Modern Set Theory and Mathematical Foundations

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What songs the Sirens sang, or what name Achilles used when he hid himself among women, though puzzling questions, are not beyond all conjecture.

Sir Thomas Browne, 1658

Overview

We have mentioned many times in this series of talks issues related to mathematical foundations and the present talk aims to expand, a little, on these topics.

Nota Bene: This one-hour talk is descriptive. If you want detailed proofs of hard results you must seek them elsewhere.

We will touch on some historical efforts and the reason for them and changes in the way scholars have thought of foundations.

Then we will present the nine Zermelo-Fraenkel axioms and these axioms with the additional Axiom of Choice. These systems are usually denoted ZF and ZFC, respectively.

A Few Words About the Greeks

The ancient Greeks, apparently, invented axiomatic geometry. They stated the simple rules of logic we usually use today, assumed properties of lines and points and drew conclusions therefrom. Euclidean geometry is a powerful and elegant theory, and was the "gold standard" for logical reasoning for 2500 years. But the Greeks were not only, or even primarily, interested in mathematics per se. They were trying to understand how to apply logical methods to derive truths about the important things in *all* areas of life.

Plato (about 400 BCE) for instance thought of our sensible world as a pale and corrupted shadow of the ideal world.

Our senses provide us with impressions of the perfect ideal things in this ideal world. The ideal world was populated with objects like "the good" and "beauty" and "integrity" and "red" and also lines, points and circles.

Things like "the good" are messy and hard to define but we feel we have a good understanding of points, lines and circles, even if there are only corrupt inexact examples in the sensible world.

Mathematics is the test case. Methods of learning facts about the ideal objects of mathematics, the Greeks hoped, would lead to methods of use in discerning the true nature of other denizens of the ideal world such as "love".

The collection of ideal *objects* corresponding to a subject area was, and is still, referred to as the **ontology** of that area. The *means by which we learn of the properties of these objects*—our senses combined with features of our minds such as our reasoning ability—is called the **epistemology** of the subject.

More modern subject areas have there own ontology and epistemology. The subject of electrodynamics posits coordinates, charged particles and electric fields. The scientific method provides means of testing, and possibly refuting, ideas about these objects.

One can imagine charged particles as tiny objects ridden by angels who beat their little wings and steer their steeds according to a glorious song from the deity. Electrodynamics is the study of angels and songs. This point of view would have seemed normal and understandable to both Christian and Islamic scholars of 1000 CE.

We, on the other hand, see no need for angels in this theory. The electrons and the song remain.

Ibn al-Haytham and Newton

Arguably the first recognizably modern scientist was Ibn al-Haytham (around 1000 CE) who was heavily influenced by Aristotle but who was far more interested in practical applications than Aristotle. He wrote *at least* 77 books, of which over 50 survive, on a vast range of topics from optics to practical engineering to mathematics. His example was imitated and developed further in the Islamic world and, after 1250 CE, in the Christian world too.

Isaac Newton (late 1600s) was at a pinnacle of accumulated knowledge from these sources, and contributed his own incredible intellect to invent Calculus, with which he proceeded to solve problem-after-problem in mathematics and the sciences.

During all this time EVERY scholar in the Islamic and Christian world studied Euclidean geometry to learn how to carry out deductive reasoning in all subject areas.

Any scholarly work, including works in theology and other areas, used the axiomatic methods enshrined in Euclidean geometry. This WAS how one did logical reasoning.

But Newton's work was different, and horrified many philosophers of the day, such as George Berkeley. Newton did not clearly define what he was doing. In particular he multiplied infinitesimal numbers by infinite numbers, or divided infinitesimals by each other to create the formulae he then used to solve other problems.

What, exactly, is the ontological status of an "infinite number?"

He never bothered to explain, nor could he, why and *exactly* when these calculations worked. They just did. *He justified the mathematics by the results.* His claim was, basically, "If my reasoning is incorrect why do my predictions, which you can see by looking out the window, always work?"

Most unsettling.

It wasn't until the 1800s that Richard Dedekind and Augustin-Louis Cauchy repaired most of this logical conundrum and re-established a measure of the rigor, lost to inventors of the new mathematics for 150 years, of Euclidean geometry.

