Gamma Matrices and the Dirac Algebra

Larry Susanka

September 26, 2023

GM and DA TOC Intro to Algebras Quaternions, Pauli Spin Matrices Gamma Matrices, Dirac Algebra Clifford Algebras

GM and DA TOC Intro to Algebras Quaternions, Pauli Spin Matrices Gamma Matrices, Dirac Algebra Clifford Algebras

Introduction

A **real algebra** is a real vector space V whose scalars are the real numbers $\mathbb{R}$ together with an additional operation on pairs of vectors, usually indicated as multiplication, with a few properties.

First, the product vw of the vectors v and w from V must produce a vector in V. This property is called **closure**, and the vocabulary applies to subsets too. If $S \subset V$ we say the product is closed in S if vw is in S whenever both v and w are in S.

To avoid a few unimportant special cases we assume **the dimension of V to be at least one** and that **the multiplication operation is nontrivial**: that is, $vw \neq 0$ for at least one pair v, w of vectors in V.

GM and DA TOC Intro to Algebras Quaternions, Pauli Spin Matrices Gamma Matrices, Dirac Algebra Clifford Algebras

Unless the vector space is a one dimensional space, essentially $\mathbb{R}$ itself, dot product on $\mathbb{R}^n$ produces a number, not a vector, so the familiar dot product is *not* an example of what we mean here except in that case.

The usual cross product on vectors in $\mathbb{R}^3$ *is* an example. The usual matrix multiplication on the vector space of n-by-n matrices is another.

The two **distributive laws** must hold for this operation:

$$(v + w)z = vz + wz \quad \text{and} \quad z(v + w) = zv + zw$$

for any vectors v, w and z.

The distributive law implies that if $\mathbf{0}$ denotes the zero vector and v is any vector then $\mathbf{0}v = (\mathbf{0} + \mathbf{0})v = \mathbf{0}v + \mathbf{0}v$ so $\mathbf{0}v = \mathbf{0}$.

We adopt the usual practice of giving multiplication priority over addition, so $\boldsymbol{z}\boldsymbol{v} + \boldsymbol{z}\boldsymbol{w} = (\boldsymbol{z}\boldsymbol{v}) + (\boldsymbol{z}\boldsymbol{w})$ and is not any of the other possible combinations, such as $(\boldsymbol{z}(\boldsymbol{v} + \boldsymbol{z}))\boldsymbol{w}$.

Both cross product and matrix multiplication satisfy these distributive laws.

Finally, if r and s are scalars (i.e. real numbers) and $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors then we have the **scalar laws**

$$(rs)(\boldsymbol{v}\boldsymbol{w}) = ((rs)\boldsymbol{v})\boldsymbol{w} = (r\boldsymbol{v})(s\boldsymbol{w}) = \boldsymbol{v}((rs)\boldsymbol{w}).$$

Cross product and matrix multiplication satisfy this one too.

Taking a look at the equation above we see three different multiplications, all indicated by "juxtaposition," i.e. putting the symbols to be multiplied next to each other. rs is the product of two real numbers. $\boldsymbol{v}\boldsymbol{w}$ is the product of two vectors. $r\boldsymbol{v}$ is scalar multiplication of number r by vector $\boldsymbol{v}$. Believe it or not, in applications this rarely causes confusion.

The combination of the distributive and scalar laws is equivalent to saying that this is a bilinear multiplication.

$$(r\boldsymbol{v} + \boldsymbol{w})\boldsymbol{z} = r\boldsymbol{v}\boldsymbol{z} + \boldsymbol{w}\boldsymbol{z} \quad \text{and} \quad \boldsymbol{z}(r\boldsymbol{v} + \boldsymbol{w}) = r\boldsymbol{z}\boldsymbol{v} + \boldsymbol{z}\boldsymbol{w}.$$

If $B = \{ \boldsymbol{b_1}, \dots, \boldsymbol{b_n} \}$ is a finite basis for the algebra V then the product $\boldsymbol{v}\boldsymbol{w}$ is determined by the products $\boldsymbol{b_i}\boldsymbol{b_j}$ of basis vectors.

