Linear Spaces and Hilbert Spaces
(Draft)

Larry Susanka

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This appendix is too brief to do more than touch on its subject matter. Most of modern and classical analysis was invented to deal with issues raised in what we now call functional analysis, the study of some aspects of linear spaces. Point-set topology and much of abstract algebra were created to deal with problems that popped up, specifically, in this context. The subject is truly central to mathematics.

The reader is invited to view the older, but still wonderful, treatments in Riesz and Sz.-Nagy, *Functional Analysis* [?] and Lorch, *Spectral Theory* [?]. The influence of that majestic compendium Dunford and Schwartz, *Linear Operators Part I* [?] can be seen on every page. I also highly recommend the beautiful short book by Arveson, *A Short Course in Spectral Theory* [?] and Reed and Simon, *Methods of Modern Mathematical Physics v. 1, Functional Analysis* [?] and Narici and Beckenstein, *Topological Vector Spaces, Second Edition* [?] among *many* others.

1. Linear Functionals and Hyperplanes

We will examine some properties of vector spaces over the fields $\mathbb{R}$ or $\mathbb{C}$, as encountered in Section **??**. Any reference to a field $\mathbb{F}$ is intended to denote one of these two fields. Particular attention will be paid to vector spaces comprised of linear transformations.

First, if a is a number, A is a nonempty set of numbers and b is a vector and B and C are nonempty sets of vectors we use the rather obvious notations

$$B + C = \{\, b + c \mid b \in B \text{ and } c \in C \,\}, \qquad B \ominus C = \{\, b - c \mid b \in B \text{ and } c \in C \,\},$$
$$AB = \{\, ab \mid a \in A \text{ and } b \in B \,\}, \qquad Ab = \{\, ab \mid a \in A \,\},$$
$$aB = \{\, ab \mid b \in B \,\}, \qquad b + C = \{\, b + c \mid c \in C \,\}.$$

The set of all $\mathbb{F}$-linear transformations from the $\mathbb{F}$-vector space V to the $\mathbb{F}$-vector space W will be denoted $\mathcal{H}_{\mathbb{F}}(V, W)$. In Appendix **??** this set was denoted $Hom_{\mathbb{F}-mod}(V, W)$.

When $W = \mathbb{F}$ this set of transformations will be denoted $\mathbf{V}^*_{\mathbb{F}}$ and called the **algebraic dual of V**. Members of the algebraic dual are also called **$\mathbb{F}$-linear functionals.** Sometimes, when no misunderstanding can arise, the "$\mathbb{F}$" subscript or hyphenated prefix is suppressed.

Since any complex vector space is also a real vector space, *this should be done carefully.* It may not be possible to deduce from context *which* field you have in mind when referring to a linear transformation or functional, or other features of a space.

For instance if B is a basis for complex vector space V and $iB = \{\, ib \mid b \in B \,\}$ then $B \bigcup iB$ is a basis for V as a real vector space. So if B is finite the real dimension of V is double its complex dimension.

And the real algebraic dual of a complex vector space is not the same as the complex algebraic dual of that space. $V^*_{\mathbb{C}} \subset V^*_{\mathbb{R}} \oplus i\, V^*_{\mathbb{R}} \subset \mathbb{C}^V$.

$V^*_{\mathbb{C}}$ is a very special subspace of the indicated direct sum, as we shall see.

1.1. ***Exercise***. *(i) Any member Θ of the complex algebraic dual of the complex vector space V can be written in a unique way as $\Theta_r + i\,\Theta_c$ where Θ_r and Θ_c are in $V^*_{\mathbb{R}}$, called the* ***real and complex parts of*** $\boldsymbol{\Theta}$*, respectively. The real and complex parts of Θ are related. Show that for all $x \in V$*

$$\Theta_r(i\,x) + i\,\Theta_c(i\,x) = -\Theta_c(x) + i\,\Theta_r(x).$$

We find, then, that

$$\Theta_c(x) = -\Theta_r(i\,x) \text{ and } \Theta_c(i\,x) = \Theta_r(x).$$

(ii) If Ψ is any member of the real algebraic dual of the complex vector space V then Θ defined by $\Theta x = \Psi x - i\,\Psi(ix)$ is a member of the complex algebraic dual of V. We find that $A\colon V^*_{\mathbb{R}} \to V^*_{\mathbb{C}}$ given by $A(\Psi)(x) = \Psi x - i\,\Psi(ix)$ is a real isomorphism onto $V^*_{\mathbb{C}}$.

1.2. ***Exercise***. *Any real vector space W can be "complexified" as follows. Let $V = W \times W$. V is an additive group in the obvious way, and a real vector space. We make V into a complex vector space by declaring $i\,(x, y) = (-y, x)$ and, generally,*

$$(a + bi)(x, y) = (ax - by, bx + ay) \text{ for real } a \text{ and } b \text{ and } (x, y) \in V.$$

(i) Show that this operation does give V the structure of a complex vector space.

(ii) Show that if $B \subset W$ is a basis for the real vector space W then $\widetilde{B} = \{\,(b, 0) \mid b \in B\,\}$ is a basis for the complex vector space V.

(iii) Suppose $\alpha, \beta \in W^*_{\mathbb{R}}$ and define $\Theta_{\alpha,\beta}$ on $V = W \times W$ by

$$\Theta_{\alpha,\beta}(x, y) = \alpha(x) - \beta(y) + i\,(\beta(x) + \alpha(y)).$$

This association is one-to-one: different ordered α, β pairs produce different $\Theta_{\alpha,\beta}$. Also, $\Theta_{\alpha,\beta}$ is obviously real linear and

$$\Theta_{\alpha,\beta}(\,i\,(x, y)\,) = \Theta_{\alpha,\beta}(-y, x) = -\alpha(y) - \beta(x) + i\,(-\beta(y) + \alpha(x)) = i\,\Theta_{\alpha,\beta}(x, y)$$

so, in fact, $\Theta_{\alpha,\beta}$ is complex linear, a member of $V^*_{\mathbb{C}}$.

(iv) Suppose Θ is a member of the complex algebraic dual of $V = W \times W$. Then

$$\Theta(x, y) = \Theta(x, 0) + \Theta(0, y) = \Theta(x, 0) + i\,\Theta(y, 0).$$

So Θ is determined in a simple way by its effect on the real vector subspace $W \times \{0\}$ of V, and Θ is real linear on that subspace.

$$\Theta(x, 0) = \Theta_r(x, 0) + i\,\Theta_c(x, 0) = \alpha(x) + i\,\beta(x)$$

for certain real valued real linear functions α and β on W. From the remark above,

$$\begin{aligned}\Theta(x, y) &= \Theta(x, 0) + \Theta(0, y) = \alpha(x) + i\,\beta(x) + i\ (\alpha(y) + i\,\beta(y)) \\ &= \alpha(x) - \beta(y) + i\ (\beta(x) + \alpha(y)) = \Theta_{\alpha,\beta}(x, y).\end{aligned}$$

So the association indicated by $\Theta_{\alpha,\beta}$ between members (α, β) of the real vector space $W^*_{\mathbb{R}} \times W^*_{\mathbb{R}}$ and members of $V^*_{\mathbb{C}}$ is a real isomorphism onto $V^*_{\mathbb{C}}$.

An $\mathbb{F}$-subspace W of the $\mathbb{F}$-vector space V is said to have **codimension n** if the dimension of the $\mathbb{F}$-vector space V/W is the cardinal number n. W is called a **maximal subspace of V** if it has codimension 1. This is field dependent: a complex subspace of complex codimension 1 will have real codimension 2, and a real subspace of real codimension 1 is not a complex subspace at all.

The intersection of k maximal subspaces could have codimension as high as k, but not more.

An **$\mathbb{F}$-hyperplane** is a coset of the $\mathbb{F}$-vector space V/W where W is a maximal subspace of V.

Select non-zero $v+W$ in V/W where W is maximal. Let $\widetilde{B}$ be a basis for W. Then $B = \{v\} \cup \widetilde{B}$ is a basis of V. Every element in V can be written in a unique way as a linear combination of members of this basis, and the function Ψ defined on V by letting Ψx be the coefficient on v in the linear combination for x is $\mathbb{F}$-linear. $W = Ker(\Psi)$, and the hyperplane $v+W$ consists exactly of those members x of V for which $\Psi(x) = \Psi(v) = 1$.

Conversely, if Ψ is *any* nontrivial linear functional then $W = Ker(\Psi)$ is maximal: since the quotient V/W is isomorphic to the image of Ψ, which is $\mathbb{F}$, it has dimension 1.

If v is chosen so that $\Psi v = 1$ then Ψ is a functional of the type in the last paragraph for W and this v. Of course, any other vector that differs from v by a member of W would work equally well but that is the only flexibility in choice of vector: it must be in the hyperplane $v+W$.

Suppose Θ and Φ are two nontrivial $\mathbb{F}$-linear functionals with the same kernel W. We saw above that if $v \notin W$ and if $x = a\,v + w$ for generic member x of V, where $w \in K$, then

$$\Theta x = a\,\Theta v \quad \text{and} \quad \Phi = a\,\Phi v \quad \text{and so} \quad \Theta x = \frac{\Theta v}{\Phi v}\,\Phi.$$

In other words, functionals with the same kernel are multiples of each other.

If Ψ is a nontrivial real linear functional on the complex vector space V then $W_{\text{real}} = Ker(\Psi)$ is a real—not a complex—vector subspace of V. $W_{\text{real}} \bigcap iW_{\text{real}}$ *is* a complex vector space, however, and is the kernel of the complex linear transformation Θ defined by $\Theta x = \Psi x - i\Psi(ix)$ for $x \in V$.

Suppose W is a maximal real subspace of the (real or complex) vector space V. Suppose $v \in V$, $v \notin W$ and $c \in \mathbb{R}$. Define open and closed **halfspaces**

$$\begin{aligned} W_{v,c}^{\geq} &= \bigcup_{t\geq c}(tv+W) \qquad & W_{v,c}^{>} &= \bigcup_{t>c}(tv+W) \\ W_{v,c}^{\leq} &= \bigcup_{t\leq c}(tv+W) \qquad & W_{v,c}^{<} &= \bigcup_{t<c}(tv+W). \end{aligned}$$

These halfspaces are agglomerations of those "parallel" real hyperplanes on one side or the other of $cv+W$.

It is important to emphasize that in the infinite dimensional setting with a variety of possible topologies on our vector spaces, these open halfspaces may not *be* topologically open, and closed halfspaces may not be topologically closed. It will give us important information when they are, but for now they are simply sets defined by algebraic means.

Note that $W^{\geq}_{v,c} = W^{\leq}_{-v,-c}$ and $W^{>}_{v,c} = W^{<}_{-v,-c}$.

1.3. ***Exercise***. *(i) If we were interested in proliferating vocabulary, we might call a set similar to* $W^{\geq}_{v,c-\varepsilon} \cap W^{\leq}_{v,c+\varepsilon}$ *a slabspace and, if* Z *is maximal and different from* W*, we might refer to* $W^{\geq}_{v,c} \cap Z^{\geq}_{u,d}$ *as a wedgespace. Think about why these names would make sense.*

(ii) Suppose that neither v *nor* w *is in the maximal real subspace* W *of* V*. Then there is a unique real non-zero constant* α *with* $v - \alpha w \in W$*. If* $\alpha > 0$ *we say that* v *and* w *are on the* ***same side of*** $\mathbf{W}$*, while if* $\alpha < 0$ *we say that* v *and* w *are on* ***opposite sides of*** $\mathbf{W}$*. Suppose* $cv + W = bw + W$*. Show that* $W^{\geq}_{v,c} = W^{\geq}_{w,b}$ *if* v *and* w *are on the same side of* W*, while* $W^{\geq}_{v,c} = W^{\leq}_{w,b}$ *if* v *and* w *are on opposite sides of* W*.*

Two subsets A and B of a vector space V are said to be **separated by the real hyperplane** $\mathbf{cv} + \mathbf{W}$ provided A is contained in one of $W^{\geq}_{v,c}$ or $W^{\leq}_{v,c}$ while B is contained in the other. Note that both A and B could be entirely contained in that hyperplane, they could even be equal, or empty, and this definition would have them separated by that hyperplane: not a very interesting case.

A and B are said to be **strictly separated by the real hyperplane** $\mathbf{cv} + \mathbf{W}$ provided A is a subset of exactly one of $W^{>}_{v,c}$ or $W^{<}_{v,c}$ while B is only a subset of the other.

A and B are said to be **strongly separated by the real hyperplane** $\mathbf{cv} + \mathbf{W}$ when there is some positive number ε for which A is a subset of exactly one of $W^{>}_{v,c+\varepsilon}$ or $W^{<}_{v,c-\varepsilon}$ and B is only a subset of the other.

We say the subsets A and B are **separated, strictly separated or strongly separated** when they can be separated in the appropriate sense by some real hyperplane.

The **non-zero** real linear functional Ψ is said to **separate A and B** if there is a number c for which A is contained in (at least) one of the sets $\Psi^{-1}(\ (-\infty, c]\ )$ or $\Psi^{-1}(\ [c, \infty)\ )$ while B is contained in the other.

Ψ is said to **strictly separate A and B** if there is a number c for which A is a subset of exactly one of the sets $\Psi^{-1}(\ (-\infty, c)\ )$ or $\Psi^{-1}(\ (c, \infty)\ )$ while B is only a subset of the other.

Finally, Ψ is said to **strongly separate A and B** if there are real numbers c and ε with $\varepsilon > 0$ and for which A is a subset of just one of the sets $\Psi^{-1}(\ (-\infty, c-\varepsilon)\ )$ or $\Psi^{-1}(\ (c+\varepsilon, \infty)\ )$ while B is only contained in the other.

1.4. ***Exercise***. *If* A *and* B *are (strictly) (strongly) separated by a hyperplane* $cv + W$ *for* $v \notin W$*, add* $\{v\}$ *to a real basis of* W *to form a real basis* C *of* V *and let* Ψ *be the real linear functional defined by letting* Ψx *be the coefficient on* v *when* x *is represented as a linear combination of members of the basis* C*. Then* $W = Ker(\Psi)$ *and* Ψ *(strictly) (strongly) separates* A *and* B*.*

Conversely, if a real linear functional Ψ (strictly) (strongly) separates A and B, select $v \in \Psi^{-1}(c)$ if $c \neq 0$, while if $c = 0$ let v be any vector not in $W = Ker(\Psi)$. In either case $cv + W$ (strictly) (strongly) separates A and B.

Finally, A and B are (strictly) (strongly) separated exactly when $A \ominus B$ and the set $\{0\}$ are (strictly) (strongly) separated.

Quite a few different properties will be discussed in combinations, and we may occasionally use the following notational contrivance in the interest of brevity. If we want to ascribe a property to an object X we might, in the first instance that X is encountered in a discussion, list that property along with X as a superscript. For example we could indicate that Y is a real vector subspace of a real vector space X by $Y^{\substack{\text{real vector}\\ \text{subspace}}} \subset X^{\substack{\text{real vector}\\ \text{space}}}$, or that P is a real linear function by $P^{\substack{\text{real}\\ \text{linear}}} \colon V \to W$.

An ordinary vector space basis will frequently be called a **Hamel basis**, whose cardinality can be called the **Hamel dimension** of the space, to distinguish it from other types of bases to be discussed later. The most important fact about Hamel basis and Hamel dimension is their existence: every spanning set in a vector space can be "pruned" to a Hamel basis. Every linearly independent set can be "expanded" to a Hamel basis. They all have the same cardinality. The general argument requires the Axiom of Choice.

If you see unadorned reference to "a basis" or "dimension" we are, most likely, discussing a Hamel basis. Check context to be sure.

2. Some Properties of Subsets of Vector Spaces

A point q is said to be an **internal point** of the subset M of the vector space V if for each $v \in V$ there is a nonempty interval of the form $[0, \varepsilon)$ for which $q + tv \in M$ whenever $t \in [0, \varepsilon)$. The interval (obviously) depends on both q and v.

2.1. ***Exercise***. *Suppose M and N are subsets of the vector space V.*

(i) If q is an internal point of M and p is any point of N then $q + p$ is an internal point of $M + N$.

(ii) If x is any point of V, q is an internal point of M exactly when $x + q$ is an internal point of $x + M$.

A subset A of the $\mathbb{F}$-vector space V is called **absorbing** if, for each $v \in V$, there is a positive real number r so that $b\,v \in A$ whenever $b \in \mathbb{F}$ and $|b| < r$.

This condition can be rephrased as follows. Let $\mathbf{S_r} = \{\, b \in \mathbb{F} \mid |b| \leq r \,\}$. So if $\mathbb{F} = \mathbb{R}$ this is the interval $[-r, r]$, while in the complex case it is the complex numbers inside and on the circle of radius r.

$$A \text{ is absorbing} \quad \Leftrightarrow \quad V = \bigcup_{r>0} \{\, v \in V \mid S_r v \subset A \,\}.$$

If A is absorbing then 0 is internal to A. More generally, if $M \ominus \{q\}$ is absorbing then q is an internal point of M. For real spaces, the converse is also true. But for complex spaces, even in dimension 1, more is required. (See Exercise 2.3.)

A set N in a vector space V is said to **absorb** a set B if there is an integer n so that if $t \in \mathbb{F}$ and $|t| \geq n$ then $B \subset t\,N$.

If $0 \in N$, which is the case of primary interest, this is equivalent to saying that there is an $\varepsilon > 0$ so that $S_\varepsilon\, B \subset N$.

So a set N is absorbing if it absorbs every one-vector set in V. On the other hand, N absorbs B if there is an upper bound on the "stretch magnitude" needed for N to absorb every scalar multiple of magnitude 1 or less of every vector in B "simultaneously."

If N absorbs B and $0 \in N$ then for each $b \in B$ and scalar α of magnitude 1 there must be some "interval" of vectors $\{\, t\,\alpha\, b \mid t \in (-\varepsilon, \varepsilon)\,\}$ contained in N, where a single $\varepsilon > 0$ can be chosen for all α and all b. And if arbitrarily large (magnitude) multiples of a non-zero vector b are in B then the whole line $\{\, t\,\alpha\, b \mid t \in \mathbb{R}\,\}$ must be in N for every α.

A is called **symmetric** if $\quad -A = A$.

A subset A is called **balanced** or **circled** if, for every $v \in A$ and $b \in \mathbb{F}$ with $|b| \leq 1$ we find that $b\,v \in A$.

$$A \text{ is balanced or circled} \quad \Leftrightarrow \quad A = S_1 A.$$

The "circled" vocabulary is more natural when $\mathbb{F} = \mathbb{C}$, while "balanced" seems more descriptive of the situation for real vector spaces, but both adjectives are employed in either case.

2.2. ***Exercise***. *(i) If A is balanced then $0 \in A$ and A absorbs itself. And if $0 \in A$ and A absorbs itself then A contains a balanced subset that also absorbs A.*

(ii) Let A be the interval $(-1, 3)$ together with the single real number 4. The real numbers are a one dimensional real vector space. A absorbs itself and $0 \in A$ but A is not balanced.

2.3. ***Exercise***. *(i) If M and N are balanced and $a \in \mathbb{F}$ then $M + N$ and aM and $M \cap N$ and $M \ominus N$ are all balanced.*

(ii) If N is balanced and 0 is internal then N is absorbing.

(iii) If either N or B is balanced and $B \subset kN$ for one k then N absorbs B.

(iv) If both M and N absorb B then $M \cap N$ absorbs B.

Suppose $x, y \in V$. The linear combination $tx + (1-t)y$ is called a **convex combination** of x and y when $0 \leq t \leq 1$. A subset M of V is called **convex** if all convex combinations of x and y are in M whenever x and y are in M.

If M is convex, any straight line segment is entirely contained in M whenever its endpoints are in M. This condition can be rephrased as follows.

$$M \text{ is convex} \quad \Leftrightarrow \quad tM + (1-t)M = M \quad \forall t \in [0,1].$$

2.4. ***Exercise***. *(i) If M and N are convex in vector space V and $a \in \mathbb{F}$ then $M + N$ and aM and $S_1 M$ and $M \cap N$ and $M \ominus N$ are all convex.*

(ii) If M is convex in V and t_i is in $[0,1]$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n t_i = 1$ and $x_i \in M$ for $i = 1, \ldots, n$ then $\sum_{i=1}^n t_i x_i \in M$.

A **disk** or **absolutely convex set** is a balanced convex set. Once again, both vocabularies are found in the literature. The term "disk" might invoke misleading imagery. For instance, the disks we work with will often contain nontrivial subspaces, and of course any subspace is a disk. Any set of the form $|f|^{-1}(\,(-\varepsilon, \varepsilon)\,)$ for linear functional f and positive ε will be a disk, and contains a maximal subspace.

All the definitions of this section apply to the empty set of vectors: every set absorbs the empty set, which is symmetric, balanced and convex. Normally, of course, we will be interested in *nontrivial* sets with these properties.

2.5. ***Exercise***. *(i) The intersection of any finite number of absorbing sets is absorbing, but the intersection of a countable number need not be.*

*(ii) Any intersection of balanced sets is balanced. So the intersection of all balanced sets containing a set $B \subset V$ is balanced, called the **circled hull of*** **B**. *The circled hull of B is just $S_1 B$.*

*(iii) The **convex hull** of a set $A \subset V$ is the intersection of all convex sets containing A. Since the intersection of any family of convex subsets of V is convex, the convex hull is the smallest convex set containing A. The notation* **convex(A)** *is used for this set. Show that*

$$convex(A) = \left\{ \sum_{i=0}^{n} t_i x_i \;\middle|\; t_i \geq 0,\ x_i \in A,\ \sum_{i=0}^{n} t_i = 1,\ n \in \mathbb{N} \right\}.$$

*(iv) The **absolutely convex hull** of A is the intersection of all disks containing A. The absolutely convex hull is itself a disk, the smallest disk containing A. The notation* **disk(A)** *is used for this set. Show that*

$$disk(A) = \left\{ \sum_{i=0}^{n} \alpha_i x_i \;\middle|\; \alpha_i \in \mathbb{F},\ x_i \in A,\ \sum_{i=0}^{n} |\alpha_i| \leq 1,\ n \in \mathbb{N} \right\}.$$

(v) Is $disk(A)$ equal to $S_1\, convex(A)$? What about $convex(S_1\, A)$?

(vi) Suppose B is a disk and absorbs A. Then B absorbs $disk(A)$.

(vii) Any intersection of halfspaces is convex.

2.6. ***Exercise***. *Suppose $\Psi\colon V \to W$ is a linear transformation.*

(i) The image and inverse image (using Ψ) of balanced sets are balanced.

(ii) The inverse image of an absorbing set is absorbing. If Ψ is onto, the image of an absorbing set is absorbing.

(iii) The image and the inverse image of convex sets are convex.

The intersection of any *finite* number of absorbing disks is an absorbing disk: it too will be balanced, convex and absorbing. In the next sections we will investigate further what can be done with such sets.

3. The Minkowski Functional

Suppose V is a vector space over the field $\mathbb{F}$ and $G\colon V \to \mathbb{R}$.

G is called **subadditive** if $G(x+y) \leq G(x) + G(y)$ for any $x, y \in V$. When G is subadditive, the defining inequality is called the **triangle inequality** or, sometimes, the **Minkowski inequality**.

Examine $G(0) \leq G(x) + G(-x)$ and conclude $G(0) \geq 0$ for any subadditive G.

G is called **positively homogeneous** if $G(\,t\,x\,) = t\,G(x)$ for any x and $t \geq 0$.

G is called **sublinear** if it is both subadditive and positively homogeneous.

