's a preliminary example.



Consider the function $f(x) = |x^3|$.

f is both continuous and differentiable everywhere.

So is its first derivative, $f^{(1)}$.

$f^{(2)}(x) = |x|$, the second derivative, is still continuous, but not differentiable at the origin.

Of course, the third derivative, $f^{(3)}(x)$, is either +1 or -1, depending on which side of 0 you're on -- not even continuous, let alone differentiable, at the origin.

- This could never happen in complex analysis. As we'll show a little bit later, **all analytic functions have infinitely many continuous derivatives**.

Consequences: All of the usual derivative rules from calculus carry over, including differentiation of power series. So z^n, e^z, $\sin z$, etc. are all analytic, if defined by their ~~pow~~ Maclaurin expansions.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots$$

$$sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

<u>What is the meaning of the complex versions of these familiar functions?</u>

- polynomials – clear, because they involve complex arithmetic
- exponentials & trig – similarly clear via power series

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = r(\cos\theta + i\sin\theta)$$

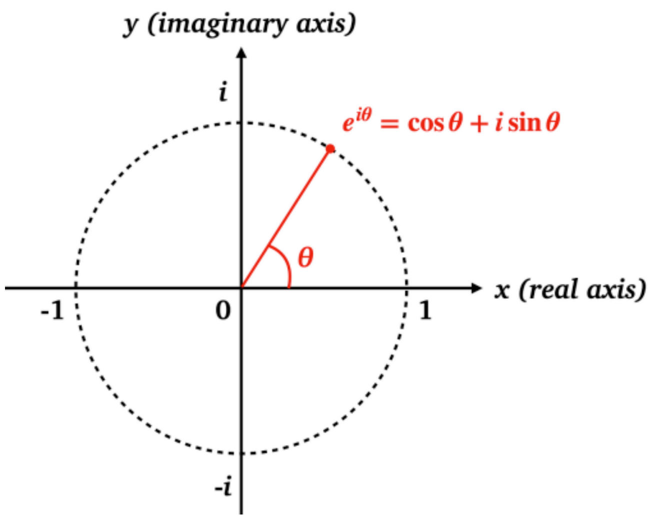
What the heck is θ in terms of y?

$$u = e^{-i\theta} \qquad v = \cos\theta + i\sin\theta$$

$$\begin{aligned}(u \cdot v)' &= u \cdot v' + u' \cdot v \\ &= e^{-i\theta}(-\sin\theta + i\cos\theta) - ie^{-i\theta}(\cos\theta + i\sin\theta) \\ &= e^{-i\theta}(-\sin\theta + i\cos\theta - i\cos\theta - i^2\sin\theta) = 0\end{aligned}$$

$$\therefore \quad e^{-i\theta}(\cos\theta + i\sin\theta) = constant$$

$$substitute \;\; \theta = 0 \implies constant = 1$$



$$\implies \boxed{e^{i\theta} = \cos\theta + i sin\theta}$$

circle unit

Euler's Formula

(one of them)

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(cosy) + i(e^x \sin y)$$

$$z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$$

$$\frac{1}{z} = \left(\frac{1}{x+iy}\right) \cdot \left(\frac{x-iy}{x-iy}\right) = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$$

$$\sqrt{z} = \sqrt{x+iy} = \left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) \quad + \quad i\frac{y}{|y|}\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)$$

$$Log(z) = Log(x+iy) = Log(re^{i\theta}) = log(r) + log(e^{i\theta}) = ln(r) + i\theta = \left(ln\sqrt{x^2+y^2}\right) \ + \ i\left(\tan^{-1}\frac{y}{x}\right)$$

Real and Imaginary Parts

$$f(z) \ = \ f(x+iy) \ = \ U(x,y) \ + \ iV(x,y)$$

THEOREM 1. $f(z) = f(x+iy) = r(x,y) + i\,c(x,y)$, where r and c are real-valued functions of 2 real variabl[es]. f is analytic if and only if $\frac{\partial r}{\partial y} = -\frac{\partial c}{\partial x}$ and $\frac{\partial r}{\partial x} = \frac{\partial c}{\partial y}$. (Cauchy-Riemann Equations)

Proof. If $\Delta z = s$ is real, then $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$

$$= \lim_{s \to 0} \frac{r(x+s,y) + i\,c(x+s,y) - r(x,y) - i\,c(x,y)}{s}$$

$$= \lim_{s \to 0} \frac{r(x+s,y) - r(x,y)}{s} + i\,\frac{c(x+s,y) - c(x,y)}{s}$$

$$= \frac{\partial r}{\partial x} + i\,\frac{\partial c}{\partial x}. \qquad (1)$$

Similarly, if $\Delta z = is$ is imaginary, then $f'(z)$

$$= \lim_{s \to 0} \frac{r(x,y+s) + i\,c(x,y+s) - r(x,y) - i\,c(x,y)}{is}$$

$$= \lim_{s \to 0} \frac{c(x,y+s) - c(x,y)}{s} - i\,\frac{r(x,y+s) - r(x,y)}{s}$$

$$= \frac{\partial c}{\partial y} - i\,\frac{\partial r}{\partial y}. \qquad (2)$$

Equating real and imaginary parts of (1) and (2), we have $\frac{\partial r}{\partial x} = \frac{\partial c}{\partial y}$ and $\frac{\partial c}{\partial x} = -\frac{\partial r}{\partial y}$.

Converse??? (later). ∎

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \;\rightarrow\; f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \;\; and \;\; f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

• If f is analytic and purely real or purely imaginary (i.e. $u \equiv 0$ or $v \equiv 0$), then f must be a constant function.

Proof. Suppose $v \equiv 0$. Then the C-R equations imply $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0.$
So u involves neither x nor y. Therefore, u must be constant. Similarly, if $u \equiv 0$, Then v must be constant. In either case, f is constant. ■

• $f(z) = |z|^2 = x^2 + y^2$ is continuous, but does not have an antiderivative.

Proof by contradiction. Suppose $f = u + iv$ does have an antiderivative: say $F = U + iV$ and $F' = f = x^2 + y^2$. Now, according to the C-R Equations:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = x^2 + y^2 \;\text{ and }\; \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = 0.$$

But $\dfrac{\partial U}{\partial y} = 0$ implies that $U(x,y) = w(x)$ (U has no y's involved).

This is a contradiction of $x^2 + y^2 = \dfrac{\partial U}{\partial x} = w'(x).$

Therefore, $f(z) = |z|^2$ has no antiderivative. ■

Maximum Modulus Theorem/Principle*

If $f(z)$ is analytic on domain $\mathcal{D}$, then $|f(z)|$ attains its maximum on the boundary of $\mathcal{D}$.

