THE SHAPE OF A HANGING ROPE
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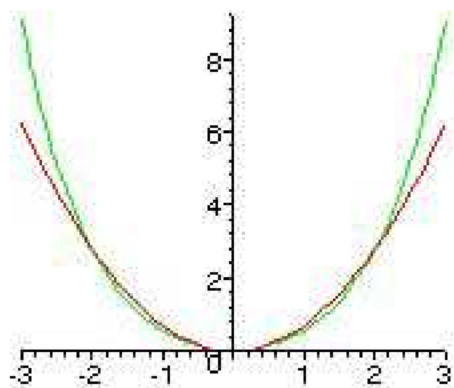
1. The Shape of a Hanging Rope: Qualitative Analysis

A heavy but "limp" rope is strung between two anchor pillars. As everyone knows, and as shown below, it will hang in a curved shape that looks a lot like a parabola.

Though it is not obvious from a rough inspection, the shape of a rope like this is not a parabola. If you cast a shadow onto a piece of graph paper the shadow point coordinates will not satisfy any second degree equation.

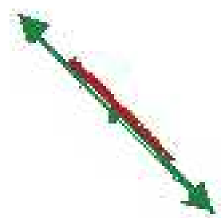
This is a golden opportunity to do an **experiment**! Make a hanging rope from a couple of feet of string. Shine a light on the string from as far away as possible, put the graph paper close to the string and put the origin at the shadow of the bottom of the arc of the string. Find the coordinates (X_0, Y_0) of a shadow-point well away from the origin. If you find a so that $Y_0 = aX_0^2$ for this point, you will find that the graph of $Y = aX^2$ does not match the shadow very well, particularly away from the center.



The shape of a rope like this is called a **catenary**, and is obviously of great importance to practical folk like engineers, some of whom spend much of their professional lives supporting and hanging structures in various ways.

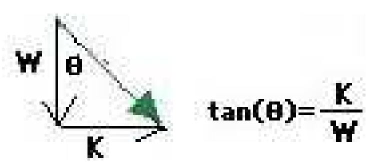
From our experience we know that a rope set up this way will start out pretty steep near where it meets the support pillar and gradually flatten until it is horizontal in the middle. But why **is** the shape as it is? What makes the angle change as you go along the rope? How can you predict what the shape will be from measuring qualities of the rope? Even if the shape had turned out to be a parabola, that fact alone would be unsatisfying without an explanation of **why**.

The complete story requires quite a bit of calculus, but we can make some serious progress by considering forces. Let's examine a small piece of the hanging rope midway between post and center.



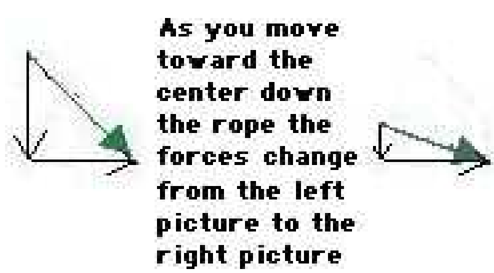
Since there is no motion the force vectors acting at each point along the rope add to the zero vector. Since the rope is floppy, these force vectors lie along the same direction as the rope at this spot.

Let's consider the force which points down and to the right. It can be thought of as the sum of vectors: a horizontally pointing vector added to a vector that points straight down.



The rightward pointing part, of magnitude K, is caused, ultimately, by the tug on the rope by the far wall. The downward pointing part, of magnitude W, is caused by the weight of the rope which hangs beneath the spot we are examining, from there to the center (each half of the rope supports half the weight.)

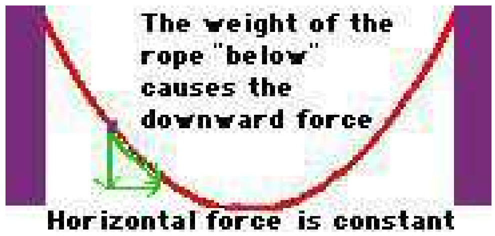
We are speculating here about why the angle of the rope changes: The horizontal component of the pull along the rope is always the same but the "weight below" is dropping - the vertical part of the force vector is getting shorter - as you move toward the middle.



There are several (possibly linked) claims about the way floppy ropes behave in the paragraphs above, and these claims constitute a "theory" about such ropes.

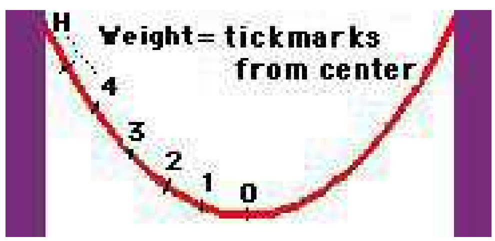
- The force is parallel to the rope at each point

- The horizontal part of the force is constant
- The vertical part is the weight of the rope from the "point" to the center.



