The Four Color Problem

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References

- Four Colors Suffice, by Robin Wilson, 2002, Princeton University Press

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- The Solution of the Four-Color-Map Problem, by Kenneth Appel and Wolfgang Haken, Scientific American 237 No. 4 (October 1977)

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- In 1890 Percy Heawood pointed out a fundamental error in Kempe's paper, but used Kempe's ideas to show that 5 colors suffice to color any map.

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- Fermat's Last Theorem was solved in 1994. The Four Color Theorem was solved in 1976. The Riemann Hypothesis remains unsolved.
- We will present the background and proof of the Five Color Theorem.
- We will give some vague indications about methods behind the proof of the Four Color Theorem.

A few simple observations

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- Any map having a vertex connecting 2 edges can simply have the vertex removed.
- Call a map cubic if every vertex of the map is the meeting point of exactly 3 edges/countries. Then (Cayley) all maps can be 4-colored if all cubic maps can be 4-colored. Proof: cover any more complex intersection with a small new country, which creates a cubic map, 4-color it, and remove the small country. The original map will be 4-colored.

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- So we can restrict our interest to cubic maps.

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- Proof: puncture one of the faces, stretch it out like rubber, and make the resulting surface a flat map. (This is what requires it to be simply connected.). Then systematically remove pairs of (E,F) or (E,V). Each operation will leave the value of $V - E + F$ unchanged, and will simplify the map into a single polygon. For a polygon, $V = E$ and $F = 2$ (inside and outside).

The Counting Formula for Cubic Maps

- For any cubic map, let C_2 the number of 2 sided countries, C_3 the number of 3 sided countries, ... C_n the number of n sided countries. Then $F = C_2 + C_3 + C_4 + \ldots$.

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- Each n sided country has n vertices, each used exactly 3 times, This means that $3V = 2C_2 + 3C_3 + 4C_4 + \dots$.
- Combined with Euler's Formula, we find $12 = 6(V - E + F) = 2(3V) - 3(2E) + 6F$ $= 4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - \dots$

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- A country that cannot be 4-colored is called a criminal. We are looking for minimal criminals.
- In particular, we want to consider whether any potential criminal can be reduced to a smaller criminal. If it can be reduced then it is not minimal.
- A reducible configuration is an arrangement of countries that cannot occur in a minimal configuration. If a map contains a reducible configuration then any 4-coloring of the rest of the map can be extended, perhaps with some recoloring, to a 4-coloring of the entire map.

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- We want to show that there are unavoidable configurations of maps (a set of configurations), one of which must appear in any map, and prove that each configuration can be reduced.
- **The 4 color problem will be solved by producing an unavoidable set of reducible configurations.**

An example of an unavoidable set of configurations

- Given that $12 = 4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - \ldots$, we immediately know that every cubic map has within it either a digon, triangle, square or pentagon.

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- Reducing a map with a digon or triangle is easy. Reducing a map with a square was done by Kempe. Kempe thought he showed how to reduce a map with a pentagon, but he didn't.
- If Kempe had properly reduced the case of the pentagon, he would have proved the Four Color Theorem.

Reducing the Digon

- We will show how to reduce any map containing a digon. Explicitly, we will show that any map containing a digon that cannot be 4 colored, cannot be so colored because a smaller map (i.e. fewer countries) also cannot be colored. This precludes the criminal map from being minimal.

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- Reforming the above statement, we show that if a map containing a digon needs to be 4-colored, and if any smaller map can be colored, then the map with the digon can be colored.

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- Reforming the above statement, we show that if a map containing a digon needs to be 4-colored, and if any smaller map can be colored, then the map with the digon can be colored.
- A digon is a country with exactly 2 neighbors. For example, Portugal is a digon sandwiched between Spain and the Atlantic. (Or, if you prefer, Andorra is squeezed between Spain and France.)

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- This new map is smaller and thus is not a criminal. Proceed to 4-color it, making the Atlantic blue and Spain red. Restore the erased boundary, recreating Portugal, squeezed between red and blue countries. With 4 colors to use and only 2 taken, Portugal can be safely colored yellow. The original map is no longer criminal.