For the next two examples, let $\mathcal{D}$ = $\{ z : |z| \leq 1 \}$, the unit disk, with boundary ∂ = $\{ z : |z| = 1 \}$.

- $\underline{f(z) = 1 - z^2}$

By the triangle inequality*, $|f(z)| \leq 1 + |z|^2 \leq 2 \;\; \forall \, z \in \mathcal{D}$, and $|f(i)| = 2$.
Note that i is on the boundary of $\mathcal{D}$.

- $\underline{g(z) = z^2 + 3z - 1}$

Again by the triangle inequality*, $|g(z)| \leq |z|^2 + 3|z| + 1 \leq 5$, but it's not clear that this upper bound is actually attained. In fact, the quadratic formula implies that if $|g(z)| = 5$ $(i.e., g(z) = \pm 5)$, then $|z| = \sqrt{11}$ or $\sqrt{10.5}$, both > 1, and thus outside $\mathcal{D}$. According to the MMT/P, to determine $\max\limits_{D}(|g(z)|)|$, we need only calculate $\max\limits_{\partial}(|g(z)|)|$. To do this parameterize $\partial(\theta) : [0,2\pi] \rightarrow \mathbb{C}$ by $z = \partial(\theta) = e^{i\theta} = cos\theta + i \cdot sin\theta$, and thus, $z^{-1} = cos\theta - i \cdot sin\theta$:

$$|g(z)| = |z^2 + 3z - 1| = |z||z + 3 - z^{-1}| = |3 + 2i \cdot sin\theta| = \sqrt{9 + 4 \cdot sin^2\theta}$$

which attains a maximum of $\sqrt{13}$ at $\theta = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ (i.e. $z = \pm i$).

* <u>This is not true for real functions.</u> For example, consider $p(x) = 1 - x^2$ defined on the interval $[-1,1]$. The maximum value $|p(0| = 1$, and 0 is in the interior of $(-1,1)$, not on the boundary (endpoints).

** See slide #42 for a proof.

Complex Fundamental Theorem of Calculus

• Let $\gamma : [a,b] \to \mathbb{C}$ be a differentiable path with endpoints $\gamma(a) = \alpha$ and $\gamma(b) = \beta$. If $f(z)$ is continuous **with a primitive*** F ($i.e.$ $F' = f$), then

$$\int_\gamma f(z)dz = F(\beta) - F(\alpha)$$

Proof.
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

(We'll formalize this on slide #17.)

$$= \int_a^b F'(\gamma(t))\gamma'(t)dt$$

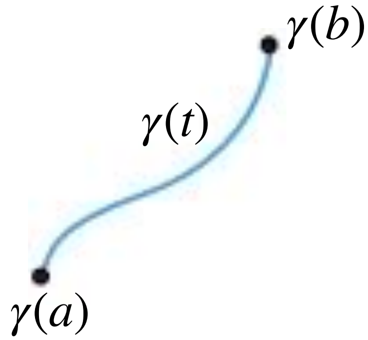
$$= \int_a^b (F \circ \gamma)'(t))dt$$

$$= F(\gamma(a)) - F(\gamma(b))$$

$$= F(\beta) - F(\alpha)\text{, by the FT of C.} \blacksquare$$

*In the calculus you're used to, every continuous function has a primitive (or antiderivative). Continuous complex functions may not have primitives. For example, as we saw on slide #12, $g(z) = |z|^2 = x^2 + y^2$ does not have a primitive.

Length of a Contour in the Complex Plane



$$L = \int_\gamma |\gamma'(t)|\,dt$$
$$= \int_a^b |\gamma'(t)|\,dt$$

<u>Why?</u>

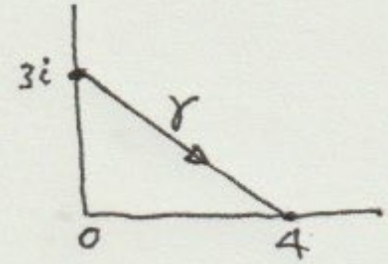
- $\gamma(t) = x(t) + iy(t) \;\rightarrow\; \gamma' = x' + iy' \;\rightarrow\; |\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

$\rightarrow\; L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt \quad$ *(recall arc length from Calc II)*

- $\Delta L = |\gamma(t+\Delta t) - \gamma(t)| = \left| \dfrac{\gamma(t+\Delta t) - \gamma(t)}{\Delta} \cdot \Delta t \right| \approx \gamma'(t)\Delta t$

Adding these up and taking the limit as $\Delta t \rightarrow 0$ *and* $n \rightarrow \infty$, *results in* $\int_a^b$.

EXAMPLE:



$$L = \int_\gamma |\gamma'(t)|\,dt = \int_0^1 \sqrt{(4)^2 + (-3)^2}\,dt = \int_0^1 5\,dt = 5t\Big|_0^1 = 5$$

$\gamma(t) = 4t + 3(1-t)i$

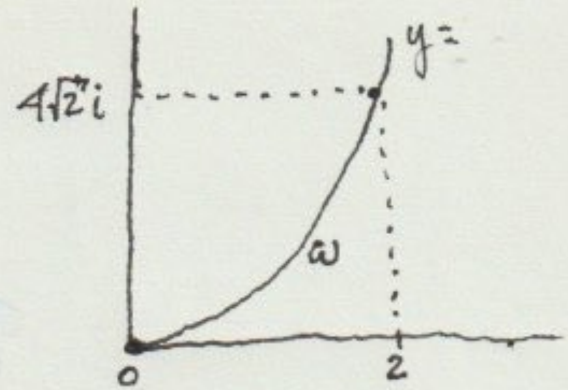
$\gamma'(t) \equiv 4 - 3i$

$0 \le t \le 1$

EXAMPLE: $\omega = t + 2t^{3/2} i$

$\omega' = 1 + 3t^{1/2} i$

$0 \le t \le 2$



$$L = \int_\omega |\omega'(t)|\,dt = \int_0^2 \sqrt{(1)^2 + (3t^{1/2})^2}\,dt$$

$$= \int_0^2 \sqrt{1 + 9t}\,dt = \frac{1}{9}\cdot\frac{2}{3}(1+9t)^{3/2}\Big|_0^2$$

$$= \frac{2}{27}\left(19\sqrt{19} - 1\right) \approx 6.06$$

Complex Integration

DEF. $\gamma : [a,b] \longrightarrow \mathbb{C}$ piecewise differentiable path in $\mathbb{C}$

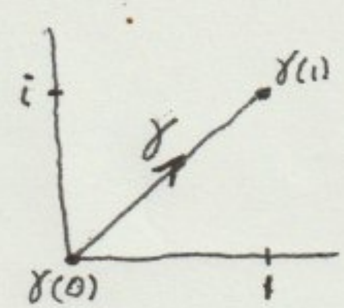
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \int_a^b \left[r(\gamma(t)) + i\,c(\gamma(t))\right]\left[\gamma_1'(t) + i\,\gamma_2'(t)\right]dt$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(\gamma(t_i))\left[\gamma(t_i) - \gamma(t_{i-1})\right]$$

This CONTOUR INTEGRAL does <u>not</u> depend on how γ is parameterized except, of course, if $-\gamma$ denotes the same path traced in the opposite direction, then $\int_{-\gamma} = -\int_\gamma$.