We can make various measurements to try to put some numbers on the table related to these forces.

First we measure the weight of the rope and tie little black threads around the rope so that the weight of the rope between thread pieces is, say, one pound or whatever force units you find handy. We count from the center of the rope. Put different color thread markers at fractional pounds along the way so you can estimate the weight of pieces of intermediate lengths. It is not necessary that the rope be of uniform density, though most ropes will be. So if H is the weight of the rope from the bottom to the edge there will be H tickmarks up to the edge.

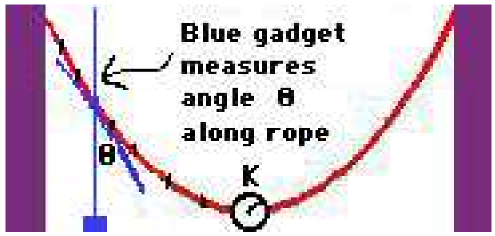


Next we need to measure the horizontal component of the force. The easiest way to do that might be to insert a (small) spring-type force measuring device into the rope at the center where there is no vertical force - all the tension in the rope comes from this horizontal force at the center, so we will be able to read off this force from the gauge.

At this point we have the means to rather thoroughly test our theory. We set up an angle-measuring device - the blue gadget in the picture found below. We then change the length of the rope and/or the pillar separation distance and the position of the gadget along the rope. With a few dozen measurements we can pretty thoroughly confirm or refute our speculations about floppy ropes. If confirmed, we could then predict the angle of the rope at each distance (that is, weight) measured along the rope from the center:

$$\theta = \tan^{-1}\left(\frac{K}{W}\right)$$

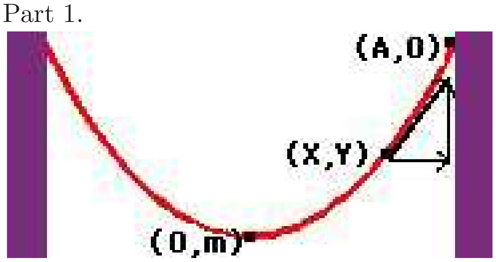
where W is the weight of the rope from "the spot" to the middle and K is the measured tension on the rope at the middle.



- How do you think the shape would change if everything else remained the same but you simply used a rope that was denser by some constant factor? How would K change in this case?
- How would K change (qualitatively) if you move the pillars closer together or farther apart?
- In case the rope has uniform density, how should the shape change if the separation of the pillars and the rope length were changed by the same multiple?

2. The Differential Equation

A heavy but "limp" rope is strung between two anchor pillars as shown below and as discussed in Part 1.



Using vectors and basic Trigonometry we found a way to predict the angle that the rope would make at various places by making some measurements involving the physical properties of the rope ... but we did not actually produce a formula for the shape. We will make an additional assumption about the density of the rope and, using calculus, go a little farther and produce such a formula. First we establish a coordinate system and name various parameters of the hanging rope.

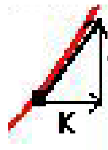
We let the origin be at the height of the anchor points and in the center measured from left to right. The distance between the two anchor points is $2A\ meters$ and the low spot on the rope has Y coordinate m. We let δ denote the mass in kilograms of the rope per unit length (meters). So the units of δ are $kilograms/meter$. δ could be a a function of X. Suppose the rope has total length $2L$. We will introduce the notation $u = Y'$ to eliminate some visual clutter.

In integral calculus we learn that the arclength along the curve Y from 0 to X is given by $\int_0^X \sqrt{1+u^2}\, dt$ and the mass of the rope from 0 to X is $\int_0^X \delta\sqrt{1+u^2}\, dt$. So the force due to gravity of this mass is $\int_0^X g\delta\sqrt{1+u^2}\, dt\ newtons$ where $g \approx 9.8\, meter/sec^2$.

We made some speculations about the nature of floppy ropes hanging like this which we discussed and, perhaps, confirmed in Part 1:

- The horizontal part of the tension is constant, of magnitude K.

- The vertical part of the tension is the weight of the rope from the "point" to the center. We will call this magnitude W.
- The tension in the rope is parallel to the rope at each point so $u = \frac{W}{K}$.



With those definitions and assumptions the magnitude of the tension on the rope is $\sqrt{W^2 + K^2}$ where W varies from place to place but K does not.