G is called **homogeneous** if $G(\,\alpha\,x\,) = |\alpha|\,G(x)$ for any $x \in V$ and $\alpha \in \mathbb{F}$.

G is called a **semi-norm** if it is both subadditive and homogeneous.

Any semi-norm must be non-negative: $G(x) = G(-x)$ and so $0 \leq G(x) + G(x)$.

A semi-norm G is called a **norm** if $G(x) = 0$ exactly when $x = 0$.

G is called **convex** if $G(\,t\,x + (1-t)\,y\,) \leq t\,G(x) + (1-t)\,G(y)$ for any choice of $x, y \in V$ and $t \in [0,1]$. Note that every sublinear function is convex.

3.1. ***Exercise***. *(i) A positive multiple of a semi-norm is a semi-norm. A finite sum of semi-norms is a semi-norm. If $G_1, \ldots, G_n$ is a finite list of semi-norms the function G_{max} defined, for each $x \in V$ by*

$$G_{max}(x) = \max\{\,G_1(x),\ \ldots\ ,\ G_n(x)\,\}.$$

is a semi-norm too.

*(ii) If G is a semi-norm, $G^{-1}(\,[0,\varepsilon)\,)$ and $G^{-1}(\,[0,\varepsilon]\,)$ are absorbing disks for each $\varepsilon > 0$. Disks of this kind are said to be **generated by G**.*

(iii) If Ψ is a member of the algebraic dual of V then $|\Psi|$ is a semi-norm on V. It is never a norm, unless V has dimension 0 or 1.

Suppose q is a specified internal point of a set M. Let $[0, \varepsilon_v)$ be the largest half-open interval for which $q + tv \in M$ whenever $t \in [0, \varepsilon_v)$. Define $P(v)$ to be ε_v^{-1} if $\varepsilon_v < \infty$ and 0 otherwise. The function P is called the **Minkowski functional for M and internal point q**. We will refer to P, synonymously, as a **Minkowski gauge**.

The Minkowski gauge applied to v is a measure of how far the "edge" of M is from q in the direction v using v itself as a yardstick. If $P(v)$ is large then it takes a small multiple of v to get to the edge in that direction. v is "big," according to M as seen from q. On the other hand, if it takes many multiples of v to get to the edge of M from q the gauge evaluates v to be a small vector. If $P(v) = 0$ for non-zero v then M is unbounded in the v direction from q.

3.2. ***Exercise***. *Suppose P is the Minkowski functional for M and internal point q in vector space V.*

(i) Prove that if t is any non-negative real number, $0 \leq P(t\,v) = t\,P(v) < \infty$, so P is positively homogeneous.

(ii) Suppose M is convex. If $t > 0$ and $q + t\,v \in M$ then $P(v) \le 1/t$. If $q + t\,v$ is, itself, an internal point of M then $P(v) < 1/t$. If $q + t\,v \notin M$ then $P(v) \ge 1/t$.

(iii) Show that if M is convex then P is subadditive, so (combined with (i)) P is sublinear and hence convex. (hint: Let c be any number exceeding $P(v) + P(w)$. Select positive numbers a and b with $a > P(v)$ and $b > P(w)$ and $c = a + b$. Since M is convex, $\frac{a}{a+b}(q + a^{-1}v) + \frac{b}{a+b}(q + b^{-1}w)$ is in M. But this member of M is $q + \frac{v+w}{a+b} = q + \frac{v+w}{c}$ which implies $P(v + w) \le c$.)

(iv) If $M \ominus \{q\}$ is a disk and q is an internal point of M (so 0 is internal to $M \ominus \{q\}$, which is balanced and therefore absorbing) then P is a semi-norm. The Minkowski gauge for the absorbing disk $P^{-1}(\,[0, 1)\,)$ and internal point 0 is P itself. Must $P^{-1}(\,[0, 1)\,)$ equal $M \ominus \{q\}$? What about $P^{-1}(\,[0, 1]\,)$? Give a condition on $M \ominus \{q\}$ that will guarantee it equals one or the other.

(v) Suppose D is an absorbing disk and Q is the Minkowski gauge for D and internal point 0. Suppose, for some $v \in V$ and $t > 0$, we find that $v \in b\,D$ if $b > t$ but $v \notin b\,D$ if $b < t$. Then $Q(v) = t$.

In this section we have seen that purely geometrical considerations involving absorbing, balanced convex sets in a vector space can be captured in a gauge, or semi-norm. Conversely, semi-norms produce these absorbing disks.

4. The Hahn-Banach Theorem

This section is devoted to the possibility of extending a function with certain properties to a larger domain while preserving those properties.

One reason we want this theorem is that it will allow us to conclude that there is a rich stock of continuous functionals whenever the theorem applies.

4.1. ***Theorem***. ***The Hahn-Banach Theorem***

If $Y^{\substack{\text{real vector}\\ \text{subspace}}} \subset X^{\substack{\text{real vector}\\ \text{space}}}$ and $P\colon X \to \mathbb{R}$ is convex

and $\Lambda \in Y^*_{\mathbb{R}}$ satisfies $\Lambda \le P|_Y$

then $\exists \Psi \in X^*_{\mathbb{R}}$ with $\Lambda = \Psi|_Y$ and $\Psi \le P$.

Proof. Suppose $\Lambda'\colon Y' \to \mathbb{R}$ is a linear extension of Λ to a proper subspace Y' of X and that Λ' is dominated by P on Y', as required in the theorem.

If $w \in X - Y'$ and α, β are positive and $u, v \in Y'$

$$\begin{aligned}\beta\,\Lambda' u + \alpha\,\Lambda' v \;&=\; (\alpha+\beta)\,\Lambda'\left(\frac{\beta}{\alpha+\beta}u + \frac{\alpha}{\alpha+\beta}v\right)\\ &\le (\alpha+\beta)\,P\left(\frac{\beta}{\alpha+\beta}(u-\alpha w) + \frac{\alpha}{\alpha+\beta}(v+\beta w)\right)\\ &\le \beta\,P(u-\alpha w) + \alpha\,P(v+\beta w).\end{aligned}$$

So $\quad \dfrac{1}{\alpha}\,[\;\Lambda' u - P(u - \alpha\; w)\;] \;\le\; \dfrac{1}{\beta}\,[\;P(v + \beta\; w) - \Lambda' v\;].$

The left side does not depend on v or β, while the right is independent of α and u. So there is a real number t with

$$\sup_{\substack{u\in Y'\\ \alpha>0}} \frac{1}{\alpha}\ [\ \Lambda' u - P(u-\alpha\ w)\ ]\ \le\ t\ \le\ \inf_{\substack{v\in Y'\\ \beta>0}} \frac{1}{\beta}\ [\ P(v+\beta\ w)-\Lambda' v\ ].$$

Define $\Lambda''\colon Y' \bigoplus \mathbb{R}w \to \mathbb{R}$ by $\Lambda''(v+rw) = \Lambda' v + rt$ for each $r\in\mathbb{R}$ and $v\in Y'$. Considering the cases of r positive, negative or zero separately, the definition of t yields

$$\Lambda''(v+rw) = \Lambda' v + rt \le \Lambda' v + P(v+rw) - \Lambda' v = P(v+rw).$$

So any function Λ' satisfying the conditions of the theorem and whose domain is not all of X can be extended to a larger subspace of X while preserving its relationship with P.

Let S be the set of all linear extensions of Λ to subspaces of X which are dominated by P on their domain. Partially order this set of extensions by $\Theta \le \Psi$ if Ψ is an extension of Θ. Chains in S have upper bounds in S and we invoke Zorn's lemma and assert that there is a maximal member Ψ of S. The domain of Ψ is X, else it could be extended by one dimension, contradicting maximality. $\square$

An inspection of the proof shows that for every vector not in the domain of a functional dominated by P there is a (nonempty) interval of real numbers for which the value of *any* functional extension of Λ must lie if it too is to be dominated by P. The point of the proof is that there always *is* a value that is consistent with the other, previously chosen, values of this function subject to domination by P, allowing for an extension to one more dimension. A typical Zorn's lemma argument finishes the job.

4.2. ***Corollary***. ***The Hahn-Banach Theorem***

If $Y^{\text{complex vector}}_{\text{subspace}} \subset X^{\text{complex vector}}_{\text{space}}$ *and* $P\colon X\to\mathbb{R}$ *satisfies*

$P(\alpha v+\beta u) \le |\alpha| P(v) + |\beta| P(u)$ *if* $u, v\in X$ *and* $|\alpha|+|\beta| = 1$

and if $\Lambda \in Y^*_{\mathbb{C}}$ *satisfies* $|\Lambda| \le P|_Y$

then $\exists \Psi \in X^*_{\mathbb{C}}$ *with* $\Lambda = \Psi|_Y$ *and* $|\Psi| \le P$.

Proof. Let L be the real part of Λ, thought of as a real linear functional. $\forall y\in Y,\ Ly \le |\Lambda y| \le P(y)$. Also, for real positive constants α and β the condition on P in the statement of this corollary reduces to convexity. So Theorem 4.1 applies: $\exists$ real linear $M\colon X\to\mathbb{R}$ extending L and with $Mx \le P(x)\ \forall x\in X$.

Let $\Psi x = Mx - iM(ix)\ \forall x \in X$. Ψ is real linear (because M is) and we check that $\Psi(ix) = i\Psi x$, so Ψ is actually a complex linear functional and extends Λ to all of X. It remains only to show that $|\Psi| \le P$.

Pick $x\in X$. Find angle θ so that $\Psi x = |\Psi x| \mathrm{e}^{i\theta}$.

$$\begin{aligned} \text{Then } |\Psi x| \ =\ (\Psi x)\mathrm{e}^{-i\theta} &= M\left(\mathrm{e}^{-i\theta}x\right) - iM\left(i\mathrm{e}^{-i\theta}x\right) \\ &= M\left(\mathrm{e}^{-i\theta}x\right) \quad \text{(the complex part must be zero)} \\ &\le P\left(\mathrm{e}^{-i\theta}x\right)\ \le \left|\mathrm{e}^{-i\theta}\right|\ P(x) = P(x). \end{aligned}$$

□

4.3. ***Lemma***. *Suppose M is a convex subset of the (real or complex) vector space V with internal point 0 and $x \notin M$. Then there is a real linear functional Ψ so that $\Psi x \geq 1$ but $\Psi y \leq 1$ for all $y \in M$. In other words, x is separated from M by this real linear Ψ.*

Proof. Let P be the Minkowski functional for M and the internal point 0 as in Exercise 3.2. Since $x \notin M$, $P(x) \geq 1$.

Define linear Λ on $\mathbb{R}x$ by $\Lambda(t\,x) = t\,P(x)$. If $t \geq 0$ then $\Lambda(t\,x) = P(t\,x) = t\,P(x) \geq t$. If $t < 0$ then $\Lambda(t\,x) = t\,P(x) < 0 \leq P(t\,x)$.

So by the Hahn-Banach Theorem, Λ can be extended to a real linear functional Ψ dominated by P on all of V. We saw in Exercise 3.2 that $P(y)$, and hence Ψy, cannot exceed 1 on M. □

4.4. ***Exercise***. *In Lemma 4.3, if $P(x) > 1$ then the real linear functional Ψ strongly separates x and M. This must happen, for instance, if $M = P^{-1}(\,[0,1]\,)$.*

Suppose M is an absorbing disk, and consider the situation of Lemma 4.3. Suppose $P(y) < 1$ for all y in M. We will call M **edgeless** in that case. The temptation to call it "open" is resisted to avoid clash with the related topological notion, and we will see later that the two ideas may not coincide.

For edgeless M, since M is balanced $y \in M$ exactly when $S_1 y \subset M$. Generally, $P(cy) = |c|P(y)$ so for those x with $P(x) = 1$ the vector cx is definitely not in M when $|c| \geq 1$, and definitely in M if $|c| < 1$.

Given real linear Ψ for any x with $P(x) = 1$, as in the lemma, note that $|\Psi|(cy)$ never exceeds $|c|P(y)$. That is because $\Psi(cy) \leq P(cy) = |c|P(y)$. And if $\Psi(cy)$ were ever less than $-|c|P(y)$ then $|c|P(y) = P(-cy) \geq \Psi(-cy) > |c|P(y) = P(cy)$, impossible.

Every member of M is in the open halfspace $\Psi^{-1}(\,(-\infty,1)\,)$ and x is not. Actually, every member of M is in the "slab" $\Psi^{-1}(\,(-1,1)\,) = |\Psi|^{-1}(\,[0,1)\,)$ which is, itself, an absorbing disk.

Finally, if z satisfies $P(z) > 1$ then choosing $x = z/P(z)$ we see that z is not in the halfspace either.

We conclude with the following proposition.

4.5. ***Proposition***. *Edgeless absorbing disks are the intersection of open halfspaces. The complement of an edgeless absorbing disk is the union of closed halfspaces.*

Proof. See the discussion above. □

So these absorbing disks are exactly the balanced absorbing sets that can be separated from the rest of the space by slicing away that which is not wanted using translates of real maximal subspaces. It might take, however, an infinite or even uncountably number of these real hyperplanes to nip off all the unwanted corners, even in the real two dimensional setting.

In that particular case, though, you can use linear combinations of just two functionals to create the necessary maximal subspaces (lines through the origin). In an infinite dimensional setting you will need ... more.

4.6. ***Exercise***. *(i) Modify the argument from above to accommodate* ***edged*** *absorbing disks: i.e. those for which* $M = P^{-1}(\,[0,1]\,)$*. These are the intersection of closed halfspaces.*

(ii) Is an edged absorbing disk the intersection of open halfspaces? Is an edgeless absorbing disk the intersection of closed halfspaces?

(iii) Can anything be done with disks that are not absorbing?

4.7. ***Exercise***. *(i)* ***(Convex Separation Theorem)*** *Suppose* M *and* N *are disjoint nonintersecting convex subsets of the vector space* V*, and the convex set* M *has an internal point. Then there is a real linear functional* Ψ *which separates* M *and* N*. (hint: Suppose* m *is internal in* M*. Then* $M \ominus \{m\}$ *and* $N \ominus \{m\}$ *are disjoint and* 0 *is internal in* $M \ominus \{m\}$*. Any* Ψ *separating* $M \ominus \{m\}$ *and* $N \ominus \{m\}$ *would also separate* M *and* N*, so without loss we assume* 0 *to be internal to* M*. Now select* $x \in N$*.* 0 *is also internal to the convex set* $M \ominus N + \{x\}$*, and* x *is not in* $M \ominus N + \{x\}$*. Look at the real linear functional of Lemma 4.3 for* $M \ominus N + \{x\}$ *and* $\{x\}$*.)*

(ii) If, following the hint in part (i), $P(x) > 1$ *where* P *is the Minkowski gauge for* $M \ominus N + \{x\}$ *with internal point* 0*, then* Ψ *strongly separates* M *and* N*.*

(iii) If V *is a complex vector space there is a complex linear functional* Θ *whose real part separates* M *and* N*. (hint: Let* $\Theta p = \Psi p - i\Psi(ip)$*.)*

5. Normed Linear and Banach Spaces

A **semi-normed linear space** is a vector space Y over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$, together with a semi-norm $\|\cdot\|\colon Y \to [0,\infty)$. If the semi-norm is a norm, Y together with this norm is called a **normed linear space**. The abbreviations **SNLS** and **NLS** are used for the phrases "semi-normed linear space" and "normed linear space" respectively.

5.1. ***Exercise***. *(i) Any semi-norm* $\|\cdot\|$ *gives rise to a pseudometric* ρ *defined by* $\rho(x,y) = \|x-y\|$*. This pseudometric is* ***translation invariant:*** $\rho(x+z,y+z) = \rho(x,y)\ \forall x,y,z \in Y$*. Also,* $\rho(\alpha x, \alpha y) = |\alpha|\,\rho(x,y)$ *for any* $\alpha \in \mathbb{F}$*. In fact, any pseudometric on a vector space that is translation invariant and also has this* ***homogeneity property*** *for scalars can be used to produce a semi-norm for which it is the pseudometric: let* $\|x\| = \rho(x,0)$*.*

Any SNLS can (and will, whenever convenient) be regarded as a pseudometric space with this pseudometric, and a topological space with the topology induced by this pseudometric.

(ii) Scalar multiplication and vector addition are jointly continuous (product topology on domain pairs) with the topology generated by this pseudometric, and this topology makes V *into a topological group with vector addition. (See Proposition 7.2 for the generalization to locally convex topologies.)*

(iii) This pseudometric is a metric exactly when the semi-norm is a norm.

The set of all continuous linear transformations in $\mathcal{H}_{\mathbb{F}}(V, W)$ will be denoted $\mathcal{CL}_{\mathbb{F}}(V, W)$. And we may use, from time to time, $\mathcal{CL}_{\mathbb{F}}(V)$ in place of $\mathcal{CL}_{\mathbb{F}}(V, V)$ to denote the set of **continuous operators** on the SNLS V. The vector space $\mathcal{CL}_{\mathbb{F}}(V)$ is also an algebra with composition as multiplication, non-commutative except in trivial cases, and this additional structure will be important later.

Here, and in later usage, topologies on a vector space V will come from different sources; there may be, for instance, more than one semi-norm on V. In that case the notation $\mathcal{CL}_{\mathbb{F}}(V, W)$ is ambiguous. Whenever this confusion is likely we will get more specific, identifying a topology $\mathcal{T}$ on V and $\mathcal{S}$ on W and then denote the continuous linear maps from V with topology $\mathcal{T}$ and W with topology $\mathcal{S}$ by a sufficiently detailed notation such as $\mathcal{CL}_{\mathbb{F}}(\,(V, \mathcal{T}),\,(W, \mathcal{S})\,)$ or $\mathcal{CL}_{\mathbb{F}}(\,(V, \|\cdot\|_1),\,(W, \|\cdot\|_2)\,)$.

If $W = \mathbb{F}$ we write $V'_{\mathbb{F}}$ instead of $\mathcal{CL}_{\mathbb{F}}(V, F)$. $V'_{\mathbb{F}}$ is called the **continuous dual of V**. It is a subset of the algebraic dual $V^*_{\mathbb{F}}$. Short forms neglecting to mention the field $\mathbb{F}$ will be used when that will not cause confusion.

We call an SNLS **complete** if it is complete with the pseudometric of the exercise above, which means that every Cauchy sequence[1] in the space converges to a point in the space. A complete NLS is called a **Banach space.**

Two (semi-)norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on Y are called **(semi)norm equivalent** if there are positive constants C_1 and C_2 with

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1 \quad \forall x \in Y.$$

This means, exactly, that the pseudometrics generated by these semi-norms are pseudometrically equivalent, and equivalent semi-norms produce the same Cauchy sequences.

If A is a nonempty subset of an $\mathbb{F}$-vector space let **span(A)** denote the set of all finite $\mathbb{F}$-linear combinations of members of A.

5.2. ***Exercise***. *(i) Two semi-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on Y are semi-norm equivalent if and only if the identity map from $(Y,\ \|\cdot\|_1)$ to $(Y,\ \|\cdot\|_2)$ is a homeomorphism.*

(ii) Every vector space has a norm. (hint: Let B be a basis of vector space Y. For $y = \sum_{i=0}^{n} y^i\, b_i$, where the b_i are distinct members of B and each y^i is a number, define $\|\, y \,\| = \sum_{i=0}^{n} |\, y^i \,|$.)

(iii) If Y is infinite dimensional not all norms on Y are equivalent. (hint: Define $\|\cdot\|$ on Y with basis B as in (ii). Let c_i for $i \in \mathbb{N}$ be an ordering of a countably infinite subset C of B and let $A = B - C$. So $Y = span(A) \oplus span(C)$. Define $\|\cdot\|_0$ on $span(C)$ by $\|\, y \,\|_0 = \sum_{i=0}^{n} \left| \frac{y^i}{i+1} \right|$ when $y = \sum_{i=0}^{n} y^i\, c_i$. Finally, for $y = y_C + y_A$ where $y_C \in span(C)$ and $y_A \in span(A)$ define $\|\, y \,\|_1 = \|\, y_C \,\|_0 + \|\, y_A \,\|$. Norm $\|\cdot\|_1$ is not equivalent to $\|\cdot\|$.)

(iv) If Y is finite dimensional all norms on Y are equivalent. (hint: Suppose Y is n dimensional with norm $\|\cdot\|$ and ordered basis $b_1, \ldots, b_n$. Let $e_1, \ldots, e_n$ be the

[1]Recall that a sequence x_n for $n \in \mathbb{N}$ is **Cauchy** exactly when, for every $\varepsilon > 0\, \exists N$ so that $m, n > N \Rightarrow \|\, x_n - x_m \,\| < \varepsilon$.

standard basis of $\mathbb{F}^n$*. Define isomorphism* $T\colon Y \to \mathbb{F}^n$ *by* $T(y) = \sum_{i=1}^n y^i\, e_i$ *when* $y = \sum_{i=1}^n y^i\, b_i$*. Define norm* $\|\cdot\|_0$ *on* $\mathbb{F}^n$ *by* $\|x\|_0 = \left\|T^{-1}x\right\|$*.*

We will show that $\|\cdot\|_0$ *is equivalent to the norm* $\|\cdot\|_1$ *on* $\mathbb{F}^n$ *given by* $\|\, a \,\|_1 = \sum_{i=1}^n \left|a^i\right|$ *for each* $a = \sum_{i=1}^n a^i\, e_i$ *and the desired result about norms on* Y *follows.*

First, if M *is the maximum value of* $\|e_i\|_0 = \|b_i\|$ *for* $i = 1, \ldots, n$ *then the triangle inequality implies*

$$\|\, a \,\|_0 \;\le\; \sum_{i=1}^n \left|a^i\right| \,\|\, e_i \,\|_0 \;\le\; M \sum_{i=1}^n \left|a^i\right| \;=\; M\|\, a \,\|_1.$$

If $\mathbb{F}^n$ *is endowed with the usual topology, the one generated by norm* $\|\cdot\|_1$*, this inequality and the fact that* $|\,\|\, x \,\|_0 - \|\, y \,\|_0\,| \le \|\, x - y \,\|_0$ *shows that* $\|\cdot\|_0$ *is a continuous function.*

The set of members of $\mathbb{F}^n$ *that satisfy* $\|\, x \,\|_1 = 1$ *is compact by the Heine-Borel theorem. Since* $\|\cdot\|_0$ *is continuous it attains a minimum value* L *on this set, and this minimum value cannot be* 0*. Then homogeneity of the norm* $\|\cdot\|_0$ *implies* $L\|\, a \,\|_1 \le \|\, a \,\|_0$ *for all* a*.)*

(v) If $\|\cdot\|_1$ *is a semi-norm on* Y *then* $N = \{\, x \in Y \mid \|x\| = 0 \,\}$ *is a closed subspace of* Y*. The function* $\|\cdot\|_2\colon Y/N \to [0, \infty)$ *defined by* $\|y + N\|_2 = \|y\|_1$ *is a norm on the quotient space* Y/N*. The quotient space is Banach exactly when* Y *is complete.*

5.3. ***Lemma***. *If* A *is a subset of an NLS* X *and* $span(A)$ *is finite dimensional then* $span(A)$ *is closed. But if* A *is countable and* $span(A)$ *is not finite dimensional and* X *is Banach then* $span(A)$ *is not closed. So an infinite dimensional Banach space has uncountable Hamel dimension.*

PROOF. $span(A)$ is itself an NLS using the restriction of the norm from X. It is easy to show that any Cauchy sequence in a finite dimensional subspace of an NLS converges in that subspace (identify that subspace with $\mathbb{F}^n$ as in Exercise 5.2) and so finite dimensional subspaces are closed in any NLS.