EXAMPLE: $f(z) = \overline{z}$, $\gamma(t) = t + ti$



$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \int_0^1 \overline{(t+ti)}\,(1+i)\,dt$$

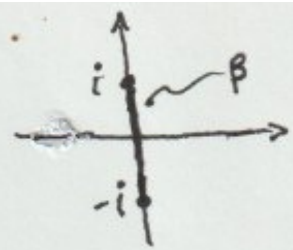
$$= \int_0^1 (t-ti)(1+i)\,dt$$

$$= (1+i)\int_0^1 (1-i)\,t\,dt$$

$$= (1+i)(1-i)\left[\frac{t^2}{2}\right]_0^1 = 2\left[\frac{1}{2}\right] = 1$$

EXAMPLE: $p(z) = p(x+iy) = 5 + 2x + 4y$

$$\beta(t) = \underset{x}{0} + i\underset{y}{t}, \quad -1 \le t \le 1$$

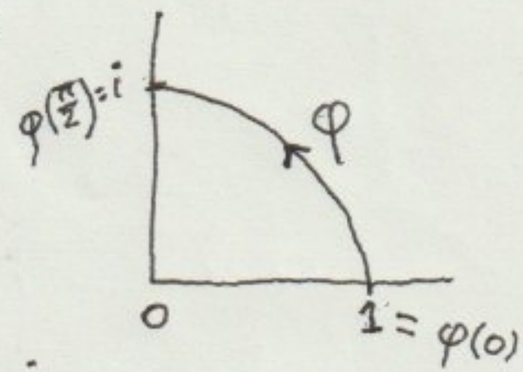


$$\int_\beta p(z)\,dz = \int_\beta (5 + 2x + 4y)\,dz$$

$$= \int_{-1}^{1} (5 + 2(0) + 4(it))\cdot(i)\,dt$$

$$= i\left[5t + 2t^2 i\right]_{-1}^{1} = i\left[(5+2i) - (-5+2i)\right] = 10i$$

EXAMPLE: $\int_{\varphi} z^2\,dz$, $\varphi(t)=e^{it}$



$$=\int_0^{\frac{\pi}{2}} (e^{it})^2\cdot(ie^{it})\,dt$$

$$= i\int_0^{\pi/2} e^{3ti}\,dt = i\left[\frac{e^{3ti}}{3i}\right]_0^{\pi/2} = \frac{1}{3}\left[e^{\frac{3\pi}{2}i} - e^0\right]$$

$$= \frac{1}{3}\left[-i-1\right] = -\frac{1+i}{3}$$

EXAMPLE: $\gamma(t)=a+e^{it}=a+\cos t+i\sin t$, $0\le t\le 2\pi$

(γ is a circle centered at a with radius 1)

$$\int_\gamma \frac{1}{z-a}\,dz = \int_0^{2\pi}\frac{1}{a+\cos t+i\sin t-a}(-\sin t+i\cos t)\,dt$$

$$= \int_0^{2\pi}\frac{i(\cos t+i\sin t)}{\cos t+i\sin t}\,dt$$

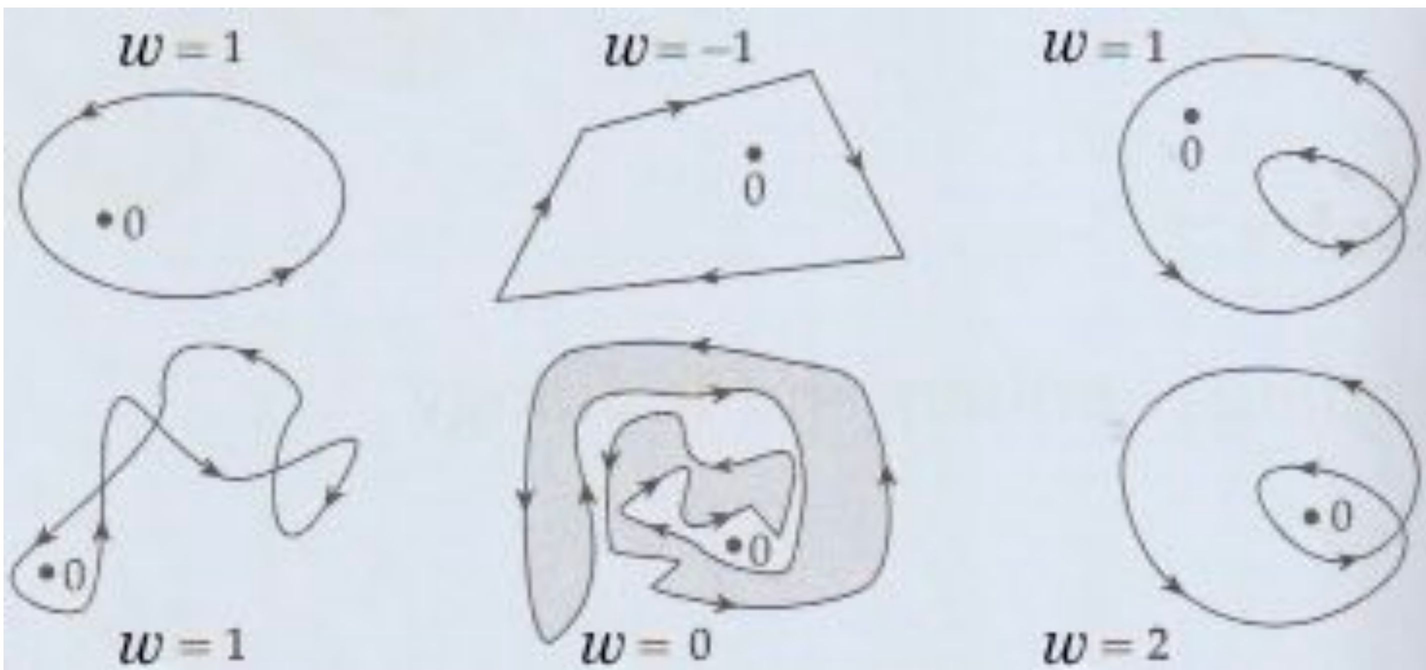
$$= \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = 2\pi i$$

Winding Numbers

<u>Theorem</u>. *For all smooth, closed paths* $w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz = an\ integer$.