Differentiating

$$W = Ku = \int_0^X g\delta\sqrt{1+u^2}\ dt$$

yields

$$Ku' = g\delta\sqrt{1+u^2} \quad \text{with initial condition } u(0) = 0.$$

3. Constant Density Series Solution

We now make an additional assumption: **that our rope has constant density everywhere along its length.**

We can differentiate both sides of the differential equation above to produce

$$u'' = \frac{g\delta}{K}\frac{1}{2}(1+u^2)^{-\frac{1}{2}}\,2\,u\,u'.$$

Replacing u' by $\frac{g\delta}{K}\sqrt{1+u^2}$ on the right, we get the differential equation

$$u'' = \frac{g^2\delta^2}{K^2}u \quad \text{with initial condition } u(0) = 0,\ u'(0) = \frac{g\delta}{K}.$$

This form is more amenable to series solution. Under the assumption that there *is* a solution of the form

$$u = \sum_{n=0}^{\infty} a_n X^n \quad \text{where } a_0 = 0 \text{ and } a_1 = \frac{g\delta}{K}$$

the differential equation implies that

$$0 = \sum_{n=0}^{\infty}\left(\frac{g^2\delta^2}{K^2}\,a_n - (n+2)(n+1)a_{n+2}\right)X^n.$$

Equating each coefficient to 0 in turn, we find that $a_n = 0$ for all even n, while $a_n = \frac{g\delta^n}{n!\,K^n}$ for odd n. So

$$u = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{g\delta}{K}\right)^{2n+1}X^{2n+1} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{g\delta\,X}{K}\right)^{2n+1}.$$

Those who are familiar with Taylor Series might recognize this as the series for

$$\sinh\left(\frac{g\delta}{K}X\right) = \frac{\mathrm{e}^{\frac{g\delta}{K}X} - \mathrm{e}^{\frac{g\delta}{K}X}}{2}.$$

Integrating this series term-by-term, we find that for some constant C

$$Y = C + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\left(\frac{g\delta}{K}\right)^{2n+1} X^{2n+2} = C + \frac{K}{g\delta}\sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\left(\frac{g\delta\, X}{K}\right)^{2n+2}$$

$$= C + \frac{K}{g\delta}\sum_{n=1}^{\infty} \frac{1}{(2n)!}\left(\frac{g\delta\, X}{K}\right)^{2n} = C - \frac{K}{g\delta} + \frac{K}{\delta}\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{g\delta\, X}{K}\right)^{2n}.$$

The last series on the right can be identified with

$$\cosh\left(\frac{g\delta}{K}X\right) = \frac{\mathrm{e}^{\frac{g\delta}{K}\,X} + \mathrm{e}^{\frac{g\delta}{K}\,X}}{2}.$$

In view of the fact that $Y(A) = 0$ we have

$$Y = \frac{K}{g\delta}\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{g\delta\, X}{K}\right)^{2n} - \frac{K}{g\delta}\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{g\delta\, A}{K}\right)^{2n}.$$

4. Constant Density Solution By Integration

In this section we also assume **that our rope has constant density everywhere along its length,** which yields the differential equation

$$\frac{u'}{\sqrt{1+u^2}} = \frac{g\delta}{K} \quad \text{with initial condition } u(0) = 0.$$

This can be integrated directly to produce

$$\sinh^{-1}(u) = \frac{g\delta}{K}X \quad \text{so} \quad u = Y' = \sinh\left(\frac{g\delta}{K}X\right).$$

Integrating the right hand equation, we have

$$Y = \frac{K}{g\delta}\cosh\left(\frac{g\delta}{K}X\right) + P$$

for a constant P. We organized things initially with $Y(A) = 0$ and so

$$P = -\frac{K}{g\delta}\cosh\left(\frac{g\delta}{K}A\right)$$

which gives

$$Y = \frac{K}{g\delta}\cosh\left(\frac{g\delta}{K}X\right) - \frac{K}{g\delta}\cosh\left(\frac{g\delta}{K}A\right).$$

Another important parameter to consider is the half-length of the rope, L.

$$L = \int_0^A \sqrt{1+\sinh^2\left(\frac{g\delta}{K}X\right)}\,dX = \int_0^A \cosh\left(\frac{g\delta}{K}X\right)dX = \frac{K}{g\delta}\sinh\left(\frac{g\delta}{K}A\right).$$

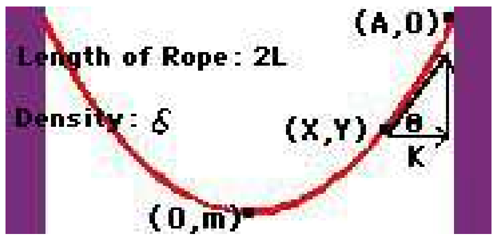
Finally, two visible parameters would be the "low spot"

$$m = Y(0) = \frac{K}{g\delta} - \frac{K}{g\delta}\cosh\left(\frac{g\delta}{K}A\right).$$

and the angle θ_X of the rope at X:

$$\tan(\theta_X) = Y'(X) = \sinh\left(\frac{g\delta}{K}X\right).$$

Note that $\tan(\theta_A)$ should equal $\dfrac{\text{half weight of the rope}}{K}$. Since the half weight of the rope is $g\delta L$ the representation of L above verifies that $\tan(\theta_A) = \frac{g\delta L}{K}$ should be $\sinh\left(\frac{g\delta}{K}A\right)$.