Now suppose X is Banach and $span(A)$ is not finite dimensional for countable A. We suppose, to obtain contradiction, that $span(A)$ is closed. In that case, $span(A)$ is itself a Banach space with restriction norm, and so is, itself, a complete metric space. Complete metric spaces are of second category, according to the Baire category theorem.

Let (a_i) be the sequence formed (without repeats) from the members of A, including all members somewhere in the sequence. Let $A_i = \{\, a_k \mid 1 \le k \le i \,\}$. So $span(A_i)$ is finite dimensional for each i, and therefore closed. Note that $span(A) = \bigcup_{i=1}^\infty span(A_i)$ so

$$\varnothing = span(A) - \bigcup_{i=1}^\infty span(A_i) = \bigcap_{i=1}^\infty \left(span(A) - span(A_i)\right).$$

But each $span(A) - span(A_i)$ is open in $span(A)$ and its closure is easily seen to be all of $span(A)$ and therefore dense. But the intersection of such sets cannot be empty in a second category space. This contradiction implies that $span(A)$ is not closed, and so in particular cannot be all of X. So no countable subset of X can span X unless X is finite dimensional. $\square$

We remark that the same argument works for any vector space with a topology making the vector operations continuous (we used this continuity twice) provided it is of second category.

5.4. ***Lemma***. *If A is a closed subspace of an NLS X and F is a finite dimensional subspace of X then $A \oplus F$ is closed.*

PROOF. Suppose, to obtain contradiction, that $A \oplus F$ is not closed. Then there must be members $a_i \in A$ and $f_i \in F$ for $i \in \mathbb{N}$ for which $y_i = a_i + f_i$ converges to a vector $y \notin A \oplus F$.

Should the f_i contain any bounded subsequence, this subsequence will itself have a subsequence f_{i_n} converging to some member $f \in F$, since the part of the unit sphere in finite dimensional F is compact. Then the sequence $y - f_{i_n} = a_{i_n}$ also converges, in this case to $a = y - f$. But A is presumed to be closed so $a \in A$ and then $y = a + f \in A \oplus F$, contrary to assumption.

So we may presume that there is an unbounded subsequence of f_i. In particular, we may assume (by going to a subsequence if necessary) that $\| f_i \| > i$ for every i. Now let's examine

$$z_i = \frac{y_i}{\| f_i \|} = \frac{f_i}{\| f_i \|} + \frac{a_i}{\| f_i \|}.$$

Then z_i converges to the zero vector, and $f_i / \| f_i \|$ has norm 1 for each i. That means $f_i / \| f_i \|$ has a convergent subsequence $f_{i_n} / \| f_{i_n} \|$ whose limit f is on the unit sphere and in F. So $a_{i_n} / \| f_{i_n} \|$ also has a limit: it is $-f$ and is on the unit sphere and in $A \cap F = \{ 0 \}$.

With this final contradiction we conclude that $A \oplus F$ must be closed. $\square$

The result from above relies on the finite dimensional nature of one of the summands: it might surprise you to learn that the statement is false without that assumption. See Exercise 14.13 for more on this.

A member $T \in \mathfrak{H}_{\mathbb{F}}\left(V^{\mathrm{NLS}}, W^{\mathrm{NLS}}\right)$ is called **bounded** provided there is a *nonnegative* number c with

$$\|Tv\| \leq c \, \|v\| \text{ for every } v \in V.$$

If T is bounded, the infimum of the numbers c with $\|Tv\| \leq c \, \|v\|$ for every $v \in V$ is denoted $\|T\|$.

If T *is not* bounded, we could find a sequence v_i for which $\|Tv_i\| \to \infty$ but $\|v_i\| = 1$ for all i. Even more, by choosing a subsequence v_{i_k} for which $\|Tv_{i_k}\| > k^2$ for all k and defining $w_k = v_{i_k}/k$ we have $\|Tw_k\| \to \infty$ but $\|w_k\| \to 0$.

T is called **bounded below** provided there is a *positive* number c with

$$\|Tv\| \geq c \, \|v\| \text{ for every } v \in V.$$

We note here that if T *is not* bounded below, we could find a sequence v_i for which $\|Tv_i\| \to 0$ but $\|v_i\| = 1$ for all i. By choosing a subsequence v_{i_k} for which $\|Tv_{i_k}\| < 1/k^2$ for all k and defining $w_k = k \, v_{i_k}$ we have $\|Tw_k\| \to 0$ but $\|w_k\| \to \infty$.

5.5. ***Exercise***. *We will see in a moment that the bounded and continuous linear transformations between normed linear spaces coincide, but for now let $\mathfrak{B}$ denote the bounded members of $\mathfrak{H}_{\mathbb{F}}(V, W)$.*

(i) $\mathfrak{B}$ is a vector subspace of $\mathfrak{H}_{\mathbb{F}}(V, W)$.

(ii) $\|\cdot\|$ as defined above is a norm on $\mathfrak{B}$, called the ***operator norm****, and unless otherwise specified $\mathfrak{B}$ will be assumed to be an NLS endowed with this norm.*

5.6. ***Proposition***. *For $\Psi \in \mathfrak{H}_{\mathbb{F}}\left(V^{NLS}, W^{NLS}\right)$, the following are equivalent:*
(i) Ψ is continuous at a particular $x \in V$.
(ii) Ψ is uniformly continuous.
(iii) Ψ is bounded.

Proof. (i) $\Leftrightarrow$ (ii): Suppose Ψ is continuous at x. So for each $\varepsilon > 0$ there is a $\delta > 0$ so that $\|y - x\| < \delta$ implies $\|\Psi(y) - \Psi(x)\| < \varepsilon$. So if $\|z - w\| = \|(z - w + x) - x\| < \delta$, we find $\|\Psi(z - w + x) - \Psi(x)\| = \|\Psi(z - w)\| < \varepsilon$ and conclude that Ψ is continuous at each $w \in W$ with a uniform choice of δ for each ε.

(ii) $\Rightarrow$ (iii): If Ψ is not bounded then there is a sequence $x\colon \mathbb{N} \to W$ with $\lim_{n\in\mathbb{N}} \|x_n\| = 0$ and $\|\Psi x_n\| > 1$ for each n. So Ψ is not continuous at 0

(iii) $\Rightarrow$ (ii): If $\|\Psi\|$ exists and is non-zero, and if $\|w - v\| < \frac{\varepsilon}{\|\Psi\|}$ then $\|\Psi(w - v)\| < \varepsilon$ so Ψ is continuous. $\square$

It is easy to show that if $F \in \mathfrak{CL}_{\mathbb{F}}(V, W)$ and $G \in \mathfrak{CL}_{\mathbb{F}}(W, Z)$ where V, W and Z are normed linear spaces that

$$\| G \circ F \| \leq \| G \| \, \| F \|.$$

The **distance** between two nonempty subsets A and B of an NLS X is

$$\mathbf{d}(\mathbf{A}, \mathbf{B}) = \inf\{ \|a - b\| \mid a \in A, \ b \in B \}.$$

It is easy to show that this is a pseudometric on the collection of nonempty subsets of X, and in fact this pseudometric has a homogeneity condition

$$d(\, cA, \, cB \,) = |c| \, d(A, B) \quad \forall c \in \mathbb{F}.$$

The **distance** between a point v and a nonempty subset B is then given by the obvious modification $d(v, B) = d(\{v\}, B)$. The nonempty set B is closed exactly when $d(v, B) = 0$ implies $x \in B$.

5.7. ***Proposition***. *Suppose A is a subspace of an NLS X and $v \in X$ and $v \notin \overline{A}$. Then there is a member ϕ of X' with $\| \phi \| = 1$ and $\phi(v) = d(v, \overline{A})$ and $\overline{A} \subset Ker(\phi)$.*

Proof. Define ϕ on the subspace $Y = \mathbb{F}v \oplus \overline{A}$ of X by

$$\phi(\lambda \, v + a) = \lambda \, d \qquad \text{where} \quad d = d(v, \overline{A}).$$

If $w = \lambda \, v + a$ then

$$\|w\| = |\lambda| \left\| v - \frac{a}{-\lambda} \right\| \geq |\lambda| \, d(v, \overline{A}) = |\lambda| \, d.$$

So the operator norm of ϕ on Y cannot exceed 1. In particular, ϕ is continuous when restricted to Y, and $| \, \phi(w) \, | \leq \| \, w \, \|$ there.

On the other hand, we can select a sequence of vectors $a_i \in \overline{A}$ for which $\| v - a_i \|$ converges to d. Then $x_i = \frac{v-a_i}{\| v-a_i \|}$ has norm 1 and so $\phi(x_i) = \frac{d}{\| v-a_i \|}$ which converges to 1. Thus $\| \phi \| \geq 1$.

Coupled with the earlier inequality we have $\| \phi \| = 1$. Also $\phi(v) = d$ and $\phi(\overline{A}) = \{ 0 \}$.

Now extend ϕ to all of X using the Hahn-Banach theorem using the norm as dominating function on all of X. □

Note that if $A = \{ 0 \}$ in this proposition we have found, as a special case, continuous ϕ for which $\| \phi \| = 1$ and $\phi(v) = \| v \|$. This useful and important observation has many consequences, as we shall see; the following lemma is one.

We will call a sequence x_n, $n \in \mathbb{N}$ in an NLS X **uniformly weakly Cauchy** if for every $\varepsilon > 0$ there is an integer N so that

$$m, n \geq N \Rightarrow | \phi(x_n - x_m) | \leq \|\phi\| \varepsilon \quad \forall \phi \in X'.$$

In other words, the functional differences can be made small simultaneously for every functional in the unit ball in X'.

5.8. ***Lemma***. *Suppose x_n, $n \in \mathbb{N}$ is a sequence in an NLS X. The sequence is Cauchy in the norm of X if and only if it is uniformly weakly Cauchy.*

PROOF. Since $| \phi(x_n - x_m) | \leq \| \phi \| \, \| x_n - x_m \|$ it is clear that if the sequence is Cauchy it will be uniformly weakly Cauchy.

On the other hand, if the sequence is *not* Cauchy, there is a $\varepsilon > 0$ so that for every N there are n_N and m_N for which $\| x_{n_N} - x_{m_N} \| > \varepsilon$. By the remark above there is a functional ϕ_N with $\| \phi_N \| = 1$ and $| \phi_N(x_{n_N} - x_{m_N}) | = \| x_{n_N} - x_{m_N} \| > \varepsilon$. So the sequence is not uniformly weakly Cauchy. □

5.9. ***Exercise***. *Suppose V is an NLS and V' is separable. That means there is a countable set of continuous functionals ϕ_i, $i \in \mathbb{N}$, which is dense in V'.*

(i) Show that the set of functionals of the form $\tau_i = \phi_i / \| \phi_i \|$ (delete any instances of the zero functional first) is dense in the unit sphere in V'.

(ii) So there is a sequence of unit vectors v_i in V for which $\tau_i(v_i) > 1/2$ for all i. Let $\overline{A}$ denote the closure of the span of these vectors.

(iii) Suppose $v \in V - \overline{A}$. By Proposition 5.7 there is a functional $\mu \in V'$ of norm one with $\mu(v) = d = d(v, \overline{A}) > 0$ and for which $\mu(v_i) = 0$ for all i.

(iv) For each i we have $1/2 < \tau_i(v_i) = \tau_i(v_i) - \mu(v_i) \leq \| \tau_i - \mu \|$. But this contradicts the assumption that the τ_i are dense in the unit sphere in V'. Therefore $V - \overline{A}$ is empty.

(v) The complex numbers with rational real and complex part form a countable dense subset of the complex numbers. So the set of finite linear combinations using these "rational" complex numbers and the vectors v_i for $i \in \mathbb{N}$ is dense in $\overline{A} = V$.

(vi) ***Conclude that if $\mathbf{V'}$ is separable so is $\mathbf{V}$.***

(vii) The converse of this statement is false. In Section **??** *we saw that $\boldsymbol{\ell^1}$, which is separable, has dual $\boldsymbol{\ell^\infty}$, which is not separable. (hint: For each subset A of $\mathbb{N}$ the characteristic function χ_A is in $\boldsymbol{\ell^\infty}$ and each characteristic function of this type is distance 1 from every other. Looking at the ball of radius $1/2$ around*

each we have an uncountable number of disjoint balls in $\boldsymbol{\ell}^{\infty}$. No countable dense set can have a member in each.)

A result, similar in flavor, is given next. It says, essentially, that there are parts of the unit sphere in any Banach space that are a substantial distance away from any proper closed subspace, nearly the maximum possible distance of 1.

5.10. ***Proposition***. ***Riesz' Lemma***
Suppose A is a subspace of a Banach space X and $X \neq \overline{A}$. Then for every t with $0 < t < 1$ there is a member $x \in X$ with $\| x \| = 1$ and $d\left(x, \overline{A}\right) \geq t$.

PROOF. Suppose $y \in X - \overline{A}$ and let $d = d\left(y, \overline{A}\right)$. So $d > 0$, else there would be a sequence in $\overline{A}$ converging to y and then $y \in \overline{A}$, contrary to assumption.

For any $\varepsilon > 0$ select $a_\varepsilon \in \overline{A}$ so that $\| a_\varepsilon - y \| < d + \varepsilon$. and then define

$$x_\varepsilon = \frac{a_\varepsilon - y}{\| a_\varepsilon - y \|}.$$

So $$d\left(x_\varepsilon, \overline{A}\right) = \inf\left\{ \| x_\varepsilon - a \| \mid a \in \overline{A} \right\} = \inf\left\{ \left\| \frac{a_\varepsilon - y}{\| a_\varepsilon - y \|} - a \right\| \mid a \in \overline{A} \right\}$$
$$= \inf\left\{ \left\| \frac{a - y}{\| a_\varepsilon - y \|} \right\| \mid a \in \overline{A} \right\} = \frac{\inf\left\{ \| a - y \| \mid a \in \overline{A} \right\}}{\| a_\varepsilon - y \|} = \frac{d}{d + \varepsilon}.$$

ε can now be chosen to satisfy the requirement for any t with $0 < t < 1$. □

Sometimes Riesz' Lemma is phrased in the following (obviously equivalent) form: If A is a subspace of a Banach space X and $X \neq \overline{A}$ then there is a sequence x_n of unit vectors in X for which $d\left(x_n, \overline{A}\right) \to 1$.

5.11. ***Exercise***. *Use Reisz' Lemma to show that the unit sphere in an infinite dimensional Banach space cannot be compact.*

5.12. ***Exercise***. *We will show here that if V is an infinite dimensional NLS then V' cannot be all of V^*: there are linear functionals which are not continuous.*

Let S be a a Hamel (i.e. ordinary vector space) basis for V and let $x_0, x_1, \ldots$ be a countably infinite ordered list of distinct members of S with $\| x_i \| = 1$ for each i. Define $f(x_i) = i$ for each $i \in \mathbb{N}$ and define $f(s) = 0$ for the remaining members of S. Extend f to all of V by linearity. This function is not bounded.

A function $T\colon V \to W$ between vector spaces V and W is called **conjugate linear** if $T(v + \alpha\, w) = T(v) + \overline{\alpha}\, T(w)$ for all $v, w \in V$ and $\alpha \in \mathbb{F}$. If a one-to-one linear or conjugate linear function T is onto W it has an inverse function, and that inverse is also linear or, respectively, conjugate linear.

5.13. ***Exercise***. *(i) A linear or conjugate linear function T between normed linear spaces is an* ***isometry*** *if and only if $\|x\| = \|Tx\|$ for each $x \in V$. Note that one must deduce from context which norm we mean (the one on V or the one on W) in an expression such as this one. An isometry is continuous and one-to-one. If T is an isometry onto W then T^{-1} is also an isometry.*

(ii) Make the necessary definitions and prove for conjugate linear functions the analogue of Proposition 5.6.

(iii) Suppose V and W are Banach spaces and A is a vector subspace (not necessarily closed) of V. $\overline{A}$ is the topological closure of A, the set of limits of all Cauchy sequences of members of A, and is itself a Banach space with norm from V. Suppose $T\colon A \to W$ is a linear isometry. Then T can be extended in a unique way to a continuous function $G\colon \overline{A} \to W$ and $G(\overline{A}) = \overline{T(A)}$. So thought of as as a map $G\colon \overline{A} \to G(\overline{A})$ the function G is an invertible isometry between two Banach spaces.

We will refer to a member of $\mathcal{CL}_{\mathbb{F}}(V, W)$ as an **isomorphism of normed linear spaces** provided it is a homeomorphism: that is, it is continuous with continuous inverse. In light of Proposition 5.6, an invertible member of $\mathcal{CL}_{\mathbb{F}}(V, W)$ is an isomorphism in this sense exactly when the inverse is in $\mathcal{CL}_{\mathbb{F}}(W, V)$.

5.14. ***Proposition***. *Suppose $\Psi \in \mathcal{CL}_{\mathbb{F}}\left(V^{Banach}, W^{NLS}\right)$. Ψ has an inverse function $\Psi^{-1} \in \mathcal{CL}_{\mathbb{F}}(W, V)$ exactly when Ψ is bounded below and $\Psi(V)$ is dense in W.*

PROOF. If Ψ has an inverse function at all then $\Psi(V)$ is not only dense in W, it must equal W. And it is easy to show that if Ψ is not bounded below then Ψ^{-1} cannot be continuous. So the necessity of the two conditions is clear.

On the other hand, suppose $\Psi(V)$ is dense in W and $\Psi(v_i)$ for $i \in \mathbb{N}$ is a Cauchy sequence in $\Psi(V)$. If Ψ is bounded below there is a greatest positive constant c for which $\|\Psi(x)\| \geq c\,\|x\|$ for all $x \in V$. It follows that $\|\Psi(v_i) - \Psi(v_j)\| \geq c\,\|v_i - v_j\|$ so the sequence of vectors v_i is Cauchy in Banach V and therefore converges to a vector $v \in V$. By continuity of Ψ we must have $\Psi(v_i)$ converging to $\Psi(v)$, so W is complete too and in fact $\Psi(V) = W$. It follows easily that $\left\|\Psi^{-1}(x)\right\|$ can approach but never exceed $\|x\|/c$ so $\left\|\Psi^{-1}\right\| = 1/c$. $\square$

For more along this line see Lemma 10.5 and the exercise that follows it.

5.15. ***Exercise***. *Suppose B is a countable Hamel basis for a vector space V and that $b_0, b_1, b_2, \ldots$ is an enumeration of the members of this basis. Every member of V has a unique representation as $\sum_{n\in\mathbb{N}} a_n b_n$ where only finitely many of the a_n are non-zero. Define $\|\sum_{n\in\mathbb{N}} a_n b_n\|$ to be the maximum value among the numbers $|a_n|$, $n \in \mathbb{N}$. Define $T\colon V \to V$ by $T\left(\sum_{n\in\mathbb{N}} a_n b_n\right) = \sum_{n\in\mathbb{N}} \frac{a_n}{n+1} b_n$.*

(i) $\|\cdot\|$ is a norm on V but V is not complete. (The sequence $v_k = \sum_{n=0}^{k} \frac{1}{n+1} b_n$ is Cauchy but does not converge to any member of V.)

(ii) T is bounded and $\|T\| = 1$.

(iii) T has an inverse but T^{-1} is not bounded.

5.16. ***Proposition***. *$\mathcal{CL}_{\mathbb{F}}\left(V^{NLS}, W^{Banach}\right)$ is Banach with operator norm. In particular, $V'_{\mathbb{F}}$ is Banach.*

Proof. We need only demonstrate completeness. Suppose $A\colon \mathbb{N} \to \mathfrak{CL}_{\mathbb{F}}(V, W)$ is Cauchy. So for each $x \in V$ the sequence formed by $A_n x$ is Cauchy in W. Since W is complete $A_n x$ converges to some point $B(x)$ in W. Because each A_n is linear, so too is the function B.

It remains to show that B is bounded and A converges to B in operator norm.

Note that $|\ \|A_n\| - \|A_m\|\ | \le \|A_n - A_m\|$. So the numbers $\|A_n\|$ converge to some non-negative number C. For each $x \in V$, $\|Bx\| = \lim_{n\to\infty} \|A_n x\| \le C\|x\|$. So B is bounded.

Select k so large that $m, n > k$ implies $\|A_m - A_n\| < \varepsilon$. So if x is non-zero,

$$\frac{\|(B - A_n)x\|}{\|x\|} \le \frac{\|(B - A_m)x\|}{\|x\|} + \frac{\|(A_m - A_n)x\|}{\|x\|}.$$

As m grows the first term on the right converges to 0 and the second cannot exceed ε. We conclude that

$$\frac{\|(B - A_n)x\|}{\|x\|} \le \varepsilon \text{ for } n > k \text{ and any non-zero } x.$$

Since ε was an arbitrary positive number, we find that $\lim_{n\to\infty} \|B - A_n\| = 0$ and A converges to B in operator norm.

Since $|\ \|B\| - \|A_n\|\ | \le \|B - A_n\|$ and the last quantity can be made as small as desired provided only that n is large enough, we observe that $\|B\| = C$. $\square$

A mapping $\Psi^{\text{Linear}}\colon X^{\text{NLS}} \to Y^{\text{Banach}}$ is called **compact** exactly when $\overline{\Psi(S)}$ is a compact subset of Y whenever S is a bounded subset of X.

If compact, a function Ψ must be bounded, and so compact linear functions are continuous.

When we discuss compact sets in a more general context (see Proposition 7.13 and Exercise 7.14) we will see that Ψ is compact if and only if it takes bounded sets to *totally* bounded sets.

For now there is an important equivalent condition, which reduces the question of whether a mapping is compact or not to an issue involving sequences.

Suppose (x_i) is a **bounded sequence** in X: that is, its range is a bounded set. Then compactness of Ψ requires $\{\, \Psi(x_i) \mid i \in \mathbb{N} \,\}$ to have compact closure in Y. This implies that the sequence of image points $(\Psi(x_i))$ has a Cauchy subsequence.

Conversely, if this last condition must necessarily hold for any bounded sequence in X then Ψ is a compact mapping.

5.17. ***Lemma***. *$\Psi\colon X^{NLS} \to Y^{Banach}$ is compact exactly when either of the following equivalent conditions pertain.*

(i) Ψ takes the unit ball in X to a set with compact closure in Y.

(ii) Ψ takes any sequence in the unit ball in X to a sequence with a convergent subsequence in Y.

PROOF. Let D be the unit ball of X. Every bounded set B in X is contained in nD for some positive n. So $\frac{1}{n}B \subset D$. So $\Psi\left(\frac{1}{n}B\right) \subset \Psi(D)$. So $\overline{\Psi\left(\frac{1}{n}B\right)} \subset \overline{D}$. A closed subset of a compact set is compact, and continuity of scalar multiplication finishes the argument for (i).

Condition (ii) now follows from (i) and the remarks preceding this lemma. □

5.18. ***Lemma***. *If $\Psi\colon X^{NLS} \to Y^{Banach}$ is compact and $A\colon Y \to Z^{Banach}$ and $B\colon W^{NLS} \to X$ are continuous then $A\,\Psi\,B$ is compact.*

PROOF. Show that under the given conditions a bounded sequence in W must be taken to a sequence with a convergent subsequence in Z. □

Define $\mathfrak{K}_{\mathbb{F}}(\mathbf{X},\mathbf{Y})$ to be the set of compact linear functions from X to Y with operator norm. The special case where $Y = X$ will be denoted $\mathfrak{K}_{\mathbb{F}}(\mathbf{X})$, the compact operators on X.