But was the job complete? The structure (the definition of numbers and the technology of limit-taking) that had been built on top of Calculus was complicated. People *believed* that the assumptions, the "axioms," involved were legitimate.

But people were . . . *not sure.*

Gotlob Frege and Bertrand Russell and David Hilbert

Attempts to *become* sure were made by numerous mathematicians and included Gotlob Frege's *The Foundations of Arithmetic*. This major work made good progress *but*. . .

. . . in 1901 Bertrand Russel discovered a flaw, an internal contradiction in Frege's initial collection of axioms.

Many mathematicians were at work on this, attempting to construct such systems not subject, provably, to this problem.

David Hilbert's famous list of 23 problems for the century included, second on the list,

Prove that the axioms of arithmetic are consistent.

Overview

People knew that the idea of "set" could provide an answer. (More about this soon.)

The Axioms of any set theory are statements about *sets*, assumed truths. Just as there is no definition of line and point in Euclidean geometry, there is no definition of set or the "element of" relation in Zermelo-Fraenkel set theory. But the entire theory is built from these two things.

The **ontology** of a set theory is the collective of all sets, which we may denote ***Set***.

You know what a set is . . . right? This knowledge comes from the "meta" world of mathematics. This meta-mathematics is what set theory is to model.

We acquire sure knowledge of sets—**epistemology**—by using the rules of our chosen logic in conjunction with the axioms of the theory.

Set theory is meaningless to humans without some very clear ideas, in advance, of the structures set theory is to model. On the most simplistic level, I think of sets as bags of distinct objects, and elements as the objects in the bags. Set theory does not tell me this. I just use this mental image to help me organize my thinking about sets.

On its own, set theory is nothing more (or less) than a juxtaposition of symbols satisfying rules. There is no *reason* for a language without something to communicate. That is why the study of subject areas—groups, rings, real numbers, measures, topological spaces, bags and so on—must, practically, come first. We will then find that set theory helps us feel more confident that inconsistency has not maneuvered its way into our forest, at least in an obvious way, while our attention was devoted to trees.

The point must be made that set theory does not "construct" anything, even if that vocabulary is ubiquitous. We *presume*, at the outset, that there is a collection of "objects-of-the-mind," a universe of discourse we will call **Set**.

The axioms authorize us to reel in our net, from the ocean of **Set**, and conclude we have caught something.

Each individual object is called "a" **set**. We will assign names to some of these sets in the course of a mathematical argument, usually letters or a combination of letters.

After much debate, mathematicians have largely agreed on the axioms this "object-of-their-collective-mind," **Set**, must possess. Axioms are expressed by reference to the names of sets. They represent assumed-truths about how **Set** works, and though they cannot be *proven*, these properties could potentially be contradictory.

Contradictions don't present much of a problem for us in most aspects of our lives but for the intended purpose of **Set** they would be deadly, and the ease with which they can be formed in ordinary human language worrisome.

Whatever axioms we adopt, we insist that it is possible to name any set without contradiction. We do *not* intend to imply that we can unambiguously describe each set—only that the act of assigning a name to a "certified" set must not produce, alone, contradiction.

If we find such a contradiction we have decisions to make.

First, if the axioms we use guarantee the existence of this set we must abandon or change one or more axioms. The choice here is only about which we modify.

Second, (and this is the usual situation) if the axioms do not *guarantee* the existence of the set we could simply accept that the set we thought to name was an illusion. The "name" we gave it actually names nothing. There are no sets with the given property.

If we cannot bear to give up this set we are back to the first case. We must give up or change one or more of the axioms. Perhaps we were mistaken in our perception of one or more of these elemental features of **Set**.

As a variation on the theme we might find not a contradiction but instead strange and horrifying sets, thrust forward perhaps by the left hand of AC, stumbling and blinking into the bright lights of mathematical center stage. We might be willing to abandon cherished axioms to expunge these stains.

The latter paths would not be taken lightly. They would represent a bifurcation in the collective vision of **Set**. Heresy is a serious matter for any primate, with consequences.

One should ponder the following items about the axioms.

- Is our list of axioms consistent, or do they conflict with each other? If they conflict, we would interpret that to mean that we had misunderstood some properties of **Set**, through wishful thinking or some other human propensity.
- Properties of **Set** are intended to mirror properties of interest to mathematicians. We want to deduce from obvious properties of **Set** those that are much less obvious. Is our list of axioms rich enough to allow us to make logical inferences about mathematically interesting topics? About *any* mathematical topic? *Decide* any interesting mathematical question?