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- Erase one edge of the triangle, merging it with the neighboring country, and reducing the size of the map. By assumption, the original map was a minimal criminal, so the new map can be 4 colored.
- Restore the erased edge, and the triangle has only 3 neighbors. This means that it has a spare color available and can be assigned a color. Now the map has been 4-colored and was not a criminal.

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- Reducing a map with a square uses a method introduced by Kempe in his 1879 paper, and was correctly done.
- To see where the difficulty lies, notice that if one edge of the square is erased, making a smaller colored map, then redrawing the erased edge may well use all 4 colors for the countries that border the square. No color is available to color the square.

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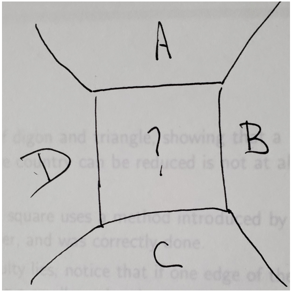
- To see where the difficulty lies, notice that if one edge of the square is erased, making a smaller colored map, then redrawing the erased edge may well use all 4 colors for the countries that border the square. No color is available to color the square.
- We need to recolor some of the countries so as to free up a color that the square can use.

The Square

- 

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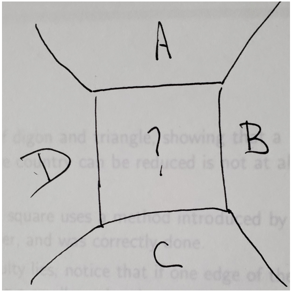
- 

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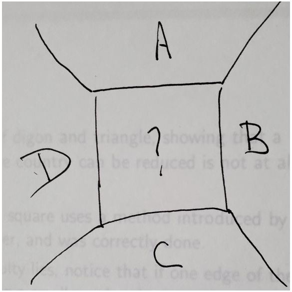
- 

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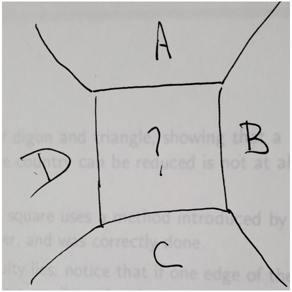
- 

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- There are 2 cases to consider.

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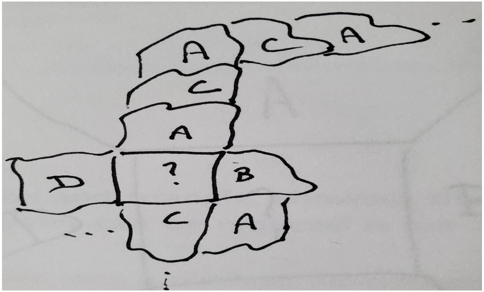
- 

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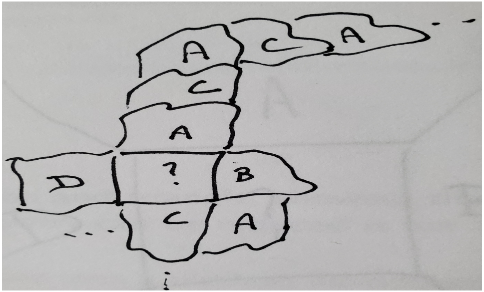
- 

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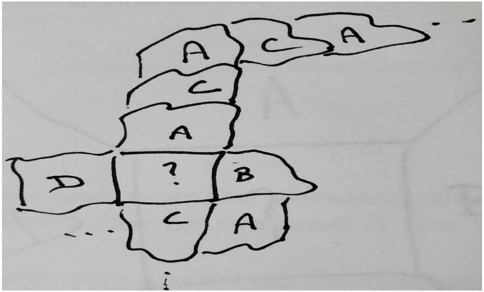
- 

- Select 2 colors on opposite sides of the square, e.g. A and C.
- Highlight all the countries that are either A or C. It may or may not be possible to connect a path of them from the original A to the original C.

