Algebras and Other Stories

Part 1

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Introduction

These notes accompany the first few of what will be a number of related talks in this colloquium series (not all on consecutive weeks of course!) on algebras and representation theory—critical topics in mathematical physics.

They comprise an outline of (some of) the things that were/will be said at the talks themselves.

Getting started, our main objects of concern are groups and vector spaces, structures listed in order of increasing complexity but, usually, introduced to students in the opposite order. We follow that tradition, introducing the main ideas of basic linear algebra here, and will follow with groups in the next set of talks.

Each of the objects we will consider in these talks is associated with functions that display or preserve defining features, and this will be true in later talks involving bilinear forms and inner products and algebras as well. Groups are associated with group-homomorphisms, vector spaces with linear transformations.

In addition, vector spaces often come equipped with a bilinear form—such as dot product or a Lorentz inner product or a symplectic form.

Certain linear transformations that preserve *whatever it is that a bilinear form measures about pairs of vectors* are important. These are called isometries and their presence is a recurring theme.

After our discussion of linear algebra we will develop a vocabulary and a small library of examples of groups. In later talks we will go further and discuss normed algebras, division algebras, the exponential and logarithm maps, Lie groups and algebras, Clifford algebras, spinors, Dirac matrices and more.

Most of the results we discuss are not particularly hard to prove, as these things go, but we will not be bashful about quoting and discussing important facts with references to the proofs and getting on with business.

Let's start!

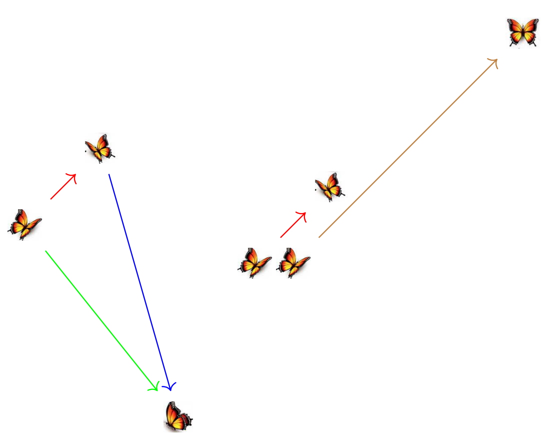
Linear Algebra Basics

Algebras are a vector space with a multiplication operation that satisfies certain properties, so first we establish a few facts and notation involving vector spaces generally.

The notion of a vector space comes from our natural perception of certain aspects of the world around us.

For instance the displacement of a butterfly in space, in front of our nose, can be visualized by an arrow floating in space with tail at the original position of the butterfly and nose at the final position.

Such displacements can be combined or stretched and yield operations on these visible arrows (at least potentially, in the metaverse) which we call vector addition and scalar multiplication.



Velocities and forces are also commonly represented as arrows in this way, and in these instances the magnitude (length) of the arrows have their own meaning attached to the particular domain of discourse.

At this point we allow the Mathematician's instinct for structure to take over and define an "abstract" representation of these common things, suitable (eventually) for doing calculations and producing numerical answers to natural questions.

A **real vector space**[1] is a set $\mathcal{V}$ together with two *binary operations*

$$\cdot \quad : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$$
$$\text{and} \quad + : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

which satisfy a short list of properties suggested by the behavior of the arrows.

The details are explored early in any linear algebra class but include such properties as, for instance,

$$r \cdot (v + w) = r \cdot v + r \cdot w \quad \text{and} \quad v + w = w + v$$
$$\text{and} \quad (rs) \cdot v = r \cdot (s \cdot v) \quad \text{and} \quad 1 \cdot v = v.$$

for real r and s and vectors v and w.

[1] Vector spaces can be defined over any field, such as the complex numbers or finite fields. It is even possible to create "vector-space-like" objects called modules whose scalars are drawn from other algebraic objects called rings, such as the quaternions. We work with real vector spaces for now.

There are numerous examples of real vector spaces studied and used throughout mathematics, engineering and physics.

For instance $\mathbb{R}^n$ which consists of the ordered n-tuples

$$v = (v^1, v^2, \ldots, v^n)$$

of real numbers is a real vector space for each positive integer n.

The cases $n = 2$ and $n = 3$ are often called (i.e. "identified with") the plane and space, while in some contexts $\mathbb{R}^4$ might be identified with space-time.

The set $\mathbb{M}_{n \times m}$ of $n \times m$ matrices

$$M = (m_{i,j})$$

with real entries and the usual addition of matrices and scalar multiplication is a real vector space, always visualized as a rectangular array of numbers with n rows and m columns.