- Does our exploration of the consequences of the axioms suggest previously unconsidered qualities which **Set** could have? These are properties that might have been placed among the axioms, but their relevance was unknown at the outset so no one thought to include them. Might there be consequences of these properties so compelling as to *change our perception* of **Set**?

Setting aside most of the ontological and epistemological issues as beyond me, let us proceed to outline set theory itself. If you are troubled by the lack of referents for some of the words, or want more precise or detailed formulations of these statements, your option is to study logic and set theory until the feeling passes.

Though the *mathematics* follows the standard path, our philosophical ruminations take one tack among many. Such matters can be the source of *spirited* debate.

The Zermelo-Fraenkel Axioms

At this point we will list the axioms of ZFC and briefly discuss each one.

Not all of the ten axioms below are independent of each other. Some combinations imply others. We will not worry here about assembling the minimal list of axioms, just a useful one, free of obvious inconsistencies and able to generate structures rich enough to model most of mathematics.

These axioms make reference to the names of sets and the binary "element of" relation between sets, usually denoted $\in$. All other relations between sets, such as $\subset$ or $=$ or $\neq$ or $\notin$ are defined in terms of this relation.

Set theory does not presume to tell us what a set is, or what this relation means. We provide the meaning ourselves, part of our vision of **Set**.

We need to point out that our description of **Set** as the collective of all sets is more a manner of speaking than the definition of an explicit object inside the theory. We will see that **Set** is not itself, cannot be, a set.

To say A is a member of **Set** is simply shorthand for saying that A satisfies an unambiguous condition: A is a set.

Collectives such as **Set**, given by an explicit, unambiguous condition such as this, are called **classes,** and classes that are not sets are called *proper classes*.

Set is simply too large, and the antinomies of pre-ZF set theory, such as Russell's paradox, occur when the axioms of *set* theory are applied to proper classes.

Set theory has nothing to say about proper classes.

Let's get on to the axioms!

(I) Axiom of the Empty Set

There exists a set, denoted $\varnothing$, which has no elements.

This axiom gets us started. At least, we're talking about *something*, even if it is nothing.

A set without members is called an **individual.** Our vision of **Set** calls for one.

$x \in \varnothing$ is always false, no matter what set x represents.

(II) The Axiom of Extensionality

Sets are equal exactly when they have the same elements.

This axiom constitutes the *definition* of the equality relation in ordinary set theory. An implication of this axiom is that sets have properties of any kind solely by virtue of the elements (also called members) they contain.

To show a set A equals a set B you need only show that $A \subset B$ and $B \subset A$. Sets with the same elements cannot be distinguished: they are the same set.

There is, for instance, but a single individual in our set theory.

(note: $A \subset B$ is shorthand for "$x \in A$ implies $x \in B$.")

(III) The Axiom of Pairing

If A and B are sets there is a set whose elements are, exactly, A and B.

It follows immediately that ordered pairs with specified first and second elements can be created, as $(a, b) = \{a, \{a, b\}\}$

Also, letting $A = B$ we can create the nested sets $\{A\}$, $\{\{A\}\}$, $\{\{\{A\}\}\}$ and so on.

At this point we can represent most of the common mathematical objects as sets. A function is a set of ordered pairs, with (a, b) corresponding to $f(a) = b$. An ordering on set A is a set of ordered pairs where (a, b) corresponds to $a \leq b$. An addition on a set A is a set of ordered pairs of the form $((a, b), c)$, which corresponds to $a + b = c$.

etc, etc, etc. But we still don't know *how many* of these interesting objects live in our embodiment of **Set**.

(IV) The Axiom of Infinity

There exists a set A with $\varnothing \in A$ and such that whenever X is a set and $X \in A$ then $X \cup \{X\} \in A$.

This axiom makes the structure we are creating **much** richer by capturing one view of an infinite set.

(V) The Axiom of Union

If S is a set of sets there is a set whose elements are exactly the elements of the members of S.