Specifically,

$$\boldsymbol{v}\boldsymbol{w} = \left(\sum_{i=1}^{n} v^i \boldsymbol{b_i} \right) \left(\sum_{j=1}^{n} w^j \boldsymbol{b_j} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \boldsymbol{b_i}\boldsymbol{b_j}$$

when $\boldsymbol{v}$ and $\boldsymbol{w}$ are given in terms of basis B as indicated.

For a vector space of dimension n there are n^2 different possible algebra products of basis vectors and each product of is specified by n constants so n^3 numbers determine the algebra.

Additional regularity properties may reduce the number of algebras possessing them to just a few, or even one or . . . none.

Requiring (or determining) commutativity ($\boldsymbol{v}\boldsymbol{w} = \boldsymbol{w}\boldsymbol{v}$) or anticommutativity ($\boldsymbol{v}\boldsymbol{w} = -\boldsymbol{w}\boldsymbol{v}$) or associativity ($(\boldsymbol{v}\boldsymbol{w})\boldsymbol{z} = \boldsymbol{v}(\boldsymbol{w}\boldsymbol{z})$) or the existence of a unit element $\boldsymbol{e}$ (an element for which $\boldsymbol{e}\boldsymbol{w} = \boldsymbol{w}\boldsymbol{e} = \boldsymbol{w}$) and the existence of multiplicative inverses are examples of such regularity properties. (Algebras with a unit element are called **unitary**)

We will be concerned entirely with matrix algebras: that is algebras that are, or can be construed as, algebras of $n \times n$ matrices for some integer n. The multiplication will often, but not always, be matrix multiplication.

$\mathbb{M}_{n \times n}$, the vector space of $n \times n$ matrices, is an algebra with matrix multiplication.

Skew_n, the vector space of skew symmetric $n \times n$ matrices, is *not* an algebra with matrix multiplication but *it is* an algebra with Lie bracket multiplication, given by

$$\boldsymbol{A} \odot \boldsymbol{B} = AB - BA.$$

(Lie bracket is, more often than not, denoted by the commutator symbol $[A, B]$ rather than $A \odot B$ but this can be awkward in more complex products.)

Lie bracket is not associative, and it is anticommutative, not commutative. That means it cannot have a multiplicative identity. However it does satisfy the **Jacobi identity** which we choose to write as

$$(A \odot B) \odot C - A \odot (B \odot C) = B \odot (C \odot A).$$

Sym_n, the vector space of symmetric $n \times n$ matrices, is *not* an algebra with matrix multiplication. For instance

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is an algebra with Jordan product, given by

$$A \bullet B = \frac{1}{2}(AB + BA).$$

Jordan product is not associative but it is commutative. It too satisfies an identity, called the **Jordan identity,**

$$A \bullet (B \bullet (A \bullet A)) = (A \bullet B) \bullet (A \bullet A).$$

Expanding to consider complex matrices, we have real vector spaces Herm_n and $\mathrm{SkewHerm}_n$, the Hermitian and skew Hermitian complex $n \times n$ matrices, respectively.

Matrix $A \in \mathrm{Herm}_n$ if and only if $A^* = A$ where A^* is the conjugate transpose of A.

Matrix $A \in \mathrm{SkewHerm}_n$ if and only if $A^* = -A$.

Jordan product makes Herm_n into an algebra while the Lie bracket operation makes $\mathrm{SkewHerm}_n$ into an algebra.

The Jordan and Jacobi identities hold here too.

In 1933 Pascual Jordan together with collaborators Johnny von Neumann and Eugene Wigner investigated the properties of algebras with Lie bracket in connection with creation and annihilation operators for particles with an even number of spin units such as mesons or photons. These particles are called bosons.

Algebras with Jordan product are associated with these operators for particles with an odd number of spin units, such as neutrons, protons and electrons, called fermions.