(See slide #43 for a proof.)

This integer, $w(\gamma, a)$ is called the winding number γ of about a .
It counts the number of times γ winds around or "encircles" a.
Counterclockwise counts positive, and clockwise counts negative.



Example: $\gamma(\theta) = e^{5\theta i}$ describes a circular path, traced around the origin 5 times: $w(\gamma, 0) = 5$.

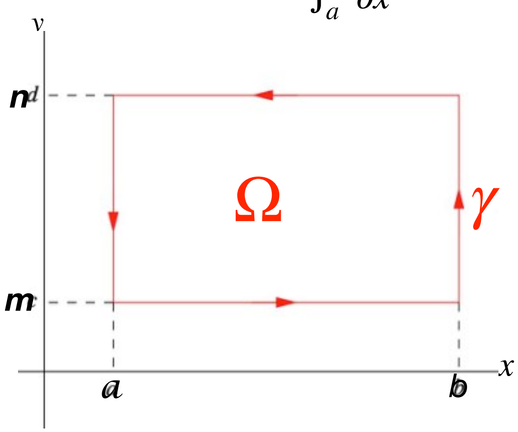
Green's Theorem

If γ is a piecewise smooth, simple curve that bounds a region Ω, $r(x,y)$ and $c(x,y)$ are real-valued, continuous, and have continuous partial derivatives along γ and throughout Ω, then

$$\oint_{\gamma} r\,dx + c\,dy \;=\; \iint_{\Omega}\left[\frac{\partial c}{\partial x} - \frac{\partial r}{\partial y}\right] dA\,.$$

<u>Proof when Ω is a rectangle.</u>

F T of C → $\displaystyle\int_a^b \frac{\partial c}{\partial x}\,dx = c(x,y)\Big|_{x=a}^{x=b} = c(b,y) - c(a,y)$ and $\displaystyle\int_m^n \frac{\partial r}{\partial y}\,dy = r(x,y)\Big|_{y=m}^{y=n} = r(x,n) - r(x,m)$



vertical $\rightarrow dx = 0$ and horizontal $\rightarrow dy = 0$

$$\oint_{\gamma} r\,dx + c\,dy = \int_m^n [c(b,y) - c(a,y)]dy - \int_a^b [r(x,n) - r(x,n)]dx$$

$$= \int_m^n \int_a^b \frac{\partial c}{\partial x}\,dxdy \quad - \quad \int_a^b \int_m^n \frac{\partial r}{\partial y}\,dydx$$

$$= \iint_{\Omega}\left[\frac{\partial c}{\partial x} - \frac{\partial r}{\partial y}\right] dA$$

■ or QED

THEOREM 2. (CAUCHY) If $f(z)$ is analytic inside the closed curve γ, then $\int_\gamma f(z)\,dz = 0$.

Proof.

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b f(\gamma_1(t)+i\gamma_2(t))\left[\gamma_1'(t)+i\gamma_2'(t)\right]dt$$

$$= \int_a^b \left[r(\gamma_1(t),\gamma_2(t)) + i\,c(\gamma_1(t),\gamma_2(t))\right]\left[\gamma_1'(t)+i\gamma_2'(t)\right]dt$$

$$= \int_a^b r(\gamma_1,\gamma_2)\gamma_1' - c(\gamma_1,\gamma_2)\gamma_2'\,dt + i\int_a^b r(\gamma_1,\gamma_2)\gamma_2' + c(\gamma_1,\gamma_2)\gamma_1'$$

$$= \int_\gamma r\,dx - c\,dy + i\int_\gamma r\,dy + c\,dx$$

Green's Thm.

$$= \iint_\Omega \frac{\partial r}{\partial y} + \frac{\partial c}{\partial x}\,dx\,dy + i\iint_\Omega \frac{\partial c}{\partial y} - \frac{\partial r}{\partial x}\,dx\,dy$$

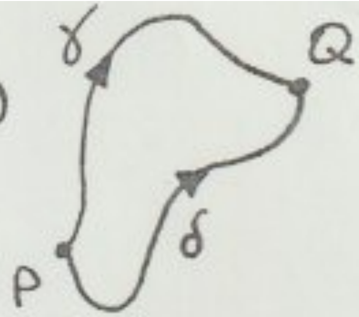
C-R Eqs.

$$= 0. \blacksquare$$

* <u>COROLLARIES.</u> (1) If γ and δ are paths with the same endpoints that can be deformed continuously into one another without crossing singularities of $f(z)$ or moving the endpts., then $\int_\gamma f(z)\,dz = \int_\delta f(z)\,dz$.

(2) If γ and δ are CLOSED paths that can be deformed into each other without crossing singularities of $f(z)$, then $\int_\gamma f(z)\,dz = \int_\delta f(z)\,dz$.

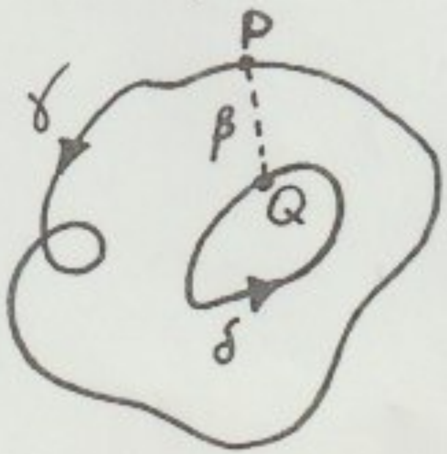
Proofs. (1)



Let C denote path that traces γ from P to Q, then δ from Q to P. Apply Cauchy's Thm.:

$$0 = \int_C f(z)\,dz = \int_\gamma f(z)\,dz + \int_{-\delta} f(z)\,dz = \int_\gamma f - \int_\delta f .$$

(2)



Let C denote path tracing γ from P to P, then along β to δ, around δ to Q and back along $-\beta$. ∎

THEOREM 3. (CAUCHY'S INTEGRAL FORMULA)

If $f(z)$ is analytic and γ is a simple closed curve, then $\forall$ a inside γ

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Proof.

$$F(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$ is analytic.

$$\therefore\ 0 = \int_\gamma F(z)\, dz = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz - f(a)\int_\gamma \frac{dz}{z-a}$$