The set of all real-valued functions on the unit interval with the pointwise operations given for real c and functions f, g by

$$(cf)(x) = cf(x) \quad \text{and} \quad (f + g)(x) = f(x) + g(x) \quad \text{for all } x \in [0, 1]$$

is a vector space too, apparently different from the previous examples.

However those earlier vector spaces really aren't much different from this example if you think of them "properly."

A member of $\mathbb{R}^3$ is an ordered triple of real numbers and that is **nothing more or less than a function from $\{1, 2, 3\}$ to $\mathbb{R}$.**

A member of $\mathbb{M}_{n \times m}$ is, similarly, just a real valued function defined on the nm "location inside" ordered pairs

$$\{ (1,1), (2,1), \ldots, (n,1), (2,1), (2,2), \ldots, (n,m) \}.$$

The way we visualize it is just a stacking of these values in a way that is convenient for us.

In fact every vector space you will see in any common application "is," or can be construed as, a subset of a function space with pointwise operations consisting of all functions

$$f : D \to \mathcal{V}$$

where D is an index set and $\mathcal{V}$ is a previously defined vector space which may be chosen to be $\mathbb{R}$ if you prefer.

The reason this is a highly convenient way of thinking about this is that it is easy to show that the ten or so properties required of scalar multiplication and vector addition hold for the set $\mathcal{V}^D$ of all these functions with pointwise operations for any D and any $\mathcal{V}$, once and for all, in your beginning linear algebra class.

So if you have any subset $\mathcal{W}$ of $\mathcal{V}^D$ and want to verify that it, independently, is a vector space you need only demonstrate closure of the two operations in $\mathcal{W}$: that is to say, whenever r is real and $v, w \in \mathcal{W}$ we have

$$rw \in \mathcal{W} \quad \text{and} \quad v + w \in \mathcal{W}.$$

The other properties required of these operations are "inherited" from $\mathcal{V}^D$ and so require no verification.

For instance if you want to know if the set of *differentiable* real functions on $[0, 1]$ is a vector space you merely observe that the sum of two differentiable functions is differentiable and a scalar multiple of a differentiable function is differentiable and you may draw that conclusion, since the set $\mathbb{R}^{[0,1]}$ of *all* real valued functions defined on $[0, 1]$ is definitely a vector space.

Functions from one vector space to another are called linear transformations if they "preserve" the vector space structure of the domain space. Specifically, $T\colon \mathcal{V} \to \mathcal{W}$ is called a linear transformation if

$$T(rv + w) = rT(v) + T(w) \quad \text{for all real } r \text{ and all } v, w \in \mathcal{V}.$$

Linear transformation T is called an isomorphism if it is invertible, an endomorphism if $\mathcal{V}=\mathcal{W}$ and an automorphism if it is an invertible endomorphism.

A linear transformation is the equivalent, in higher dimensions, of a direct variation. It's graph "goes through" the origin and restricted to any "line" through the origin it actually *is* a direct variation.

If v is a vector in the domain and $z(t) = tv$ for real t is the line through the origin with "velocity vector" v then $T(tv) = tT(v)$. The vector $T(v)$ plays the role of a direct variation constant, in the higher dimensional case a constant *vector*.

Every vector space has a basis: that is, a linearly independent spanning set.

Subset B of vector space $\mathcal{V}$ is a basis if and only if for each $v \in \mathcal{V}$ there is a finite set of scalars $v^1, \ldots, v^n$ and a collection $b_1, \ldots, b_n$ of *distinct* members of B for which

$$v = v^1 b_1 + v^2 b_2 + \cdots + v^n b_n$$

and with the key property that this representation is unique except for order of terms.

If $\mathcal{V}$ is vector space with basis B and $T\colon \mathcal{V} \to \mathcal{W}$ is linear then T is completely determined by what it does to members of B.

If there is a finite basis for $\mathcal{V}$ then $\mathcal{V}$ is called finite dimensional and **the number of vectors in a basis for a finite dimensional vector space does not vary.** This number is called the dimension of the space.

In practice it is convenient to give members of a finite basis an ordering and we will presume this, with another ordering of the same vectors regarded as a different ordered basis.

An example of a basis for the familiar vector space $\mathbb{R}^n$ is the set

$$E_n = \{\, e_1,\ e_2,\ \ldots,\ e_n \,\}$$

where each e_i has a 1 in the i th spot and zeroes elsewhere. For instance the basis E_3 of $\mathbb{R}^3$ is

$$\{\, (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1) \,\}$$

with basis vectors ordered as listed.