Interestingly, at the very time when he was closely collaborating with these partners in writing the foundational papers on the subject Jordan was also writing vigorous pro-nazi polemical tracts under a nom de plume. Wigner and von Neuman are Jewish.

Quaternions and the Pauli Spin Matrices

Last year we discussed a real 4-dimensional algebra $\mathbb{H}$ called the **quaternions.** Members of this vector space can be written in the form

$$w + x\vec{i} + y\vec{j} + z\vec{k} \quad \text{for } w, x, y, z \in \mathbb{R}.$$

The symbols $\vec{i}$, $\vec{j}$ and $\vec{k}$ are characterized by

$$\vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = -1. \qquad \vec{i}\vec{j} = \vec{k}, \qquad \vec{j}\vec{k} = \vec{i}, \qquad \vec{k}\vec{i} = \vec{j}.$$

Points in ordinary three dimensional space are identified with the pure quaternions $x\vec{i} + y\vec{j} + z\vec{k}$ and rotations are handled in space using Hamilton product (the product implied by the multiplication table above) in a way that is similar to how complex numbers are used implement rotations in the plane.

We saw that the quaternions can be realized as a matrix vector space with

$$1 \leftrightarrow \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \vec{\boldsymbol{i}} \leftrightarrow \boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\text{and } \vec{\boldsymbol{j}} \leftrightarrow \boldsymbol{B} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \vec{\boldsymbol{k}} \leftrightarrow \boldsymbol{C} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Thus any quaternion can be represented as

$$w + x\vec{\boldsymbol{i}} + y\vec{\boldsymbol{j}} + z\vec{\boldsymbol{k}} \quad \longleftrightarrow \quad \begin{pmatrix} w+yi & -x+zi \\ x+zi & w-yi \end{pmatrix}.$$

The three **Pauli spin matrices** σ_1, σ_2 and σ_3 (and "auxiliary" matrix σ_0) can be defined in terms of this representation of the quaternions (and conversely) and are given by

$$\sigma_0 = \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \sigma_1 = -i\boldsymbol{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\text{and } \sigma_2 = i\boldsymbol{A} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = -i\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\sigma_1\sigma_2 = i\sigma_3$ and $\sigma_2\sigma_3 = i\sigma_1$ and $\sigma_1\sigma_3 = -i\sigma_2$
and $\sigma_1\sigma_2\sigma_3 = i\boldsymbol{I}$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0 = \boldsymbol{I}$
with $\sigma_k\sigma_m = -\sigma_m\sigma_k$ for nonzero unequal m and k.

The three Pauli spin matrices (and σ_0 too) are Hermitian, which means they are their own conjugate transpose, and so correspond to "observables" in quantum mechanics. They represent spin with respect to the coordinate axes in quantum mechanical descriptions of certain particles.

Any 2 by 2 Hermitian matrix can be found (in one way) as the real span of these four matrices:

$$w\,\sigma_0 + x\,\sigma_1 + y\,\sigma_2 + z\,\sigma_3 = \begin{pmatrix} w+z & x-iy \\ x+iy & w-z \end{pmatrix}$$

with real w, x, y and z.

So the real span of the Pauli spin matrices is Herm_2. This 4-dimensional real vector space is not an algebra with matrix multiplication or Lie product.

It is, however, a Jordan algebra with Jordan product

$$A \bullet B = \frac{1}{2}(AB + BA).$$

The eight matrices

$$\sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_1\sigma_2 \quad \sigma_1\sigma_3 \quad \sigma_2\sigma_3 \quad \sigma_1\sigma_2\sigma_3$$

are real-linearly independent and so form a basis of $\mathbb{M}_{2\times 2}(\mathbb{C})$, the complex 2×2 matrices, as a real vector space.

And it follows that the smallest *real* algebra containing σ_1, σ_2 and σ_3 using matrix (rather than Jordan) product is the 8-dimensional real algebra of all 2×2 matrices with complex entries.

