Comparing The Size Of (Large) Sets

Steve Ziskind

What are we measuring?

What are we measuring?

- We will adopt a simple notion of set: a collection of things

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people
- We will be interested in sets of numbers, or sets of sets

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people
- We will be interested in sets of numbers, or sets of sets
- Our most important sets of numbers are $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$,

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people
- We will be interested in sets of numbers, or sets of sets
- Our most important sets of numbers are $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$,
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers (fractions), and $\mathbb{R}$ denotes the real numbers (i.e. the points on the line)

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people
- We will be interested in sets of numbers, or sets of sets
- Our most important sets of numbers are $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$,
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers (fractions), and $\mathbb{R}$ denotes the real numbers (i.e. the points on the line)
- We identify a real number with its decimal representation

What are we measuring?

- We will adopt a simple notion of set: a collection of things
- A set might be a collection of numbers, or fingers, or people
- We will be interested in sets of numbers, or sets of sets
- Our most important sets of numbers are $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$,
- $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers (fractions), and $\mathbb{R}$ denotes the real numbers (i.e. the points on the line)
- We identify a real number with its decimal representation
- Note that $\mathbb{N}$ is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$. And the last two are different because not all reals are rational ($\sqrt{2}$)

How are we measuring size?

How are we measuring size?

- Just what size means is not clear

How are we measuring size?

- Just what size means is not clear
- To start, consider small/finite sets. Let $A = \{1, 2, 3\}$ and $B = \{algebra, banana, covid\}$.

How are we measuring size?

- Just what size means is not clear
- To start, consider small/finite sets. Let $A = \{1, 2, 3\}$ and $B = \{algebra, banana, covid\}$.
- We say that A and B have the same size because they each have 3 things in them

How are we measuring size?

- Just what size means is not clear
- To start, consider small/finite sets. Let $A = \{1, 2, 3\}$ and $B = \{algebra, banana, covid\}$.
- We say that A and B have the same size because they each have 3 things in them
- The set of all of the subsets of a set is called the Power Set of that set. For the set A above, its power set has 8 members:

How are we measuring size?

- Just what size means is not clear
- To start, consider small/finite sets. Let $A = \{1, 2, 3\}$ and $B = \{algebra, banana, covid\}$.
- We say that A and B have the same size because they each have 3 things in them
- The set of all of the subsets of a set is called the Power Set of that set. For the set A above, its power set has 8 members:
- $\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Note that $8 = 2^3$. Because of this, the notation for the power set of A is 2^A, even when A is infinite.

Simple counting works well with finite sets, but won't work for infinite sets. We need something more general.

The meaning of size

- A way of comparing size that works equally well for both finite and infinite sets was introduced by Georg Cantor in 1873.

The meaning of size

- A way of comparing size that works equally well for both finite and infinite sets was introduced by Georg Cantor in 1873.
- Cantor proposed that two sets have the same size when their elements can be paired in a direct way. Explicitly we say that sets A and B have the same <u>cardinality</u> when there exists a function f, from A to B, which is <u>one-to-one</u> and <u>onto</u>.

The meaning of size

- A way of comparing size that works equally well for both finite and infinite sets was introduced by Georg Cantor in 1873.
- Cantor proposed that two sets have the same size when their elements can be paired in a direct way. Explicitly we say that sets A and B have the same cardinality when there exists a function f, from A to B, which is one-to-one and onto.
- The function f is **one-to-one** when distinct inputs from A give distinct values in B. That is, f(x)=f(y) only when x=y.

The meaning of size

- A way of comparing size that works equally well for both finite and infinite sets was introduced by Georg Cantor in 1873.
- Cantor proposed that two sets have the same size when their elements can be paired in a direct way. Explicitly we say that sets A and B have the same cardinality when there exists a function f, from A to B, which is one-to-one and onto.
- The function f is **one-to-one** when distinct inputs from A give distinct values in B. That is, f(x)=f(y) only when x=y.
- The function f is **onto** when every member of B is the value of f(x) for some x in A.

The meaning of size

- A way of comparing size that works equally well for both finite and infinite sets was introduced by Georg Cantor in 1873.
- Cantor proposed that two sets have the same size when their elements can be paired in a direct way. Explicitly we say that sets A and B have the same cardinality when there exists a function f, from A to B, which is one-to-one and onto.
- The function f is **one-to-one** when distinct inputs from A give distinct values in B. That is, f(x)=f(y) only when x=y.
- The function f is **onto** when every member of B is the value of f(x) for some x in A.
- Such a function is said to establish a one-to-one correspondence between the elements of A and B.

The meaning of size

- As an example, let $A = [0,1]$ and $B = [0,2]$. The function $f(x)=2x$ from A to B shows that card(A)=card(B). Nevertheless, A and B have different lengths as subsets of the line. This shows that cardinality is measuring something different from length.

The meaning of size

- As an example, let A = [0,1] and B = [0,2]. The function f(x)=2x from A to B shows that card(A)=card(B). Nevertheless, A and B have different lengths as subsets of the line. This shows that cardinality is measuring something different from length.
- Even more extreme: if $A = (\pi/2, \pi/2)$ and $B = \mathbb{R}$, and $f(x) = tan(x)$ then $f : A \to B$ takes a finite interval to the entire line. So we should not jump too quickly to judge cardinality.