Rephrasing, this axiom allows us to infer that the union of any family of sets is itself a set ***so long as this family can be indexed by a set.***

This condition is a "size" restriction, and in conjunction with other size restrictions in the two axiom schema (below) *seems* to have struck a balance between the desire to infer the existence of—or "select" or "create"—sets based on any clear criterion and the need to avoid contradictory axioms.

(VI) The Axiom of the Power Set

For any set A there is a set $\mathbb{P}(A)$ consisting of all, and only, the subsets of A.

The Axiom of the Power Set is not constructive. It has nothing to say about which sets these are, or how many there are. It only tells you how a set can be identified as a member of the class $\mathbb{P}(A)$, and that this class is a *set*, eligible to participate as a first-class citizen in building other sets.

The effect of including this axiom is to guarantee that if, *by any means*, we ever find ourselves in possession of a member of **Set** all of whose members are in a set A then that set is an element of the *set* $\mathbb{P}(A)$.

In combination with other axioms several of which provide methods of identifying subsets one can show that $\mathbb{P}(A)$ can be **huge** in comparison to A. Exactly *how huge* is an interesting question.

The size of a set can be defined by its cardinality, as in the discussion of Dr. Ziskind last week.

One has the feeling that "$cardinal(A) < cardinal(B)$ should imply $cardinal(\mathbb{P}(A)) < cardinal(\mathbb{P}(B))$", yet it is known that this statement is independent of ZFC.

By this we mean that it can neither be proven nor refuted using the axioms of ZFC.

This is just the kind of unsettling situation that the circa-1900 Logicians and Set Theorists were working to banish. Interesting but independent statements of this kind are all too common, and many Set Theorists claim that this is an invitation to change our notion of set by adding more axioms.

(VII) The Axiom Schema of Subset Selection

Suppose P is an explicit property of sets. If A is any set then the class of all $y \in A$ for which $P(y)$ is true is a set.

Thoralf Skolem, in 1922, proposed that the "explicit properties" mentioned in the Axiom Schema (here and in the Axiom Schema of Replacement below) be drawn from carefully formed statements called predicate formulae. These must be unambiguous and refer in their statements to sets, not classes. (Clarifying further the form of such statements is important, but we will not go further here.)

This is a "restricted" selection axiom. It puts a restriction on P and requires the elements it is gathering to be inside a *set*.

The Axiom Schema of Subset Selection asserts that the class determined by an explicit property of sets is, when restricted to a set, itself a set. It is called a "schema" to indicate it is not really a single axiom. Rather it is a different axiom for each property P.

This axiom can be used, for instance, to deduce directly that if $\mathfrak{C}$ is a proper class then no set can contain all the elements of $\mathfrak{C}$. So classes, if they are not sets, are "larger" than any set.

As another example of the usage of this axiom, suppose it is a known that a member of nonempty class of sets, $\mathfrak{C}$, defined by property P is a subset of a known set A. Then the intersection of all the sets in class $\mathfrak{C}$ is a set. To see this define predicate formula Q by

$$\forall y \; (\; (P(y) \text{ is true}) \Rightarrow (x \in y)\;).$$

The Axiom Schema of Subset Selection using Q and requiring $x \in A$ yields the desired intersection. It is easy to show that this definition does not depend on *which* known set A is used, only that there is one such set in hand.

It might not be common to encounter a proper class without being able to deduce that some member of the class is a subset of a **known set.** But if this were to occur, ZFC provides no means to deduce that the elements shared by all members of the class comprise a set.

We can use this last axiom to define the integers, $\mathbb{N}$.

If you recall, the Axiom of Infinity asserts that there is a set A with $\varnothing \in A$ and such that whenever X is a set and $X \in A$ then $X \cup \{X\} \in A$.

Given one such A let $\mathcal{C}$ be the class of all sets with this same property. We see that the intersection of all members of the class, which must be a subset of A, is a set.

And this "minimal" set satisfies the same property that A does. We define this minimal set to be $\mathbb{N}$.

What does $\mathbb{N}$ look like, and what does this set have to do with numbers?

We define $0 = \varnothing \in \mathbb{N}$. We define $1 = 0 \cup \{0\} = \{0\}$. The integer 2 is defined to be $1 \cup \{1\} = \{0, 1\}$ and more generally $n + 1 = n \cup \{n\} = \{0, 1, \ldots, n\}$.