A finite basis for $\mathcal{V}$ let's us "represent" members of $\mathcal{V}$, and linear transformations between two such finite dimensional spaces, as matrices.

This is super-convenient: an abstract vector space is just that—abstract. Presumably $\mathbb{R}^n$ is (more) familiar.

And matrices allow us to perform calculations (usually with hardware assistance) in a uniform manner, not dependent on the vagueries of the specific vector spaces involved.

Facts about these matrices are then interpreted in the original setting.

We define the coordinate map for ordered basis B containing n basis vectors, denoted $[\cdot]_B : \mathcal{V} \to \mathbb{R}^n$, to be the isomorphism that sends

$$v = v^1 b_1 + v^2 b_2 + \cdots + v^n b_n$$

$$\text{to} \qquad (v^1, v^2, \ldots, v^n) = v^1 e_1 + v^2 e_2 + \cdots + v^n e_n \in \mathbb{R}^n.$$

Note that in particular we have $[b_i]_B = e_i$ for each i.

We use the following convention:
Members of $\mathbb{R}^n$ are columns, i.e. $n \times 1$ matrices. Displaying these as rows is convenient as above for purely typographical reasons, to avoid taking up too much space on the page, but when doing calculations with other matrices they "are" columns.

Of course someone else might have chosen ordered basis C for $\mathcal{V}$ and created coordinates for members of $\mathcal{V}$ using $[\cdot]_C$ and we need to see how these coordinates are related.

Each $c_i \in C$ has coordinates $[c_i]_B$ in basis B. Create $n \times n$ matrix

$$P_{B \leftarrow C} = \left([c_1]_B \ \ldots [c_n]_B \right)$$

called the matrix of transition from basis C to basis B.

Columns of $P_{B \leftarrow C}$ are the B-coordinates of the C-basis vectors.

A calculation (check on members of the basis C) shows that

$$[v]_B = P_{B \leftarrow C}\, [v]_C \qquad \text{for any } \ v \in \mathcal{V}.$$

$P_{B \leftarrow C}$ is the "translator" from language C to language B.

And $P_{B \leftarrow C}$ is invertible with $P_{B \leftarrow C}^{-1} = P_{C \leftarrow B}$.

The Matrix of a Linear Transformation

If $f : \mathcal{V} \to \mathcal{W}$ is a linear transformation between finite dimensional $\mathcal{V}$ with basis B of dimension n and finite dimensional $\mathcal{W}$ with basis C of dimension m define the $m \times n$ matrix $f_{C \leftarrow B}$ to be

$$f_{C \leftarrow B} = ([f(b_1)]_C \ [f(b_2)]_C \cdots [f(b_n)]_C).$$

Verifying on the basis vectors in B we find that for every $v \in \mathcal{V}$

$$[f(v)]_C = f_{C \leftarrow B}\, [v]_B$$

so the coordinates of $f(v)$ in basis C can be calculated by left-multiplying the matrix $f_{C \leftarrow B}$ on the B-coordinates of v.

For linear transformation $f: \mathcal{V} \to \mathcal{W}$ between finite dimensional $\mathcal{V}$ with basis B of dimension n and finite dimensional $\mathcal{W}$ with basis C of dimension m the individual entries of

$$A = (a_{i,j}) = f_{C \leftarrow B} = (\, [f(b_1)]_C \; [f(b_2)]_C \cdots [f(b_n)]_C \,)$$

have a meaning in terms of direct variation of *coordinates* in these bases.

The entry $a_{i,j}$ is the direct variation constant of the changes induced by f on the ith range coordinate values with respect to changes in the jth domain coordinate values. Since there are n domain coordinate variables and m range coordinate variable there must be (and are) mn of these variation constants.

Purely in terms of coordinates in these bases we have

$$y = Ax \qquad \text{and} \qquad \frac{\partial y^i}{\partial x^j} = a_{i,j}.$$

If D is another basis for $\mathcal{W}$ and A another basis for $\mathcal{V}$ we have

$$f_{D \leftarrow A}\,[v]_A = [f(v)]_D = P_{D \leftarrow C}\,[f(v)]_C = P_{D \leftarrow C} f_{C \leftarrow B}\,[v]_B$$
$$= P_{D \leftarrow C} f_{C \leftarrow B}\, P_{B \leftarrow A}\,[v]_A.$$

This means the matrix $f_{D \leftarrow A}$ can be calculated as

$$f_{D \leftarrow A} = P_{D \leftarrow C} f_{C \leftarrow B}\, P_{B \leftarrow A}.$$

We do calculations in bases where the work is easiest and very often these are not coordinates which are natural or in which the vector spaces or transformations are originally specified. Then transfer back to the original context via the (inverse of the) coordinate map.