5.19. ***Proposition***. *The compact linear maps $\mathfrak{K}_{\mathbb{F}}(X,Y)$ from NLS X to Banach space Y are closed in $\mathfrak{CL}_{\mathbb{F}}(X,Y)$ and therefore form, themselves, a Banach space with operator norm.*

PROOF. Suppose $\Psi_n\colon X \to Y$, $n \in \mathbb{N}$, is a sequence of compact maps and $\lim_{n\to\infty} \Psi_n = A$, where the limit is taken with respect to operator norm.

Suppose (x_i) is a sequence in X for which $\|x_k\| < B$ for all k: that is, it is a bounded sequence. There is a subsequence $\left(x_i^0\right)$ with $\|\Psi_0(x_j^0 - x_i^0)\| < 1$ for all i and j because Ψ_0 is compact.

Having found subsequence (x_i^n) of the original bounded sequence for which $\|\Psi_n(x_j^n - x_i^n)\| < 1/(n+1)$ for all i and j select subsequence $\left(x_i^{n+1}\right)$ of (x_i^n) for which $\|\Psi_{n+1}(x_j^{n+1} - x_i^{n+1})\| < 1/(n+1+1)$ for all i and j. And now define sequence (y_i) by $y_k = x_0^k$ for all $k \in \mathbb{N}$.

Then the sequence $(A(y_k))$ converges in Y. To see this, we suppose $\varepsilon > 0$ and choose n so large that $1/n < \varepsilon$ and also that $\|A - \Psi_n\| < \varepsilon$. Then we have for all k and m exceeding n

$$\begin{aligned}\|A(y_k - y_m)\| &= \|A(y_k - y_m) - \Psi_n(y_k - y_m)\| + \|\Psi_n(y_k - y_m)\| \\ &< \|A - \Psi_n\|\,\|y_k - y_m\| + \|\Psi_n(y_k - y_m)\| < 2B\varepsilon + \varepsilon.\end{aligned}$$

Since B is fixed and ε can be chosen to be arbitrarily small, $A(y_k)$ is a Cauchy sequence in complete Y which therefore converges in Y.

Our conclusion is that A is compact too. □

If a continuous linear map $\Psi\colon X^{\text{Banach}} \to Y^{\text{Banach}}$ has **finite rank** (that is, $\Psi(X)$ is finite dimensional) it is clearly compact. (Apply the Heine-Borel theorem to $\Psi(X)$ with restriction norm.)

The range of any finite rank mapping is separable—look at the rational linear combinations of any Hamel basis. But this is also true of any compact mapping.

5.20. ***Lemma***. *If $\Psi \in \mathfrak{K}\left(X^{Banach}, Y^{Banach}\right)$ then $\Psi(X)$ is separable.*

PROOF. Let B be the unit ball in X and D the unit ball in Y. Suppose Ψ to be compact. The vector subspace $\Psi(X)$ of Y can be written as

$$\Psi(X) = \bigcup_{n=1}^{\infty} \Psi(nB) = \bigcup_{n=1}^{\infty} n\Psi(B).$$

By Exercise 7.14 each $\Psi(nB)$ is totally bounded. So for each n there are members $y_1^n, \ldots, y_{i_n}^n$ of $\Psi(nB) = n\Psi(B)$ for which

$$n\Psi(B) \subset \bigcup_{k=1}^{i_n} \left(y_k^n + \frac{1}{n} D \right).$$

If y is any member of $\Psi(X)$ then it is in all $\Psi(nB)$ for large enough n. If $\mathcal{O}$ is any open set containing y then $y + \frac{2}{n} D$ is inside $\mathcal{O}$ for all n exceeding some value. And $y \in y_k^n + \frac{1}{n} D$ for one of the y_k^n. But then

$$y_k^n + \frac{1}{n} D \subset y + \frac{2}{n} D \subset \mathcal{O}.$$

The set of all these y_k^n is countable, and form the necessary countable dense subset.
$\square$

A Banach space Y is said to have the **approximation property** if the identity operator can be uniformly approximated by continuous finite rank maps on each compact subset of Y.

Specifically, this means that for each compact $B \subset Y$ and each $\varepsilon > 0$ there is a continuous finite rank map $F\colon Y \to Y$ (depending on B and ε) for which

$$\sup\{\{\|x - F(x)\| \mid x \in B\} < \varepsilon.$$

We will show that when Y has this property every member of $\mathcal{K}(X, Y)$ is an operator norm limit of continuous finite rank operators, a desirable feature referred to as the **approximation property** for these compact operators.

It is known that some Banach spaces *do not* have the approximation property. We will show that if Y is a Hilbert space (see Section 14) or any Banach space with a Schauder basis (see Section 11) then Y *does* have this property.

The approximation property is key to finding solutions to some of the problems that generated the field of functional analysis to begin with, including study of Fredholm integral equations, and we will bring the matter up again later.

6. Topological Vector Spaces

If, for a given topology on vector space V, the vector space operations are continuous we say that V is a **topological vector space**, or simply a **TVS**.

Continuity of scalar multiplication is not implied by continuity of vector addition. All topological vector spaces are path connected—scalar multiplication yields a continuous path from any point to 0—but continuity of vector addition alone does not require this. The discrete topology provides the extreme example.

On that note, the same continuous path shows that 0 is an internal point of every open neighborhood of 0 which, in the real case, is enough to conclude that every open neighborhood of 0 is absorbing.

But actually, more is true. In the complex case scalar multiplication is a continuous function from $\mathbb{C} \times V \to V$. So the inverse image of any neighborhood $\mathcal{O}$ of 0 is a neighborhood of $(0, 0)$ in $\mathbb{C} \times V$. So there is an $\varepsilon > 0$ and an open neighborhood $\mathcal{O}_1$ of 0 in V for which basic open $S_\varepsilon \times \mathcal{O}_1$ is contained in this inverse image. Then $S_\varepsilon \mathcal{O}_1 \subset \mathcal{O}$. And $S_\varepsilon \mathcal{O}_1$ is balanced. Coupled with the fact that 0 is interior, we find that $S_\varepsilon \mathcal{O}_1$ is absorbing. Containing an absorbing set, we find that every neighborhood of 0 is absorbing in the complex case too.

6.1. ***Lemma***. *Every neighborhood of* 0 *in a TVS contains a balanced open neighborhood of* 0*, which is interior, so every neighborhood of* 0 *is absorbing.*

PROOF. See the remarks above □

The subject of topological vector spaces can draw on the material developed in Section **??** on topological groups. There the group operation is given in multiplicative notation, but here the group operation is vector addition. Commutativity of vector addition simplifies many of the results and changes their appearance somewhat. We will prove again a few of these as a reminder, and to take advantage of simplifications found in our setting.

In a topological vector space V, if S is an open neighborhood base or subbase at 0 then the set of translates of the form $x + B$ for $B \in S$ is a neighborhood base or subbase at $x \in V$. **So we just need this one neighborhood base or subbase to define a topology on all of V compatible with the vector operations.**

Moreover, if $A \in S$ then $-A$ is also open, and therefore so is $B = A \cap (-A)$. And B is symmetric. The collection of sets of this form is also an open neighborhood base at 0, so we may assume, whenever we need to, that our neighborhood base consists of symmetric sets. Even more, our earlier observation about continuity of scalar multiplication shows that there is actually an open neighborhood base at 0 consisting of *balanced* sets.

6.2. ***Lemma***. *Every TVS has an open neighborhood base at* 0 *consisting of balanced absorbing sets.*

PROOF. See the remarks above □

A net n converges to $x \in V$ precisely when the net $n - x$ (here, x is taken to be the constant net) is eventually in $\mathcal{O}$ for every $\mathcal{O} \in S$, where S is any neighborhood subbase at 0.

Suppose $f : V \to W$ is a function between two topological vector spaces and S is an open neighborhood base at 0 for V and T is an open neighborhood subbase at 0 for W.

f is called **continuous at x** $\in$ **V** if $f^{-1}(f(x) + \mathcal{O})$ is open in V for every $x \in V$ and $\mathcal{O} \in T$.

f is called **continuous** if it is continuous at each $x \in V$. This implies that the inverse image under f of any open set in W is open in V, and is equivalent to the condition: If n is any net in V that converges to 0 then for each $x \in V$ the net $f(x+n)$ converges to $f(x)$.

There are useful alternative phrasings of the continuity condition.

For instance, f is continuous if for each $x \in V$ and member $\mathcal{O}$ of a neighborhood subbase at 0 in W there is a member $\mathcal{B}$ of S for which $f(x+\mathcal{B}) \subset f(x)+\mathcal{O}$.

Also, f is continuous if, whenever n is any net in V that converges to 0 then for each $x \in V$ and member $\mathcal{O}$ of subbase T the net $f(x+n)$ is eventually in $f(x)+\mathcal{O}$.

f as above is called **uniformly continuous** if there is no x dependence in the alternative definitions of continuity mentioned above.

Specifically, f is uniformly continuous if for each member $\mathcal{O}$ of a neighborhood subbase at 0 in W there is a member $\mathcal{B}$ of S for which $f(x+\mathcal{B}) \subset f(x)+\mathcal{O}$ for every $x \in V$. This single $\mathcal{B}$ "works" for every x.

Adapting the net version, this means that if D is the directed set for net n and n converges to 0 then for each member $\mathcal{O}$ of subbase T there is a member d of D so that $t \geq d$ implies $f(x+n(t))-f(x) \in \mathcal{O}$ for all $x \in V$.

For *linear* functions, continuity and uniform continuity are equivalent concepts. An inspection of the last two paragraphs shows that linear f is uniformly continuous exactly when it is continuous at 0, just as in the normed case. For such functions, we can check continuity at 0 by verifying that for each $\mathcal{O}$ in T there is a member $\mathcal{B}$ of S with $f(\mathcal{B}) \subset \mathcal{O}$.

We now generalize the definition of $\mathcal{CL}_{\mathbb{F}}(V,W)$ and $V'_{\mathbb{F}}$ to denote the continuous members of $\mathcal{H}_{\mathbb{F}}(V,W)$ and $V^*_{\mathbb{F}}$, respectively, when V and W are *any* two topological vector spaces.

6.3. ***Exercise***. *Suppose V is a complex TVS and $f \in V^*$, the algebraic dual of V. Then $f(v)=\alpha(v)+i\,\alpha(iv)$ for all $v \in V$, where α is the real part of f.*
f is continuous if and only if α is continuous.

Once again, $\mathcal{CL}_{\mathbb{F}}(V,W)$ is a vector subspace of $\mathcal{H}_{\mathbb{F}}(V,W)$.

We saw before that when V and W are normed we could define the operator norm $\|\cdot\|$ on $\mathcal{CL}_{\mathbb{F}}(V,W)$.

In the normed case, a net T_ν of these functions converges to linear S in the **operator norm topology** provided that for all $\varepsilon > 0$ there is an index d so that $m \geq d$ implies

$$\|S-T_\nu\| = \sup\{\, \|S(x)-T_\nu(x)\| \mid x \in B_1 \,\} < \varepsilon$$

where B_1 is the unit ball in V.

There are numerous other notions of convergence in $\mathcal{CL}_{\mathbb{F}}(V,W)$. These are often useful when V and W are normed but are required when V and W are more general topological vector spaces. We will list the two most common of these alternative convergence criteria here.

We say that the net T_ν converges to S in the **strong operator topology** when $T_\nu(x)$ converges to $S(x)$ in W for every $x \in V$. If W is a normed space this means $\|S(x)-T_\nu(x)\|$ converges to 0 for each x, one at a time. This is just the topology of pointwise convergence for these functions.

We say that T_ν converges to S in the **weak operator topology** when the numbers $\phi(\,T_\nu(x)\,)$ converge to the number $\phi(\,S(x)\,)$ for every $x \in V$ and $\phi \in W'$. This means $|\,\phi(\,S(x)\,) - \phi(\,T_\nu(x)\,)\,|$ converges to 0 for each x and each continuous functional ϕ.

It is obvious that these three notions of convergence are progressively weaker, in the sense that it becomes easier for a net to converge as you go down the list. So an operator norm closed set is strong operator closed, and a strong operator closed set is weak operator closed. So the operator norm topology contains the strong operator topology, which itself contains the weak operator topology.

We return now to features of the topology on the topological vector space V.

The interior of a subset Q of V, denoted Q^o, is a neighborhood of each of its points and is the largest open subset of Q.

If S is an open neighborhood base at 0, for every $x \in Q^o$ there is an open set $\mathcal{O}_x \in S$ so that $x + \mathcal{O}_x \subset Q^o$. In that case,

$$Q^o = \bigcup_{x \in Q^o} (x + \mathcal{O}_x).$$

If A is any set and B is open, then of course $A + B$ and $A \ominus B$ and $B \ominus A$, unions of open sets, are open. But if A and B are closed $A + B$ need not be closed, even in dimension 1. (hint: let $A = \{\, n + \frac{1}{n} \mid n \text{ is an integer exceeding } 1\,\}$ and let $B = \{\, -n \mid n \text{ is an integer exceeding } 1\,\}$.)

6.4. ***Exercise***. *Suppose V is a TVS and A and B are compact. Suppose also that C is closed and α is a number.*

(i) αA and $A + B$ and $A \ominus B$ are all compact. (hint: $A \times B$ is compact in the product space so its image under addition is compact.)

(ii) αC and $B + C$ and $B \ominus C$ and $C \ominus A$ are closed.

Suppose $y \notin \overline{A}$, the smallest closed set containing A. Then there is a member $\mathcal{O}$ in the open neighborhood base S of 0 with $(y + \mathcal{O}) \cap A = \varnothing$. If $\mathcal{O}$ is symmetric this means $y \notin \mathcal{O} + A$. If $\mathcal{O}$ is not symmetric, then it contains a symmetric subset, which itself contains an even smaller member of S. We find that $\overline{A} \supset \bigcap_{\mathcal{O} \in S} (\mathcal{O} + A)$. On the other hand, if $y \in \overline{A}$ and $\mathcal{O} \in S$ let $\mathcal{P}$ be a symmetric open subset of $\mathcal{O}$. There is a member $a = y + p$ in $A \cap (y + \mathcal{P})$, where $a \in A$ and $p \in \mathcal{P}$. Since $\mathcal{P}$ is symmetric $-p \in \mathcal{P}$. So $y = a - p = a + (-p) \in A + \mathcal{P} \subset A + \mathcal{O}$.

So we have shown:

6.5. ***Lemma***. *If S is an open neighborhood base at 0 in a TVS then for any set A of vectors*

$$\overline{A} = \bigcap_{\mathcal{O} \in S} (\mathcal{O} + A).$$

PROOF. See the remarks above. $\square$

As an aside, if the members of S and A itself are balanced, so too will be each term in the intersection of the last lemma, so $\overline{A}$ will be too. If the members of S and A itself are convex, $\overline{A}$ will be too.

Continuity of addition has other interesting consequences. For instance, if A is any open neighborhood of 0, the inverse image of vector addition is a neighborhood of each point $(x, -x)$ in the product space $V \times V$. Then there are basic open sets $(x + B) \times (-x + C)$ contained in this inverse image, where B and C are basic open neighborhoods of 0. This implies that $B + C \subset A$. It follows that for any neighborhood A of 0 there is a symmetric (even balanced if you wish) neighborhood D with $D + D = D \ominus D \subset A$.

Note that since $\overline{D} = \bigcap_{\mathcal{O} \in S}(\mathcal{O} + D)$, and there is a member of S which is a subset of D, the closed neighborhood $\overline{D}$ satisfies $\overline{D} \subset D + D \subset A$.

6.6. ***Lemma***. *If A is a neighborhood of 0 in a TVS then there is a balanced open neighborhood D of 0 with $D + D \subset A$ and $\overline{D} \subset A$. This implies that there is a balanced open neighborhood base at 0 for which the closures of its members form, themselves, a balanced closed neighborhood base at 0.*

Proof. See the remarks above □

6.7. ***Exercise***. *Suppose V is a TVS. Prove, consecutively:*
(i) V is T_3: that is, if $x \notin A$ and A is closed then x and A can be separated by open sets. (hint: There are open symmetric H and D for which $x \in H \subset \overline{H} \subset D \subset \overline{D} \subset V - A$.)
(ii) $y \in \overline{\{x\}}$ if and only if $x \in \overline{\{y\}}$ if and only if $x - y \in \overline{\{0\}}$.
(iii) If V is T_0 then V is T_2.
(iv) $\overline{\{0\}}$ is a closed subspace of V.
(v) $V/\overline{\{0\}}$ with quotient topology is a TVS and T_2.

A subset B of V is called **bounded** if it is absorbed by every neighborhood of 0. Given a neighborhood subbase S at 0, B is bounded if every member of S absorbs B.

6.8. ***Exercise***. *Suppose that V is a T_2 topological vector space and there is a bounded open set B containing 0. Let S be a neighborhood base at 0 for the topology. Then for each member A of S there is a positive integer k for which $\frac{1}{k}B \subset A$, and $\frac{1}{k}B$ is itself open. So sets of the form $\frac{1}{k}B$ constitute a neighborhood base at 0: that is, V is C_I. In Exercise **??** we show that when V is C_I there is a translation invariant metric which generates the topology on V. So unless we are dealing with a metrizable TVS, no open set in a T_2 TVS can be bounded. If the T_2 condition is dropped can a bounded open set be used to produce a translation invariant pseudometric which generates the topology?*

The set B is called **totally bounded** if for every neighborhood $\mathcal{O}$ of 0 there are a finite number of elements $b_1, \ldots, b_n$ in B for which $B \subset \bigcup_{i=1}^{n}(b_i + \mathcal{O})$.

6.9. ***Lemma***. *Totally bounded sets are bounded.*

Proof. Suppose B to be totally bounded and let $\mathcal{O}$ be a neighborhood of 0. Select open balanced D for which $D + D \subset \mathcal{O}$. Find members $b_1, \ldots, b_n$ in B for which $B \subset \bigcup_{i=1}^{n}(b_i + D)$. Select number n so that $|t| \geq n$ implies $b_i \in tD$ for all i. Require, if necessary, that n be at least 1. So for $t \geq n$

$$B \subset tD + D = tD + t(1/t)D \subset tD + tD \subset t\mathcal{O}.$$

□

6.10. ***Exercise***. *(i) If c is a scalar, v a vector and sets A and B are (totally) bounded then cA, $v + A$, $A + B$ and $A \ominus B$ are (totally) bounded.*

(ii) If $\mathcal{O}$ is a neighborhood of 0, since $B \subset B + \mathcal{O}$ we easily conclude that every compact set is totally bounded.

(iii) In an infinite dimensional NLS the unit ball is bounded but not totally bounded.

6.11. ***Exercise***. *In an SNLS a set B is bounded when it is absorbed by the unit ball D. B is totally bounded if, for every $\varepsilon > 0$, there are a finite number of elements $b_1, \ldots, b_n$ in B for which $B \subset \bigcup_{i=1}^{n}(b_i + \varepsilon D)$.*

A *function* between two topological vector spaces is called **bounded** if the image of every bounded set is bounded. This does agree with the definition in semi-normed spaces, where bounded sets are just those inside some ball defined by the semi-norm.

6.12. ***Exercise***. *(i) Continuous functions (not necessarily linear) between topological vector spaces are bounded.*

(ii) A uniformly continuous function (again, not necessarily linear) between topological vector spaces takes totally bounded sets to totally bounded sets in the image space.

7. Locally Convex and Frechét Spaces

There are various ways that topologies can be defined on a vector space. Earlier we examined semi-norms. In important cases, a vector space V comes equipped with a family $\mathcal{F}$ of semi-norms rather than a single semi-norm which, together, define convergence in the space.

The family of semi-norms generates a family $\mathcal{S}_{\mathcal{F}}$ of pseudometrics and family $\mathcal{B}_{\mathcal{F}}$ of open *unit* balls, one pseudometric and one ball for each semi-norm.

The topology we intend is the one whose open sets are all unions of finite intersections of open balls of all possible radii produced by any of these pseudometrics.

A topology of this kind, which comes from (or could have come from) a family of semi-norms will be called a **locally convex topology**.

Since each pseudometric is translation invariant, the topology can be given by specifying a neighborhood base or subbase S at 0. Members of a neighborhood base or subbase at x are then given as $x + B$ where $B \in S$.

Since the semi-norms (and, analogously, the pseudometrics) are homogeneous, any open ball of radius $r > 0$ created from a semi-norm can be given as $r\,B$ where $B \in \mathcal{B}_{\mathcal{F}}$.

Note that any ball B (open or closed) from semi-norm g is an absorbing disk. If B is open of radius 1, this semi-norm is the Minkowski gauge for B and internal point 0.

The set of finite intersections of open disks formed from different members of $\mathcal{F}$ with rational radius $1/n,\ n \geq 1$ would constitute one neighborhood base at 0.

7.1. ***Lemma***. *Suppose g is a semi-norm with open unit disk B and $r > 0$. The Minkowski gauge for the open disk rB of radius r and internal point 0 is just $(1/r)\,g$.*

PROOF. See the remarks above. □

7.2. ***Proposition***. *A vector space with a locally convex topology is a topological vector space: that is, a topological group with addition, and scalar multiplication is jointly continuous*

PROOF. Suppose vector space V has family of semi-norms and B is the open unit disk with respect to one of the semi-norms, g.

Suppose $0 < \varepsilon < 1$ and $s = (s_1, s_2)$ is a net on directed set Δ that is eventually in subbasic open set $(a + S_\varepsilon) \times (x + \varepsilon B)$ in the product space $\mathbb{F} \times V$. Suppose too that $t = (t_1, t_2)$ is a net that is eventually in subbasic open set $(x + \varepsilon B) \times (y + \varepsilon B)$ in the product space $V \times V$.

So for large enough members of Δ

$$\begin{aligned} g(s_1 s_2 - ax) &\leq g(s_1 s_2 - a s_2) + g(a s_2 - ax) \leq |s_1 - a|\, g(s_2) + |a|\, g(s_2 - x) \\ &\leq \varepsilon(g(x) + \varepsilon) + |a|\varepsilon \leq \varepsilon(g(x) + 1 + |a|). \end{aligned}$$

That is, $s_1 s_2 \in ax + \varepsilon(g(x) + 1 + |a|)B$.

Similarly, for large enough members of Δ

$$g(\ (t_1 + t_2) - (x + y)\ ) \leq g(t_1 - x) + g(t_2 - y) \leq \varepsilon + \varepsilon.$$

So $t_1 + t_2 \in x + y + 2\varepsilon B$.

By adroit choice of ε and B, the nets $s_1 s_2$ and $t_1 + t_2$ are seen to be, eventually, in any specified subbasic open neighborhood of ax and $x + y$, respectively. The desired conclusions follow. □

So the open sets provided by the semi-norms are rich enough (in the right way) to allow the vector operations to be continuous.

7.3. ***Exercise***. *In any vector space V each absorbing disk can be used to create a semi-norm through its Minkowski gauge, so the set of all absorbing disks is a neighborhood base at 0 for the finest possible locally convex topology on V. This is called the* ***fine topology*** *on V. Every fine topology is T_2 and, of course, every locally convex topology is a subset of the fine topology. For the fine topology, among locally convex topologies, it is hardest for nets to converge, hardest for a given set to be compact, easiest for sets to be open.*

The initial topology for a particular semi-norm g, the weakest topology with respect to which g is continuous, consists of unions of open disks and complements of closed disks and their intersections (open "annuli" centered at 0) and will never be T_2. It fails to distinguish between x and y if $g(x) = g(y)$. However this initial topology does contain the open neighborhood base of disks for the metric created from g. And—this is important—if $f\colon X \to V$ is any function from topological space X to V with the topology produced by the family of semi-norms as above and if $f(p) = 0$, then f is continuous at p exactly when $g \circ f$ is continuous at p for *every* semi-norm $g \in S$.