The meaning of size

- As an example, let A = [0,1] and B = [0,2]. The function f(x)=2x from A to B shows that card(A)=card(B). Nevertheless, A and B have different lengths as subsets of the line. This shows that cardinality is measuring something different from length.
- Even more extreme: if $A = (\pi/2, \pi/2)$ and $B = \mathbb{R}$, and $f(x) = tan(x)$ then $f : A \to B$ takes a finite interval to the entire line. So we should not jump too quickly to judge cardinality.
- Centuries before Cantor, Galileo noted that there are the same number of positive whole numbers as the set of perfect squares. Cantor's idea wasn't entirely original, but he pursued it further.

The meaning of size

- As an example, let A = [0,1] and B = [0,2]. The function f(x)=2x from A to B shows that card(A)=card(B). Nevertheless, A and B have different lengths as subsets of the line. This shows that cardinality is measuring something different from length.
- Even more extreme: if $A = (\pi/2, \pi/2)$ and $B = \mathbb{R}$, and $f(x) = tan(x)$ then $f : A \to B$ takes a finite interval to the entire line. So we should not jump too quickly to judge cardinality.
- Centuries before Cantor, Galileo noted that there are the same number of positive whole numbers as the set of perfect squares. Cantor's idea wasn't entirely original, but he pursued it further.
- The function f(n)=n+1 establishes a 1-1 correspondence between $\{0, 1, 2, 3, \dots\}$ and $\mathbb{N}$. This is a shift function, pairing each number with the number to its right.

Challenge Problems

- Challenge Problem 1: The set of all integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ has the same cardinality as $\mathbb{N}$. Find an explicit 1-1 correspondence function (a formula for f).

Challenge Problems

- Challenge Problem 1: The set of all integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ has the same cardinality as $\mathbb{N}$. Find an explicit 1-1 correspondence function (a formula for f).
- Challenge Problem 2: The closed interval A=[0,1] and the half open interval B=[0,1) have the same cardinality. Find an explicit 1-1 correspondence function (a formula $f : A \to B$).

$\mathbb{N}$ versus $\mathbb{Q}$

- Theorem: card($\mathbb{N}$) = card($\mathbb{Q}$)

$\mathbb{N}$ versus $\mathbb{Q}$

- Theorem: card($\mathbb{N}$) = card($\mathbb{Q}$)
- We will prove the slightly simpler statement using the positive rationals. The method of Challenge Problem 1 will extend the result to all of $\mathbb{Q}$.

$\mathbb{N}$ versus $\mathbb{Q}$

- <u>Theorem:</u> card($\mathbb{N}$) = card($\mathbb{Q}$)
- We will prove the slightly simpler statement using the positive rationals. The method of Challenge Problem 1 will extend the result to all of $\mathbb{Q}$.
- <u>Proof:</u> What is required is a way to systematically list all the positive rationals. Note that we can first order these rationals via the sum of their numerators and denominators. Thus there are none whose sum is 1, one whose sum is 2, 2 whose sum is 3, and generally n whose sum is n+1.

$\mathbb{N}$ versus $\mathbb{Q}$

- Theorem: card($\mathbb{N}$) = card($\mathbb{Q}$)
- We will prove the slightly simpler statement using the positive rationals. The method of Challenge Problem 1 will extend the result to all of $\mathbb{Q}$.
- Proof: What is required is a way to systematically list all the positive rationals. Note that we can first order these rationals via the sum of their numerators and denominators. Thus there are none whose sum is 1, one whose sum is 2, 2 whose sum is 3, and generally n whose sum is n+1.
- Within each sum, order from 1/n to n/1, with the numerator increasing and the denominator decreasing. Thus, the sequence begins: 1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, etc.

$\mathbb{N}$ versus $\mathbb{Q}$

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
 & & \cdots & & &
\end{array}
$$

$\mathbb{N}$ versus $\mathbb{Q}$

- Our listing of fractions contains duplicates: $1/1 = 2/2 = 3/3$ etc.

$\mathbb{N}$ versus $\mathbb{Q}$

- Our listing of fractions contains duplicates: $1/1 = 2/2 = 3/3$ etc.
- It is straightforward (though tedious) to remove duplicates. They are the ones that can be reduced. Once this is done, the list begins: $1/1$, $1/2$, $2/1$, $1/3$, $3/1$, $1/4$, $2/3$, $3/2$, $4/1$, $1/5$, $5/1$, etc. Now we have our desired 1-1 matching of the members of $\mathbb{N}$ with the positive members of $\mathbb{Q}$. The idea of challenge problem 1 be extends this to all of $\mathbb{Q}$. QED

$\mathbb{N}$ versus $\mathbb{Q}$

- Our listing of fractions contains duplicates: $1/1 = 2/2 = 3/3$ etc.
- It is straightforward (though tedious) to remove duplicates. They are the ones that can be reduced. Once this is done, the list begins: $1/1$, $1/2$, $2/1$, $1/3$, $3/1$, $1/4$, $2/3$, $3/2$, $4/1$, $1/5$, $5/1$, etc. Now we have our desired 1-1 matching of the members of $\mathbb{N}$ with the positive members of $\mathbb{Q}$. The idea of challenge problem 1 be extends this to all of $\mathbb{Q}$. QED
- Challenge Problem 3: Write a computer program to list the first 50 members of $\mathbb{Q}$ described by our algorithm. (Include negatives and zero.)

