Dirac γ Matrices

How Physicists
Think of, and Use the Dirac Algebra

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October 3, 2023

Three Approaches in Quantum Theory.

Matrix Mechanics (1925)

M.Born, W. Heisenberg, and P. Jordan

Gőttingen, Germany (The "Drei-Männer Arbeit")

In 1928 all three authors were nominated for Nobel Prize by A. Einstein. Only Heisenberg won it in 1932

Wave Mechanics (1926)

E. Schrődinger,

Zűrich, Switzerland

(followed the steps of L. de Broglie)

Max Born

W. Heisenberg

P. Jordan

Louis de Broglie

E. Schrődinger

R. Feynman

The Path Integral Formulation (1948)

R. Feynman

Princeton, NJ, USA.

Sources of Schrödinger’s Inspiration

1923

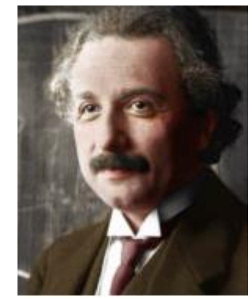
De Broglie’s Thesis

“In de Broglie’s dissertation one finds a very remarkable geometrical interpretation of the Bohr-Sommerfeld quantum rule”
(A. Einstein, February 9, 1925)

Louis de Broglie
(1892 – 1987)
Doctoral Student of
P. Langeven

Paul Langeven
1872 - 1946

P. Langevin sends De Broglie’s Thesis for reference to Einstein



Albert Einstein
1879 - 1955

“A few days ago, I read with the greatest interest the ingenious thesis of de Broglie”
(Letter to Einstein of **November 3, 1925**)

Peter Debye
1884 - 1966

November 23, 1925. Debye asked Schrödinger to give a talk on de Broglie’s work.

E. Schrödinger
(1987-1961)
full professor of
theoretical physics of the
University of Zürich

In 1925, in Zürich, in Swiss Federal Institute of Technology every fortnight **Peter Debye** (1884 – 1966) ran joint colloquium on theoretical physics.

Non-Relativistic Quantum Mechanics: Schrödinger's Equation

$$\left(i\hbar\frac{\partial}{\partial t}\right)\Psi = H\Psi \quad\Rightarrow\quad E \to i\hbar\frac{\partial}{\partial t}$$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + U\left(x, y, z\right) \quad\Rightarrow\quad p_j \to -i\hbar\frac{\partial}{\partial x_j}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + U\left(x, y, z\right)\right]\Psi$$

(E. Schrődinger, **January 1926**)

In non-relativistic quantum mechanics, $\Delta x \Delta p_x \geq \frac{\hbar}{2} \quad \to \quad \Delta x \geq \frac{\hbar}{2\Delta p_x}$

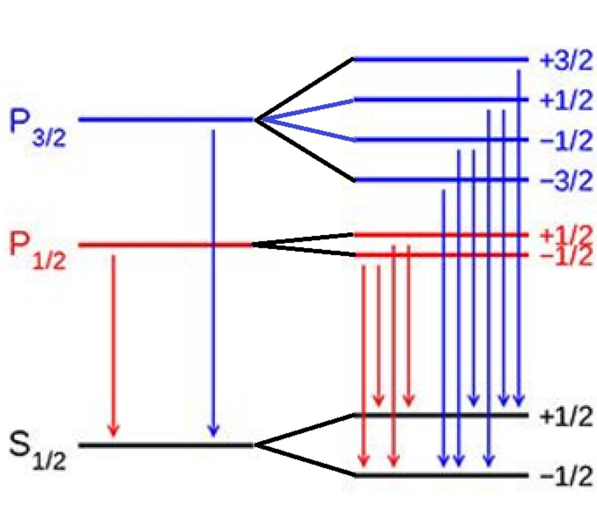
In special relativity, $\Delta p_x \leq 2mc$. Therefore,

$$\Delta x \geq \frac{\hbar}{4mc} \quad \Rightarrow \quad \Delta t \geq \frac{\Delta x}{c} \approx \frac{\hbar}{4mc^2}$$

The wave function as solution of the Schrődinger's equation does not make sense.