This will be pertinent for linear f, for which $f(0) = 0$, and for which continuity at 0 implies continuity everywhere.

7.4. ***Proposition***. *A linear function $f\colon V \to W$ from topological vector space V to topological vector space W, where W has a locally convex topology generated by family of semi-norms $\mathcal{F}$, is continuous if and only if $g \circ f$ is continuous for every $g \in \mathcal{F}$.*

PROOF. Examine the remarks above. □

7.5. ***Exercise***. *Suppose that V has a locally convex topology and there is a bounded open disk B in V. Then the Minkowski gauge for B with internal point 0 is a single semi-norm on V that generates the original locally convex topology.*

In spaces with locally convex topology every neighborhood of 0 contains an open disk. So in such spaces, unless V is actually an SNLS no open set can be bounded.

Two families of semi-norms on the same vector space are called **topologically equivalent** if they generate the same topology.

When g_i is a semi-norm on V for $i = 1, \ldots, n$, define g^{max} and g^{sum} for each $x \in V$ by

$$g^{max}(x) = \sup\{\, g(x)_1,\ g(x)_2,\ \ldots,\ g(x)_n \,\} \text{ and } g^{sum}(x) = g(x)_1 + \cdots + g(x)_n.$$

Both are, themselves, semi-norms, and

$$\frac{1}{n}\, g^{max} \le \frac{1}{n}\, g^{sum} \le g^{max} \le g^{sum}.$$

So there is a multiple of the unit disk B_{sum} for g^{sum} inside any multiple of the unit disk B_{max} for g^{max}, and conversely. Specifically, for unit disks,

$$n\, B_{max} \supset n\, B_{sum} \supset B_{max} \supset B_{sum}.$$

That means g^{max} and g^{sum} are interchangeable in the formation of all open sets from a family of semi-norms for a locally convex topology.

Now suppose $\mathcal{F}$ is a family of semi-norms on V with family of open unit disks $\mathcal{B}_{\mathcal{F}}$. Let $\mathcal{G} = \{\, n\, f \mid f \in \mathcal{F}$ and n is a positive integer$\,\}$. And let $\mathcal{H}$ be the set of all finite sums of members of $\mathcal{G}$.

We will call a family of semi-norms **sufficient** if the open unit disks of the family constitute a neighborhood base for the topology at 0.

Note that if $B \in \mathcal{B}_{\mathcal{F}}$ then $\frac{1}{n}B \in \mathcal{B}_{\mathcal{G}}$ for every positive integer n. So $\mathcal{B}_{\mathcal{G}}$ is a subbase for the topology at 0. That means finite intersections of members of $\mathcal{B}_{\mathcal{G}}$ form a neighborhood base at 0.

A finite intersection $\bigcap_{i=1}^{n} B_i$ of such unit disks is exactly the unit disk for the semi-norm g^{max}, as discussed above. We can swap out all such disks in this base for unit disks for the related g^{sum} without affecting the resulting topology. But these "swapped" disks are exactly the disks for semi-norms in $\mathcal{H}$.

In an LCS, the interior of a disk, if nonempty, is also a disk. A **barrel** in an LCS is a closed absorbing disk. In an LCS many barrels have nontrivial interior (see the proposition below) but in some rather common spaces some barrels have empty interior. An example may be found in Exercise 8.6.

7.6. ***Proposition***. *Suppose $\mathcal{F}$ is a family of semi-norms on V and $\mathcal{H}$ is the family of semi-norms consisting of finite sums of positive integer multiples of members of $\mathcal{F}$, with open unit disks $\mathcal{B}_{\mathcal{H}}$.*

(i) Then $\mathcal{H}$ is a sufficient family of semi-norms, and topologically equivalent to $\mathcal{F}$: that is, $\mathcal{B}_{\mathcal{H}}$ constitutes an open neighborhood base at 0 for the topology generated by $\mathcal{F}$.

(ii) Moreover, for each $B \in \mathcal{B}_{\mathcal{H}}$ there is a member D of $\mathcal{B}_{\mathcal{H}}$ for which $D + D \subset B$ and $\overline{D} \subset B$.

(iii) The closed unit disks, the collection of neighborhoods formed by closing each disk in $\mathcal{B}_{\mathcal{H}}$, forms a neighborhood base at 0 consisting of barrels.

Proof. The justification is in the remarks above and Lemma 6.6. □

A locally convex topology on a vector space V need not be T_2. We now discuss a condition that guarantees the T_2 property.

A family $\mathcal{F}$ of semi-norms on V is said to be **separating** if the set of pseudometrics they generate is separating. We mean by this that for each pair of distinct points $x, y \in V$ there is a semi-norm which can tell they are different: that is, $\exists g \in \mathcal{F}$ with $g(x - y) \neq 0$. This amounts to saying that for each non-zero v there is a g for which $g(v) \neq 0$.

7.7. ***Exercise***. *Suppose $\mathcal{F}$ is the family of semi-norms for a locally convex topology on V and for each $g \in \mathcal{F}$ define $O_g = \{\, v \in V \mid g(v) = 0 \,\}$. Each O_g is a closed subspace of V. That means $\mathcal{O} = \bigcap_{g \in \mathcal{F}} O_g$ is a closed subspace too. Give $V/\mathcal{O}$ the quotient topology. Create a family of semi-norms on $V/\mathcal{O}$ that generates this quotient topology, and show that this family is separating.*

A vector space equipped with a *separating* family of semi-norms, together with the locally convex topology generated by these semi-norms, is called a **locally convex space.** The abbreviation **LCS** is used for these.

We emphasize that, by definition, ***locally convex spaces are topological vector spaces with a $\mathbf{T_2}$ locally convex topology.***

For each LCS we will presume given a separating family $\mathcal{F}$ of semi-norms with unit disks $\mathcal{B}_{\mathcal{F}}$, and the sufficient family $\mathcal{H}$ of semi-norms, the finite sums of positive integer multiples of members of $\mathcal{F}$. The open unit disks $\mathcal{B}_{\mathcal{H}}$ then form a neighborhood base of open disks at 0.

A normed linear space is a locally convex space with a separating family consisting of a single norm, and this will be a common type of locally convex space.

For normed spaces, boundedness is equivalent to continuity. The replacement for that result can be found below.

7.8. ***Proposition***. *Suppose $\Psi \in \mathcal{H}_{\mathbb{F}}\left(V^{LCS}, W^{LCS}\right)$, where V has separating family of semi-norms $\mathcal{S}$ and W has separating family of semi-norms $\mathcal{T}$. The following are equivalent:*

(i) Ψ is continuous at 0.
(ii) Ψ is uniformly continuous.
(iii) For each $t \in \mathcal{T}$ there are members $s_1, s_2, \ldots, s_n \in \mathcal{S}$ and real $K > 0$ for which

$$t(\Psi(v)) \leq K(s_1(v) + \cdots + s_n(v)) \quad \forall v \in V.$$

Proof. The equivalence of (i) and (ii) follows as in Proposition 5.6 for each semi-norm.

It is obvious that (iii) implies (i): For each $t \in \mathcal{T}$, if (iii) holds $t(\Psi(\nu))$ must converge to 0 whenever ν is a net in W converging to 0.

We now assume Ψ to be continuous and $t \in \mathcal{T}$.

Let B be the open unit disk for t. Then $\Psi^{-1}(B)$ is open. So there is a closed unit disk D for some semi-norm p in $\mathcal{H}$, where $\mathcal{H}$ is the sufficient family of semi-norms built from $\mathcal{S}$, with $D \subset \Psi^{-1}(B)$. That is, $\Psi(D) \subset B$.

Translating from subsets to semi-norms, $t(\Psi(v)) < 1$ whenever $p(v) \leq 1$.

If $p(v) = 0$ then $kv \in D$ for any positive number k. But then $t(\Psi(kv)) = k\, t(\Psi(v)) < 1$ for all such k requires $t(\Psi(v)) = 0$ and we conclude (since both are 0) that $t(\Psi(v)) \leq p(v)$.

On the other hand if $p(v) \neq 0$ then

$$t\left(\Psi\left(\frac{v}{p(v)}\right)\right) \leq 1 \quad \text{so} \quad t\left(\Psi\left(v\right)\right) \leq p(v) \quad \text{here too.}$$

Every p is of the form $k_1\, s_1 + \cdots + k_n\, s_n$ for certain positive integers k_i and members s_i of $\mathcal{S}$. Letting K be the maximum of the k_i, the inequality of (iii) follows. □

7.9. ***Exercise***. *Suppose V is an LCS and W is a vector subspace of V.*

(i) W with subspace topology is also an LCS. (hint: Restrict the members of the family of semi-norms on V to W.)

(ii) If $T \in W'$ there is at least one member $S \in V'$ with $S|_W = T$. (hint: By Proposition 7.8 there is a constant K and members $s_1, s_2, \ldots, s_n$ of the family of semi-norms on V so that $Tw \leq |Tw| \leq K(s_1(w) + \cdots + s_n(w))$ for all $w \in W$. By the Hahn-Banach Theorem T can be extended to all of V while preserving this relationship.)

(iii) Suppose h is any semi-norm on V, not necessarily from the specified family of semi-norms $\mathcal{S}$, with open unit disk D. h will be continuous at 0 if for each $r > 0$ the set rD contains a basic open neighborhood of 0. Actually, if D itself contains

basic open disk B then rD contains rB, so only one basic open set B must be found to check continuity at 0.

Examining the leftmost inequality for semi-norms below

$$|\,h(x) - h(y)\,| \leq h(x - y) \leq h(x) + h(y)$$

we see that continuity at 0 *is sufficient to find h to be continuous everywhere, uniformly, in V.*

Show that h is continuous if it is dominated by any continuous semi-norm. In particular, h is continuous if and only if there is a positive integer n and members $s_1, \ldots, s_n$ of $\mathcal{S}$ and positive constant K for which

$$h \leq K(s_1 + \cdots + s_n).$$

7.10. ***Exercise***. *Different topologies compatible with vector space operations pop up in applications. Here are two.*

Let H denote the set of those real valued sequences which form absolutely convergent series: that is, those $a\colon \mathbb{N} \to \mathbb{R}$ for which $\sum_{n\in\mathbb{N}} |a_n| < \infty$. Then $\|a\| = \sum_{n\in\mathbb{N}} |a_n|$ is a norm on the real vector space H.

For each $k \in \mathbb{N}$ define $\|a\|_k$ to be $|a_k|$. Each $\|\cdot\|_k$ is a semi-norm on H and the family $\mathcal{F} = \{\, \|\cdot\|_k \mid k \in \mathbb{N}\,\}$ is separating but this family does not generate the same topology as does the single norm $\|\cdot\|$: each $\mathcal{F}$-open neighborhood of 0 *is $\|\cdot\|$-open, but not conversely as we shall see.*

Consider the sequence δ^k defined by $\delta^k(n) = 1$ if $n = k$ but $\delta^k(n) = 0$ if $n \neq k$. So δ^k converges to 0 *in the $\mathcal{F}$ topology, but it does not converge in the norm topology.*

Let $I\colon (H, \mathcal{F}) \to (H, norm)$ denote the identity function from H with the $\mathcal{F}$-topology to H with the norm topology.

Using the example above, δ^k converges in the domain to the zero sequence but $I(\delta^k) = \delta_k$ does not converge in the range of I. So this function fails to be continuous: there are actually more open sets in the range than in the domain.

Suppose B is any set bounded in the domain space. So for each n, the set of numbers $\{\, b_n \mid b \in B\,\}$ must be bounded. But that doesn't imply that B is norm-bounded. For instance the set of sequences $\{\, k\,\delta^k \mid k \in \mathbb{N}\,\}$ is not norm bounded.

We mentioned above that boundedness is not sufficient to guarantee continuity for functions from one LCS to another, and gave a similar but more complicated equivalent condition in Proposition 7.8. It would be nice to know that this is actually necessary: that even in fairly nice cases boundedness does not imply continuity. We discuss this in the next exercise.

7.11. ***Exercise***. *Suppose V is an infinite dimensional NLS with dual V'. Give V a second topology generated by the family of all semi-norms $|f|$ for $f \in V'$. This is called the weak topology on V, and is a T_2 locally convex topology. (We discussed a similar example in Exercise 7.10, which has an an even weaker topology than this one.)*

In Lemma 8.2 we conclude that this is strictly weaker than the norm topology: there are norm open sets that are not weakly open, the open unit ball for the *norm being an example. In Exercise 9.4 we conclude that weakly bounded sets must be norm bounded, relying on an important theorem we will prove later called the Banach-Steinhaus Theorem for this. (In the even weaker example in Exercise 7.10 this is not the case: we find there a bounded set that is not norm bounded.)*

Let $I\colon (V, weak) \to (V, norm)$ *denote the identity map between these two topological spaces.* I *is bounded but not continuous.*

A net in a locally convex space is called **Cauchy** if it is Cauchy with respect to every one of the pseudometrics. Rephrasing, a net ν is Cauchy if, for every open disk A in neighborhood base $\mathcal{B}_{\mathcal{H}}$, there is a member d of the directed set of the net so that $\nu(s) - \nu(t) \in A$ whenever both s and t exceed d.

The net is called **bounded** if its range is a bounded set.

An LCS is called **complete** if every Cauchy net converges.

The concepts of Cauchy and completeness are not purely topological concepts. In a metric space setting, topologically equivalent metrics can have different Cauchy sequences. Here that is not the case. Topologically equivalent families of semi-norms produce the same Cauchy nets and bounded Cauchy nets.

7.12. ***Exercise***. *The concept of bounded net requires a different intuition than that provided by bounded sequence. In particular it is possible to have a convergent (and therefore Cauchy) net which is not bounded and, in fact, for which no tail net is bounded.*

Consider again the real sequence space H *with separating family of semi-norms* $\mathcal{F}$ *from Example 7.10. Let* $\mathcal{H}$ *be an equivalent but sufficient family of semi-norms.* $\mathcal{B}_{\mathcal{H}}$ *is the set of unit disks for* $\mathcal{H}$*.*

Direct the set $D = \mathbb{N} \times \mathcal{B}_{\mathcal{H}}$ *by* $(n, B_f) \geq (m, B_g)$ *when* $B_f \subset B_g$*. Each* B_g *contains a non-zero vector* x_g *and subspace* $\mathbb{R}x_g$*. Define the net* $\nu(m, B_g) = m\, x_g$*.*

If A *is any open neighborhood of* 0 *there is a* $B_g \subset A$ *and so* $\nu\left(\mathcal{T}_{(0,B_g)}\right) \subset A$*; that is, this net converges to* 0*.*

Now select specific $(m, B_g) \in D$ *producing tail* $\mathcal{T}_{(m,B_g)}$ *of those members of* D *greater than or equal to* (m, B_g)*. Since* $\mathcal{H}$ *is separating there is an* $h \in \mathcal{H}$ *with* $h(x_g) \neq 0$*. But then* $h\left(\nu\left(\mathcal{T}_{(0,B_g)}\right)\right)$ *is not bounded.*

7.13. ***Proposition***. *In a complete LCS a set* B *is compact when and only when it is closed and totally bounded.*

Proof. In any T_2 space compact sets must be closed, and since $B + V$ can be thought of as an open cover of B for any open V, compactness implies total boundedness.

We need to show that B totally bounded and closed implies B compact. We will proceed to show that, under those conditions, any universal net in B is Cauchy. Since the LCS is complete, that net must converge. Since B is closed the limit point must be a point of B. Our conclusion, that B is compact, will then follow from what is (essentially) a version of the **Bolzano-Weierstrass Theorem.** See Exercise **??** for more on this.

Suppose ν is a universal net in B and A is an open disk. We have seen that there is an open disk D with $D \ominus D \subset A$. Then there is an integer n and elements $x_1, \ldots, x_n$ in B for which

$$B \subset \bigcup_{i=1}^{n} (x_i + D).$$

Since ν is universal and entirely in B, it must be, eventually, in either $x_1 + D$ or its complement, and in the latter case it is eventually in

$$\left(\bigcup_{i=2}^{n} (x_i + D) \right) - (x_1 + D).$$

If not eventually in $x_1 + D$ it must, then, eventually be in $x_2 + D$ or

$$\left(\bigcup_{i=3}^{n} (x_i + D) \right) - \left((x_1 + D) \cup (x_2 + D) \right).$$

Proceeding in this manner (at most $n - 1$ steps) we find that the net is eventually in some $(x_i + D)$. So for large enough indices s and t we have

$$\nu(s) = x_i + d_s \quad \text{and} \quad \nu(t) = x_i + d_t \qquad \text{for certain } d_s, d_t \in D.$$

Then $\nu(s) - \nu(t) = d_s - d_t \in D \ominus D \subset A$. So ν is Cauchy. □

7.14. ***Exercise***. *By Lemma 6.5, if V is an open neighborhood of 0 in any TVS then $B + V$ contains $\overline{B}$. So if B is totally bounded, so too will be $\overline{B}$.*

On the other hand, suppose $\overline{B}$ is totally bounded and V is an open neighborhood of 0. There is a symmetric open neighborhood D of 0 for which $D + D \subset A$. Then there are vectors $q_i, \ldots, q_n$ in $\overline{B}$ with $B \subset \overline{B} \subset \bigcup_{i=1}^{n} (q_i + D)$. In every set $q_i + D$ there is a vector $p_i = q_i + d_i \in B$. And so $q_i - p_i \in D$. So $q_i + D \subset p_i + D + D \subset p_i + A$. So $B \subset \bigcup_{i=1}^{n} (p_i + A)$. So B is totally bounded too.

Conclude that in a complete LCS a set $\overline{\mathbf{B}}$ is compact when and only when $\mathbf{B}$ is totally bounded.

Locally convex spaces have appealing separation properties as shown in the next proposition, following the discussion in Reed and Simon [**?**].

7.15. ***Proposition***. *Suppose V is an LCS and S and T are disjoint nonempty convex subsets.*

(i) If S is open then S and T can be separated.

(ii) If S and T are open then S and T can be strictly separated.

(iii) If S is compact and T is closed then S and T can be strictly separated.

(iv) If S and T are both compact then S and T can be strongly separated.

Proof. The proof of (i) follows by appeal to Exercise 4.7, since every point of an open set is internal.

If both S and T are open, we use (i) to infer that there is a real linear functional Ψ separating open S and T. Easily, both $\Psi(S)$ and $\Psi(T)$ are open intervals and cannot overlap: $\Psi(S) \cap \Psi(T) = \varnothing$ and we have (ii).

Now suppose S is compact and T closed. So $T \ominus S$ is closed, convex and does not contain 0. So there is an open disk Y and so that $Y + Y$ is disjoint from $T \ominus S$. The sets $\widetilde{S} = S + Y$ and $\widetilde{T} = T \ominus Y$ are convex and open and contain S and T respectively, and can be strictly separated by a hyperplane by (ii). So we have shown (iii).

If both S and T are compact (hence closed in T_2 V) we apply (iii) to find real linear Ψ that strictly separates S and T. So there is a real c with S and T in different sets $\Psi^{-1}(\,(-\infty, c)\,)$ and $\Psi^{-1}(\,(c, \infty)\,)$. But $\Psi(S)$ and $\Psi(T)$ are compact and connected in $\mathbb{R}$: one must be of the form $[a, b]$ and the other of the form $[s, t]$ with $b < c < s$. (iv) now follows. $\square$

We now consider the issue of metrizability.

Since metrizable spaces are first countable, there is a countable base at 0 for the topology of any metrizable LCS, and this countable base can be chosen to consist of absorbing disks. If $\mathcal{F} = \{\, g_i \mid i \in \mathbb{N} \,\}$ is the set of corresponding Minkowski gauges, this must be a separating family of semi-norms. The set of unit disks $\mathcal{B}_{\mathcal{F}}$ constitute the neighborhood base.

Even better, we can let, for $i \in \mathbb{N}$, the semi-norm h_i^{max} be given by

$$h_i^{max}(v) = \max\{\, g_0(v),\, \ldots,\, g_i(v) \,\}.$$

Then $\mathcal{F}^{max} = \{\, h_i^{max} \mid i \in \mathbb{N}\}$ is a topologically equivalent set of semi-norms, $h_{i+1}^{max} \geq h_i^{max}$ for all $i \in \mathbb{N}$, and the unit disks for *this* set form a nested neighborhood base at 0.

Any countable set $\{\, \|\cdot\|_i \mid i \in \mathbb{N} \,\}$ of semi-norms which is topologically equivalent to the specified family can be used to produce a metric defined by

$$d(x, y) = \sum_{i \in \mathbb{N}} \frac{\|x - y\|_i}{2^i(1 + \|x - y\|_i)}.$$

Though translation invariant, this metric is not homogeneous, so cannot be used to produce a single generating semi-norm.

This metric does not tell us everything we might want to know about V. For instance the diameter of V itself is 2 in this metric, so the metrical notion of boundedness is not pertinent. We stick with the TVS notions of boundedness for sets and functions.

Suppose D is a finite intersection $(\varepsilon_0 B_0) \cap \cdots \cap (\varepsilon_n B_n)$ where each B_i is the unit disk for the ith semi-norm and $0 < \varepsilon_i \leq 1$ for each i. Suppose for each $\delta > 0$ that A_δ is the metric ball of radius δ centered at 0.

If $x \in D$ and ε_{max} is the largest of the ε_i

$$\begin{aligned} d(x, 0) &= \sum_{i \in \mathbb{N}} \frac{\|x\|_i}{2^i(1 + \|x\|_i)} = \sum_{i=0}^{n} \frac{\|x\|_i}{2^i(1 + \|x\|_i)} + \sum_{i > n} \frac{\|x\|_i}{2^i(1 + \|x\|_i)} \\ &< \sum_{i=0}^{n} \frac{\varepsilon_i}{2^i} + \sum_{i > n} \frac{1}{2^i} \leq \left(2 - \frac{1}{2^n}\right) \varepsilon_{max} + \frac{1}{2^n} < 2\,\varepsilon_{max} + \frac{1}{2^n}. \end{aligned}$$

So D is contained in A_δ where $\delta = 2\,\varepsilon_{max} + \frac{1}{2^n}$. For *any* $\delta_1 > 0$, the ε_i and n can be chosen so that $\delta < \delta_1$, in which case $D \subset A_\delta \subset A_{\delta_1}$. This means that the locally convex topology is at least as fine as the metric topology.

On the other hand, suppose given subbasic open εB_n for some ε with $0 < \varepsilon \leq 1$. For $x \in V$ with $d(x,0) = \delta < \frac{\varepsilon}{2^{n+1}}$ we find

$$\frac{\varepsilon}{2^{n+1}} > \delta = d(x,0) \geq \frac{\|x\|_n}{2^n(1+\|x\|_n)} \quad \text{so} \quad \varepsilon\,(1+\|x\|_n) > 2\,\|x\|_n.$$

It follows that $\varepsilon > (2-\varepsilon)\|x\|_n > \|x\|_n$. So the metric ball of radius δ centered at 0 is contained in εB_n, so the metric topology is at least as fine as the locally convex topology.

The conclusion, then, is that this metric does generate the original locally convex topology on V.

7.16. ***Proposition***. *A locally convex space V is metrizable if and only if there is a countable set of semi-norms on V which generate the same topology.*

PROOF. The proof is contained in the remarks above. $\square$

7.17. ***Exercise***. *Show that a net is metrically Cauchy, with the metric described above, exactly when it is Cauchy in the TVS sense.*

The fact of metrizability, even without use of an explicit metric, makes a space easier to work with. For instance, in a metrizable space the concepts of sequence and subsequence are sufficient to define all topological entities: open sets, closed sets, boundaries, compact sets and so on. Nets are not required.