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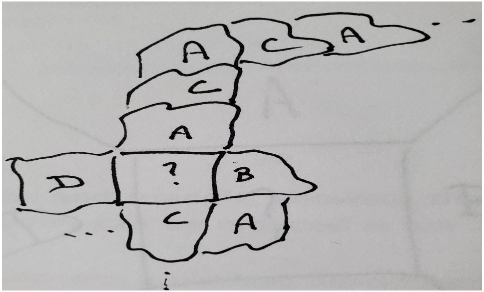
- 

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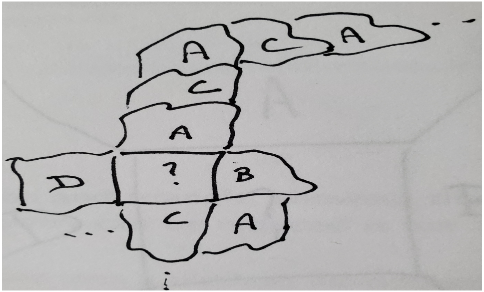
- 

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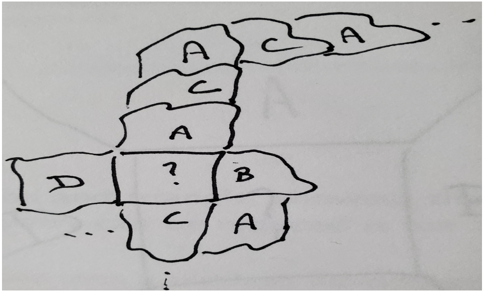
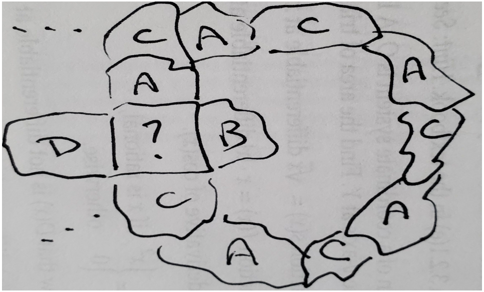
- 

- In case 1 it is not possible. We simply take all the countries that connect to the original A and switch the colors of A and C.
- This removes A as a boundary color of the square, and A can become the color of the square.

The Square: Case 2

- 

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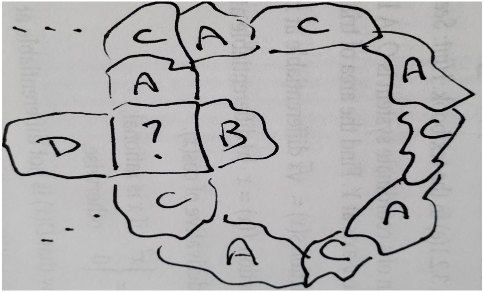
- 

- Highlight all the countries that are either A or C. This time it is possible to connect a path from original A to original C. Such a path is called a Kempe Chain.

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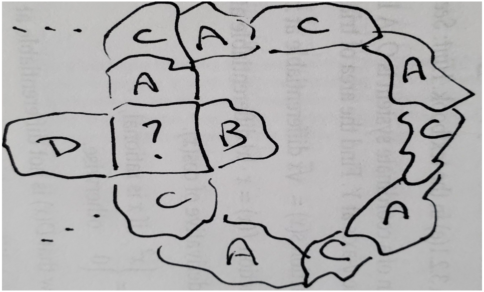
- 

- Highlight all the countries that are either A or C. This time it is possible to connect a path from original A to original C. Such a path is called a Kempe Chain.
- Switching A and C in this case is useless: the square will still have 4 different colors on its borders.

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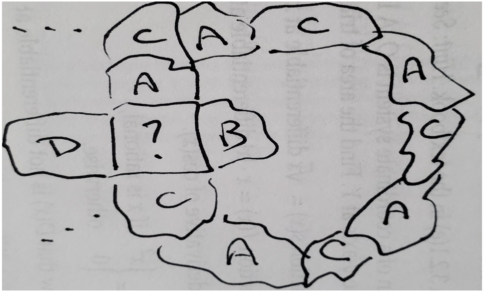
- 