$\mathbb{N}$ versus $\mathbb{Q}$

- Our listing of fractions contains duplicates: $1/1 = 2/2 = 3/3$ etc.
- It is straightforward (though tedious) to remove duplicates. They are the ones that can be reduced. Once this is done, the list begins: $1/1$, $1/2$, $2/1$, $1/3$, $3/1$, $1/4$, $2/3$, $3/2$, $4/1$, $1/5$, $5/1$, etc. Now we have our desired 1-1 matching of the members of $\mathbb{N}$ with the positive members of $\mathbb{Q}$. The idea of challenge problem 1 be extends this to all of $\mathbb{Q}$. QED
- Challenge Problem 3: Write a computer program to list the first 50 members of $\mathbb{Q}$ described by our algorithm. (Include negatives and zero.)
- We say that the set $\mathbb{Q}$ is countable or denumerable.

$\mathbb{N}$ versus $\mathbb{Q}$

- Our listing of fractions contains duplicates: $1/1 = 2/2 = 3/3$ etc.
- It is straightforward (though tedious) to remove duplicates. They are the ones that can be reduced. Once this is done, the list begins: $1/1$, $1/2$, $2/1$, $1/3$, $3/1$, $1/4$, $2/3$, $3/2$, $4/1$, $1/5$, $5/1$, etc. Now we have our desired 1-1 matching of the members of $\mathbb{N}$ with the positive members of $\mathbb{Q}$. The idea of challenge problem 1 be extends this to all of $\mathbb{Q}$. QED
- Challenge Problem 3: Write a computer program to list the first 50 members of $\mathbb{Q}$ described by our algorithm. (Include negatives and zero.)
- We say that the set $\mathbb{Q}$ is countable or denumerable.
- The idea behind this array shows that: A countable union of countable/finite sets is countable.

$\mathbb{N}$ versus $\mathbb{R}$

- Theorem: $\text{card}(\mathbb{N}) \neq \text{card}(\mathbb{R})$

$\mathbb{N}$ versus $\mathbb{R}$

- Theorem: card($\mathbb{N}$) $\neq$ card($\mathbb{R}$)
- To prove this, Cantor showed that any supposed listing of the reals must necessarily be incomplete. The remarkable method is called Cantor's Diagonal Argument.

- Theorem: $\text{card}(\mathbb{N}) \neq \text{card}(\mathbb{R})$
- To prove this, Cantor showed that any supposed listing of the reals must necessarily be incomplete. The remarkable method is called Cantor's Diagonal Argument.
- Proof: Suppose that a list of the reals between 0 and 1 could be made: $\{x_1, x_2, x_3, \ldots\}$

$\mathbb{N}$ versus $\mathbb{R}$

- Theorem: card($\mathbb{N}$) $\neq$ card($\mathbb{R}$)
- To prove this, Cantor showed that any supposed listing of the reals must necessarily be incomplete. The remarkable method is called Cantor's Diagonal Argument.
- Proof: Suppose that a list of the reals between 0 and 1 could be made: $\{x_1, x_2, x_3, \ldots\}$
- Each x can be expressed as an infinite decimal:

  $x_1 = 0.x_{11}x_{12}x_{13}x_{14}\ldots$

  $x_2 = 0.x_{21}x_{22}x_{23}x_{24}\ldots$

  $x_3 = 0.x_{31}x_{32}x_{33}x_{34}\ldots$

  $\ldots$

$\mathbb{N}$ versus $\mathbb{R}$

- Theorem: $\text{card}(\mathbb{N}) \neq \text{card}(\mathbb{R})$
- To prove this, Cantor showed that any supposed listing of the reals must necessarily be incomplete. The remarkable method is called Cantor's Diagonal Argument.
- Proof: Suppose that a list of the reals between 0 and 1 could be made: $\{x_1, x_2, x_3, \ldots\}$
- Each x can be expressed as an infinite decimal:

  $x_1 = 0.x_{11}x_{12}x_{13}x_{14}\ldots$

  $x_2 = 0.x_{21}x_{22}x_{23}x_{24}\ldots$

  $x_3 = 0.x_{31}x_{32}x_{33}x_{34}\ldots$

  $\ldots$
- Choose a sequence of digits: $a_1 \neq x_{11}, a_2 \neq x_{22}, a_3 \neq x_{33}\ldots$

- Define the number $a = 0.a_1 a_2 a_3 \ldots$

$\mathbb{N}$ versus $\mathbb{R}$

- Define the number $a = 0.a_1a_2a_3\ldots$
- The number a differs from x_1 in the first decimal place, it differs from x_2 in the second decimal place, etc. It differs from each of them, so it isn't on the original list

$\mathbb{N}$ versus $\mathbb{R}$

- Define the number $a = 0.a_1 a_2 a_3 \ldots$
- The number a differs from x_1 in the first decimal place, it differs from x_2 in the second decimal place, etc. It differs from each of them, so it isn't on the original list
- This means that the list cannot have been complete, and no list can be complete. QED

$\mathbb{N}$ versus $\mathbb{R}$

- Define the number $a = 0.a_1 a_2 a_3 \ldots$
- The number a differs from x_1 in the first decimal place, it differs from x_2 in the second decimal place, etc. It differs from each of them, so it isn't on the original list
- This means that the list cannot have been complete, and no list can be complete. QED
- Because $\mathbb{Q}$ and $\mathbb{R}$ have different cardinalities, they cannot be the same set. This proves the existence of irrational numbers.