A **Frechét space** V is a complete metrizable LCS. Banach spaces provide examples, but not the only ones. Arguments that depend on the Baire Category Theorem, as many of our most important theorems do, hold for Frechét spaces.

8. Duality and Various Topologies on a Locally Convex Space

We now suppose V to be any LCS with topology $\mathbb{T}$ generated by family $\mathcal{F}$ of semi-norms and with continuous dual V'.

If A is a nonempty subset of V', we define $\sigma(\mathbf{V},\mathbf{A})$ to be the weakest topology on V for which $|h|$ (and therefore h itself) is continuous for each member h of A. This is a locally convex topology on V: for each $h \in A$, $|h|$ is the Minkowski gauge for the set $|h|^{-1}(\,[0,1)\,)$.

8.1. ***Exercise***. *If A is a nontrivial subspace (rather than a subset) of V' then the set of all $|h|^{-1}(\,[0,1)\,)$ for $h \in A$ is an open neighborhood subbase of absorbing disks for the locally convex topology. Finite intersections of these form a base. In fact all finite sums $|h_0| + \cdots + |h_n|$ for $h_i \in A$ and $n \in \mathbb{N}$ form a sufficient family of semi-norms for this topology.*

$\sigma(V, A)$ is called the **weak topology on V generated by** subspace **A** of V'.

Note that if $A \subset B$ then $\sigma(V, A) \subset \sigma(V, B) \subset \mathbb{T}$. Weak topologies are coarser that the original topology, so convergence is easier, there are fewer open sets, more bounded sets. Though there are fewer closed sets too, it is easier for those that remain closed to be compact.

A linear functional is a pretty crude instrument for exploring the detailed topology on V. Each $|h|^{-1}(\,[0, 1)\,)$ is a "slab" in which only one dimension is restricted. So finite intersections of these, the neighborhood base for $\sigma(V, A)$ at 0, control finitely many dimensions.

No set can be a neighborhood of 0 in this topology unless it contains a subspace of finite codimension. But open sets in a semi-norm topology face no such restrictions. They can be confined in (potentially, depending on the dimension of V) uncountably many independent directions.

8.2. ***Lemma***. *Suppose V is an infinite dimensional NLS. The weak topology on V is strictly weaker than the norm topology.*

PROOF. See the remarks above. □

The **weak topology on V** is, specifically, $\sigma(V, V')$.

Even though the weak topology on an infinite dimensional normed space contains fewer open sets than norm topology, the semi-norms that generate the weakly open sets can, in some important cases, be used to deduce norm convergence to a specified member of the space.

Recall Section **??**, where we determined that the set of absolutely summable sequences $\boldsymbol{\ell^1}$ with norm $\|\, f \,\| = \sum_{k=0}^{\infty} |f(k)|$ has dual $\boldsymbol{\ell^\infty}$, the set of bounded sequences with supremum norm.

8.3. ***Proposition***. *Suppose given a sequence $f(n)$ of members of $\boldsymbol{\ell^1}$. The sequence converges weakly to the 0 sequence exactly when it converges to 0 in $\boldsymbol{\ell^1}$ norm.*

PROOF. If the sequence converges to 0 in $\boldsymbol{\ell^1}$ norm then, of course, it must converge to 0 weakly: that is, for each $g \in \boldsymbol{\ell^\infty}$ we must have

$$g(f(n)) = \sum_{k=0}^{\infty} \overline{g_k} f_k(n) \to 0.$$

Suppose, conversely, that $g(f(n))$ converges to 0 for every bounded sequence g. Choosing for g the sequence that is 1 in the "mth spot" and zero elsewhere, we see that for each m the numerical sequence $f_m(n)$ converges to 0, and in fact $\sum_{k=0}^{m} |f_k(n)| \to 0$ for each m.

The same fact is true for each subsequence of the $f(n)$.

Now suppose that our sequence fails to converge to 0 in norm. So there is a number $a > 0$ so that $\|\, 0 - f(n) \,\| = \sum_{k=0}^{\infty} |f_k(n)| > a$ for infinitely many n.

By moving to a subsequence if necessary and multiplying by a^{-1} we may assume that $\sum_{k=0}^{\infty} |f_k(n)| > 1$ for *every* n.

Let $L_0 = 0 = M_0$.

Having found M_{j-1} select $L_j > L_{j-1}$ so large that $\sum_{k=0}^{M_{j-1}} |f_k(z)| < 1/5$ for all $z \geq L_j$ and select $M_j > M_{j-1}$ so large that $\sum_{k=M_j+1}^{\infty} |f_k(L_j)| < 1/5$.

Consider

$$\sum_{k=0}^{\infty} |f(L_j)| = \sum_{k=0}^{M_{j-1}} |f_k(L_j)| + \sum_{k=M_{j-1}+1}^{M_j} |f_k(L_j)| + \sum_{k=M_j+1}^{\infty} |f_k(L_j)|.$$

The magnitude of the middle term on the right is at least $1/5$ greater than the sum of the other two terms on the right, and all terms are positive. That means for *any* sequence of complex numbers α_k for which $|\alpha_k| \leq 1$ for all k the magnitude of

$$\sum_{k=0}^{M_{j-1}} \alpha_k\, f_{L_j}(k) + \sum_{k=M_{j-1}+1}^{M_j} |f_{L_j}(k)| + \sum_{k=M_j+1}^{\infty} \alpha_k\, f_{L_j}(k)$$

must be *at least* $1/5$.

Define bounded sequence g by

$$g(k) = \begin{cases} \frac{\overline{f_{L_j}(k)}}{|f_{L_j}(k)|} & \text{if } f_{L_j}(k) \neq 0 \text{ and } M_{j-1}+1 \leq k \leq M_j \\ 0 & \text{if } f_{L_j}(k) = 0 \text{ and } M_{j-1}+1 \leq k \leq M_j. \end{cases}$$

The numerical series $g\,(f_{L_n})$ is a series of the type considered above for each n. We conclude that there is a subsequence $f_{L_0}, f_{L_1}, f_{L_2}, \ldots$ for which the magnitude of $g\,(f_{L_n}) > 1/5$. So the sequence $f(n)$ does not converge to 0 weakly. □

8.4. ***Proposition***. *Suppose V is an LCS and A is a subspace of V' that separates points: that is, if $x \neq 0$ in V there is a member g of A for which $g(x) \neq 0$.*

(i) $\sigma(V, A)$ is a T_2 topology on V.

(ii) A member h of V^* is continuous with the topology $\sigma(V, A)$ on V when and only when it is in A. So when V is given the topology $\sigma(V, A)$ its continuous dual is A and this is the coarsest topology for which this is true.

Proof. Suppose $x \neq y$ in V and select $g \in A$ for which $g(x - y) = \varepsilon > 0$. So $x - y \notin C$ where $C = g^{-1}(\,[0, \varepsilon/2)\,) \in \sigma(V, A)$. There is an open disk $D \in \sigma(V, A)$ with $D + D \subset C$. Then $x + D$ and $y + D$ are disjoint open sets in $\sigma(V, A)$ which is, therefore, T_2 and we have (i).

By Proposition 7.8 h continuous with respect to $\sigma(V, A)$ for each $h \in A$. It remains to show that any function which is continuous with respect to $\sigma(V, A)$ must be in A.

Suppose h is continuous in this topology. Then there are members $f_1, \ldots, f_n$ in A for which $|h| \leq c_1|f_1| + \cdots + c_n|f_n|$ on all of V, again by Proposition 7.8. We may, and will, assume the f_i are linearly independent: if one of the f_i is a combination of the rest we can, using the triangle inequality, modify the other c_j and eliminate f_i, retaining domination of $|h|$ by a new combination involving the remaining $|f_j|$.

Observe that the inequality $|h| \leq c_1|f_1| + \cdots + c_n|f_n|$ implies that the kernel of h contains $\bigcap_{i=1}^{n} Ker(f_i)$. And to avoid triviality assume we can select $x \in V$ with $h(x) = 1$.

We now define $Z\colon V \to \mathbb{F}^{n+1}$ by

$$Z(v) = (\, h(v), f_1(v), \ldots, f_n(v)\,) \text{ for } v \in V.$$

So Z is not onto $\mathbb{F}^{n+1}$: we know, for example, that $(\,1, 0, \ldots, 0\,) \notin Z(V)$. So there is a non-zero vector p in $\mathbb{F}^{n+1}$ which is perpendicular to every vector in the nontrivial subspace $Z(V)$ of $\mathbb{F}^{n+1}$. This gives, for every $v \in V$,

$$p_0\, h(v) + p_1\, f_1(v) + \cdots + p_n\, f_n(v) = 0.$$

If $p_0 = 0$ we would have a nontrivial relation among the f_i, contrary to assumption.

$$\text{So} \quad h = \sum_{i=1}^{n} \frac{-p_i}{p_0}\, f_i \quad \in \quad A.$$

□

8.5. ***Exercise***. *Note the results from Exercises 7.12 and 9.4. A weakly convergent net in a locally convex space need not be weakly bounded. Must a weakly convergent sequence be weakly bounded?*

8.6. ***Exercise***. *Suppose V is an infinite dimensional NLS with closed unit ball B and unit sphere S. By the Hahn-Banach theorem, for each $x \in S$ there is an $f_x \in V'$ whose real part r_x satisfies $r_x(x) = 1$ and for which B is contained in the halfspace $H_x = r_x^{-1}(\,(-\infty, 1]\,)$. That means $B = \cap_{x \in S} H_x$.*

Give V the weak topology. B has empty weak interior, because it contains no nontrivial subspace. B is weakly closed: each halfspace H_x is closed because each r_x is weakly continuous. And B is absorbing and balanced. We have shown that this barrel in the weak topology has empty weak interior.

We've also shown that the weak closure of the sphere $\mathbf{S}$ is all of $\mathbf{B}$.

Every member v of the locally convex space V can be regarded as a member of V^{**} by defining $v(f) = f(v)$ for all $f \in V^*$. Appealing to the Hahn-Banach theorem, if $f(v) = f(w)$ for all $f \in V'$ then $v = w$.

So the **evaluation map** given by $\mathbf{E(v)(f) = f(v)}$ can be construed as a one-to-one map

$$E\colon V \to \left(V'\right)^*.$$

The **weak* topology on $\mathbf{V}'$** is defined to be $\sigma(V', V)$. This is the coarsest topology on V' for which all the members of V, or rather the members $E(v)$ for $v \in V$, are continuous.[2]

[2]There are a dozen or more topologies on various function spaces in *common* use and the naming scheme for these topologies is not the model of clarity one should emulate, were one starting from scratch. It features a variety of combinations of the words and symbols: strong, weak, $*$, norm, operator, uniform, ultra, and σ, among others.

As an example, the strong topology on V' is the operator norm topology. But the set of continuous linear operators $\mathcal{CL}_{\mathbb{F}}(V)$ on the NLS V with topology given by operator norm is called the uniform operator topology. The strong operator topology on this set is weakest topology for which the evaluation map $f \to f(v)$ is continuous for every $v \in V$. By analogy with the situation with V' described above we would be tempted, erroneously, to call this the weak* topology.

Suppose D is a directed set and $f_d \in V'$ for all $d \in D$. So this net in V' is Cauchy when and only when the net of numbers $f_d(x)$ form a Cauchy net for each $x \in V$. Since $\mathbb{F}$ is complete a Cauchy net in $\mathbb{F}$ converges to some value $h(x)$ for each x. Since each f_d is linear, it is easy to see that h is linear. This is nothing more than the topology of pointwise convergence of a net of functionals.

Without more conditions on V that limit function might fail to be continuous. Even if V is normed, so that V' is Banach with operator norm, this weak* limit functional might be discontinuous.

8.7. ***Exercise***. *Let c_{00} be the set of real sequences which are non-zero at only finitely many places. Give this set the norm $\| x \|_{\sup} = \sup\{ x(n) \mid n \in \mathbb{N} \}$. For each $n \in \mathbb{N}$ define $g_n \colon c_{00} \to \mathbb{R}$ by*

$$g_n(x) = \sum_{i=0}^{n} x(i).$$

Each g_n is in c_{00}' and the sequence of functionals is weak* Cauchy since, for each x, the sequence of numbers $g_n(x)$ is eventually constant. But the limit functional is not continuous. c_{00}' is not weak* complete.

Note also that the operator norm of each g_n is $n+1$ so this sequence is not norm bounded.

As weak as it is, the weak* topology is still T_2 since if $f(x) = g(x) \forall x \in V$ then $f = g$.

V' with its weak* topology is called the **weak dual of V**. In case of possible confusion with other topologies on V' we will denote V' with this locally convex topology by $\mathbf{V'_w}$.

By Proposition 8.4, the dual of V'_w is exactly V, or rather $E(V) \subset (V')^*$.

If D is a directed set a net f_d, $d \in D$, of continuous functionals converges in this topology precisely if $f_d(x)$ converges for every $x \in V$.

This locally convex topology is generated by semi-norms of the form $\| \cdot \|_x$ where for each $\phi \in V'$ we have $\| \phi \|_x = |\phi(x)|$.

8.8. ***Theorem***. ***The Banach-Alaoglu Theorem***
If V is an NLS the operator norm closed unit ball in V' is weak* compact.

Proof. For each $v \in V$ define K_v to be the compact interval $[\,-\|v\|, \|v\|\,]$. Give the product space

$$P = \Pi_{v \in V} K_v$$

the product topology. By Tychonoff's Theorem this product space is compact.

P consists of all functions f from V to $\mathbb{F}$ for which $|f(v)| \le \|v\|$ for every $v \in V$. The linear members of P, that is those whose values are coordinated so that $f(av + w) = a\,f(v) + f(w)$ for every scalar a and $v, w \in V$, automatically have operator norm bound not exceeding 1.

Let B' denote the operator norm closed unit ball in V'. We give B' the topology it inherits from the weak* topology, the topology from V'_w.

For $\phi \in B'$ define $I\colon B' \to P$ by $I(\phi) = \phi$. So I is one-to-one (obviously): as a function between these two *sets* I is the identity function on B'. But it is not onto P and we will be interested in its topological properties.

A generic weak* basic neighborhood of ϕ restricted to B' consists of all those members τ in B' for which $|\,\phi(v_i) - \tau(v_i)\,| < \varepsilon$ for some $\varepsilon > 0$ and some finite list $v_1, \ldots, v_n$ of members of V.

Give $I(B')$ the subspace topology inherited from the product space P. A generic basic neighborhood of $I(\phi)$ in $I(B')$ consists of those points $I(\tau)$ for which $|\,\phi(v_i) - \tau(v_i)\,| < \varepsilon$ for some $\varepsilon > 0$ and some finite list $v_1, \ldots, v_n$ of members of V.

In other words, the relatively open basic neighborhoods of ϕ in B' and $I(\phi)$ in $I(B')$ are taken to each other by the functions

$$I\colon B' \to I(B') \quad \text{and} \quad I^{-1}\colon I(B') \to B'$$

which are, therefore, homeomorphisms between these two topological spaces.

The point is that the two restriction topologies, B' situated in V'_w and B' situated in P, match.

So suppose f_d is a net in B' with directed set D converging to a point $p \in P$.

If a is a scalar and $v, w \in V$ then for every d we have $f_d(av + w) = a\, f_d(v) + f_d(w)$. The left side converges to $p(av+w)$ while the right converges to $a\, p(v)+p(w)$. In other words, p is linear. We have already mentioned that every linear member of P has operator norm not exceeding 1. So $p \in B'$ which is, therefore, a closed subset of compact P. This implies B' is itself compact. □

8.9. ***Exercise***. *If V is separable then the operator norm closed unit ball B' in V' with the weak* topology is metrizable. Since B' is weak* compact, every sequence in B' has a subsequence that is weak* convergent. (hint: Suppose x_i, $i \in \mathbb{N}$, is dense in V. For any $x \in V$ let $\|\cdot\|_x$ denote the semi-norm $\|\phi\|_x = |\phi(x)|$ on V'. Let $B_{\varepsilon,x}$ denote the ball in V' centered at the origin of radius $\varepsilon > 0$ produced by this semi-norm. Show that for each $B_{\varepsilon,x}$ there is a $\delta > 0$ and an i so that $B_{\delta,x_i} \cap B' \subset B_{\varepsilon,x} \cap B'$. Then apply Proposition 7.16.)*

Note also that this metric does not produce the weak topology on all of V' unless V is finite dimensional: the unit ball of a metric on V' cannot contain an infinite dimensional subspace and so cannot be open in infinite dimensional V' with weak* topology.*

The **strong topology**, $\beta(\mathbf{V}', \mathbf{V})$ on V' is the topology of uniform convergence on bounded sets in V.

If B is a bounded subset of V we define g_B on V' by

$$g_B(y) = \sup\{\, |y|(x) \mid x \in B\,\}.$$

Each g_B is a semi-norm on V', and the set $\mathcal{G}$ of all these semi-norms generates the strong topology and makes V' into an LCS, the **strong dual of V**.

Unless otherwise mentioned, we will always give V' this strong dual topology.

Since points in V are definitely bounded, there are more semi-norms involved in defining the strong topology than used to define the weak* topology; $\beta(V', V)$ contains the weak* topology $\sigma(V', V)$.

If V is a normed space the strong topology is the topology of the operator norm: this single norm is equivalent to the whole family of semi-norms.

If $f_d \in V'$ for all d in directed set D, the net of functionals converges in the strong topology on V' when and only when the net of numbers $g_B(f_d) = \sup\{\, |f_d|\,(x) \mid x \in B\,\}$ converges for every bounded subset B of V.

With the strong topology, V' itself has a strong dual, called the **strong bidual of $\mathbf{V}$** and denoted $\mathbf{V}''$.

Since every weak* continuous functional is strongly continuous, $E(V) \subset V''$. So we can and will consider E to have V'' as range space, rather than $(V')^*$:

$$E\colon V \to V''.$$

The image of V under the evaluation map is, generally, a proper subspace of V'' so $\sigma(V', V) \subset \sigma(V', V'')$. Any members of V'' not in V are *not* continuous when V' is given the weak* topology.

So the weak* topology on V' is strictly coarser than the weak topology on V' unless $E(V) = V''$.

There are (many) interesting topologies on V and V' and, now, V'' created for specific purposes, and we will only touch lightly on these matters.

If E happens to be onto V'' then it does have a linear inverse function. If V'' is given *its* strong topology, it may be that E is continuous with continuous inverse: E may be a homeomorphism.

Locally convex spaces for which E is a homeomorphism onto the strong bidual V'' are very important, and are called **reflexive**.

V and V'' are not usually distinguished when V is reflexive. We shall see that Hilbert spaces are reflexive, but other Banach spaces may not be.

We will now walk through this material again in a common case to see how these new ideas interact with what we have seen before.

Suppose V is an NLS with norm $\|\cdot\|$.

Then V' is Banach with operator norm, which corresponds to the strong topology mentioned above: in the normed case bounded sets are nothing more than sets which can be absorbed by the unit ball.

Suppose x is non-zero in this NLS. The linear transformation Λ defined on $\mathbb{F}x$ by $\Lambda(a\,x) = a\,\|x\|$ satisfies the condition of Theorem 4.1 (or its Corollary) where $\|\cdot\|$ is the norm on V. In this one dimensional case we have equality:

$$|\,\Lambda(a\,x)\,| = |a|\,\|x\| = \|a\,x\|.$$

So Λ can be extended to linear Ψ defined on all of V and for which

$$|\Psi(v)| \le \|v\| \quad \text{for all} \quad v \in V.$$

This means that the operator norm $\|\Psi\|$ cannot exceed 1.

But $\Psi(x/\|x\|) = \Lambda(x/\|x\|) = 1$, So $\|\Psi\| = 1$.

To recap, for each $x \in V$ there is a functional Ψ in V' with

$$\|\Psi\| = 1 \quad \text{and} \quad \Psi x = \|x\|.$$

With operator norm (that is, with the strong topology) V' also has an algebraic dual $(V')^*$ and continuous dual $V'' \subset (V')^*$ which is *also* a normed space.

Every member of $(V^*)^*$ produces a member of $(V')^*$ by restriction and the evaluation map $E(x)$ applied to $x \in V$ is such a member.

And if $f \in V'$ and $\|f\| = 1$ then $|E(x)(f)| = |f(x)| \leq \|f\|\,\|x\| = \|x\|$.

So each $E(x)$ is bounded as a function from Banach space V' with operator norm to $\mathbb{F}$. In other words, $E(V) \subset V''$ and we may and will consider the evaluation map E to be a function

$$E\colon V \to V''.$$

But by the result above there is a functional $\Psi \in V'$ with operator norm 1 for which $|\Psi(x)| = \|x\|$. So $|E(x)|$ actually attains its maximum value, $\|x\|$, on the members of V' with operator norm 1.

8.10. ***Proposition***. *If V is an NLS the evaluation map E is not only continuous and one-to-one. In fact*

$$E\colon V \to V'' \quad \text{is an isometry.}$$

Proof. See the remarks above. □

The image of E with operator norm and V itself with its norm are not only isomorphic as vector spaces but are interchangeable in any calculation involving norms as well.

***But the proposition above does not imply that* $\mathbf{V}$ *is reflexive: the function* $\mathbf{E}$ *might not be onto* $\mathbf{V}''$.**

8.11. ***Exercise***. *(i) A reflexive NLS must be Banach. If V is not complete in the discussion above then E is not onto V'' and V cannot be reflexive.*

(ii) If V is a reflexive normed space, the weak and weak* topologies coincide. In such spaces the closed unit balls in V and V' are both compact with weak topology.

(iii) ***In a reflexive space every norm bounded sequence has a weakly convergent subsequence.***

The following construction is an example of how reflexiveness and our other constructs comes into play.

Suppose $F \in \mathcal{H}(V, W)$, where domain V and range W are vector spaces. These functions themselves comprise a vector space.

Define for each $w^* \in W^*$ the member $F^*(w^*) \in V^*$ given by

$$F^*(w^*)(v) = w^*(F(v)).$$

So $F^* \in \mathcal{H}(W^*, V^*)$, and F^* is called the **adjoint** of F. This is a very common and useful type of operation on functions, appearing in topology, group theory, differential geometry and in many other contexts.

We will primarily be interested in the case where V and W are normed spaces and we will assume that now.

Suppose $F \in \mathcal{CL}\left(V^{\text{NLS}}, W^{\text{NLS}}\right)$, the bounded linear functions with domain V and range W. These functions themselves comprise a normed linear space with operator norm, and a Banach space when W is Banach.

We will further restrict attention by defining F^* solely on members of the continuous dual W' rather than the whole algebraic dual W^*.

So if $w' \in W'$ and $v \in V$ we have $\|F^*(w')(v)\| \leq \|w'\|\,\|F\|\,\|v\|$.

So, in fact

$$\|F^*(w')\| \leq \|w'\|\,\|F\|$$

and this means $F^*(w')$ is actually in V', not just V^*: that is, $F^* \in \mathcal{CL}(W', V')$.

Looking at the same line again, we see that $\|F^*\| \leq \|F\|$.

8.12. ***Exercise***. *Review the following calculation to conclude* $\|F^*\| = \|F\|$.

$$\|F^*\| = \sup\{\, \|F^*(\phi)\| = \|\phi \circ F\| \;\mid \phi \in W' \text{ and } \|\phi\| = 1\,\}.$$

Find sequence x_n *of members of* V *with* $\|x_n\| = 1$ *for each* $n \geq 1$ *and*

$$\|F\| - \frac{1}{n} < \|F(x_n)\| \leq \|F\|.$$

As we've seen before, by Hahn-Banach there is, for each n*, a member* $\phi_n \in W'$ *with* $\|\phi_n\| = 1$ *and* $\phi_n(F(x_n)) = \|F(x_n)\|$*. So*

$$\|F\| - \frac{1}{n} < \phi_n\left(F(x_n)\right) = F^*(\phi_n)(x_n) \leq \|F\|.$$

8.13. ***Exercise***. *(i) If* F *is an isomorphism of normed spaces (i.e. a continuous linear mapping with continuous inverse) then so is* F^**, and* $\left(F^{-1}\right)^* = (F^*)^{-1}$.