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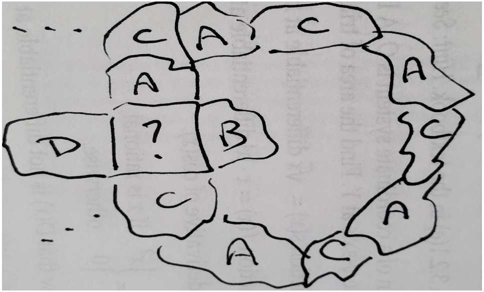
- 

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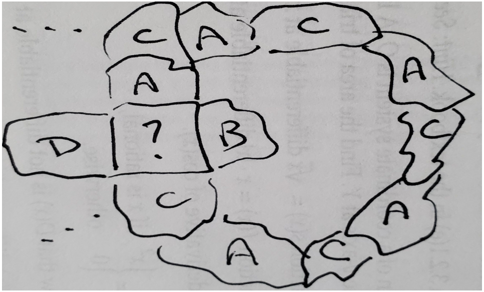
- 

- Notice now that the countries colored B and D within the path are isolated from those outside the path.
- Switch the colors B and D for those countries inside the path. B is no longer a color adjoining the square, and the square can utilize it.

The Pentagon and the Five Color Theorem

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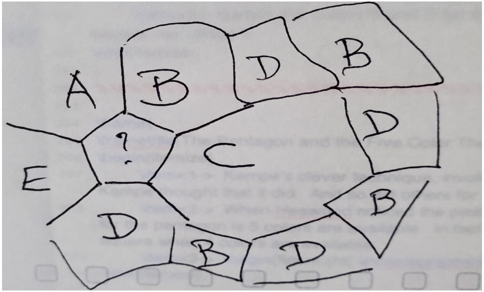
- 

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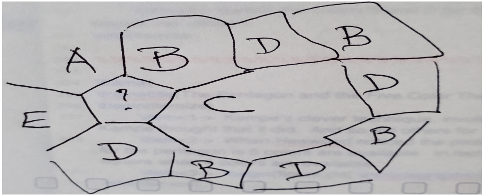
- 

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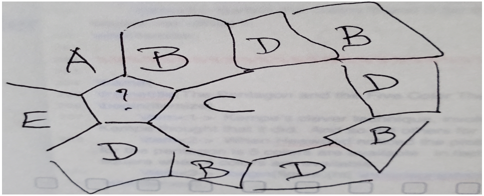
- 

- As before, choose 2 non-adjacent colors bordering the pentagon, e.g. those colored with B and D, and focus on all the countries using those two colors.

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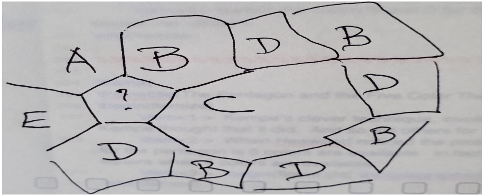
- 

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- In one case (not shown), there is no chain of countries that link the initial B to the initial D. We can simply flip colors on the countries that directly connect to the initial B, freeing up B for use by the pentagon.

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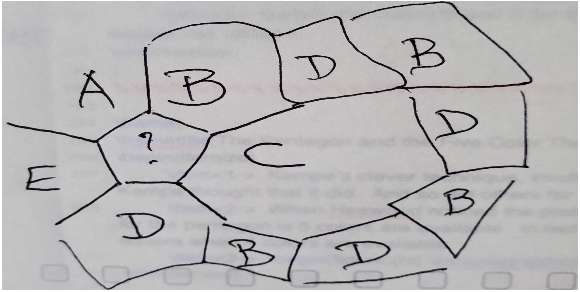
- 

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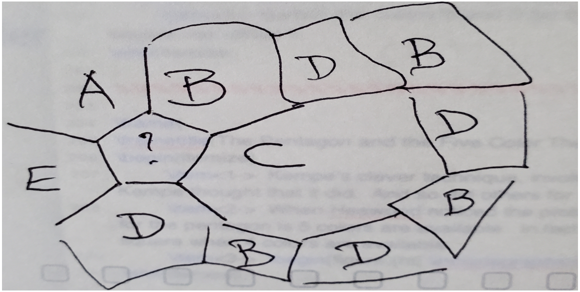
- 

- In the other case, shown here, there is a Kempe chain connecting initial countries B and D. This means that one of the other 3 neighbors of the pentagon, here the one colored C, is isolated from the initial A and E neighbors. Choosing either A or E, (use E here), focus on all the C and E countries within the B/D chain. They can have their colors flipped, making C available for the pentagon.