To reiterate, we have:

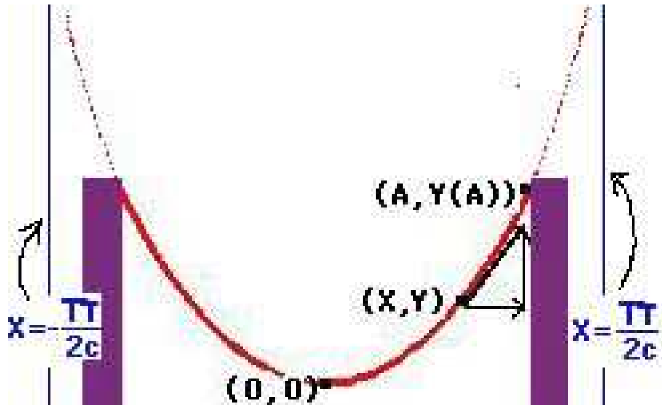
- $Y = \dfrac{K}{g\delta}\cosh\left(\dfrac{g\delta}{K}X\right) - \dfrac{K}{g\delta}\cosh\left(\dfrac{g\delta}{K}A\right)$ and so $m = \dfrac{K}{g\delta} - \dfrac{K}{g\delta}\cosh\left(\dfrac{g\delta}{K}A\right)$.
- $L = \dfrac{K}{g\delta}\sinh\left(\dfrac{g\delta}{K}A\right)$.
- $\tan(\theta_X) = \sinh\left(\dfrac{g\delta}{K}X\right)$ and specifically, $\dfrac{g\delta L}{K} = \sinh\left(\dfrac{g\delta}{K}A\right)$.

There are still two (related) items that could be addressed here, using only the most modest accouterments from mechanics, and also one topic using the series representation for $\sinh(X)$.

- If δ changes the graph stays the same. This will follow if $\frac{\delta}{K}$ is constant for fixed values of A and L, which seems obvious. Can you prove it?
- If the $Y = g(X)$ is the solution we just calculated and we replace L by sL and A by sA for a positive scaling constant s and let $Y = f(X)$ be the new solution then $f(X) = sg\left(\frac{X}{s}\right)$. Essentially, this says that the picture "scales" so that if you increase L and A proportionately the graph is "similar" to the old graph. This will follow if the magnitude of new horizontal component of the tension is sK which, once again, seems pretty clear. Can you prove it?
- It would be useful to people building structures to be able to predict K from L, δ and A. However K cannot be solved for explicitly. Here is a case where the first few terms of the Taylor series for $\sinh(X)$ with error bounds will be useful.

5. A Rope of Variable Density

In this section we use notation and conventions similar to those used in Section 2. In this section we will think about what happens when we adjust the density as we go along the rope proportionately with increasing magnitude of tension.



You will recall that we had arrived at the equation

$$\frac{u'}{\sqrt{1+u^2}} = \frac{g\delta}{K}$$

where $Y'(X) = u$ and then made the constant δ assumption. Here we will presume instead that density changes with position and

$$\delta_X = c\sqrt{W_X^2 + K^2}.$$

where W_X is the weight of the rope from the center to a given X value.

$\sqrt{{W_X}^2 + K^2}$ is the magnitude of the tension on the rope at each X and we will use a stronger (and proportionately heavier) rope where that tension is higher. This is a natural assumption if you are trying to connect the pillars while being parsimonious with rope material. The rope will be thinnest at the bottom where it has minimum weight per unit length $\delta_0 = cK$ and get thicker as needed.

The constant $c = \dfrac{\delta_0}{K}$ $(kg/(newton\ meter)$ is the mass per unit distance of a rope thick enough to withstand K newtons of tension and reflects the strength of the material. We might prefer a smaller c so long as the rope doesn't snap, with an adequate safety margin. For example, a uniform cable of a certain very strong material of density 1 $kg/meter$ might plausibly support 10^6 newtons, so $c = 10^{-6}\ kg/(newton\,meter)$ for this rope.

In this section we presume that the origin is at the middle of the rope as indicated in the picture with support pillars at $X = \pm A$.