- Define the number $a = 0.a_1a_2a_3\ldots$
- The number a differs from x_1 in the first decimal place, it differs from x_2 in the second decimal place, etc. It differs from each of them, so it isn't on the original list
- This means that the list cannot have been complete, and no list can be complete. QED
- Because $\mathbb{Q}$ and $\mathbb{R}$ have different cardinalities, they cannot be the same set. This proves the existence of irrational numbers.
- Of course, we already knew this because of $\sqrt{2}$. But neither our method nor the Greek result says how common irrationals are relative to rationals.

A set larger than $\mathbb{R}$?

- Let A $=$ [0,1] and B $= \{(x, y)|x \in A, y \in A\}$

A set larger than $\mathbb{R}$?

- Let A = [0,1] and B = $\{(x, y)|x \in A, y \in A\}$
- Because A is one-dimensional and B is two-dimensional it seems clear that $card(A) < card(B)$. In fact card(A)=card(B).

A set larger than $\mathbb{R}$?

- Let A = [0,1] and B $= \{(x, y) | x \in A, y \in A\}$
- Because A is one-dimensional and B is two-dimensional it seems clear that $card(A) < card(B)$. In fact card(A)=card(B).
- Proof: Every point in A has a decimal expansion $a = 0.a_1 a_2 a_3 a_4 \ldots$. Let $x = 0.a_1 a_3 a_5 \ldots$ and $y = 0.a_2 a_4 a_6 \ldots$ This sends every point on the edge to a point in the interior. And every interior point arises from an edge point like this: given $x = 0.x_1 x_2 x_3 \ldots$ and $y = 0.y_1 y_2 y_3 \ldots$, the interior point (x,y) arises from the edge point $0.x_1 y_1 x_2 y_2 x_3 y_3 \ldots$. QED

A set larger than $\mathbb{R}$?

- Let A = [0,1] and B = $\{(x, y) | x \in A, y \in A\}$
- Because A is one-dimensional and B is two-dimensional it seems clear that $card(A) < card(B)$. In fact card(A)=card(B).
- Proof: Every point in A has a decimal expansion $a = 0.a_1a_2a_3a_4\ldots$. Let $x = 0.a_1a_3a_5\ldots$ and $y = 0.a_2a_4a_6\ldots$ This sends every point on the edge to a point in the interior. And every interior point arises from an edge point like this: given $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$, the interior point (x,y) arises from the edge point $0.x_1y_1x_2y_2x_3y_3\ldots$. QED
- Morals: First, increasing the dimension does not change cardinality. Second, don't trust your intuition.

Almost all reals are irrational

- We say that a set of real numbers has measure zero if it can be contained within a set of open intervals, the sum of whose lengths is arbitrarily small.

Almost all reals are irrational

- We say that a set of real numbers has measure zero if it can be contained within a set of open intervals, the sum of whose lengths is arbitrarily small.
- Theorem: Any countable set of reals has measure zero. In particular, $\mathbb{Q}$ has measure zero.

Almost all reals are irrational

- We say that a set of real numbers has measure zero if it can be contained within a set of open intervals, the sum of whose lengths is arbitrarily small.
- Theorem: Any countable set of reals has measure zero. In particular, $\mathbb{Q}$ has measure zero.
- Proof: Given the list of reals $\{x_1, x_2, x_3, \ldots\}$ and given your favorite small number $\epsilon > 0$, define a corresponding set of intervals like this:

  $I_1 = (x_1 - \epsilon/4, x_1 + \epsilon/4)$
  $I_2 = (x_2 - \epsilon/8, x_2 + \epsilon/8)$
  $I_3 = (x_3 - \epsilon/16, x_3 + \epsilon/16)$
  $\ldots$

Almost all reals are irrational

- We say that a set of real numbers has measure zero if it can be contained within a set of open intervals, the sum of whose lengths is arbitrarily small.
- Theorem: Any countable set of reals has measure zero. In particular, $\mathbb{Q}$ has measure zero.
- Proof: Given the list of reals $\{x_1, x_2, x_3, \ldots\}$ and given your favorite small number $\epsilon > 0$, define a corresponding set of intervals like this:
  $I_1 = (x_1 - \epsilon/4, x_1 + \epsilon/4)$
  $I_2 = (x_2 - \epsilon/8, x_2 + \epsilon/8)$
  $I_3 = (x_3 - \epsilon/16, x_3 + \epsilon/16)$
  $\ldots$
- The set of x values is contained within the union of the intervals, and the sum of the lengths of those intervals is ϵ, which can be as small as we wish. QED

Almost all reals are irrational

- Theorem: If the interval [0,1] is contained within a set of open intervals, then the sum of their lengths must be at least 1. (i.e. [0,1] does not have measure zero.)

Almost all reals are irrational

- Theorem: If the interval [0,1] is contained within a set of open intervals, then the sum of their lengths must be at least 1. (i.e. [0,1] does not have measure zero.)
- Proof: Sorry. This relies on a technical property of [0,1] known as compactness. You need to take my word. QED

Almost all reals are irrational

- Theorem: If the interval [0,1] is contained within a set of open intervals, then the sum of their lengths must be at least 1. (i.e. [0,1] does not have measure zero.)
- Proof: Sorry. This relies on a technical property of [0,1] known as compactness. You need to take my word. QED
- The standard language for this is: almost all real numbers are irrational.

Almost all reals are irrational

- Theorem: If the interval [0,1] is contained within a set of open intervals, then the sum of their lengths must be at least 1. (i.e. [0,1] does not have measure zero.)
- Proof: Sorry. This relies on a technical property of [0,1] known as compactness. You need to take my word. QED
- The standard language for this is: almost all real numbers are irrational.
- One way to make this seem obvious is to recall that rational numbers are those whose decimal expansions are repeating. If you randomly select digits to choose a real number, the chance of it repeating at some point is zero (not the same as impossible).