Now, since γ can be deformed into the circle of radius 1 centered at a (without crossing a), $\int_\gamma \frac{dz}{z-a} = 2\pi i$ (see example on slide #19), and consequently,

$$\int_\gamma \frac{f(z)}{z-a}\, dz = f(a) \cdot 2\pi i \quad \blacksquare$$

* **COROLLARY.** If $f(z)$ is analytic and γ is a simple closed curve, then $\forall a$ inside γ

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Proof. $n=1$: $\displaystyle \frac{f(z)-f(a)}{z-a} = \frac{1}{z-a}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}d\zeta\right]$

$$= \frac{1}{2\pi i}\cdot\frac{1}{z-a}\int_\gamma \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta-a}\, d\zeta$$

$$= \frac{1}{2\pi i}\cdot\frac{1}{\cancel{z-a}}\int_\gamma \frac{f(\zeta)[\cancel{\zeta - a - \zeta + z}]}{(\zeta-z)(\zeta-a)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)(\zeta-a)}\, d\zeta$$

$$\longrightarrow \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^2}\, d\zeta = f'(a)$$

as $z \longrightarrow a$.

$n>1$: by induction. ∎

NOTE: 1. Cauchy's Theorem is the primary ingredient in establishing Cauchy's Integral Formula. The hypothesis of Cauchy's theorem can be weakened to "$f(z)$ satisfies the Cauchy-Riemann Equations" (that's all that was used), so Cauchy's Integral Formula only requires that "$f(z)$ satisfy the C-R Eqs." Therefore, the converse of Theorem 1 can be argued in the following way:

C-R eqs. $\Rightarrow$ Int. Form. $\Rightarrow$ analytic.

2. Analytic functions are INFINITELY differentiable

* **Corollary.** If $f(z)$ is analytic on the open disk $D = \{z \in \mathbb{C} \mid |z-a| < r\}$, then $\forall z \in D$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Proof. Let $\gamma = a + r\cos t + ri\sin t$, so that $|\gamma(t) - a| = r$.

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{1}{1-\frac{z-a}{\zeta-a}} \cdot \frac{1}{\zeta - a} \cdot f(\zeta)\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n \frac{f(\zeta)}{\zeta-a}\,d\zeta, \quad \text{since } \left|\frac{z-a}{\zeta-a}\right| < 1 \text{ for } \zeta \text{ on } \gamma$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\gamma \frac{(z-a)^n}{(\zeta-a)^{n+1}} \cdot f(\zeta)\,d\zeta$$

$$= \sum_{n=0}^{\infty}(z-a)^n \cdot \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta = \sum_{n=0}^{\infty}(z-a)^n \frac{f^{(n)}(a)}{n!}.$$

■

* All analytic functions have valid Taylor Series.

THEOREM 4. (FUNDAMENTAL THEOREM OF ALGEBRA)
Every polynomial with complex coefficients and whose degree is ≥ 1 has at least 1 complex root.

Proof. Suppose $p(z)$ has no roots. Then $f(z) = 1/p(z)$ is analytic. $|p(z)| \to \infty$ as $|z| \to \infty$, so $|f(z)| \to 0$ as $|z| \to \infty$. $\therefore |f(z)| \leq 1$ for $|z| > R$ if R is chosen large enough. Since $f(z)$ is continuous, it attains a maximum value on the closed and bounded disk $\{z \in \mathbb{C} \mid |z| \leq R\}$; say $|f(z)| \leq m$ for $|z| \leq R$. If $M = \max\{1, m\}$, then we have $|f(z)| \leq M$ $\forall z$. Now, let γ_r be the circle of radius r around z.

$$|f'(z)| = \left|\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta\right| \leq \frac{1}{2\pi}\int_{\gamma_r} \frac{|f(\zeta)|}{|\zeta - z|^2}\,d\zeta$$

$$\leq \frac{1}{2\pi}\int_{\gamma_r} \frac{M}{r^2}\,d\zeta = \frac{1}{2\pi}\cdot\frac{M}{r^2}\cdot 2\pi r = \frac{M}{r};$$

i.e. $|f'(z)| \leq \frac{M}{r}$. But r can be arbitrarily large, so $f'(z) = 0 \Rightarrow f(z) \equiv \text{constant} \Rightarrow p(z) \equiv \text{constant}$, which contradicts $\deg(p(z)) \geq 1$. ∎

What is complex analysis used for in the real world?

- propagation of acoustic waves
- fluid dynamics
- signal processing
- telecommunications
- heat transfer
- quantum mechanics
- computational biology
- other mathematics

(integral calculus, differential equations, number theory, algebraic geometry, probability & statistics, etc.)

Examples Of Chaos/Weirdness Associated With **REAL** Analysis

- Size of the Rationals
- Cantor Set
- Koch Snowflake
- Peano Space-filling Curve
- No Valid Maclaurin Series
- Lots of Corners
- Banach-Tarski Paradox

The collection of all rational numbers is pretty big.

• You can get arbitrarily close to every real number with rational numbers. That is, they're packed real tightly onto the real line. That's called being "**dense**." Think about decimal expansions.

e.g. $\sqrt{2} \approx 1.4, 1.41, 1.414, 1.4142, 1.41421; 1.414213, \ldots \rightarrow 1.41421356237309504880168872 42 \ldots.$

Well, maybe the collection of all rational numbers is not so big.

• The reals are uncountable, while the rationals are only **countable**. That is, the rationals can be listed, but the reals can't: $\mathbb{Q} = \{a_1, a_2, a_3, a_4, \ldots\}$.

• Let ϵ be an arbitrarily small positive number.

Define the open interval $A_n = \left(a_n - \frac{1}{2} \cdot \frac{\epsilon}{2^n}, a_n + \frac{1}{2} \cdot \frac{\epsilon}{2^n}\right)$.

The measure of A_n is $m(A_n) = \frac{\epsilon}{2^n}$. Define $B_\epsilon = \bigcup_{n=1}^{\infty} A_n$.

$m(B_\epsilon) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$. (≤ because the $A_n s$ may overlap)

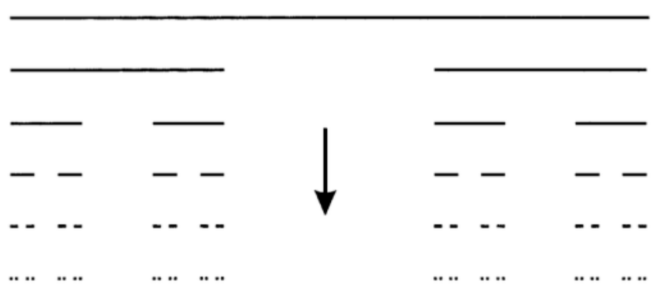
$\mathbb{Q} \subseteq B_\epsilon \rightarrow m(\mathbb{Q}) \leq \epsilon$

Since ϵ was arbitrarily small, $m(\mathbb{Q}) = 0$.