From Part 2 we have

$$W_X' = g\delta_X\sqrt{1+u^2} \qquad and \qquad W_X = Ku.$$

These facts yield

$$Ku' = g\delta_X\sqrt{1+u^2}.$$

We combine this with our density assumption to find

$$Ku' = gc\sqrt{{W_X}^2 + K^2}\sqrt{1+u^2} = gc\sqrt{K^2u^2 + K^2}\sqrt{1+u^2} = gcK(1+u^2).$$

Separating variables gives

$$\frac{u'}{u^2+1} = gc \qquad \text{and so} \qquad \tan^{-1}(u) = gcX + E.$$

Using the fact that $u(0) = 0$ we find that $E = 0$ and so

$$u = Y'(X) = \tan(gcX).$$

This means that θ_X, the angle between the rope and the X axis, is gcX. A rope built this way will hang in a shape whose angle increases linearly with the X coordinate.

If gcX were ever $\pm\frac{\pi}{2}$ the rope would hang vertically at that X, which would imply unbounded rope weight, a non-phsical situation. So we must have $-\frac{\pi}{2gc} < X < \frac{\pi}{2gc}$ and, in particular, $A < \frac{\pi}{2gc}$. This says something about the strength of the material of the rope if we want to build it this way: We must use a stronger material (which is what a smaller c implies) if we want to build it wider.

Integrating once more gives

$$Y(X) = \frac{\ln(\sec(gcX))}{gc} + G \quad \text{for a constant } G.$$

We organized the rope so it passed through $(0,0)$ so $G = 0$ and

$$Y(X) = \frac{1}{gc}\ln(\sec(gcX)).$$

The height at the pillar is $Y(A) = \dfrac{1}{gc}\ln(\sec(gcA))$.

The length of the rope from the center to positive X is

$$\begin{aligned} L_X &= \int_0^X \sqrt{1+\tan^2(gct)}\; dt = \int_0^X \sec(gct)\; dt \\ &= \frac{1}{gc}\; \ln(\sec(gct)+\tan(gct))\,\Big|_0^X = \frac{1}{gc}\; \ln(\sec(gcX)+\tan(gcX)). \end{aligned}$$

In particular, the half-length of the rope (center to pillar) is $L_A = \frac{1}{gc}\ln(\sec(gcA) + \tan(gcA))$. So a product gcA that is close to $\dfrac{\pi}{2}$ corresponds to a **very** long rope.

The natural parameters here are not the half-rope length L and a choice of pillar distance $A < L$ as in the Part 1 and 2. Here the natural parameters are gc and the range of A values allowed by gc. If you want to link two points $2A$ apart using rope material with strength parameter c built this way the length is determined by those conditions, not specified at our convenience. The possibility of building any rope like this at all is determined by a combination of rope strength and pillar separation: gcA must be less than $\frac{\pi}{2}$.

The weight W_X of a rope built this way from the center to positive X is

$$W_X = Ku = K\tan(gcX).$$

Once again, if gcX is near $\dfrac{\pi}{2}$ we have a huge weight.

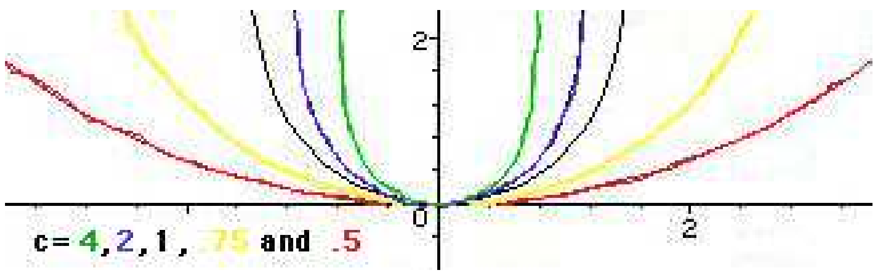
The function δ_X is

$$\delta_X = c\sqrt{{W_X}^2 + K^2} = cK\sec(gcX) = \delta_0\sec(gcX)$$

which becomes unbounded as gcX nears $\dfrac{\pi}{2}$.

Putting this all together we get:

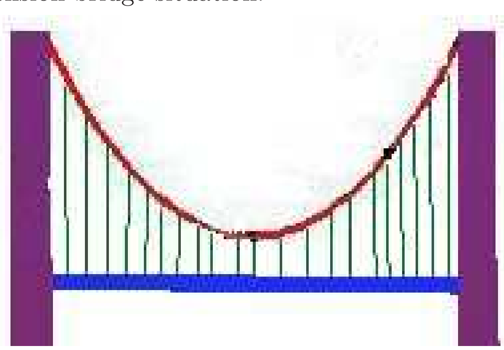
- $Y'(X) = \tan(gcX)$ so the angle of the rope at X is $\theta_X = gcX = \dfrac{g\delta_0}{K}X$.
- $Y(X) = \dfrac{1}{gc}\ln(\sec(gcX))$ and in particular $Y(A) = \dfrac{1}{gc}\ln(\sec(gcA))$.
- The rope length from center is $L_X = \dfrac{1}{gc}\ \ln(\sec(gcX) + \tan(gcX))$.
- The rope weight from center to X is $W_X = K\tan(gcX) = \dfrac{\delta_0}{gc}\tan(gcX)$.
- The required density of the rope at X is $\delta_X = \delta_0\sec(gcX)$.