The 16 matrices of the form

$$\pm\sigma_k \text{ or } \pm i\,\sigma_k \text{ for } k = 0, 1, 2 \text{ or } 3$$

form a multiplicative subgroup (with ordinary matrix multiplication, not Jordan product) of this algebra, the **Pauli spin group**.

The **quaternion group** is a subgroup of the Pauli spin group, and consists of the 8 elements $\pm\sigma_0, \pm i\sigma_1, \pm i\sigma_2$ and $\pm i\sigma_3$.

$$\sigma_0 = I, \quad i\sigma_1 = C, \quad i\sigma_2 = -A, \quad i\sigma_3 = B.$$

The 4-dimensional quaternion sub-algebra corresponds to rotations in space; other members of the 8-dimensional algebra generated by the Pauli matrices rotate and reflect.

The Gamma Matrices and the Dirac Algebra

When considering spin and other features of particle interactions in relativistic space-time, a 4-dimensional affine space with Lorentz inner product, larger matrices must be used.

The correct notion of "distance" in this space is given by the symmetric nondegenerate bilinear form

$$g(\boldsymbol{v}, \boldsymbol{w}) = v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3.$$

Like the situation in Euclidean space the Lorentz inner product can be used to define projection onto subspaces of dimension three and reflections, and the composition of two reflections is (defined to be) a rotation with respect to this inner product.

And according to the Cartan-Dieudonné Theorem any isometry (a map from space-time to itself that preserves the Lorentz inner product) taking origin to origin is the composition of no more than 4 reflections, and if the isometry preserves orientation it must be the composition of 1 or 2 rotations.

To understand these we need efficient ways of handling reflections and rotations and this is what gamma matrices are good for.

There are gamma matrices of various types (for instance **Dirac**, **chiral** or **Majorana**) and whatever their type there are always four of them denoted $\gamma^0, \gamma^1, \gamma^2, \gamma^3$. There is a fifth "auxiliary" gamma matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ used in various calculations for each collection of gamma matrices. **The four gamma matrices are linearly independent and may be associated with the standard basis vectors of $\mathbb{R}^4$.**

The **contravariant Dirac gamma matrices** are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\text{fifth auxiliary} \qquad \gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

Note that the γ^0 and γ^5 are symmetric while the others are skew symmetric.

Note also that

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$
$$\text{fifth auxiliary} \qquad \gamma_5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$$

where the zeros in these matrices are 2×2 zero blocks.

The smallest unitary algebra (with matrix multiplication) containing the Dirac gamma matrices is called the **Dirac algebra.** The 4-dimensional subspace spanned by the gamma matrices themselves, identified with space-time, is acted upon by the other members of this algebra via matrix product. In this setting real multiples of I_4 are associated with real numbers.

$\mathbb{M}_{4\times 4}(\mathbb{C})$ has real dimension 32, so the Dirac algebra cannot have dimension higher than that. We will see it actually has real dimension 16, a fact that is not entirely obvious.

We are going to be doing calculations with products and sums of 4×4 matrices. There is no virtue—none whatsoever—in doing these multiplications by hand. MATLAB code to make this easy will be posted on the website, and we will see that code in action here in a moment. We will show that

$$(\gamma^0)^2 = I_4, \quad (\gamma^1)^2 = -I_4, \quad (\gamma^2)^2 = -I_4, \quad (\gamma^3)^2 = -I_4$$

and $\gamma^i\gamma^j = -\gamma^j\gamma^i$ whenever $i \neq j$.

Assuming this it is easy to calculate that if $Y = a\gamma^0 + b\gamma^1 + c\gamma^2 + d\gamma^3$ for real coefficients then

$$Y^2 = \left(a^2 - b^2 - c^2 - d^2\right) I_4.$$

This matches the Lorentz inner product $g(Y, Y)$.

The algebra generated by the Dirac matrices is spanned by the set of all possible products $\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_k}$.