All these sets must be in every one of the sets whose intersection *is* $\mathbb{N}$, and so they must be in $\mathbb{N}$ too. And the union of all sets of this form satisfies the property of the Axiom of Infinity so this union *is* $\mathbb{N}$.

The order relation on the integers is defined by containment and integer arithmetic is defined after using $\mathbb{N}$ to create the "proof by induction" technique.

Once we define the natural numbers $\mathbb{N}$ we define non-negative rational p/q for nonzero q as
$\{(m,n) \mid mq = np \text{ where } m,n \in \mathbb{N} \text{ and } m \neq 0\}$.

Once we define non-negative rationals we define the non-negative real numbers to be the set of Dedekind cuts. These are bounded intervals of non-negative rationals which start at 0 and have no largest member.

(VIII) The Axiom Schema of Replacement

Suppose P is an explicit property of ordered pairs of sets for which $P(x,y)$ is true for at most one set y for each set x.

If A is any set then the class of all y for which $P(x,y)$ is true for *some* $x \in A$ is a set.

The word "replacement" in the name of this axiom schema comes from one of its applications. Suppose P provides a way of associating a single set $f(x)$ to set x for certain sets x. There is no need to insist that f is defined for all $x \in \mathbf{Set}$. The property P we have in mind is given by:

$P(x,y)$ is true if $y = f(x)$ and $P(x,y)$ is false otherwise.

For a set A let B be the class formed by "replacing" those $x \in A$ upon which $f(x)$ is defined by $f(x)$. A set z is in the class B precisely if z satisfies the property

$(\ z \in A$ and $f(z)$ is not defined $)$ or $(\ z = f(x)$ for some $x \in A\ )$.

We can conclude that B is a set by invoking the Axiom Schema of Replacement.

Classes identified as sets by the Axiom Schema of Replacement are not, necessarily, subsets of any previously known set. However the elements of the new set are *associated* with elements of another set so in this sense the newly formed class is not "larger" than some previously defined set. This loose size restriction *should be* enough for this class to be a set. This axiom declares that it is.

It was the inclusion of this axiom in 1922 by Adolf Fraenkel (along with independent contributions of Thoralf Skolem) that finished the job begun in 1908 by Ernst Zermelo, leading to the modern formulation of set theory shortly thereafter.

(IX) The Axiom of Foundation

Every nonempty set A has an element which contains no element of A.

To set the stage for an example of this axiom in action, we first discuss the reason for the careful distinction between classes and sets.

Consider the following argument, associated with "**Russell's Paradox**" from a critique by Bertrand Russell of Frege's pre-ZF attempts to reform set theory on a more rigorous basis.

The property of sets "$x \notin x$" is either true or not for each set x, and is quite explicit. Let $\mathfrak{C}$ denote the class defined by this property.

Russell noted that $\mathfrak{C}$ cannot be a set. For if $\mathfrak{C}$ **were** a set then it would either contain itself or not, and both cases lead to a contradiction. The act of assigning a set name to the class leads to this contradiction. If $\forall x(x \in \mathfrak{C} \Leftrightarrow x \notin x)$ is true then $\mathfrak{C} \in \mathfrak{C} \Leftrightarrow \mathfrak{C} \notin \mathfrak{C}$ must be true, which it obviously is not.

To see how the Axiom of Foundation dispatches Russell's class $\mathfrak{C}$ from the ranks of **Set**, consider $\{\mathfrak{C}\}$. If $\mathfrak{C}$ is a set the Axiom of Foundation tells us that $\mathfrak{C} \notin \mathfrak{C}$ which implies $\mathfrak{C} \in \mathfrak{C}$, a contradiction.

As a side effect, if x is a set and $x \in x$ we could form the set $\{x\}$ which is not well founded. So no set can be an element of itself. For instance, **Set** cannot be a set.

We should point out that classes of this kind, formed by an "impredicative" property of this type[1] can be shown (by contradiction) to be proper classes using the Axiom of Subset Selection. Though efficient at dealing with these classes, banishing contradiction is not the function of the Axiom of Foundation in set theory.

The earlier axioms asserted that we may speak, in our mathematical arguments, of reasonable or necessary objects from our vision of **Set**. This last axiom seems unnatural in that it *restricts* our purview directly.

The main purpose of this axiom, to my way of thinking, is related to the Zermelo hierarchy of sets which we will not discuss here.