Any finite dimensional space is isomorphic (via the coordinate map) to $\mathbb{R}^n$. **There is, essentially, only ONE VECTOR SPACE of each given dimension!!! And we may use matrices to describe ALL LINEAR TRANSFORMATIONS between them!!!**

Affine Maps

Linear transformations are the equivalent of direct variation functions in the context of vector spaces. Affine maps, on the other hand, are the equivalent of those functions from elementary algebra whose graphs are straight lines—whether they pass through the origin or not.

Specifically, a function $g: \mathcal{V} \to \mathcal{W}$ is called affine if there is a vector $w \in \mathcal{W}$ and linear transformation $f: \mathcal{V} \to \mathcal{W}$ with

$$g = w + f.$$

In later examples (of groups) we will be particularly interested in the case of $\mathcal{W} = \mathcal{V}$ where f is invertible.

Euclidean Isometries

A function $T: \mathbb{R}^n \to \mathbb{R}^n$ is called a Euclidean isometry if $(T(v) - T(0)) \cdot (T(w) - T(0)) = v \cdot w$ for all $v, w \in \mathbb{R}^n$. So T preserves the notion of Euclidean distance in $\mathbb{R}^n$.

Any rotation about an axis is a Euclidean isometry and so is inversion: the map $T(x) = -x$.

The Mazur-Ulam Theorem tells us that *any* Euclidean isometry is an affine map: it is linear except for translation of the origin. The Cartan-Dieudonné Theorem says that any nontrivial linear isometry on $\mathbb{R}^n$ is the composition of no more than n consecutive reflections, and each pair of reflections forms a rotation about some axis. The case of dimension 2 will be featured in some of our examples of groups.

Projection and Reflection

For nonzero vector $v \in \mathbb{R}^n$ the function

$$\mathrm{Proj}_v \colon \mathbb{R}^n \to \mathbb{R}^n \text{ given by } \mathrm{Proj}_v(x) = \frac{v \cdot x}{v \cdot v} v$$

is called a projection. The map $\mathrm{CoProj}_v \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\mathrm{CoProj}_v(x) = x - \mathrm{Proj}_v(x)$$

is called a coprojection.

$\mathrm{CoProj}_v(x)$ is always orthogonal to $\mathrm{Proj}_v(x)$ (i.e. their dot product is 0) and for $a = \mathrm{CoProj}_v(x)$ and $b = \mathrm{Proj}_v(x)$

$$x \cdot x = a \cdot a + b \cdot b \qquad \text{(the Pythagorean Theorem).}$$

A calculation shows that reflection, defined by

$$\mathrm{Refl}_v(x) = x - 2\,\mathrm{Proj}_v(x)$$

is a Euclidean isometry. It is the reflection across the (hyper) plane perpendicular to v.

Functionals and the Dual

A linear functional on a vector space $\mathcal{W}$ is a linear function $f \colon \mathcal{W} \to \mathbb{R}$. The set of all of these functionals is itself a vector space denoted $\mathcal{W}^*$ and called the dual of $\mathcal{W}$.

If B is a basis for $\mathcal{W}$ define for each i in the index set for B the functional $\boldsymbol{b}^i \colon \mathcal{W} \to \mathbb{R}$ by specifying its values on the basis members from B.

Specifically, $\boldsymbol{b}^i(\boldsymbol{b}_j) = \delta_{i,j}$ where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

(Note: δ is called the Kronecker delta function and the matrix $(\delta_{i,j}) = I_n$, the $n \times n$ identity matrix.)

$B^* = \{\, \boldsymbol{b}^1, \ldots, \boldsymbol{b}^n \,\}$ is a linearly independent set in $\mathcal{W}^*$ and spans $\mathcal{W}^*$ so is a basis. The dimension of $\mathcal{W}^*$ is the same as $\mathcal{W}$. In a basis functionals are usually represented as row matrices, not columns. So if row $\boldsymbol{p}$ represents functional $\boldsymbol{\sigma}$ and column $\boldsymbol{q}$ represent vector $\boldsymbol{x}$ then $\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{pq} \in \mathbb{R}$.