This can be modified to an equivalent expression obtained by permuting neighboring factors and introducing a minus sign after each "switch."

By a sequence of such switches we can obtain a new product of the form $\pm\gamma^{j_1}\gamma^{j_2}\cdots\gamma^{j_k}$ where the j_m are in non-decreasing order.

By replacing pairs of equal neighbors by ± 1 we can reduce any such product to a product of *no more* than four factors listed in order of strictly increasing index with a possible sign adjustment.

There are 15 of these products and, together with I_4, put the dimension of the Dirac algebra at no more than 16.

For convenience in representing the sum we label

$$a^0 = \gamma^0 \quad a^1 = \gamma^1 \quad a^2 = \gamma^2 \quad a^3 = \gamma^3$$
$$a^4 = \gamma^0\gamma^1 \quad a^5 = \gamma^0\gamma^2 \quad a^6 = \gamma^0\gamma^3 \quad a^7 = \gamma^1\gamma^2$$
$$a^8 = \gamma^1\gamma^3 \quad a^9 = \gamma^2\gamma^3$$
$$a^{10} = \gamma^0\gamma^1\gamma^2 \quad a^{11} = \gamma^0\gamma^1\gamma^3 \quad a^{12} = \gamma^0\gamma^2\gamma^3$$
$$a^{13} = \gamma^1\gamma^2\gamma^3 \quad a^{14} = \gamma^0\gamma^1\gamma^2\gamma^3$$
$$a^{15} = I_4$$

Consider the sum $x_i a^i$ for real x_i. Expressed as a single matrix this linear combination is

$$\begin{pmatrix} x_0 - ix_{10} + x_{15} - ix_7 & x_{11} - ix_{12} + x_8 - ix_9 & -ix_{13} - ix_{14} + x_3 + x_6 & x_1 - ix_2 + x_4 - ix_5 \\ -x_{11} - ix_{12} - x_8 - ix_9 & x_0 + ix_{10} + x_{15} + ix_7 & x_1 + ix_2 + x_4 + ix_5 & -ix_{13} - ix_{14} - x_3 - x_6 \\ ix_{13} - ix_{14} - x_3 + x_6 & -x_1 + ix_2 + x_4 - ix_5 & -x_0 + ix_{10} + x_{15} - ix_7 & -x_{11} + ix_{12} + x_8 - ix_9 \\ -x_1 - ix_2 + x_4 + ix_5 & ix_{13} - ix_{14} + x_3 - x_6 & x_{11} + ix_{12} - x_8 - ix_9 & -x_0 - ix_{10} + x_{15} + ix_7 \end{pmatrix}$$

Notice that the real constants come in pairs in each entry. For instance x_0 and x_{15} are always paired. There are 8 such pairs and each occurs four times. x_0 and x_{15} are in entries $(1,1), (2,2), (3,3)$ and $(4,4)$. Note that the real part of these entries are $x_0 + x_{15}$ and $-x_0 + x_{15}$, each repeated twice.

So if this linear combination is the zero matrix then $x_0 = x_{15} = 0$.

The same is true (set the real and complex parts of the entries equal to zero separately) for the other 7 pairs of coefficients. All the x_i must be 0. So these 16 matrices are linearly independent and therefore form a basis of the Dirac algebra.

As an exercise (slightly modify the MATLAB code provided) consider the algebra generated by

$$\beta^0 = \gamma^0 \quad \beta^1 = i\gamma^1 \quad \beta^2 = \gamma^2 \quad \beta^3 = \gamma^3.$$