Almost all reals are irrational

- Theorem: If the interval [0,1] is contained within a set of open intervals, then the sum of their lengths must be at least 1. (i.e. [0,1] does not have measure zero.)
- Proof: Sorry. This relies on a technical property of [0,1] known as compactness. You need to take my word. QED
- The standard language for this is: almost all real numbers are irrational.
- One way to make this seem obvious is to recall that rational numbers are those whose decimal expansions are repeating. If you randomly select digits to choose a real number, the chance of it repeating at some point is zero (not the same as impossible).
- Challenge Problem 4: What rational number equals 1.37626262626262...?

The Cantor Set

- A remarkable set, known as the Cantor Set ($\mathbb{K}$), is built by starting with the unit interval, [0,1], removing the middle open third, and then repeatedly removing the middle open third from each of the remaining components.

The Cantor Set

- A remarkable set, known as the Cantor Set ($\mathbb{K}$), is built by starting with the unit interval, [0,1], removing the middle open third, and then repeatedly removing the middle open third from each of the remaining components.
- After the first removal the set is $[0, 1/3] \cup [2/3, 1]$.

The Cantor Set

- A remarkable set, known as the Cantor Set ($\mathbb{K}$), is built by starting with the unit interval, [0,1], removing the middle open third, and then repeatedly removing the middle open third from each of the remaining components.
- After the first removal the set is $[0, 1/3] \cup [2/3, 1]$.
- After the next removal the set is $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.

The Cantor Set

- A remarkable set, known as the Cantor Set ($\mathbb{K}$), is built by starting with the unit interval, [0,1], removing the middle open third, and then repeatedly removing the middle open third from each of the remaining components.
- After the first removal the set is $[0, 1/3] \cup [2/3, 1]$.
- After the next removal the set is $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.
- The next removal eliminates the middle third of each of the 4 intervals. Continuing indefinitely, each set has only 2/3 of the length of the set at the prior stage, so the final set has measure zero.

The Cantor Set

- A remarkable set, known as the Cantor Set ($\mathbb{K}$), is built by starting with the unit interval, [0,1], removing the middle open third, and then repeatedly removing the middle open third from each of the remaining components.
- After the first removal the set is $[0, 1/3] \cup [2/3, 1]$.
- After the next removal the set is $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.
- The next removal eliminates the middle third of each of the 4 intervals. Continuing indefinitely, each set has only 2/3 of the length of the set at the prior stage, so the final set has measure zero.
- It seems that $\mathbb{K}$ consists of the endpoints of the intervals that remain after each stage of removal, which is a countable set, and this explains why it has measure zero. WRONG

The Cantor Set

- If we express each number in [0,1] in base-3 instead of the usual base-10, then the members of $\mathbb{K}$ are those whose expansions never use the digit 1.

The Cantor Set

- If we express each number in [0,1] in base-3 instead of the usual base-10, then the members of $\mathbb{K}$ are those whose expansions never use the digit 1.
- A typical member of $\mathbb{K}$, expressed in base-3, will look like 0.0200220220200222202....

The Cantor Set

- If we express each number in [0,1] in base-3 instead of the usual base-10, then the members of $\mathbb{K}$ are those whose expansions never use the digit 1.
- A typical member of $\mathbb{K}$, expressed in base-3, will look like 0.02002202202002202....
- This can be put into a 1-1 correspondence with 0.01001101101001101.... by replacing each occurrence of 2 with 1. Such numbers, when interpreted as base-2 expressions correspond to every member of [0,1].

- If we express each number in [0,1] in base-3 instead of the usual base-10, then the members of $\mathbb{K}$ are those whose expansions never use the digit 1.
- A typical member of $\mathbb{K}$, expressed in base-3, will look like 0.0200220220200222202....
- This can be put into a 1-1 correspondence with 0.0100110110100111101.... by replacing each occurrence of 2 with 1. Such numbers, when interpreted as base-2 expressions correspond to every member of [0,1].
- In other words, card($\mathbb{K}$)=card($\mathbb{R}$). So the Cantor Set is an example of an uncountable set with measure zero.

The Size of the Power Set

- Theorem: No set can be put into a 1-1 correspondence with its power set. Thus $card(A) < card(2^A)$ for any set A.

The Size of the Power Set

- Theorem: No set can be put into a 1-1 correspondence with its power set. Thus $card(A) < card(2^A)$ for any set A.
- Proof: This subtle argument, due to Cantor, supposes that such a function f, from A to 2^A does exist, and then shows that a logical inconsistency results.

The Size of the Power Set

- Theorem: No set can be put into a 1-1 correspondence with its power set. Thus $card(A) < card(2^A)$ for any set A.
- Proof: This subtle argument, due to Cantor, supposes that such a function f, from A to 2^A does exist, and then shows that a logical inconsistency results.
- Let B denote the set of all elements of A which are not members of their images under f: B=$\{a \in A | a \notin f(a)\}$.

The Size of the Power Set

- Theorem: No set can be put into a 1-1 correspondence with its power set. Thus $card(A) < card(2^A)$ for any set A.
- Proof: This subtle argument, due to Cantor, supposes that such a function f, from A to 2^A does exist, and then shows that a logical inconsistency results.
- Let B denote the set of all elements of A which are not members of their images under f: B=$\{a \in A | a \notin f(a)\}$.
- Because B is a subset of A, it is a member of 2^A. Because f is onto, B=f(b) for some $b \in A$.