6. Suspension Bridge and Cable-Supporting-Cable

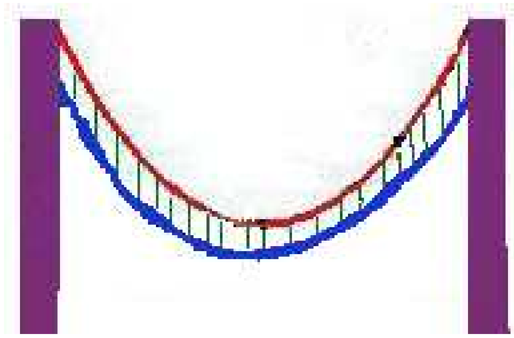
The purpose in hanging a rope in the ways we have been thinking about is usually to support something rather than simply to tie the pillars together with a certain horizontal force. We will think about two ways of hanging things off our ropes.

First is the suspension bridge situation:



We will presume that the bridge roadbed has uniform linear density β and the wires from cable to roadbed are very numerous and very light in comparison to everything else.

Next is the "parallel to the rope" situation. The assumption is that the hanging object is tied close to the support cable but does not, itself, participate as a support structure. It merely hangs by small lightweight wires from the cable. One sometimes sees powerlines or other utilities hung between poles like this.



We will presume that the suspended object has uniform linear density β here too.

We will use the notation of the last sections to create differential equations which describe four different cases which I would like to compare to see how massive the cable must be to support the roadbed or powerline. However I don't know how to solve three out of the four - yet. Perhaps numerical solutions will be necessary for one (or all) of these three.

Case I : Suspended with Density β, Rope with Constant Density δ

$$W_X = Ku \qquad \delta \text{ constant} \qquad W_X = g\int_0^X \delta\sqrt{1+u^2} \; + \; \beta \; dt.$$

This gives

$$W_X' = Ku' = g\delta\sqrt{1+u^2} + g\beta.$$

This can be transformed by a linear speed change to the differential equation

$$V' = \sqrt{1+V^2} + B.$$

I don't know how to solve this DE (yet.)

Case II : Parallel with Density β, Rope with Constant Density δ

$$W_X = Ku \qquad \delta \text{ constant} \qquad W_X = \int_0^X g(\delta+\beta) \; \sqrt{1+u^2} \; dt.$$

This gives

$$W_X' = Ku' = g(\delta+\beta) \; \sqrt{1+u^2}.$$

This is essentially the case we considered before with new density $\overline{\delta} = \delta + \beta$.

This is the one I can solve now.

Case III : Suspended with Density β, Rope with Variable Density

$$\delta_X = c\sqrt{W_X^2 + K^2}$$

$$W_X = Ku \qquad \delta_X = c\sqrt{W_X^2 + K^2} \qquad W_X = g\int_0^X \delta_t\ \sqrt{1+u^2}\ +\ \beta\ dt.$$

This gives

$$\begin{aligned} W_X' =& Ku' = g\left(\delta_X\sqrt{1+u^2}+\beta\right) \\ =& gc\sqrt{W_X^2+K^2}\sqrt{1+u^2}+g\beta = gcK(1+u^2)+g\beta. \end{aligned}$$

This cleans up to

$$u' = gcu^2 + \left(gc + \frac{g\beta}{K}\right).$$

This can be transformed by a linear speed change to the differential equation

$$V' = V^2 + B.$$

I don't know how to solve this DE (yet.)

Case IV : Parallel with Density β, Rope with Variable Density $\delta_X = c\sqrt{W_X^2 + K^2}$

$$W_X = Ku \qquad \delta_X = c\sqrt{W_X^2 + K^2} \qquad W_X = \int_0^X g(\delta_t + \beta)\ \sqrt{1+u^2}\ dt.$$

This gives

$$W_X' = Ku' = g(\delta_X+\beta)\sqrt{1+u^2} = g\left(c\sqrt{W_X^2+K^2}+\beta\right)\sqrt{1+u^2} = gcK(1+u^2)+g\beta\sqrt{1+u^2}.$$

Rearranging constants gives

$$u' = gc\left(1+u^2\right) + \frac{g\beta}{K}\sqrt{1+u^2}.$$

The substitution $u = \tan(\theta)$ followed by a linear speed change yields:

$$\theta' = \cos(\theta) + B.$$

I don't know how to solve this one either.

7. A Vertical Hanging Rope

In this final section we consider a rather different scenario. We are interested in the tension along a rope hanging vertically. We presume, first, that the rope is sufficiently short that the gravitational acceleration does not change over its length, though ultimately we will want to consider ropes sufficiently long so that gravitational change over its length should be considered.