Note that $\boldsymbol{pq} = \boldsymbol{p}P_{B\leftarrow C}P_{C\leftarrow B}\boldsymbol{q} = (\boldsymbol{p}P_{B\leftarrow C})(P_{C\leftarrow B}\boldsymbol{q})$ so coordinates of functions change in a different way (right multiplication by $P_{B\leftarrow C}$ rather than left-multiplication by $P_{C\leftarrow B}$) when changing to a new basis. Functionals are called covariant and vectors contravariant to describe this difference[2].

$\mathcal{W}^{**}$ is "essentially the same as" (i.e. naturally isomorphic to) finite-dimensional $\mathcal{W}$. The isomorphism is through the evaluation map: if $f \in \mathcal{W}^{**}$ there is a unique $\boldsymbol{x} \in \mathcal{W}$ for which $f(\boldsymbol{\sigma}) = \boldsymbol{\sigma}(\boldsymbol{x})$ for every $\boldsymbol{\sigma} \in \mathcal{W}^*$. So f is "identified with" this $\boldsymbol{x}$.

[2]Commonly represented as vectors, electric fields are best thought of as functionals and exhibit this covariant behavior under coordinate changes.

In a certain sense if you understand linear functionals on a vector space you have gone a long way toward understanding *any* linear transformation on that space. Suppose $f \colon \mathcal{W} \to \mathcal{V}$ is a linear transformation and C is a finite basis for $\mathcal{V}$.

Then

$$f(w) = g^1(w)c_1 + \cdots g^i(w)c_i + \cdots g^m(w)c_m$$

for linear functionals $g^i \colon \mathcal{W} \to \mathbb{R}$, called the coordinate functionals for f in basis C.

If A is a finite basis for n-dimensional $\mathcal{W}$ the ith row of the $m \times n$ matrix $f_{C\leftarrow A}$ is the row matrix $g^i_{E_1\leftarrow A}$ for the coordinate functional g^i.

In any basis f is determined by these m linear functionals.

Product of Two Vector Spaces

If $\mathcal{V}$ and $\mathcal{W}$ are two vector spaces their product $\mathcal{V} \times \mathcal{W}$ is the set of ordered pairs $(\boldsymbol{v}, \boldsymbol{w})$ for all $\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{W}$ with scalar multiplication and vector addition defined by

$$a(\boldsymbol{v}, \boldsymbol{w}) = (a\boldsymbol{v}, a\boldsymbol{w}) \quad \text{and} \quad (\boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{v} + \boldsymbol{p}, \boldsymbol{w} + \boldsymbol{q}).$$

If A is a finite basis for $\mathcal{V}$ and B is a finite basis for $\mathcal{W}$ then $\{ (0, \boldsymbol{q}), (\boldsymbol{p}, 0) \mid \boldsymbol{p} \in A \text{ and } \boldsymbol{q} \in B \}$ so the dimension of a product vector space is the sum of the dimensions of the factor spaces. Example: $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}$.

A function $g\colon \mathcal{V} \times \mathcal{W}$ is called a bilinear form if both $g(\boldsymbol{p}, \cdot)$ and $g(\cdot, \boldsymbol{q})$ are linear for each $\boldsymbol{p} \in \mathcal{V}$ and $\boldsymbol{q} \in \mathcal{W}$.

Unless it is the zero function a bilinear form *is not* linear. We will encounter these often.

Tensor Product of Two Vector Spaces

If $\mathcal{V}$ and $\mathcal{W}$ are two vector spaces their tensor product $\mathcal{V} \otimes \mathcal{W}$ is the set of bilinear functions on $\mathcal{V}^* \times \mathcal{W}^*$.

This set is made into a vector space with the usual pointwise operations, and its members are called tensors.

If $\boldsymbol{q} \in \mathcal{V}$ and $\boldsymbol{p} \in \mathcal{W}$ the element $\boldsymbol{q} \otimes \boldsymbol{p}$ defined by $(\boldsymbol{q} \otimes \boldsymbol{p})((\boldsymbol{\sigma}, \boldsymbol{\tau})) = \boldsymbol{\sigma}(\boldsymbol{q})\boldsymbol{\tau}(\boldsymbol{p})$ is called a simple tensor.

If A is a finite basis of $\mathcal{V}$ and B is a finite basis of $\mathcal{W}$ the set of simple tensors of the form $\boldsymbol{a} \otimes \boldsymbol{b}$ for $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$ span $\mathcal{V} \otimes \mathcal{W}$ and form a linearly independent set and so constitute a basis.

So the dimension of a tensor product vector space is the product of the dimensions of the factor spaces.