(ii) If F *is an isometry onto* W *then* F^* *is an isometry onto* V'.

The map $F^*\colon W' \to V'$ is called the **Banach adjoint** of F and we have just shown (among other things) that for normed spaces the **Banach adjoint operator** is an isometry.

8.14. ***Proposition***.

$$^* \;:\; \mathcal{CL}(V, W) \to \mathcal{CL}(W', V') \quad \text{is an isometry onto its range.}$$

Proof. The proof is from the last exercise. □

8.15. ***Exercise***. *Since it is an isometry, the adjoint operator* * *is continuous when*
$\mathcal{CL}(V, W)$ *and* $\mathcal{CL}(W', V')$ *are given operator norm.*

Generally, the adjoint operator might not be onto $\mathcal{CL}(W', V')$, but if V and W are reflexive it is, and we note the very important facts below.

For the normed spaces V and W in the statement of the proposition below let $E(V)$ and $E(W)$ denote the embedded images of V and W in V'' and W'', respectively, using the evaluation isometry E.

8.16. ***Proposition***. *If V and W are normed spaces and $F \in \mathfrak{CL}(V, W)$ then the restriction of F^{**} to $E(V)$ is $F \circ E^{-1}$.*
When V and W are reflexive (and hence Banach) we may therefore identify F^{**} with F, and the adjoint operator is an isomorphism of normed linear spaces.

PROOF. For $x \in V$ let $\overline{x}$ be the member of V'' corresponding to x and let $\overline{F(x)}$ denote the member of W'' corresponding to $F(x)$.

We want to show that $F^{**}(\overline{x}) = \overline{F(x)}$, which will mean, essentially, that $F^{**} \colon V'' \to W''$ "is" F, at least when evaluated on $E(V)$.

By definition, $F^{**}(\overline{x}) = \overline{x} \circ F^*$ so if $\phi \in W'$

$$F^{**}(\overline{x})(\phi) = \overline{x} \circ F^*(\phi) = \overline{x}(\phi \circ F) = \phi(F(x)) = \overline{F(x)}(\phi).$$

The remaining remarks are the content of the exercise above. □

8.17. ***Proposition***. *Suppose $\Psi \in \mathfrak{CL}\left(V^{Banach}, W^{NLS}\right)$.*

The following are equivalent conditions on Ψ.

(i) $\Psi \in \mathfrak{CL}(V, W)$ has an inverse function $\Psi^{-1} \in \mathfrak{CL}(W, V)$.
(i) $\Psi^* \in \mathfrak{CL}(W', V')$ has an inverse function $(\Psi^*)^{-1} \in \mathfrak{CL}(V', W')$.
(iii) Both Ψ and Ψ^* are bounded below.

PROOF. You proved in Exercise 8.13(i) that if Ψ is invertible so is Ψ^*: that is, (i) implies (ii).

Also, we know by Proposition 5.14 that a linear transformation between normed spaces with Banach domain, such as V (by assumption) or W' (always) has an inverse if and only if it is bounded below and has image dense in its range.

Now suppose Ψ^* has an inverse. So Ψ^* is one-to-one. That means if $\phi \circ \Psi = \tau \circ \Psi$ for $\phi, \tau \in W'$ then $\phi = \tau$. If $\Psi(V)$ were not dense in W there would be a vector x not in closed $\overline{\Psi(V)}$ and a functional $\phi \in W'$ with $\phi(\Psi(V)) = \{\,0\,\}$ but $\phi(x) \neq 0$ so ϕ is not the zero functional. This contradiction implies that Ψ has dense range.

We need to show that if Ψ^* has an inverse then Ψ is bounded below. If x is any non-zero member of V there is a functional $\lambda \in V'$ for which $\|\lambda\| = 1$ and $\lambda(x) = \|x\|$. But now

$$\begin{aligned}\|x\| = \lambda(x) &= \Psi^* \circ (\Psi^*)^{-1}(\lambda)(x) = (\Psi^*)^{-1}(\lambda)(\Psi(x)) \\ &\leq \left\|(\Psi^*)^{-1}\right\| \, \|\Psi(x)\|.\end{aligned}$$

That means $\|\Psi\| \geq \left\|(\Psi^*)^{-1}\right\|^{-1}$ and so Ψ is bounded below.

So with these two facts in hand we know that Ψ has an inverse, and have shown that (ii) implies (i).

It is obvious now that either (i) or (ii) implies (iii).

Finally, we suppose that (iii) holds. We know that Ψ^* and Ψ are both one-to-one, else they would not be bounded below. And in the calculation above we used only the fact that Ψ^* was one-to-one to show that $\Psi(V)$ was dense. Now (i) follows. □

8.18. ***Exercise***. *Suppose V and W are Banach spaces and V is reflexive.*

The adjoint operator $\quad ^*\colon \mathfrak{CL}\,(V,W) \to \mathfrak{CL}\,(W',V')$ $\quad$ *is continuous when* $\mathfrak{CL}\,(V,W)$ *and* $\mathfrak{CL}\,(W',V')$ *are both given weak operator topologies.*

To see this, suppose T_ν is a net in $\mathfrak{CL}\,(V,W)$ converging to S in weak operator topology. That means for each $\phi \in W'$ and each $x \in V$ the sequence of numbers $|\phi(S(x)) - \phi(T_\nu(x))|$ converges to 0.

Members of V'' are just $E(x)$ where E is the evaluation map and $x \in V$. So T_ν^* will converge top S^* in the weak topology on $\mathfrak{CL}\,(W',V')$ if the numbers $|E(x)(S^*(\phi)) - E(x)(T_\nu^*(\phi))|$ converge to 0 for each $x \in V$ and $\phi \in W'$. But

$$|E(x)(S^*(\phi)) - E(x)(T_\nu^*(\phi))| = |\phi(S(x)) - \phi(T_\nu(x))|$$

which converges to 0 by assumption.

8.19. ***Exercise***. *Suppose V and W are Banach spaces and $T\colon V \to W$ is continuous and of finite rank. Then $T^*\colon W' \to V'$ is also continuous and of finite rank.*

In the following, we see that the adjoints of compact maps are compact.

8.20. ***Proposition***. *If V and W are Banach and $F \in \mathfrak{K}(V,W)$ then $F^* \in \mathfrak{K}(W',V')$.*

Proof. Let D_V and D'_W be the open unit disks in V and W', respectively.

We know $F(D_V)$ is totally bounded in W and we need to show $F^*(D'_W) = \{\, \phi \circ F \mid \phi \in D'_W \,\}$ is totally bounded in V'.

That means we need to find $\phi_1, \ldots, \phi_k$ in D'_W so that for each $\phi \circ F \in F^*(D'_W)$ the operator norm of $\phi \circ F - \phi_t \circ F$ does not exceed ε for some t.

That means we need to have, for this t,

$$\sup\{\, |\phi \circ F(v) - \phi_t \circ F(v)| \mid v \in D_V \,\} < \varepsilon.$$

Suppose $\varepsilon > 0$ and let $v_1, \ldots, v_n$ be members of D for which each member of $F(D_V)$ is within $\varepsilon/3$ of one of the $F(v_i)$.

Define $H\colon W' \to \mathbb{F}^n$ by $\quad H(\phi) = (\, \phi(F(v_1)), \ldots, \phi(F(v_n))\,)$.

Each of the finitely many coordinate functions of H is the composition of the evaluation map and F and so is continuous. Therefore H itself is continuous when $\mathbb{F}^n$ is given the usual Euclidean norm. A set has compact closure in $\mathbb{F}^n$ with this norm if it is bounded and H is a bounded linear function, and so H is a compact mapping.

That tells us that $H(D'_W)$ is totally bounded, so there are functionals $\phi_1, \ldots, \phi_k$ in D'_W so that every $H(\phi) \in H(D'_W)$ is within $\varepsilon/3$ of one of $H(\phi_1), \ldots, H(\phi_k)$.

Writing this out explicitly we have for some t

$$\|H(\phi) - H(\phi_t)\|^2 = \sum_{i=1}^k |\phi(F(v_i)) - \phi_t(F(v_i))|^2 = \sum_{i=1}^k |(\phi - \phi_t)(F(v_i))|^2 < \left(\frac{\varepsilon}{3}\right)^2$$

and in particular *none* of the $|(\phi - \phi_t)(F(v_i))|$ can exceed $\varepsilon/3$.

Now suppose $v \in D_V$ and $\phi \in D'_W$. Select i so that $F(v)$ is within $\varepsilon/3$ of $F(v_i)$. Select ϕ_t for this ϕ as above. Recall that the operator norms of both ϕ and ϕ_t are limited by 1. We now have

$$\begin{aligned}
&|(\phi-\phi_t)(F(v))| \\
&\qquad \le |\phi(F(v))-\phi(F(v_i))| + |\phi(F(v_i))-\phi_t(F(v_i))| + |\phi_t(F(v_i))-\phi_t(F(v))| \\
&\qquad \le \|\phi\|\,\|F(v)-F(v_i)\| + |(\phi-\phi_t)(F(v_i))| + \|\phi_t\|\,\|F(v)-F(v_i)\|
\end{aligned}$$

None of the three terms in the last line exceed $\varepsilon/3$.

So $|(\phi-\phi_t)(F(v))| = |(\phi\circ F(v)-\phi_t\circ F(v)|$ is no more than ε for any $v \in D_V$.

That was what we needed to show to conclude that $F^*(D'_W)$ is totally bounded, and finally that F^* is compact. □

9. The Open Mapping, Banach-Steinhaus and Closed Graph Theorems

The results of this section are direct consequence of the Baire Category Theorem. There are numerous generalizations and variations on these theorems, positing combinations of properties on domain and range spaces of the linear transformations to which they refer. We prove the results in typical generality, and mention a few extensions with references.

9.1. ***Proposition***. ***The Open Mapping Theorem***

If $\Psi \in \mathfrak{CL}_{\mathbb{F}}\left(V^{Banach}, W^{Banach}\right)$ and $\Psi(V)=W$ then Ψ is an open map.

Proof. Let $B_V(x,\varepsilon)$ and $B_W(y,\varepsilon)$ denote open balls of radius ε centered at x and y in V and W, respectively.

Since $\bigcup_{n\in\mathbb{N}} \Psi B_V(0,n) = W$, the Baire Category Theorem implies that there exists an integer n, $y \in W$ and $\delta > 0$ so that $B_W(y,\delta) \subset \overline{\Psi B_V(0,n)}$.

Pick an element x of $\Psi^{-1}(-y)$ and integer $k > \|x\| + n$. So

$$B_W(0,\delta) \subset \Psi x + \overline{\Psi B_V(0,n)} = \overline{\Psi B_V(x,n)} \subset \overline{\Psi B_V(0,k)}.$$

Let $\varepsilon = \delta/k$. $B_W(0,\varepsilon) \subset \overline{\Psi B_V(0,1)}$ so for each integer n,

$$B_W\left(0,\frac{\varepsilon}{2^n}\right) \subset \overline{\Psi B_V\left(0,\frac{1}{2^n}\right)}.$$

Now suppose z is any member of $B_W(0,\varepsilon)$. We will show that $z \in \Psi B_V(0,2)$, and conclude that $B_W(0,\varepsilon) \subset \Psi B_V(0,2)$.

Select $x_1 \in B_V(0,1)$ for which $z - \Psi x_1 \in B_W\left(0,\frac{\varepsilon}{2}\right)$. Having selected $x_1,\ldots,x_n$ for which

$$z - \sum_{i=1}^{n} \Psi x_i = z - \Psi\left(\sum_{i=1}^{n} x_i\right) \in B_W\left(0,\frac{\varepsilon}{2^{n+1}}\right)$$

find $x_{n+1} \in B_V\left(0,\frac{1}{2^{n+1}}\right)$ with

$$z - \sum_{i=1}^{n+1} \Psi x_i = z - \Psi\left(\sum_{i=1}^{n+1} x_i\right) \in B_W\left(0,\frac{\varepsilon}{2^{n+2}}\right).$$

The sequence $y_n = \sum_{i-1}^{n} x_i$ constructed by this process is Cauchy and converges to a point p in $B_V(0,2)$. The sequence $\Psi(y_n)$ was constructed to converge to z. Continuity of Ψ requires that $z = \Psi(p)$.

Let $\mathcal{O}$ be a nonempty open subset of V with $p \in \mathcal{O}$. Select μ so small that $p + B_V\,(0,\mu) \subset \mathcal{O}$.

$$\text{So}\quad \Psi p + B_W\left(0, \frac{\varepsilon\mu}{2}\right) \subset \Psi p + \Psi B_V\,(0,\mu) = \Psi B_V(p,\mu).$$

So $\Psi\mathcal{O}$ is a neighborhood of Ψp for each $p \in \mathcal{O}$, and is therefore open in W. □

A typical variation on this theorem is the following. If $\Psi \in \mathfrak{H}_{\mathbb{F}}(V^{\text{Frechét}}, W^{\text{barreled}})$ has a closed graph (a necessary condition for continuity, see Proposition 9.8 below) and is onto W then Ψ is an open map. For a proof go to Narici and Beckenstein [?] p. 468.

9.2. ***Corollary***. *Any bounded one-to-one linear map $\Psi\colon V^{Banach} \to W^{Banach}$ onto W has a continuous linear inverse function.*

Proof. The proof is an immediate consequence of the last proposition, and will hold whenever a version of that theorem holds. □

9.3. ***Proposition***. ***The Banach-Steinhaus Theorem****, also called* ***The Principle of Uniform Boundedness***

Suppose $\mathcal{A} \subset \mathfrak{CL}_{\mathbb{F}}\left(V^{Banach}, W^{NLS}\right)$.

Then one of two alternatives apply.

Either $\exists M < \infty$ *for which* $\|\Psi\| \le M\ \forall \Psi \in \mathcal{A}$

or *the function $\lambda\colon V \to [0,\infty]$ given by $\lambda(x) = \sup_{\Psi\in\mathcal{A}} \|\Psi x\|$ is infinite on a dense G_δ subset of V.*

Proof. Since λ is the supremum of continuous functions, it is lower semicontinuous. So the sets $A_n = \lambda^{-1}(\ (n,\infty]\ )$ are open.

If every A_n is dense then, once again by the Baire Category Theorem, so is the G_δ set $\bigcap_{n=1}^{\infty} A_n$.

On the other hand, if some A_n is *not* dense, then $\exists\varepsilon > 0$ and $y \in V$ for which $A_n \cap B_V(y,\varepsilon) = \varnothing$. So $\lambda\,(B_V(x,\varepsilon)) \subset [0,n]$. So for any $z \in B_V(0,\varepsilon)$ we find

$$\lambda(z) \le \sup_{\Psi\in\mathcal{A}} (\|\Psi(y+z)\| + \|\Psi y\|) \le \sup_{\Psi\in\mathcal{A}} \|\Psi(y+z)\| + \sup_{\Psi\in\mathcal{A}} \|\Psi y\| \le 2n.$$

So if $M = \frac{2n}{\varepsilon}$, then $\|\Psi\| \le M\ \forall \Psi \in \mathcal{A}$. □

A variation of this result (see Narici and Beckenstein [?] p. 400) is the following. If $\mathcal{A} \subset \mathfrak{CL}_{\mathbb{F}}\left(V^{\text{barreled}}, W^{\text{LCS}}\right)$ and $\lambda(x) = \sup_{\Psi\in\mathcal{A}} \|\Psi x\| < \infty$ for each $x \in V$ then $\mathcal{A}$ is **equicontinuous**: that is, for any open neighborhood R of 0 in the range there is an open neighborhood D of 0 in the domain for which $\Psi(D) \subset R\ \forall \Psi \in \mathcal{A}$.

9.4. ***Exercise***. ***Weakly bounded sets in an NLS are norm bounded.***

Suppose V is an NLS with continuous dual V'. Suppose nonempty subset B of V is bounded when V is given the weak topology: that is, $|g|(B)$ is a bounded set of numbers for all $g \in V'$. Then B is norm-bounded in V. (hint: The dual V' is Banach with operator norm. The evaluation functionals $E_x\colon V' \to \mathbb{F}$ given by $E_x(f) = f(x)$ have norm $\|x\|$ as members of Banach space V'', by an application of the Hahn-Banach theorem. And $\lambda(f) = \sup\{\, |E_x(f)| = |f(x)| \mid x \in B \,\}$ is just $\sup(|f|(B))$ which is finite for each f by assumption. We invoke Banach-Steinhaus to conclude that there is a real M for which $\|E_x\| = \|x\| \leq M$ for all $x \in B$.)

9.5. ***Exercise***. ***Pointwise limits of sequences of continuous operators from a Banach space to an NLS are continuous.***

*(Note: Some sources refer to this application as the **Banach-Steinhaus Theorem**, and reserve the name **Uniform Boundedness Principle** for the earlier more general theorem.)*

Suppose $f_n \in \mathcal{CL}_{\mathbb{F}}\left(V^{Banach}, W^{NLS}\right)$ for each $n \in \mathbb{N}$ and the sequence $(f_n(x))$ converges to a member $g(x) \in W$ for each $x \in V$. (That means the sequence (f_n) is Cauchy in the strong operator topology, the topology of pointwise convergence of these operators, though this property is not enough to imply pointwise convergence unless W is Banach.) It is easy to show that the function g produced by these pointwise limits is linear, but we want to show that g is continuous so the sequence converges to g in $\mathcal{CL}_{\mathbb{F}}(V, W)$ with strong operator topology. Though this is the only operator the sequence might converge to in operator norm, we specifically are not implying that operator norm convergence is present.

Define $\lambda(x) = \sup_{n\in\mathbb{N}} \|f_n(x)\|$ for each $x \in V$. By assumption, the sequence of numbers used to form $\lambda(x)$ converges, so each $\lambda(x)$ is bounded. That implies, by the Uniform Boundedness Principle, that there is a number M for which $\|f_n\| \leq M$. That is, the operator norms of these operators are uniformly bounded. But then for each $x \in V$, $\|g(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n(x)\| \leq M\,\|x\|$. So g is bounded, and therefore continuous.

9.6. ***Exercise***. ***If* g *is the pointwise limit of a sequence of continuous operators with Banach domain and NLS range, then convergence to* g *is uniform on compact subsets of the domain.***

Suppose $f_n \in \mathcal{CL}_{\mathbb{F}}\left(V^{Banach}, W^{NLS}\right)$ for each $n \in \mathbb{N}$ and that the sequence converges to g in strong operator topology, as in the last exercise. We showed there that g is continuous, and $\|g\|$ and all the $\|f_n\|$ are uniformly bounded by a number M. Suppose K is a compact subset of V and $\varepsilon > 0$. K is totally bounded so there are a finite number of members $x_1, \ldots, x_n$ of K for which $K \subset \bigcup_{i=1}^{n} (x_i + \varepsilon B)$ where B is the unit ball in V. Choose N so large that $\|g(x_j) - f_i(x_j)\| < \varepsilon$ whenever $i \geq N$ and for every j. If $x \in K$ pick x_j for which $x \in x_j + \varepsilon\, B$. Then if $i \geq N$

$$\begin{aligned}\|g(x) - f_i(x)\| &\leq \|g(x) - g(x_j)\| + \|g(x_j) - f_i(x_j)\| + \|f_i(x_j) - f_i(x)\| \\ &\leq M\,\varepsilon + \varepsilon + M\,\varepsilon.\end{aligned}$$

9.7. ***Exercise***. ***A bilinear functional on the product of a Banach space with an NLS which is continuous in each factor separately is jointly continuous.***

Suppose V is an NLS and W is Banach space, both over the field $\mathbb{F}$. Give $V\times W$ the product norm, $\|(x,y)\| = \|x\| + \|y\|$, which generates the product topology.

Suppose $\Psi\colon V \times W \to \mathbb{F}$ and for each $v \in V$ and $w \in W$ the functions $\Psi(\,\cdot\,,w)\colon V \to \mathbb{F}$ and $\Psi(v,\,\cdot\,)\colon W \to \mathbb{F}$ are $\mathbb{F}$-linear. A function of this kind is called ***bilinear*** *on the product space.*

Show that if every $\Psi(\,\cdot\,,w)$ is in V' and every $\Psi(v,\,\cdot\,)$ is in W' then Ψ itself is continuous on the product space.

Ψ is continuous precisely when the convergence of the two sequences x_n to 0 in V and y_n to 0 in W implies that $\Psi(a+x_n, b+y_n)$ converges to $\Psi(a,b)$ in $\mathbb{F}$. But

$$\Psi(x_n + a, y_n + b) = \Psi(x_n, y_n) + \Psi(x_n, b) + \Psi(a, y_n) + \Psi(a, b)$$

by the bilinearity condition, and our assumptions imply that the two middle terms on the right converge to 0. So Ψ will be continuous exactly when the convergence of those two sequences to their respective zero vectors implies that $\Psi(x_n, y_n)$ converges to the number 0.

For each fixed $y \in W$ the convergence of the sequence (x_n) and continuity of $\Psi(\,\cdot\,,y)$ implies that $\sup_{n\in\mathbb{N}} \|\Psi(x_n,y)\|$ is finite. The Principle of Uniform Boundedness implies there is a constant M for which $\sup_{n\in\mathbb{N}} \|\Psi(x_n,y)\| \leq M\|y\|$ for all $y \in W$. The result now follows.

Recall that a relation is defined to be a set of ordered pairs. The domain of the relation is the set of all first components of any pair in the relation. And a function is a relation for which there is exactly one pair in the set with any domain member as first component.

If $f\colon X \to Y$ where X and Y are topological spaces, the **graph of f** is the set $\{\,(x, f(x)) \mid x \in X\,\} \subset X \times Y$ with subspace topology from the product topological space. The graph of f will be denoted $\gamma(\mathbf{f})$.

So the difference between a function f and its graph $\gamma(f)$ is simply the additional structure of a topological space possessed by the latter. As *sets* they are the same and the properties of $\gamma(f)$ as a topological space can usually be phrased in terms of properties of X and Y, and we will take that approach in the following proposition.

If Y is T_2 and f is continuous, then f must be closed as a subset of $X \times Y$, essentially by definition of continuity. Sometimes the converse implication holds.

9.8. ***Proposition. The Closed Graph Theorem***

Suppose $\Psi \in \mathfrak{H}_{\mathbb{F}}(V^{Banach}, W^{Banach})$.

$\Psi \in \mathfrak{CL}_{\mathbb{F}}(V,W)$ if and only if
Ψ is a closed subset of $V \times W$ with product topology.

Proof. As implied by Exercise 9.7 the graph of continuous Ψ must be closed.

We now assume Ψ to be a closed subset of $V \times W$.

Consider the coordinate projections $\pi_1\colon \Psi \to V$ defined by $\pi_1(x, \Psi x) = x$ and $\pi_2\colon \Psi \to W$ defined by $\pi_2(x, \Psi x) = \Psi x$. Both coordinate projections are continuous when Ψ has subspace topology from $V \times W$. π_1 is one-to-one and onto V so the Open Mapping Theorem has π_1^{-1} continuous. So $\pi_2 \circ \pi_1^{-1} = \Psi$ is continuous. $\square$

We remark that the theorem remains valid if the domain of Ψ is a barreled LCS and the range is Frechét. See Narici and Beckenstein [**?**] p. 465.