First Difficulty: Zeeman Splitting of Energy Levels. Pauli Matrices

Emission spectrum of alkali metals



Wolfgang Pauli
1900 - 1958

Pauli's phenomenological theory of spin:
"two-valuedness
not describable classically"
(1924)

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{p^2}{2m} + U + \lambda \vec{L} \cdot \vec{S} + \mu_B \left(\vec{L} + \vec{S} \right) \cdot \vec{B} \right] \Psi$$

$$\vec{L} = \vec{r} \times \vec{p}, \qquad \vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

$$H = \frac{1}{2m} \left[\vec{\sigma} \cdot \left(\vec{p} - q\vec{A} \right)^2 + q\varphi \right] \text{ with } \langle \varphi, A_x, A_x, A_x \rangle$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Pauli Matrices

1926: Klein-Gordon Equation for a Zero-Spin Free Particle

Oscar Klein
1894 - 1977

?

Walter Gordon
1893 - 1939

Vladimir Fock
1898 - 1974

In special relativity, for a free particle, $\frac{E^2}{c^2} = \vec{p}^2 + m^2c^2$.

Replacing $E = i\hbar\frac{\partial}{\partial t}$ and $p_j = -i\hbar\frac{\partial}{\partial x_j}$, we come to

$$\left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \left(\hbar^2\nabla^2 - m^2c^2\right)\Psi$$

Introducing the 4-vector $\langle x, y, z, ict\rangle$, the Klein-Gordon equation can be presented in the covariant form:

$$\left(\sum_\mu p_\mu^2 + m^2c^2\right)\Psi = 0$$

Continuity equation: $$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$$\vec{j} = \frac{e\hbar}{2mi}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right)$$

$$\rho = \frac{e\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

$$\sum_\mu \frac{\partial j_\mu}{\partial x_\mu} = 0 \quad \text{with} \quad j_\mu = \frac{\hbar}{2mi}\left(\Psi^*\frac{\partial\Psi}{\partial x_\mu} - \Psi\frac{\partial\Psi^*}{\partial x_\mu}\right)$$

$$j = \langle ic\rho, j_x, j_x, j_x\rangle$$

Solving the K-G Equation

$$\Psi(x,y,z,t) = Ae^{(i/\hbar)(\vec{p}\cdot\vec{r}-Et)} \quad \Rightarrow \quad E = \pm E_p \quad \text{with} \quad E_p = c\sqrt{p^2+m^2c^2}$$

$$\Psi_+ = A_1 e^{(i/\hbar)(\vec{p}\cdot\vec{r}-E_p t)}, \qquad \Psi_- = A_1 e^{(i/\hbar)(\vec{p}\cdot\vec{r}+E_p t)}$$

$$\rho_+ = \frac{eE_p}{mc^2}\Psi_+^*\Psi_+, \qquad \rho_- = -\frac{eE_p}{mc^2}\Psi_-^*\Psi_-$$

Every type of particle is associated with an **antiparticle** with the same mass but with opposite physical charges such as electric charge. (Dirac, 1928)

$$\Psi = \psi + \chi, \qquad i\hbar\frac{\partial\Psi}{\partial t} = mc^2(\psi - \chi)$$

General solution is: $\Psi = C_+\Psi_+ + C_-\Psi_-$

$$\begin{cases} i\hbar\dfrac{\partial\psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2(\psi+\chi) + mc^2\psi \\ i\hbar\dfrac{\partial\chi}{\partial t} = \dfrac{\hbar^2}{2m}\nabla^2(\psi+\chi) - mc^2\psi \end{cases}$$

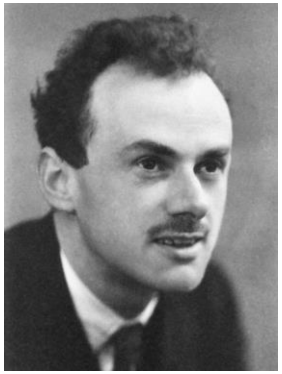
In matrix form it is:

$$\left(i\hbar\tau_0\frac{\partial}{\partial t} - \mathbf{H}\right)\mathbf{\Psi} = 0$$

$$\mathbf{\Psi} = \begin{pmatrix}\psi \\ \chi\end{pmatrix}, \qquad \mathbf{H} = (\tau_3 + i\tau_2)\frac{\vec{p}^2}{2m} + mc^2\tau_3$$

$$\tau_0 = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \quad \tau_x = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \quad \tau_y = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \quad \tau_z = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