Show that the pairs β^i and β^j for $i \neq j$ anti-commute and $(\beta^i)^2 = 1$ for $i = 0, 1$ and $(\beta^i)^2 = -1$ for $i = 2, 3$. Therefore the 16 matrices, denoted a^i above formed from these β^i as we did for the Dirac algebra, span this new algebra. And, by similar reasoning to the earlier case, they are real linearly independent. So the real dimension of this algebra is 16. Verify also that if $Y = a\beta^0 + b\beta^1 + c\beta^2 + d\beta^3$ for real coefficients then

$$Y^2 = \left(a^2 + b^2 - c^2 - d^2\right) I_4.$$

Now try this one: Consider the algebra generated by

$$\zeta^0 = \gamma^0 \quad \zeta^1 = i\gamma^1 \quad \zeta^2 = i\gamma^2 \quad \zeta^3 = i\gamma^3.$$

Show that with this new definition the pairs ζ^i and ζ^j for $i \neq j$ still all anti-commute and $(\zeta^i)^2 = 1$ for all i. This means that the 16 matrices a^i span this algebra. Show, as above, that they are real linearly independent. The real dimension of this algebra is therefore 16. If $Y = a\zeta^0 + b\zeta^1 + c\zeta^2 + d\zeta^3$ for real coefficients then

$$Y^2 = \left(a^2 + b^2 + c^2 + d^2\right) I_4.$$

Clifford Algebras

Any finite-dimensional unitary algebra generated by elements γ_i for $i = 1, \ldots, n$, matrices or not, and for which these generators anti-commute and for which $(\gamma_i)^2 = \pm e$ for each i where e is the identity is called a Clifford algebra.

If the dimension of this algebra is 2^n it is called a universal Clifford algebra. Any two such with the same number of "plus" and "minus" choices are algebra-isomorphic, hence the word "universal."

Showing existence, and how to represent such algebras in terms of matrices, is slightly involved and (possibly) the subject of another talk. Uniqueness is quite easy but we will not show that here.

Any element of any Clifford algebra $\mathcal{C}$ can be written as a sum of elements of the form $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}$, just as we did with the gamma matrices, where these are products of no more than n distinct gammas and the subscripts are in increasing order. Thus any $\alpha \in \mathcal{C}$ has a unique representation as

$$\alpha = \alpha^+ + \alpha^-$$

where α^+ consists of the sum involving basis elements with an even number of gammas and α^- is a linear combination involving basis elements with an odd number of gammas, called the even and odd parts of α.

If $\alpha = \alpha^+$ or $\alpha = \alpha^-$ we say α is "homogeneous" or "pure" and otherwise α is "mixed" or "inhomogeneous."

So $\mathcal{C}$ is the direct sum of subspaces

$$\mathcal{C} = V^+ \oplus V^-$$

where V^+ consists of the even members of the algebra and V^- is the odd members (together with zero.)

For instance the quaternions are the subspace V^+ in the Clifford algebra generated by the Pauli matrices.

If $\mathcal{C}$ is *any* algebra with any direct sum decomposition $\mathcal{C} = V^+ \oplus V^-$ and if the four containments involving the sets formed from all suggested products

$$V^+V^+ \subset V^+ \quad V^-V^- \subset V^+ \quad V^-V^+ \subset V^- \quad V^+V^- \subset V^-$$

hold $\mathcal{C}$ is called a superalgebra with reference to this specific decomposition. So all Clifford algebras are superalgebras with this even and odd decomposition.

In particle physics calculations involving parity—i.e. for instance when considering bosons or fermions—superalgebras provide a way of dealing with both in a single setting.

The prefix "super" comes from the association of these algebras with physical theories featuring supersymmetry, or "SUSY," in which they appear. There are many such theories, which attempt to reconcile the requirements of relativistic mechanics with quantum mechanics.

Since I don't actually understand these theories I will stop talking about them here . . .

Conclusion

The Pauli matrices and the Dirac matrices provide examples of matrices that generate universal Clifford algebras. These may be used to study reflections and rotations in various dimensions and are associated with nondegenerate symmetric bilinear (or sesquilinear) forms. These algebras have dimension 2^n where n is the dimension of the space upon which they act. The real numbers and the complex numbers are also Clifford algebras, generated by 1 and $1, i$ respectively.

Come back next week to see Victor Polinger talk about how physicists think of, and use, the Dirac algebra.