The Size of the Power Set

- Theorem: No set can be put into a 1-1 correspondence with its power set. Thus $card(A) < card(2^A)$ for any set A.
- Proof: This subtle argument, due to Cantor, supposes that such a function f, from A to 2^A does exist, and then shows that a logical inconsistency results.
- Let B denote the set of all elements of A which are not members of their images under f: B=$\{a \in A | a \notin f(a)\}$.
- Because B is a subset of A, it is a member of 2^A. Because f is onto, B=f(b) for some $b \in A$.
- Question: Is $b \in B$? If yes, then it is a member of its image and cannot be in B. If no, the it is not in its image and must be in B. No matter our answer, we are contradicting ourselves. Conclusion: no such f really existed. QED

Algebraic Numbers

- Saying that a number x is rational, say $x = 2/3$, is the same as saying that it solves the equation $3x - 2 = 0$. So the rationals are the numbers that solve first degree equations with integer coefficients. They are called algebraic of degree 1.

Algebraic Numbers

- Saying that a number x is rational, say $x = 2/3$, is the same as saying that it solves the equation $3x - 2 = 0$. So the rationals are the numbers that solve first degree equations with integer coefficients. They are called algebraic of degree 1.
- The number $\sqrt{2}$ is irrational, but it does solve the equation $x^2 - 2 = 0$. A number that solves a 2nd degree polynomial with integer coefficients (but of no lower degree) is called algebraic of degree 2.

Algebraic Numbers

- Saying that a number x is rational, say $x = 2/3$, is the same as saying that it solves the equation $3x - 2 = 0$. So the rationals are the numbers that solve first degree equations with integer coefficients. They are called algebraic of degree 1.
- The number $\sqrt{2}$ is irrational, but it does solve the equation $x^2 - 2 = 0$. A number that solves a 2nd degree polynomial with integer coefficients (but of no lower degree) is called algebraic of degree 2.
- Claim: There are countably many algebraic numbers of degree 2.

Algebraic Numbers

- Saying that a number x is rational, say $x = 2/3$, is the same as saying that it solves the equation $3x - 2 = 0$. So the rationals are the numbers that solve first degree equations with integer coefficients. They are called algebraic of degree 1.
- The number $\sqrt{2}$ is irrational, but it does solve the equation $x^2 - 2 = 0$. A number that solves a 2nd degree polynomial with integer coefficients (but of no lower degree) is called algebraic of degree 2.
- <u>Claim</u>: There are countably many algebraic numbers of degree 2.
- <u>Proof</u>: If the polynomial is $ax^2 + bx + c$, then we may assume that $a > 0$. For all those with a=1, the rest of $bx + c$ correspond to the countable first degree polynomials. For all those with a=2, the rest also correspond to the first degree polynomials. etc.

Algebraic Numbers

- We can now see that the set of 2nd degree polynomials is a countable union of countable sets, hence is countable. QED

Algebraic Numbers

- We can now see that the set of 2nd degree polynomials is a countable union of countable sets, hence is countable. QED
- This same argument readily extends to show that there are countably many integer polynomials of any given degree, and each such polynomial can have only finitely many zeros.

Algebraic Numbers

- We can now see that the set of 2nd degree polynomials is a countable union of countable sets, hence is countable. QED
- This same argument readily extends to show that there are countably many integer polynomials of any given degree, and each such polynomial can have only finitely many zeros.
- Taking them all together, we have shown that only countably many numbers can be the zeros of a polynomial with integer coefficients. Such numbers are called <u>algebraic numbers</u>.

Algebraic Numbers

- We can now see that the set of 2nd degree polynomials is a countable union of countable sets, hence is countable. QED
- This same argument readily extends to show that there are countably many integer polynomials of any given degree, and each such polynomial can have only finitely many zeros.
- Taking them all together, we have shown that only countably many numbers can be the zeros of a polynomial with integer coefficients. Such numbers are called algebraic numbers.
- Arguing as before, most real numbers are not algebraic. They are called transcendental.

Algebraic Numbers

- We can now see that the set of 2nd degree polynomials is a countable union of countable sets, hence is countable. QED
- This same argument readily extends to show that there are countably many integer polynomials of any given degree, and each such polynomial can have only finitely many zeros.
- Taking them all together, we have shown that only countably many numbers can be the zeros of a polynomial with integer coefficients. Such numbers are called <u>algebraic numbers</u>.
- Arguing as before, most real numbers are not algebraic. They are called <u>transcendental</u>.
- What is an example of a transcendental number?

Liouville's Number

- Liouville (1844) gave the first explicit example of a number known to be transcendental. Decades later, both e (Hermite, 1873) and π (Lindemann, 1882) were also shown to be transcendental, but with more difficult arguments. Even later (Gelfond-Schneider, 1934), $2^{\sqrt{2}}$ was shown to be transcendental, a deep result that answered a famous question of Hilbert.