Dirac Equation for a Spin-1/2 Free Particle



P.A.M. Dirac
1902-1984

Starting from the same point, but with spin ½ included:

$$\left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \left(\hbar^2\nabla^2 - m^2c^2\right)\Psi$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \frac{m^2c^2}{\hbar^2}\Psi$$

In special relativity, for a free particle, $E = c\sqrt{\vec{p}^2 + m^2c^2}$. To make it possible to extract the square root, Dirac wanted to present $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ as a perfect square:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D\frac{i}{c}\frac{\partial}{\partial t}\right)\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D\frac{i}{c}\frac{\partial}{\partial t}\right)$$

On multiplying out the right side it is apparent that, in order to get all the cross-terms such as $\frac{\partial}{\partial x}\frac{\partial}{\partial y}$ to vanish, one must assume $AB + BA = 0$ with $A^2 = B^2 = C^2 = D^2 = 1$. Dirac, who had just then been intensely involved with working out the foundations of matrix quantum mechanics, immediately understood that these conditions could be met if A, B, C and D are *matrices*, with the implication that the wave function has *multiple components*. However, one needs at least 4 × 4 matrices to set up a system with the properties required — so the wave function had *four* components, not two, as in the Pauli theory, or one, as in the bare Schrödinger theory.

1928: Dirac Matrices. Positron.

This immediately explained the appearance of two-component wave functions in Pauli's phenomenological theory of spin, something that up until then had been regarded as mysterious, even to Pauli himself.

$$\left(\mathbf{A}\frac{\partial}{\partial x}+\mathbf{B}\frac{\partial}{\partial y}+\mathbf{C}\frac{\partial}{\partial z}+\mathbf{D}\frac{i}{c}\frac{\partial}{\partial t}-\frac{mc}{\hbar}\mathbf{I}\right)\mathbf{\Psi}=0$$

$$\mathbf{A}=i\gamma^{1},\quad \mathbf{B}=i\gamma^{2},\quad \mathbf{C}=i\gamma^{3},\quad \mathbf{D}=\gamma^{0}$$

$$\gamma^{0}=\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix},\gamma^{1}=\begin{pmatrix}0 & \sigma_x\\ -\sigma_x & 0\end{pmatrix},\gamma^{2}=\begin{pmatrix}0 & \sigma_y\\ -\sigma_y & 0\end{pmatrix},\gamma^{3}=\begin{pmatrix}0 & \sigma_z\\ -\sigma_z & 0\end{pmatrix},$$

Here $\mathbf{\Psi}$ is a 4-spinor or bi-spinor. Its components have the physical meaning of the four options: particle-spin up, particle-spin down, antiparticle-spin up, and antiparticle-spin down.

In the limit of a low speed, $v << c$, Dirac's system of four coupled equations decouples into two Pauli's equations, one pair for the particle, spin and spin down, and another one for the antiparticle, spin and spin down.

Dirac’s Legacy

Dirac’s equation also implied the existence of a new form of matter, *antimatter*, previously unsuspected and unobserved and which was experimentally confirmed several years later. It also provided a *theoretical* justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin. The wave functions in the Dirac theory are vectors of four complex numbers (known as bispinors), two of which resemble the Pauli wavefunction in the non-relativistic limit, in contrast to the Schrödinger equation which described wave functions of only one complex value. Moreover, in the limit of zero mass, the Dirac equation reduces to the Weyl equation.

Although Dirac did not at first fully appreciate the importance of his results, the entailed explanation of spin as a consequence of the union of quantum mechanics and relativity—and the eventual discovery of the positron—represents one of the great triumphs of theoretical physics. This accomplishment has been described as fully on a par with the works of Newton, Maxwell, and Einstein before him.