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- Not so. Considering the cubic map counting formula again, we find that a map not containing either a digon, triangle or square must actually contain at least 12 pentagons. So there is more information to utilize.

Whither the Pentagon

- Returning to the 4 color problem, we have produced a set of unavoidable configurations {digon, triangle, square, pentagon}, but no one knows how to reduce a map that simply contains a pentagon. This would appear to make the problem unsolvable.
- Not so. Considering the cubic map counting formula again, we find that a map not containing either a digon, triangle or square must actually contain at least 12 pentagons. So there is more information to utilize.
- It would be unlikely to help if we could only suppose the 12 pentagons were widely separated. (... here a pentagon, there a pentagon, everywhere a pentagon, pentagon ...)

A New, Improved Set of Unavoidable Configurations

- Using a technique invented by Heinrich Heesch in 1969 known as discharching, we will show that a different set of unavoidable configurations is: {digon, triangle, square, pair of adjacent pentagons, a pentagon adjacent to a hexagon}.

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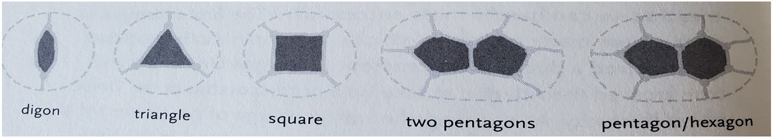

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- Suppose that none of the supposed configurations occur. We will produce a contradiction.

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- An octagon, with initial charge of -2, would need 11 neighboring pentagons, which is absurd.
- Overall, discharging would leave the map with a total negative charge. Our assumption was wrong, and the set of configurations is unavoidable.

Looking for configurations

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- The combination of these two ideas, ring size and discharging, are the key methods behind the ultimate solution to the 4 color problem.

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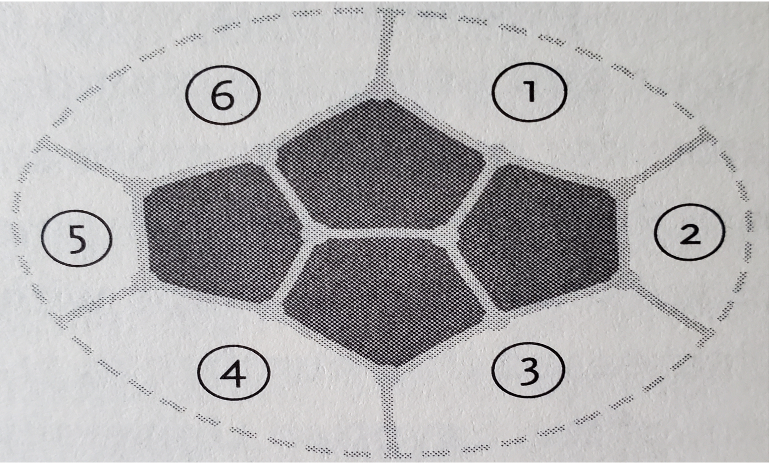
- 

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- There is another method for handling situations like this (which I found obscure). See reference 1.

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- For a given ring size a large number of configurations can be formed, using those that are unavoidable but not known to be reducible.
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- Doing this will enlarge the set of unavoidable configurations, in the hope of replacing irreducible ones with a sets that are reducible.

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- Working out of the University of Illinois, they had free access to the fastest machine: ILLIAC. And others.

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- In 1976 the announcement was made: Four Colors Suffice.
- There is much unpleasant about the proof. It is ugly. It is incomprehensible. It could be wrong because of programming errors, though Kempe's original paper was also wrong and not detected for 11 years.

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- They were able to generate a set of only 633 configurations with just 32 discharging rules, and were able to reduce them all.
- Now all the computer related calculations can be done on a mac/PC in just a few CPU hours