Liouville's Number

- Liouville (1844) gave the first explicit example of a number known to be transcendental. Decades later, both e (Hermite, 1873) and π (Lindemann, 1882) were also shown to be transcendental, but with more difficult arguments. Even later (Gelfond-Schneider, 1934), $2^{\sqrt{2}}$ was shown to be transcendental, a deep result that answered a famous question of Hilbert.
- The construction of Liouville is based on this result:

Liouville's Number

- Liouville (1844) gave the first explicit example of a number known to be transcendental. Decades later, both e (Hermite, 1873) and π (Lindemann, 1882) were also shown to be transcendental, but with more difficult arguments. Even later (Gelfond-Schneider, 1934), $2^{\sqrt{2}}$ was shown to be transcendental, a deep result that answered a famous question of Hilbert.
- The construction of Liouville is based on this result:
- Liouville's Theorem: If α is algebraic of degree $n > 1$, and if $\frac{s}{t} \in \mathbb{Q}$ satisfies $|\frac{s}{t} - \alpha| < 1$, then $|\frac{s}{t} - \alpha| > \frac{C}{t^n}$ for some constant C, independent of s and t.

Liouville's Number

- Proof:Let α be the root of the minimal degree polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} \dots a_0$. Then $p(\frac{s}{t}) = a_n (\frac{s}{t})^n + a_{n-1} (\frac{s}{t})^{n-1} \dots a_0 = \frac{\text{integer}}{t^n}$.

Liouville's Number

- Proof:Let α be the root of the minimal degree polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} \ldots a_0$. Then $p(\frac{s}{t}) = a_n(\frac{s}{t})^n + a_{n-1}(\frac{s}{t})^{n-1} \ldots a_0 = \frac{\text{integer}}{t^n}$.
- Because p has minimal degree for α this cannot equal zero, which means that $\frac{1}{t^n} < |p(\frac{s}{t}) - p(\alpha)| = |\frac{s}{t} - \alpha||\frac{dp}{dx}(\beta)|$, where β is in $[\alpha - 1, \alpha + 1]$ because of the Mean Value Theorem.

- Proof:Let α be the root of the minimal degree polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} \dots a_0$. Then $p(\frac{s}{t}) = a_n(\frac{s}{t})^n + a_{n-1}(\frac{s}{t})^{n-1} \dots a_0 = \frac{\text{integer}}{t^n}$.
- Because p has minimal degree for α this cannot equal zero, which means that $\frac{1}{t^n} < |p(\frac{s}{t}) - p(\alpha)| = |\frac{s}{t} - \alpha||\frac{dp}{dx}(\beta)|$, where β is in $[\alpha - 1, \alpha + 1]$ because of the Mean Value Theorem.
- Let C to be 1 over Max[$|\frac{dp}{dx}|$] on $[\alpha - 1, \alpha + 1]$. QED

Liouville's Number

- Proof:Let α be the root of the minimal degree polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} \ldots a_0$. Then $p(\frac{s}{t}) = a_n(\frac{s}{t})^n + a_{n-1}(\frac{s}{t})^{n-1} \ldots a_0 = \frac{\text{integer}}{t^n}$.
- Because p has minimal degree for α this cannot equal zero, which means that $\frac{1}{t^n} < |p(\frac{s}{t}) - p(\alpha)| = |\frac{s}{t} - \alpha||\frac{dp}{dx}(\beta)|$, where β is in $[\alpha - 1, \alpha + 1]$ because of the Mean Value Theorem.
- Let C to be 1 over Max[$|\frac{dp}{dx}|$] on $[\alpha - 1, \alpha + 1]$. QED
- Define $L = \sum_1^\infty 10^{-j!}$, which is Liouville's Number. Writing out the first digits, $L = 0.110001000000000\ldots$, where the unit digits occur in places 1, 2, 6, 24, 120, etc.

Liouville's Number

- Rewrite $L = \sum_1^k 10^{-j!} + \sum_{k+1}^\infty 10^{-j!}$. The first sum can be put over the common denominator of $10^{k!}$, so it is a rational approximation to L.

- Rewrite L $= \sum_1^k 10^{-j!} + \sum_{k+1}^{\infty} 10^{-j!}$. The first sum can be put over the common denominator of $10^{k!}$, so it is a rational approximation to L.
- If α were algebraic of degree n, then Liouville's Theorem would bound the second sum from below by $|C/(10^{k!})^n|$, for all values of k.

- Rewrite L $= \sum_1^k 10^{-j!} + \sum_{k+1}^{\infty} 10^{-j!}$. The first sum can be put over the common denominator of $10^{k!}$, so it is a rational approximation to L.
- If α were algebraic of degree n, then Liouville's Theorem would bound the second sum from below by $|C/(10^{k!})^n|$, for all values of k.
- But the second sum is $< 2/10^{(k+1)!}$, so that $\frac{2}{10^{(k+1)!}} > \frac{C}{(10^{k!})^n}$. Rewriting this we find $\frac{2}{C} > \frac{10^{(k+1)!}}{(10^{k!})^n} = (10^{k!})^{k+1-n}$.

Liouville's Number

- Rewrite L $= \sum_1^k 10^{-j!} + \sum_{k+1}^{\infty} 10^{-j!}$. The first sum can be put over the common denominator of $10^{k!}$, so it is a rational approximation to L.
- If α were algebraic of degree n, then Liouville's Theorem would bound the second sum from below by $|C/(10^{k!})^n|$, for <u>all</u> values of k.
- But the second sum is $< 2/10^{(k+1)!}$, so that $\frac{2}{10^{(k+1)!}} > \frac{C}{(10^{k!})^n}$. Rewriting this we find $\frac{2}{C} > \frac{10^{(k+1)!}}{(10^{k!})^n} = (10^{k!})^{k+1-n}$.
- But as k increases, this last number $\rightarrow \infty$, a contradiction. Hence L is transcendental.