The Cantor Set

• Start with the closed unit interval I = [0 ,1] and remove the middle third of it; i.e. throw away (1/3 , 2 /3) and keep C_1 = [0 , 1 /3] U [2/3 , 1].

• Next remove the open middle thirds of the two parts of C_1, leaving C_2 = [0 , 1 /9] U [2/9 , 3 /9] U [6/9 , 7 /9] U [8/9 , 1].



• Continue this process indefinitely: at each step C_n will consist of a union of 2^n closed subintervals each of length $(1/3)^{n-1}$.

The Cantor Set is C = { what's left after an infinite number of steps }.

• So what is left? At least the end points of the open intervals we removed -- a countable infinity of them. Anything else?

• The length of the stuff removed is the sum of a geometric series with common ratio 2/3:

$$(1/3) + 2\cdot(1/3)^2 + 2^2\cdot(1/3)^3 + 2^3\cdot(1/3)^4 + \cdots$$

$$= (1/3)\left[1 + (2/3) + (2/3)^2 + (2/3)^3 + (2/3)^4 + \cdots\right] = (1/3)\cdot\left[1/(1 - 2/3)\right] = \mathbf{1}$$

Therefore, the "length" (size/measure) of the Cantor Set is **0**. That is, it can't contain any intervals, it's just a bunch of "dust," and from this perspective is small.

• Look at the members of the Cantor Set in base-3 : (no 1s in ternary representations)

$$C = \{\, x \in [0,1] : x = 0.c_1c_2c_3c_4 \ldots c_n \ldots,\ \text{where } c_n = 0 \text{ or } 2 \,\}$$

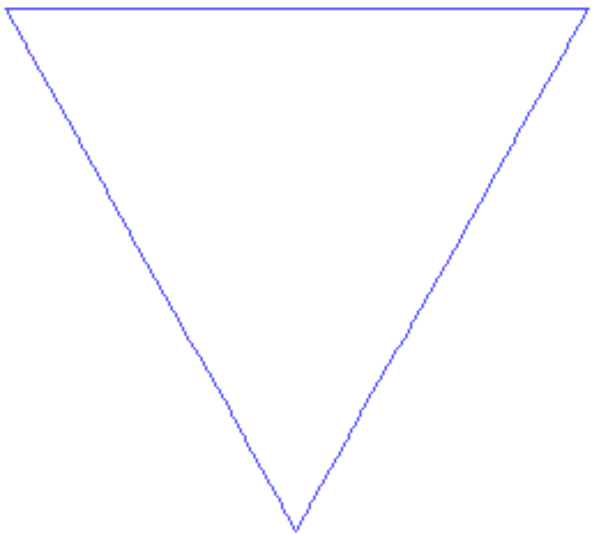
Define a function $f : C \longrightarrow I$ by $f(0.c_1c_2c_3 \ldots c_n \ldots_{(3)}) = 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2} \ldots \frac{c_n}{2} \ldots_{(2)}$

f is onto all of I, so C is uncountable, and in this sense big.

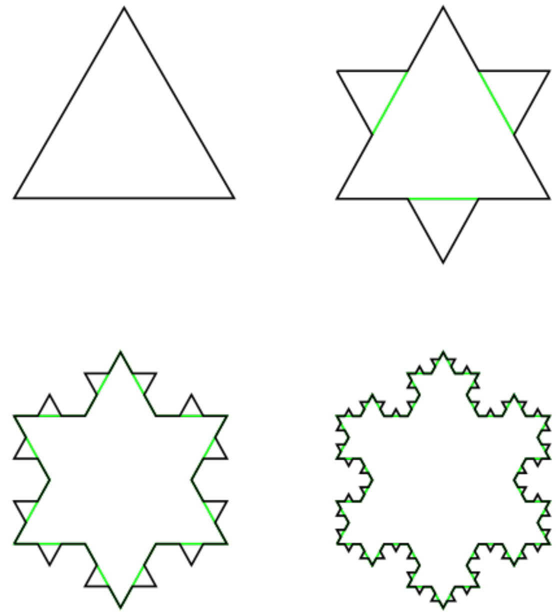
The Norwegian University of Science and Technology has produced a really nice web page that explains these, and even more, properties of the Canton Set very clearly. You can view it at

https://wiki.math.ntnu.no/_media/tma4225/2015h/cantor_set_function.pdf

Koch Snowflake Curve



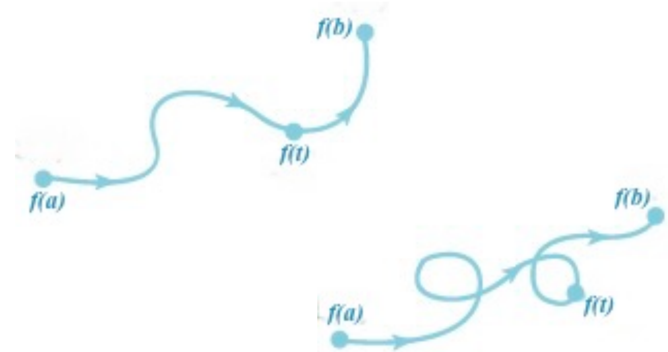
click on the blue triangle



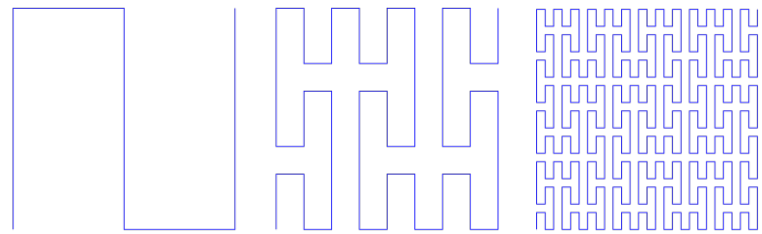
Use geometric series to show that this fractal (self-similar) curve has **infinite length** but **finite area**.

