The Leibniz Integral Rule

Larry Susanka

The Leibniz Integral Rule posits conditions under which one can "differentiate past an integral sign" and has many versions. We prove here the easiest version I know, patterned after the "hint" in a problem from Michael Spivak's book *Calculus on Manifolds.*

This proof requires that you know the meaning of (i.e. the definition of) uniform continuity of a real-valued function defined on a plane region.

Uniform Continuity.

Specifically, a function

$$g\colon [a,b]\times[c,d]\to\mathbb{R}$$

defined on the rectangle $[a,b]\times[c,d]$ in the plane is said to be uniformly continuous if for each $\varepsilon>0$ you can find a $\Delta>0$ so that $|g(x_0,y_0)-g(x_1,y_1)|<\varepsilon$ whenever the distance between the domain points (x_0,y_0) and (x_1,y_1) does not exceed Δ.

For such a function you can guarantee that function values are as close as you like to each other simply by requiring domain points to be *close enough* to each other.[1] This is in contrast to simple continuity, where the Δ for each ε is allowed to vary from one domain location to another. Here, the same Δ "works" *everywhere.*

The Multivariable Chain Rule.

We will also need, for our proof of a variant form of the integral rule in which the domain of integration is slightly more general than a rectangle, knowledge of the chain rule from multivariable calculus.

The specific chain rule case we have in mind is the following. If a real function H defined on a region in the plane has continuous partial derivatives D_1H and D_2H and if $x(t)$ and $y(t)$ are real valued differentiable functions defined on the same interval then the composite function $H(x(t),y(t))$ is differentiable and

$$\frac{d}{dt}H(x(t),y(t)) = D_1H(x(t),y(t))\,x'(t) + D_2H(x(t),y(t))\,y'(t)$$

whenever $(x(t),y(t))$ is in the domain of H.

The Basic Form of the Leibniz Integral Rule.

Suppose $f\colon [a,b]\times[c,d]\to\mathbb{R}$ is continuous at each (x,y) in $[a,b]\times[c,d]$, so the integrals we discuss can be ordinary Riemann integrals, and that the partial derivative of f with respect to its second variable $D_2f=\frac{\partial f}{\partial y}$ is a uniformly continuous function on the rectangle. Then

$$\frac{d}{dy}\int_{x=a}^{x=b} f(x,y)\,dx = \int_{x=a}^{x=b}\frac{\partial f}{\partial y}(x,y)\,dx.$$

[1] It is a fact that any continuous real-valued function defined on a closed and bounded region (i.e. a *compact* region) such as $[a,b]\times[c,d]$ is uniformly continuous. Any continuous function is bounded on a compact region.

We have used here the traditional sloppy notation where domain variable values and names and variables of integration and differentiation are intermingled.

In an attempt to clarify this a little we define $F(y) = \int_{x=a}^{x=b} f(x,y)\,dx$ and re-write the rule we wish to prove as

$$\frac{d}{dy}F(y_0) = \int_{x=a}^{x=b} D_2 f(x,y_0)\,dx.$$

Proof of the Leibniz Integral Rule.

By the Fundamental Theorem of Calculus we have, for each fixed x and each y_0,

$$f(x,y_0) = \int_{y=c}^{y=y_0} D_2 f(x,y)\,dy + f(x,c).$$

Pick $\varepsilon > 0$ and select real number Δ so that $D_2 f(x,y_0)$ is within ε of $D_2 f(x,y_1)$ whenever (x,y_0) is within Δ of (x,y_1). Suppose members y_1 and y_0 of $[c,d]$ have been chosen to satisfy this restriction, and let $y_1 = y_0 + \delta$. So $|\delta| \le \Delta$.

An approximation to the derivative $\frac{d}{dy}F(y_0)$ is given by

$$\frac{F(y_1) - F(y_0)}{\delta} = \frac{1}{\delta}\int_{x=a}^{x=b} f(x,y_1) - f(x,y_0)\,dx$$

$$= \frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=c}^{y=y_1} D_2 f(x,y)\,dy + f(x,c) - \int_{y=c}^{y=y_0} D_2 f(x,y)\,dy - f(x,c)\right] dx$$

$$= \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_1} D_2 f(x,y)\,dy\,dx.$$

Since (x,y_0) is within Δ of (x,y_1) for each x we have this approximation to $\frac{d}{dy}F(y_0)$ trapped between the values

$$\frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=y_0}^{y=y_1} D_2 f(x,y_0) - \varepsilon\,dy\right] dx$$

$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_1} D_2 f(x,y)\,dy\,dx$$

$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=y_0}^{y=y_1} D_2 f(x,y_0) + \varepsilon\,dy\right] dx.$$

The left and right inner integrands don't depend on y and can be integrated as constants with respect to y on the interval $[y_0, y_1]$.