[1] Impredicative properties are statements where a set is described with reference to another set of which it is a member.

(X) The Axiom of Choice

If J and X are sets and $A\colon J \to \mathbb{P}(X)$ is an indexed collection of nonempty sets then there is a function $f\colon J \to X$ such that $f(\beta) \in A_\beta$ for all $\beta \in J$. f is called a **choice function** for A.

Our vision of **Set** allows us to reach around in a known nonempty set B, without peeking, and grab one member. AC provides for this through a choice function f on $\{B\}$. Membership in B is the only known property of $f(B)$, which is quite different from the identification of a single element by some property of that element, possessed by it alone. But this "choice" axiom goes far beyond this (very) finite case. It calls for simultaneous selection of elements from *any set of nonempty sets*. The function values of a choice function can then be gathered to form a set whose elements have no linking property except through this mysterious function.

AC cannot be proven or refuted in ZF. It is something "extra" and in the past its acceptance in the formation of mathematical arguments was *quite* controversial, particularly when its use was not concealed by less-controversial but equivalent reformulations. To this day many mathematicians regard it as a victory if a proof can be concocted which avoids the use of this axiom on infinite sets of sets.

However AC cannot be dispensed with, without sacrificing lots of very compelling mathematics. Without going into the interesting details (and using words some of which are defined elsewhere) we list here just a few theorems of ZFC that cannot be proven in ZF.

Facts Dependent on or Equivalent to AC

- *(i)* An infinite set has a countable subset.
- *(ii)* A vector space always has a basis.
- *(iii)* The union of countably many countable sets is countable.
- *(iv)* Any pair of sets have comparable cardinality.
- *(v)* Any set has the same cardinality as an ordinal.
- *(vi)* Every set can be well-ordered.
- *(vii …)* Many more interesting facts.

(A set is well-ordered if it is linearly ordered and every subset contains a least member. Try to create a well-ordering of the real numbers.)

All of these—and *many* more—simply must go without the Axiom of Choice *or some other powerful axiom to replace it*.

Even when AC is not *necessary* in a proof, it often provides a means of deducing important facts using *very short* arguments.

But AC also implies that certain monstrous objects must exist such as non-measurable sets or bizarre decompositions of a three-dimensional sphere[2] whose (finite number of) pieces can be rotated and translated (with a finite number of steps) to create two complete spheres, each of the same volume as the original sphere. This kind of infestation disturbs people. You can't escape results like this in ZFC.

[2] The Banach-Tarski paradox, 1924

Some objections to AC were essentially esthetic in nature, boiling down to folks asking themselves "Do I want to spend my professional life debating the properties of objects whose existence can *only* be inferred by appeal to this axiom?"

Good question, though no one is forcing these folks to engage in that debate. Others focussed on the practical, expecting that AC would prove inconsistent with ZF, or conflict with other aspects of our vision of **Set** so central that they simply could not be abandoned.

A portion of this controversy subsided when Gödel proved that if ZFC contained an internal contradiction then an inconsistency could be found in ZF alone. If you use ZFC to prove some fact about a practical object you may "safely" assume it to be true. No doubter, who resolutely forgoes AC, will be able to provide any contradictory evidence unless ZF itself is inconsistent.

But there is, in fact, the potential for a presentation of a direct internal contradiction in ZF. It is at least conceivable (though regarded as unlikely) that one day using these axioms someone will prove a statement to be both true and false. That would be an exciting day to be a mathematician.

Gödel's proof that there are sensible statements in **any** system (not just ours) which could model integer arithmetic and which cannot be proven to be true or false from within that system implies that either the statement or its negation could be added as a new axiom without introducing new internal contradiction.

This put a definitive end to Hilbert's original goal for set theory which was, in part, to invent a provably consistent axiomatization of mathematics within which every meaningful mathematical statement could be decided.

From a Student of Set Theory

Might not a mouse
in iron grip of owl, review
his forest world
in wonder 'midst his fear?

And see his meadow home below,
and tree and stream as new,
and think
"how beautiful from here?"

Recommended Reading

Doxiadis, Apostolos and Papadimetriou, Christos
LOGICOMIX An Epic Search for Truth.
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Marcel Dekker, Inc, New York, 1999

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North-Holland Publishing Company, Amsterdam London, 1973