Space-Filling Curves

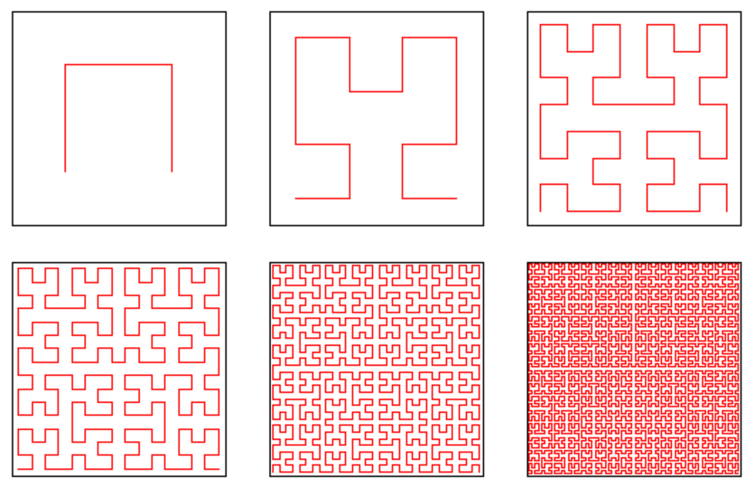
• We usually think of a curve as the path of a continuously moving point, a squiggle or a piece of string tossed carelessly on a table. More formally, if I is the closed interval [a,b] and f is continuous real-valued function on I, then the graph of f is a curve. In any case a curve is a one dimensional object with no area.



• In 1890 Giuseppe Peano produced a continuous function on the unit interval whose graph filled up a whole square. **HUH ? !**



• A year or so later David Hilbert published another example of a space-filling curve.



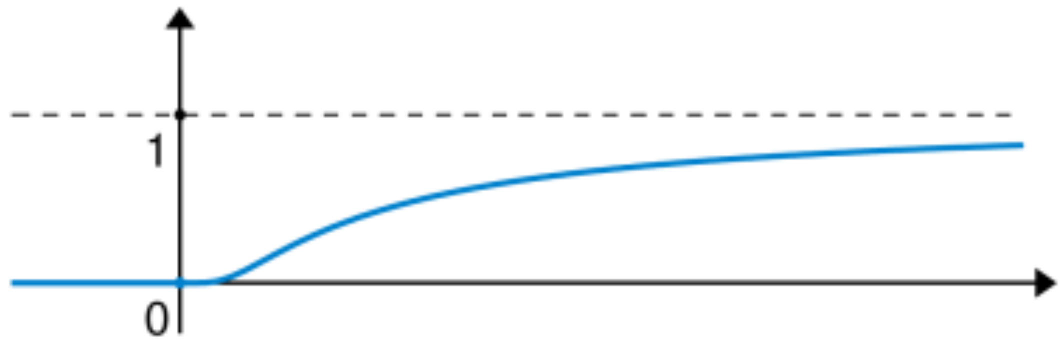
Lack of Power Series

In complex analysis every analytic function is infinitely differentiable and has a valid power series representation.

In real analysis a function may not have a valid power series representation even if it is infinitely differentiable. Here's an example.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

has continuous derivatives of all orders.



$$f^{(k)}(x) = \begin{cases} \frac{p_k(x)}{x^{2k}} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$p_k(x)$ = a polynomial of degree $k-1$

$$Maclaurin\ series\ for\ f(x) \ = \ \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k \ \equiv \ 0 \quad \forall\, x,$$

But $f(x) \ne 0$, *when* $x > 0$,
so the Maclaurin series is NOT a valid representation of $f(x)$.

Smoothness

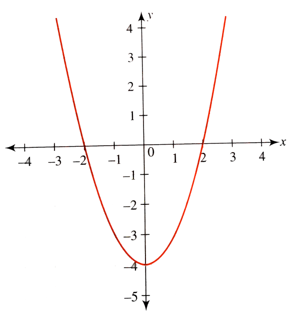
Continuous ≈ “no breaks or gaps”

Differentiable ≈ “no sharp corners”

Differentiable functions are automatically continuous.

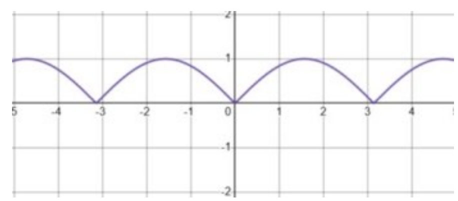
The level of “smoothness” of a function is measured by the number of continuous derivatives it has; the more derivatives, the smoother.

Here are a few examples for you to ponder. How many points of discontinuity do they have? At how many points are they not differentiable? How many derivatives do they have? How smooth are they?



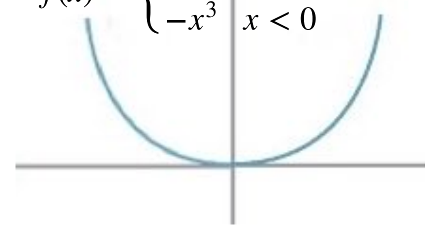
$p(x) = x^2 - 4$

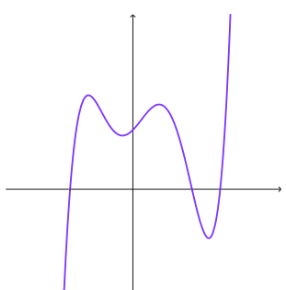
$v(x) = |x^2 - 4|$



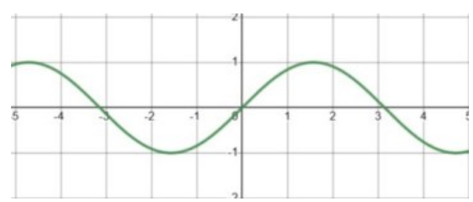
$a(x) = |sin(x)|$

$f(x) = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$

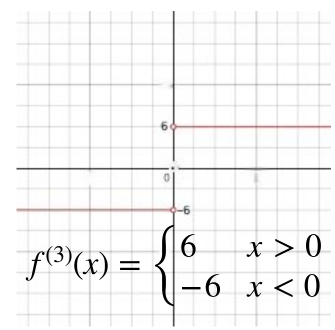




$y = g(x)$



$s(x) = sin(x)$



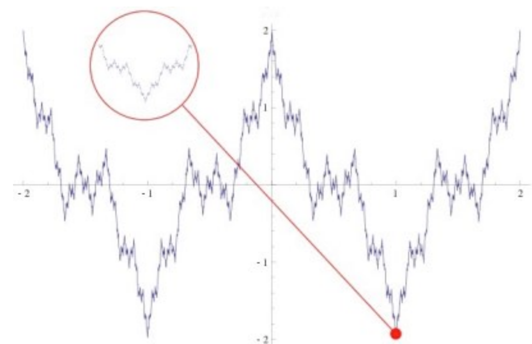
$f^{(3)}(x) = \begin{cases} 6 & x > 0 \\ -6 & x < 0 \end{cases}$

More Sharp Corners Than You Can Imagine

In 1872 Weierstrass published an example of a function that is continuous at every point on the real line, but fails to have a derivative at every point of the real line, the so called "**everywhere continuous but nowhere differentiable function**." How unsmooth can you get? Every point on the "graph" is a sharp corner!!!