A long rope is hanging vertically, supporting a mass $W\,(kg)$ attached to the bottom. Put the origin at the bottom of the rope. Assume constant linear density $\delta\,(kg/meter)$ of the rope.

So the tension at height X meters is $g(W + X\delta)$ newtons.

Let's assume that this rope material has "strength constant" $s\,(newton\,meter\,/kg)$. The numerical value of s is the number of newtons that can be safely supported by a rope of this material constructed to have linear density $1\,kg/meter$. We will let $c = s^{-1}$ where that is convenient. A currently available material based on Kevlar has a strength constant around $2 \times 10^6\,newton\,meter\,/kg$, corresponding to $c = 5 \times 10^{-7}\,kg/(newton\,meter)$. A rope of this material of linear density $1\,kg/meter$ would have a radius of roughly $1.5\,cm$.

So if $W = 100\,kg$, $\delta = 0.1\,kg/meter$ and with this s (so the diameter of this rope is roughly 1 cm) the maximum safe height X is given by

$$(0.1)(2 \times 10^6) = 9.8(100 + 0.1\,X)$$

So X is about 203 kilometers.

We will now change the scenario to allow the linear density δ to change, proportionally with the tension. Near the bottom it is thinner than at the top.

We are parsimonious with material, so $\delta(0) = g\,c\,W$. This is exactly the linear density which will allow the bottom bit of rope to support the mass W under the influence of normal gravity. At height X,

$$\delta(X) = g\,c\,\left(W + \int_{t=0}^{X} \delta(t)\,dt\right).$$

This implies $\delta' = g\,c\,\delta$ so

$$\delta(X) = g\,c\,W\,\mathrm{e}^{gcX}.$$

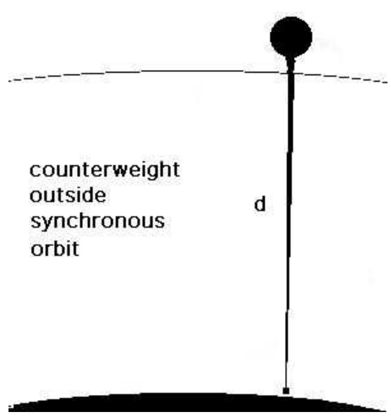
A huge s—tiny c—representing a stronger material, helps both in the exponent and as a multiplier. However growth in density with height is exponential.

Suppose, for example, that you wanted to support 100 kg on a rope constructed this way with $s = 2 \times 10^6\,newton\,meter\,/kg$ as above. The height at which a density of $0.1\,kg/meter$ would be needed satisfies

$$0.1 = 9.8 \times 5 \times 10^{-7}\,100\,\mathrm{e}^{9.8 \times 5 \times 10^{-7} X}.$$

So X could be as big as 1085 kilometers, over five times as long as the "unshaped" rope. Note that these are ropes constructed from commercially available materials.

These huge distances lead one to speculate about the density "at the top" required to support this weight over even greater lengths of rope.



With the same s but with $X = 3.6 \times 10^7$ meters (roughly the distance from Earth's surface to geosynchronous orbit) we find that $\delta(X)$ is unreasonably large: a density of around $2 \times 10^{73}\, kg/meter$ is required at the top. The exponential growth in density as a function of length is the issue.

Still, perhaps we were too crude in our calculation. The force of gravity is hardly constant over such distances. That was one assumption in our calculation and that assumption acted against us. Let's try to get a better estimate.

The radius of the Earth is roughly $d = 6.4 \times 10^6\, meters$ and the acceleration due to gravity diminishes proportionally as the inverse square of the distance from an object to the center of mass of the Earth. At the surface it is roughly $9.8\, meter/sec^2$ so

$$9.8 = \frac{\alpha}{(6.4 \times 10^6)^2} \quad \text{so} \quad \alpha \approx 4 \times 10^{14}\ meter^3/sec^2.$$

From this, our improved density formula becomes

$$\delta(X) = c\left(g\,W + \int_{t=0}^{X} \frac{\alpha}{(X+d)^2}\,\delta(t)\,dt\right).$$

This gives the differential equation

$$\delta' = \frac{c\,\alpha}{(X+d)^2}\,\delta(t) \quad \text{with initial condition } \delta(0) = c\,g\,W$$

or

$$\frac{\delta'}{\delta} = \frac{c\,\alpha}{(X+d)^2} \quad \text{with initial condition } \delta(0) = c\,g\,W.$$

Integrating, we have

$$\delta = c\,g\,W\,\mathrm{e}^{\frac{c\alpha}{d} - \frac{c\alpha}{X+d}} = c\,g\,W\,\mathrm{e}^{c\alpha\left(\frac{1}{d} - \frac{1}{X+d}\right)}.$$