After cancelling δ and integrating ε with respect to x on $[a,b]$ we have

$$\int_{x=a}^{x=b} D_2 f(x,y_0)\,dx - \varepsilon(b-a)$$

$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_1} D_2 f(x,y)\,dy\,dx$$

$$\le \int_{x=a}^{x=b} D_2 f(x,y_0)\,dx + \varepsilon(b-a).$$

Taking the limit as $\delta \to 0$ produces the estimate

$$\int_{x=a}^{x=b} D_2 f(x, y_0)\, dx - \varepsilon(b-a) \leq \frac{d}{dy} F(y_0) \leq \int_{x=a}^{x=b} D_2 f(x, y_0)\, dx + \varepsilon(b-a).$$

But ε can be chosen to be arbitrarily small, and the result follows.

A More General Form of the Leibniz Integral Rule.

Define $H(y, B) = \int_{x=a}^{x=B} f(x, y)\, dx$. By our previous calculation we know that

$$D_1 H(y, B) = \frac{\partial}{\partial y} \int_{x=a}^{x=B} f(x, y)\, dx = \int_{x=a}^{x=B} \frac{\partial f}{\partial y}(x, y)\, dx.$$

And by the fundamental theorem of calculus we have

$$D_2 H(y, B) = f(B, y).$$

Both of these partial derivatives are continuous: the first because it is an integral of a uniformly continuous function[2] and the second is continuous by assumption.

Suppose now that $z\colon [c, d] \to [a, b]$ is differentiable. Using the chain rule we calculate a slightly more general version of the Leibniz Integral Rule

$$\frac{d}{dy} \int_{x=a}^{x=z(y)} f(x, y)\, dx = \int_{x=a}^{x=z(y)} \frac{\partial f}{\partial y}(x, y)\, dx + f(z(y), y)\, z'(y).$$

To see that this equation holds, observe that the derivative on the left is

$$\frac{d}{dy} H(y, z(y)) = D_1 H(y, z(y)) \frac{dy}{dy} + D_2 H(y, z(y)) \frac{dz}{dy}$$

which is the boxed result.

Breaking $\frac{d}{dy} \int_{x=q(y)}^{x=z(y)} f(x, y)\, dx$ into pieces as

$$\frac{d}{dy} \int_{x=q(y)}^{x=z(y)} f(x, y)\, dx = \frac{d}{dy} \int_{x=a}^{x=z(y)} f(x, y)\, dx - \frac{d}{dy} \int_{x=a}^{x=q(y)} f(x, y)\, dx$$

we easily have the result for an integral with two edge functions.

It is worth remarking that less stringent requirements than continuity of f and uniform continuity of the partial derivative suffice for this theorem to hold. Further variations of this result are left for other treatments.

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[2]Choose δ_2 so small that $|f(x, y_0) - f(x, y_1)| < \varepsilon$ whenever $|y_0 - y_1| < \delta_2$, and we suppose y_0 and y_1 are points in $[c, d]$ subject to this restriction. Suppose δ_1 is a real number and both B and $B + \delta_1$ are in $[a, b]$. Let M denote the maximum magnitude of f on its domain rectangle.

$$|D_1 H(y_0, B + \delta_1) - D_1 H(y_1, B)| = \left| \int_{x=a}^{x=B+\delta_1} f(x, y_0)\, dx - \int_{x=a}^{x=B} f(x, y_1)\, dx \right|$$

$$\leq \left| \int_{x=a}^{x=B+\delta_1} f(x, y_0)\, dx - \int_{x=a}^{x=B+\delta_1} f(x, y_1)\, dx \right| + \left| \int_{x=a}^{x=B+\delta_1} f(x, y_1)\, dx - \int_{x=a}^{x=B} f(x, y_1)\, dx \right|$$

$$\leq \int_{x=a}^{x=B+\delta_1} |f(x, y_0) - f(x, y_1)|\, dx + \int_{x=B}^{x=B+\delta_1} |f(x, y_1)|\, dx \leq \varepsilon(b-a) + M|\delta_1|.$$

This can be made arbitrarily small by choosing δ_1 and δ_2 small enough, so $D_1 H$ is continuous.