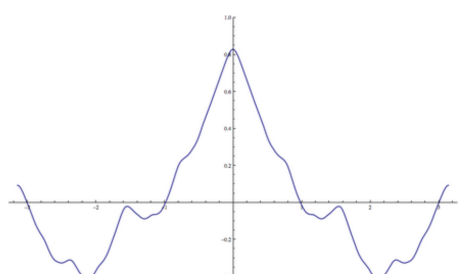
$$W(x) = \sum_{n=o}^{\infty} \frac{\cos(3^n x)}{2^n}$$

(Notice that $\cos(3^n x)$ wiggles really rapidly for large values of n.)

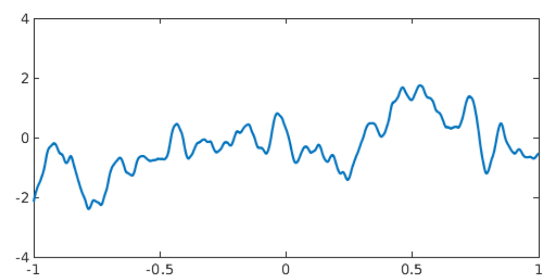


According to Paley and Wiener a Fourier series with random coefficients produces Brownian motion on [0,2π]. In the early 2000s, while fooling around in the coffee room, a group at Oxford used these random Fourier series to invent a function $S(x)$ that possesses Taylor series at every point of a closed interval, but not one of these Taylor series coverages to $S(x)$. Since then, a whole class of such function have been produced. With apologies to the fruit and beverage industry, they call them a *smoothies*. Here are a couple of samples.

$$S(x) = \sum_{k=o}^{\infty} e^{-\sqrt{2^k}} \cos(2^k x)$$



2^{500} *terms*



Banach-Tarski Paradox*

A bowling ball can be "cut up" into a small finite number of pieces in such a way that the pieces can be carefully reassembled into two bowling balls identical to the original one!

Another version of the paradox asserts that "a marble can be chopped into chunks that can be put back together to form the Moon."

*In these procedures, the **Axiom of Choice** is used to create the fragments of bowling ball and marble, and it takes an uncountable number of steps. The fragments themselves are not ordinary solids, but scattered, **"nonmeasurable"** sets of points.

Exercises

1. Compute $\int_\gamma z\, dz$, where γ is
   a. the line $a t + b(it)$, $0 \le t \le 1$
   b. the circle e^{it}, $0 \le t \le 2\pi$

2. Compute $\int_\gamma \frac{1}{z}\, dz$, where γ is the
   a. top half of e^{it}
   b. bottom half of e^{it}

3. Compute $\int_{\{e^{it} | 0 \le t \le 2\pi\}} 1/z^n\, dz$.

4. Check C-R eqs. for $|z|^2 = |x+iy|^2 = x^2+y^2$.

5. Show that if $f = r + ic$ is analytic, then r and c are both HARMONIC (i.e. they satisfy Laplace's eq. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$).

6. Show that the only analytic functions with imaginary part $c(x,y) = xy$ are $f(z) = \frac{1}{2}z^2 + k$ where k is an arbitrary constant.

7. Convince yourself that $g(z) = |z|$ is not analytic at any point. (think about polar form)

8. Use geometric series to show that the Koch Snowflake has infinite length but finite area.

9. Show that the winding number of $\gamma(\theta) = e^{5\theta i}$ about 0 is $w(\gamma,0) = 5$.

10. Finish the induction in the proof of the corollary to Cauchy's Integral Formula on slide #26.

Triangle Inequality for Complex Numbers

$$|z+w| \le |z| + |w|$$

<u>Proof</u>.
$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= |z|^2 + z\overline{w} + \bar{z}w + |w|^2 \\ &= |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 \\ &= |z|^2 + 2Re(z\overline{w}) + |w|^2 \\ &\le |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

Since $|z+w|$ and $|z|+|w|$ are both positive, taking square roots of both sides yields the result.

<u>Proof of</u>: *For all smooth, closed paths* $w(\gamma, a) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z-a}dz = \textit{an integer}$.

Just to keep notation as clean as possible, let's assume that $\gamma : [0,1] \to \mathbb{C}$ and that $a = 0$.

Define $G(t) = \displaystyle\int_0^t \frac{\gamma'(s)}{\gamma(s)}ds$, and note that $G(0) = 0$, $G(1) = \displaystyle\int_\gamma \frac{1}{z}dz$, and $G'(t) = \dfrac{\gamma'(t)}{\gamma(t)}$.

Thus,
$$\begin{aligned}\frac{d}{dt}(e^{-G}\cdot\gamma) &= e^{-G}\cdot\gamma' - G'\cdot e^{-G}\cdot\gamma \\ &= e^{-G}\cdot[\gamma' - \frac{\gamma'}{\gamma}\cdot\gamma] \\ &= 0\end{aligned}$$

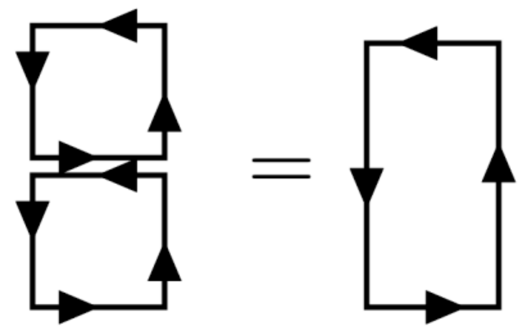
This implies that $e^{-G}\cdot\gamma$ is a constant function. So $e^{-G(0)}\cdot\gamma(0) = e^{-G(1)}\cdot\gamma(1)$.

Since $\gamma(0) = \gamma(1)$, $e^{-G(1)} = 1$ and hence, $e^{G(1)} = 1$.

By Euler's formula ($e^{i\theta} = cos(\theta) + i\cdot sin(\theta)$), $G(1)$ must be an integer multiple of $2\pi i$. ■

Sketch of completion of proof of Green's Theorem

The sum of line integrals around abutting rectangles is just the single line integral around the outer boundary, since common edges are traversed in opposite directions. and the line integrals along them cancel.



Thus, Green's Theorem is true for "sums" of abutting rectangles.

But any region can be approximated by "sums" of abutting rectangles, Green's Theorem holds for arbitrary regions.