This is a completely different kind of solution. As X increases, so the acceleration due to the Earth's gravity drops to 0, the required density asymptotically rises toward the constant value

$$c\,g\,W\,\mathrm{e}^{\frac{c\alpha}{d}}.$$

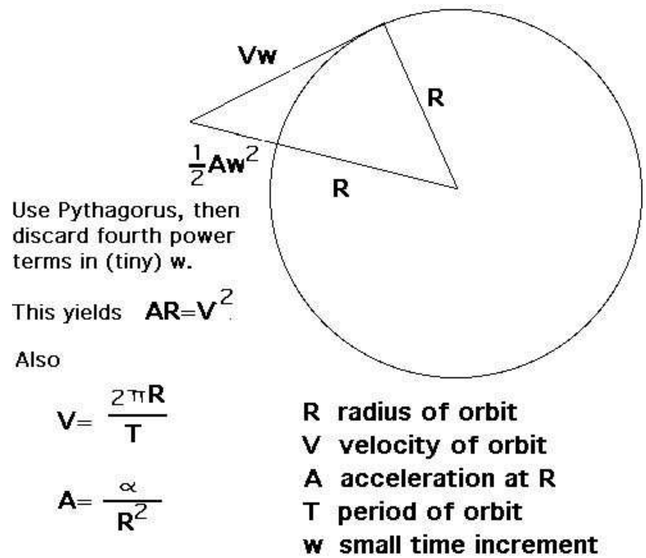
With $c = 5 \times 10^{-7}\ kg/(newton\ meter)$ and $\alpha = 4 \times 10^{14}\ meter^3/sec^2$ and $W = 100\ kg$ and $d = 6.4 \times 10^6\ meters$ we find this asymptotic density to be about $1.8 \times 10^{10}\ kg/meter$. Going to geosynchronous orbit is, practically speaking, no better: the density there is about $1.6 \times 10^8\ kg/meter$.

So how strong must the cable be to support this weight from geosynchronous orbit using an *imaginable* cable density at the top—perhaps $10^3\ kg/meter$? This gives

$$10^3 = c\,980\,\mathrm{e}^{c\alpha\left(\frac{1}{d} - \frac{1}{X+d}\right)} \approx c\,980\,\mathrm{e}^{c\,5.3\times 10^7}.$$

Roughly, this gives $1 \approx c\,\mathrm{e}^{c\,5.3\times 10^7}$. Checking values of this increasing function we find that a value of c of around 2.8×10^{-7}, or s of around 3.5×10^6 $newton\,meter/kg$ will do the trick. This is less than double the strength of the material we were working with before. Evidently the practical possibility of such a cable is *very* sensitive to the strength of the material used. And we will soon have materials able to withstand the tensions under consideration here.

If we come into possession of materials five times stronger than kevlar, so for this material $s = 10^7$ $newton\,meter\,/kg$, the "thread" at the bottom has density $10^{-4}\,kg/meter$ and the density at the top to support this 100 kg mass at Earth's surface is just $.02\,kg/meter$. If the material were similar in density to Kevlar, this rope would have radius in the neighborhood of $2\,mm$!



The formula relating the relevant quantities for stationary orbit is

$$R = \left(\frac{T^2\alpha}{4\pi^2}\right)^{\frac{1}{3}}$$

where $R\,(meters)$ is the radius of the orbit, $T\,(seconds)$ is the length of day and α is the acceleration constant as calculated for earth above.

For Mars, the radius of the planet is about 3.4×10^6 $meters$, acceleration due to gravity at the surface about 3.7 $meters/sec^2$, $\alpha \approx 4.3 \times 10^{13}$ $meter^3/sec^2$ and $T \approx 2.13 \times 10^6\,seconds$ corresponding to about 24.6 $hours$. The radius of the stationary orbit is, therefore, about 2.04×10^7 $meters$. So the distance from Mars surface to stationary orbit is roughly 1.7×10^7 $meters$. On Mars, the formula for variable density ropes with W (kg) hanging on the end is

$$\delta = c\,3.7\,W\,\mathrm{e}^{c\left(\frac{4.3\times10^{13}}{3.4\times10^{6}} - \frac{4.3\times10^{13}}{X+3.4\times10^{6}}\right)}.$$

Given $c = 5 \times 10^{-7}$ $kg/(newton\,meter)$ and at stationary orbit and $W = 100$ this gives a density at the top of

$$\delta \approx 370\,c\,\mathrm{e}^{c\,1.0539\times10^{7}} \approx 3.6 \times 10^{-2}\ kg/meter.$$

We seem to have the basis, with Kevlar ropes, for space elevators at Mars. Phobos, however, will have to go.