The Continuum Hypothesis

- In 1900, David Hilbert gave a celebrated lecture at a mathematics congress. He proposed 23 deep problems for mathematicians to address in the coming century. The first was to settle the Continuum Hypothesis (CH).

The Continuum Hypothesis

- In 1900, David Hilbert gave a celebrated lecture at a mathematics congress. He proposed 23 deep problems for mathematicians to address in the coming century. The first was to settle the Continuum Hypothesis (CH).
- This can be stated in several equivalent ways. The simplest is: Is $\mathbb{R}$ (the continuum) the smallest uncountable set?

The Continuum Hypothesis

- In 1900, David Hilbert gave a celebrated lecture at a mathematics congress. He proposed 23 deep problems for mathematicians to address in the coming century. The first was to settle the Continuum Hypothesis (CH).
- This can be stated in several equivalent ways. The simplest is: Is $\mathbb{R}$ (the continuum) the smallest uncountable set?
- Answering this question required a starting point of axioms for set theory. The standard set is called the Zermello-Frankel System (ZF).

The Continuum Hypothesis

- In 1900, David Hilbert gave a celebrated lecture at a mathematics congress. He proposed 23 deep problems for mathematicians to address in the coming century. The first was to settle the Continuum Hypothesis (CH).
- This can be stated in several equivalent ways. The simplest is: Is $\mathbb{R}$ (the continuum) the smallest uncountable set?
- Answering this question required a starting point of axioms for set theory. The standard set is called the Zermello-Frankel System (ZF).
- In 1940, Kurt Godel proved that CH is fully consistent with ZF. In 1963, Paul Cohen proved that the negation of CH is consistent with ZF. This means that CH is independent of the other axioms of set theory, much like the Parallel Postulate is independent of the other axioms of classical Euclidean geometry.

The Schroeder-Bernstein Theorem

- Theorem: Given A and B, let $f : A \to B$ and $g : B \to A$ both be one-to-one. Then there is a function $h : A \to B$ which is one-to-one and onto. In particular, card(A)=card(B).

The Schroeder-Bernstein Theorem

- Theorem: Given A and B, let $f : A \to B$ and $g : B \to A$ both be one-to-one. Then there is a function $h : A \to B$ which is one-to-one and onto. In particular, card(A)=card(B).
- Proof: (Birkoff and MacLaine) The problem is that neither f nor g is known to be onto. The very clever proof considers ancestors of elements. We say that $y \in B$ is a parent of $x \in A$ if $x = g(y)$. If that $y = f(z)$ for some $z \in A$ then x has a grandparent. Similar for members of B.

The Schroeder-Bernstein Theorem

- Theorem: Given A and B, let $f : A \to B$ and $g : B \to A$ both be one-to-one. Then there is a function $h : A \to B$ which is one-to-one and onto. In particular, card(A)=card(B).
- Proof: (Birkoff and MacLaine) The problem is that neither f nor g is known to be onto. The very clever proof considers ancestors of elements. We say that $y \in B$ is a parent of $x \in A$ if $x = g(y)$. If that $y = f(z)$ for some $z \in A$ then x has a grandparent. Similar for members of B.
- Partition A into 3 parts, according to whether a member has an even number of ancestors, and odd number, or an infinite number: A_E, A_O, A_∞. LIkewise for B. The function f sends A_E onto B_O, and sends A_∞ onto B_∞. g^{-1} sends A_O onto B_E.

The Schroeder-Bernstein Theorem

- Theorem: Given A and B, let $f : A \to B$ and $g : B \to A$ both be one-to-one. Then there is a function $h : A \to B$ which is one-to-one and onto. In particular, card(A)=card(B).
- Proof: (Birkoff and MacLaine) The problem is that neither f nor g is known to be onto. The very clever proof considers ancestors of elements. We say that $y \in B$ is a parent of $x \in A$ if $x = g(y)$. If that $y = f(z)$ for some $z \in A$ then x has a grandparent. Similar for members of B.
- Partition A into 3 parts, according to whether a member has an even number of ancestors, and odd number, or an infinite number: A_E, A_O, A_∞. LIkewise for B. The function f sends A_E onto B_O, and sends A_∞ onto B_∞. g^{-1} sends A_O onto B_E.
- Define h to be f on $A_E \cup A_\infty$, and g^{-1} on A_O. QED.

The Schroeder-Bernstein Theorem

- Theorem: Given A and B, let $f : A \to B$ and $g : B \to A$ both be one-to-one. Then there is a function $h : A \to B$ which is one-to-one and onto. In particular, card(A)=card(B).
- Proof: (Birkoff and MacLaine) The problem is that neither f nor g is known to be onto. The very clever proof considers ancestors of elements. We say that $y \in B$ is a parent of $x \in A$ if $x = g(y)$. If that $y = f(z)$ for some $z \in A$ then x has a grandparent. Similar for members of B.
- Partition A into 3 parts, according to whether a member has an even number of ancestors, and odd number, or an infinite number: A_E, A_O, A_∞. LIkewise for B. The function f sends A_E onto B_O, and sends A_∞ onto B_∞. g^{-1} sends A_O onto B_E.
- Define h to be f on $A_E \cup A_\infty$, and g^{-1} on A_O. QED.
- Challenge Problem 5: Let A=[0,1], B=[0,1), f(x)=x/2, and g(x)=x. Using the above construction, solve Challenge Problem 2.