10. Closed Operators

Some of the main applications of functional analysis will involve linear functions defined on a subspace, and not all of, a particular normed space. Many of these *cannot* be extended to a continuous function defined on the whole ambient space but have other properties that will, nonetheless, give us some traction.

To that end we consider linear $T\colon \mathcal{D}_T \to W$ where $\mathcal{D}_T = \mathit{Domain}(T)$ is a subspace of V and both V and W are normed spaces. Let $\mathcal{R}_T$ denote the range $T(\mathcal{D}_T)$ of T. So $\mathcal{D}_T$ and $\mathcal{R}_T$ are themselves normed spaces with restriction norms.

Recall the nature of a function: in our case, a function T *is* a set of ordered pairs of the form $(\,x, T(x)\,) \in V \times W$ where x is restricted to come from $\mathcal{D}_T$. We will not distinguish here between a function and its graph.

T is called a **closed linear operator** if T, thought of as this set of ordered pairs, is a closed subset of $V \times W$ where $V \times W$ is given product norm. That means that whenever $(x_n, y_n) \in T$ for all $n \in \mathbb{N}$, and if this sequence converges to (a, b) in $V \times W$, then $a \in \mathcal{D}_T$ and $b = T(a)$.

If T is closed, this does *not* imply that $\mathcal{D}_T$ is closed.

It implies that if each x_n is in $\mathcal{D}_T$ and if this sequence converges to a limit $a \in V$ then a must be in $\mathcal{D}_T$ *provided that the sequence* $T(x_n)$ *converges to a limit in* W. Only in that case does the requirement that a belong to $\mathcal{D}_T$ and $b = T(a)$ apply.

This definition does not involve sequences x_n for which $T(x_n)$ is not Cauchy, or when $T(x_n)$ has no limit because W fails to be complete, whether or not x_n converges. If there are two sequences x_n and y_n which both converge to a and $T(x_n)$ converges to b but $T(y_n)$ fails to converge it is only the limit of convergent $T(x_n)$ that is involved in this definition.

10.1. ***Corollary***. *Suppose $T\colon \mathcal{D}_T \to W$ is linear where $\mathcal{D}_T$ is a subspace of V, and both V and W are normed spaces.*

(i) If T is closed and $\mathcal{D}_T$ and W are Banach then T is continuous.

(ii) If T is closed and continuous and W is Banach then $\mathcal{D}_T$ is closed in V.

(iii) If T is continuous and $\mathcal{D}_T$ is closed then T is closed.

(iv) Continuity of T alone does not imply T is closed.

Proof. (i) restates (half of) Proposition 9.8. (ii) and (iii) are left to the reader. A counterexample showing (iv) is given by the identity map restricted to a dense subspace of infinite dimensional V. (Can you produce such a subspace? The finite linear combinations of members of any Schauder basis will do it, but we don't investigate these until a later section.) $\square$

10.2. ***Exercise***. *Suppose $T\colon \mathcal{D}_T \to \mathcal{R}_T$ where $\mathcal{D}_T$ and $\mathcal{R}_T$ are subspaces of normed spaces.*

The Inverse of a One-to-One Closed Operator is Closed
(i) If T is one-to-one and closed then $T^{-1}\colon \mathcal{R}_T \to \mathcal{D}_T$ is closed.

The Kernel of a Closed Operator is Closed
(ii) If T is closed then $Ker(T)$ is a closed subspace of the domain space.

10.3. ***Corollary***. *Suppose $T\colon \mathcal{D}_T \to W$ is linear where $\mathcal{D}_T$ is a subspace of V, and both V and W are normed spaces.*

T can be extended to a closed operator whose graph is $\overline{T}$ if and only if $(0, y) \in \overline{T}$ implies $y = 0$.

In terms of T and the original domain, the second condition is equivalent to: For every sequence x_n converging to 0 in $\mathcal{D}_T$, either $T(x_n)$ converges to 0 or $T(x_n)$ fails to converge at all.

Proof. The necessity of the second condition is obvious.

So suppose $(0, y) \in \overline{T}$ implies $y = 0$. Since the norm closure of a subspace is itself a subspace, this condition implies that if (a, b) and (a, d) are in $\overline{T}$ then $b = d$: that is, $\overline{T}$ is the graph of some function, which must then be an extension of T. Since $\overline{T}$ is a subspace that function is linear. □

10.4. ***Lemma***. ***The Open Mapping Theorem for Closed Operators***

Suppose $T\colon \mathcal{D}_T \to W$ is linear where $\mathcal{D}_T$ is a subspace of Banach V, and suppose also that W is Banach.

If T is closed and $\mathcal{R}_T$ is closed in W then T takes relatively open subsets of $\mathcal{D}_T$ to relatively open subsets of $\mathcal{R}_T$.

In particular, if T as above is also one-to-one then T^{-1} is continuous.

Proof. The proof of the Open Mapping Theorem, Proposition 9.1, used linearity of the operator and the fact that the image was second category to construct a sequence x_n in the domain and corresponding $T(x_n)$ in the range which were both Cauchy and invoked continuity and completeness of domain to conclude that $x_n \to x$ and $T(x_n) \to T(x)$.

We assume that that $\mathcal{R}_T$ is closed, and therefore a Banach space in its own right, and for purposes of the proof we may then assume that $\mathcal{R}_T = W$. With that assumption we use relatively open balls in domain in place of the open balls of that theorem. We use closedness of T in place of continuity to deduce the required convergence of x_n to a member x of $\mathcal{D}_T$ and $T(x_n) \to T(x)$. The rest of that proof is unchanged.

The comment about T^{-1} is now immediate. □

10.5. ***Lemma***. ***The Closed Range Theorem for Closed Operators***

Suppose $T\colon \mathcal{D}_T \to \mathcal{R}_T$ is linear where $\mathcal{D}_T$ is a subspace of Banach V, and $\mathcal{R}_T$ is a subspace of NLS W. If T is closed and bounded below then $\mathcal{R}_T$ is closed in W.

Proof. Since T is bounded below, there is a positive number c for which $\| T(x) \| \geq c \| x \| \; \forall x \in \mathcal{D}_T$. Suppose $T(x_n)$ is a sequence in $\mathcal{R}_T$ converging to a point $y \in W$. The bounded below condition implies x_n is Cauchy in Banach V and so converges to a point $x \in V$. The closedness assumption on T implies $x \in \mathcal{D}_T$ and $T(x) = y$. So $\mathcal{R}_T$ is closed in W. □

10.6. ***Exercise***. *If $T \colon \mathcal{D}_T \to \mathcal{R}_T$ is linear, closed and one-to-one and $\mathcal{D}_T$ and $\mathcal{R}_T$ are subspaces of Banach spaces V and W, respectively, we know that $T^{-1} \colon \mathcal{R}_T \to \mathcal{D}_T$ is also closed. So if we know that $\mathcal{R}_T$ is closed then it is Banach so T^{-1} is continuous.*

If, on the other hand, T^{-1} is continuous, which is equivalent to T bounded below, we know that $\mathcal{R}_T$ is closed. So if, somehow, we know that $\mathcal{R}_T$ is dense in W we know that $\mathcal{R}_T = W$; that is, T is onto W.

10.7. ***Exercise***. *Suppose $T \colon \mathcal{D}_T \to W$ where $\mathcal{D}_T$ is a subspace of V, and both V and W are Banach spaces.*

(i) Show that T is closed if and only if $\mathcal{D}_T$ is Banach when given the norm

$$\| x \|_* = \| x \| + \| T(x) \|.$$

(ii) Show that $\| \cdot \|_\#$ defined on $\mathcal{D}_T$ by

$$\| x \|_\# = \sqrt{\| x \|^2 + \| T(x) \|^2}$$

is also a norm and T is closed if and only if $\mathcal{D}_T$ is Banach when given this norm. Are these two norms equivalent?

An operator T for which $\overline{T}$ is a function, $\overline{T} \colon S \to W$, is called **closeable**. $\overline{T}$ is called the **closure** of T. We emphasize that for closeable T the subspace $S = \mathcal{D}_{\overline{T}}$ need not be closed in V.

10.8. ***Exercise***. *Suppose $T \colon \mathcal{D}_T \to W$ where $\mathcal{D}_T$ is a subspace of V, and both V and W are Banach spaces.*

(i) Show that T is closeable if whenever sequence x_n in $\mathcal{D}_T$ converges to 0 and $T(x_n)$ converges to $y \in W$ then $y = 0$.

(ii) If T is closeable and $\mathcal{D}_{\overline{T}} = V$ then $\overline{T}$ (and hence T itself) is bounded.

(iii) If T is closeable and $\mathcal{D}_{\overline{T}} = \overline{\mathcal{D}_T}$ can you conclude that T is bounded?

We finish this section with an interesting example of a particular operator, the derivative operator. Operators of this general type are, arguably, among the most important operators from the applications.

Let $C[a, b]$ denote the space of continuous functions on a closed interval $[a, b]$. For any function, continuous or not, we define supremum norm $\| \cdot \|$ by $\| f \| = \sup\{ f(x) \mid x \in [a, b] \}$. Convergence in this norm is called uniform convergence.

Suppose (f_n) is a sequence of bounded functions on interval $[a, b]$ and suppose (f_n) is Cauchy in supremum norm. This means that

$$\forall \varepsilon > 0 \text{ there is an integer } N \text{ so that } m, n \geq N \Rightarrow \|f_n - f_m\| < \varepsilon.$$

If (f_n) is Cauchy in supremum norm then $(f_n(x))$ is a Cauchy sequences of numbers for each x, so there is a functions f with

$$f_n \underset{\sup}{\longrightarrow} f$$

where the indicated "sup" convergence means uniform convergence.

We will review several proofs (these are special cases of more general facts proved elsewhere in the appendices) involving continuity, uniform continuity and equicontinuity.

Continuity on a compact interval implies uniform continuity: that is, if $v\colon [a,b] \to \mathbb{R}$ is continuous, then for each $\varepsilon > 0$ there is a $\delta > 0$ so that $x, y \in [a,b]$ and $|x-y| < \delta$ implies $|v(x) - v(y)| < \varepsilon$.

Proof: For each $x \in [a,b]$ find $\delta_x > 0$ so that $y \in [a,b]$ and $|x-y| < \delta_x$ implies $|v(x) - v(y)| < \varepsilon/2$. The set of intervals of the form $[x - \delta_x/2, x + \delta_x/2]$ covers $[a,b]$. Extract a finite subcover $[x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2]$ for $i = 1, \ldots, k$. Let δ_n be the least of the $\delta_{x_i}/2$ and suppose $|x - y| < \delta_n$.The number y is in one of the $[x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2]$ so both x and y are in $[x_i - \delta_{x_i}, x_i + \delta_{x_i}]$.

$$\begin{aligned} \text{Then} \quad |v(x) - v(y)| &\le |v(x) - v(x_i)| + |v(x_i) - v(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Next we show that the limit f of a sequence (f_n) of continuous functions defined on a compact interval is continuous.

Proof: Choose N so large that $\|f - f_N\| < \varepsilon/3$. Choose δ so small that $x, y \in [a,b]$ and $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \varepsilon/3$.

Now we have the necessary inequality: for $x, y \in [a,b]$ and $|x-y| < \delta$

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

We have just shown that $C[a,b]$ is a Banach space with supremum norm.

It will be convenient to know that convergence of a sequence (f_n) of continuous functions on a compact interval implies that the sequence is equicontinuous: that is, for each $x \in [a,b]$ we can find $\delta_x > 0$ so that $y \in [a,b]$ and $|x - y| < \delta_x$ then $|f_n(x) - f_n(y)| < \varepsilon$ for every n.

In fact, we have more. Each convergent sequence of continuous functions defined on a compact interval is *uniformly* equicontinuous: the same number δ can be chosen for each $x \in [a,b]$ and every n.

Proof: Choose N so large that $n, m \ge N$ implies $\|f_n - f_m\| < \varepsilon/3$. Choose δ^* so small that if $x, y \in [a,b]$ and $|x - y| < \delta^*$ then $|f_N(x) - f_N(y)| < \varepsilon/3$. Now we have for $m \ge N$

$$\begin{aligned} |f_m(x) - f_m(y)| &\le |f_m(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_m(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Each one of the $f_1, \ldots, f_{N-1}$ is uniformly continuous, so there are positive numbers δ_i so that $x, y \in [a,b]$ and $|x - y| < \delta_i$ implies $|f_i(x) - f_i(y)| < \varepsilon$ for each $i = 1, \ldots, N-1$. So now let δ be the least number in $\{\, \delta^*, \delta_1, \ldots, \delta_{N-1} \,\}$ and uniform equicontinuity of (f_n) follows from this.

Let $\frac{d}{dx}$ be the differentiation operator defined here, specifically, on the subspace $C^1[a,b]$ of $C[0,1]$, consisting of continuously differentiable functions on $[a,b]$, where we use one-sided limits to calculate derivatives at the endpoints.

Suppose (f_n) is a sequence of continuously differentiable functions on interval $[a,b]$ and suppose that (f_n') is Cauchy in supremum norm. Suppose further that $(f_n(c))$ is a Cauchy sequence of numbers for some $c \in [a,b]$, which therefore converges to some number which we denote $f(c)$.

$(f_n'(x))$ is a Cauchy sequence of numbers for each x so there is a function g, which we saw above must be continuous, for which

$$f_n' \xrightarrow[\sup]{} g.$$

We note that, for any continuous $v\colon [a,b] \to \mathbb{R}$, if $\|v\| < \varepsilon$ then basic facts about the Riemann integral imply that for every $x, c \in [a,b]$

$$\left| \int_c^x v(y)\, dy \right| < \varepsilon\, |x - c|.$$

Now consider the situation with our sequences of functions and derivatives. For each n

$$f_n(x) = f_n(c) + \int_c^x f_n'(y)\, dy$$

Since f_n' converges uniformly to continuous g, and $f_n(c)$ converges to the number $f(c)$, the right hand side converges (using the remark of the preceding paragraph) to $f(c) + \int_c^x g(y)\, dy$. That implies the sequence of numbers on the left side converges to a number we denote $f(x)$ for each $x \in [a,b]$.

By the Fundamental Theorem of Calculus, f is differentiable and $f'(x) = g(x)$.

$$\begin{aligned} \text{Also} \quad |f_n(x) - f(x)| &= \left| \int_c^x f_n'(y)\, dy f_n(x) - \int_c^x g(y)\, dy \right| \\ &= \left| \int_c^x (\, f_n'(y) - g(y)\,)\, dy \right| \le \|f_n' - g\|\, (b - a). \end{aligned}$$

Since the last term converges to 0 we have $\lim_{n\to\infty} \|f_n - f\| = 0$.

So under the indicated circumstances both (f_n) and (f_n') converge uniformly to continuous functions f and g, respectively. The function f is continuously differentiable, and in fact $f' = g$. Thus

$$f_n' \xrightarrow[\sup]{} f'.$$

We have just shown that the operator $\frac{d}{dx}\colon C^1[a,b] \to C[a,b]$ is closed.

Actually, we have shown more: closure requires only that *in case* $f_n \xrightarrow[\sup]{} f$ for a sequence f_n of continuously differentiable functions and some *continuous function* f and *if we know* the sequence (f_n') converges *to a continuous function* g, then f must be continuously differentiable and $f' = g$.

We actually showed that for any sequence of continuously differentiable functions, if the derivative sequence is just Cauchy then we concluded that the derivative sequence *must* converge uniformly to a continuous limit g. And if the derivative sequence is Cauchy and the sequence (f_n) converges at just a single domain value then we *concluded* it converges uniformly to a continuous function f. We then showed the pertinent point: f is differentiable and $f' = g$.

The calculations above show that $\gamma\left(\frac{d}{dx}\right)$ is closed in $C[a,b] \times C[a,b]$, but of course $C^1[a,b]$, the domain of $\frac{d}{dx}$, is not closed in $C[a,b]$. The Weierstrass Approximation Theorem tells us that $C^1[a,b]$ is a dense subspace of $C[a,b]$ with supremum norm.

10.9. ***Exercise***. *The sequence of functions given by* $f_n(x) = \frac{\sin(nx)}{n}$ *converges uniformly to* 0 *but* $\frac{df_n}{dx} = \cos(nx)$ *does not converge to* 0 *uniformly. There are plenty of sequences* g_n *of continuously differentiable functions converging uniformly to* 0 *(the sequence that is constantly the zero function, for instance) and for which* $\frac{dg_n}{dx}$ *also converges to* 0. *Why does this not interfere with our conclusion that this operator is closed?*

Now that we are here, we might as well gather in the following interesting fact involving term-by-term differentiation of a convergent series. The following would apply directly, for instance, to Taylor series on a closed subinterval of their interval of convergence and to trigonometric series whose coefficients are known to converge to zero "fast enough."

Suppose (h_n) is a sequence of continuously differentiable functions on $[a,b]$ and $\sum_{i=1}^{\infty} h_i(c)$ converges for some $c \in [a,b]$.

Suppose also that the numerical series $\sum_{i=1}^{\infty} \|h_i'\|$ converges.

Define sequences $f_n = \sum_{i=1}^{n} h_i$ and then $f_n' = \sum_{i=1}^{n} h_i'$ for each n.

If $n > m$ we have

$$\|f_n' - f_m'\| = \left\| \sum_{i=m+1}^{n} h_i' \right\| \leq \sum_{i=m+1}^{n} \|h_i'\|$$

which implies that the sequence of derivatives (f_n') is Cauchy with supremum norm, and therefore converges to a continuous $g\colon [a,b] \to \mathbb{R}$.

And by assumption $(f_n(c))$ converges.

So by the last section $\sum_{i=1}^{\infty} h_i$ converges uniformly to a continuously differentiable function and

$$\left(\sum_{i=1}^{\infty} h_i \right)' = \sum_{i=1}^{\infty} h_i'.$$

10.10. ***Exercise***. *Suppose* $\| \cdot \|_*$ *is defined on* $C^1[a,b]$ *by*

$$\| f \|_* = |f(c)| + \| f' \|$$

where the norm indicated on the right is the supremum norm on the interval $[a,b]$ *and* c *is some fixed element of* $[a,b]$.

(i) Show that $\| \cdot \|_*$ *is a norm on* $C^1[a,b]$, *and with this norm* $C^1[a,b]$ *is a Banach space.*

(ii) Consider the derivative operator $\frac{d}{dx}\colon C^1[a,b] \to C[a,b]$ *where* $C^1[a,b]$ *is given this new norm while* $C[a,b]$ *retains the usual supremum norm. Show that* $\frac{d}{dx}$ *is bounded with operator norm.*

(iii) Consider a second norm $\| \cdot \|_{\#}$ *on* $C^1[a,b]$ *given by*

$$\| f \|_{\#} = \| f \| + \| f' \|$$

where again the norm indicated on the right is the supremum norm on the interval $[a,b]$*. Show that this norm is equivalent to* $\| \cdot \|_*$ *but not equivalent to* $\| \cdot \|$*.*

11. Schauder Bases in a Banach Space

The Baire category theorem implies that no infinite dimensional Banach space can have a countable Hamel basis. However, if we can't have a countable *Hamel* basis, we can do almost as well with a Schauder basis, defined below, which uses concepts of limit and continuity provided by a norm to get most of what a true basis provides in the finite dimensional setting.

The ideas to follow make sense in more general settings but we will confine consideration here to Banach spaces. Much of what we do follows the discussion in Singer, *Bases in Banach Spaces I* [**?**].

We will first prove some results about the convergence of series. We will use $(\mathbf{s_i})$ to denote a sequence of members of Banach space X.

The symbol $\sum_{\mathbf{n=1}}^{\infty} \mathbf{s_n}$ denotes the limit in norm of the **sequence of partial sums** $S_k = \sum_{n=1}^{k} s_n$, when that limit exists.

The sequence of partial sums is called the **series created from** $(\mathbf{s_i})$. When the limit exists we say the series **converges.** Otherwise the series is said to **diverge.**

Since Banach spaces are complete, the limit will exist exactly when the series is Cauchy; that is, $\forall \varepsilon > 0$ $\exists$ integer N so that

$$\| S_n - S_m \| = \left\| \sum_{i=m+1}^{n} s_i \right\| < \varepsilon \text{ whenever } n > m \geq N.$$

The series is called **absolutely convergent** provided the series of numbers

$$\sum_{n=1}^{\infty} \|s_n\|$$

converges. And the series is said to **converge unconditionally** if

$$\sum_{n=1}^{\infty} s_{\sigma(n)}$$

converges for every permutation σ of the positive integers. The series is said to **converge conditionally** if it converges, but there is a permutation of terms for which the resulting series does not converge.

11.1. ***Exercise***. *Absolute convergence implies unconditional convergence.*

We are going to direct the set F of all *nonempty, finite* subsets of positive integers by inclusion: that is, $B \geq A$ provided $B \supset A$.

We use F with this order to define a limiting process on series. Specifically, we say $y = \lim_F \sum s_i$ when,

$$\forall \varepsilon > 0 \; \exists \; A \in F \text{ so that } \left\| y - \sum_{n \in B} s_n \right\| < \varepsilon \text{ whenever } B \supset A.$$

For each $A \in F$ let $\min(\mathbf{A})$ and $\max(\mathbf{A})$ denote the least and greatest elements, respectively, of the nonempty finite set A.

11.2. ***Lemma***. *Suppose (s_n) is a sequence in Banach X.*

The following are equivalent.

(i) There is a $y \in X$ with $y = \lim_F \sum s_i$.

(ii) $\forall \varepsilon > 0 \; \exists$ integer N

$$\textit{so that } \left\| \sum_{n \in B} s_n \right\| < \varepsilon \textit{ whenever } B \in F \textit{ and } \min(B) > N.$$

(iii) $\sum_{n=1}^{\infty} s_n$ is unconditionally convergent.

Proof. (i)⇒(ii). Suppose $y = \lim_F \sum s_i$. Select A so that

$$\left\| y - \sum_{n \in B} s_n \right\| < \frac{\varepsilon}{2} \text{ whenever } B \supset A.$$

For any $B \supset A$ let $B_{high} = \{ \, n \in B \mid n > \max(A) \, \}$ and let $B_{low} = B - B_{high}$. Presume, to avoid triviality below, that B_{high} is nonempty. $A \subset B_{low}$, so B_{low} is also nonempty. Then

$$0 \leq \left| \left\| y - \sum_{n \in B_{low}} s_n \right\| - \left\| \sum_{n \in B_{high}} s_n \right\| \right| \leq \left\| y - \sum_{n \in B} s_n \right\| < \frac{\varepsilon}{2}.$$

That means $\left\| \sum_{n \in B_{high}} s_n \right\|$ cannot exceed ε, and B_{high} could be any finite set of positive integers whose least member exceeds $\max(A)$. Since ε was an arbitrary positive number, we have used (i) to show (ii).

(ii)⇒(iii). Assume (ii). Suppose σ is any permutation of positive integers.

For $\varepsilon > 0$ select N so that $\left\| \sum_{n \in B} s_n \right\| < \varepsilon$ whenever $B \in F$ and $\min(B) > N$.

Let M be the greatest value of j for which $\sigma(j) \leq N$. Now suppose $n > m > M$.

If $B = \{ \, \sigma(m+1), \ldots, \sigma(n) \, \}$ then $\min(B) > N$.

So $\left\| \sum_{j=m+1}^{n} s_{\sigma(j)} \right\| = \left\| \sum_{j \in B} s_j \right\| < \varepsilon$.

Then $\sum_{n=1}^{\infty} s_{\sigma(n)}$ is Cauchy and must converge. We have shown that (iii) holds.

(iii)⇒(i). We will assume that $\sum_{n=1}^{\infty} s_n$ is unconditionally convergent and, in particular, converges in the unpermuted order to $y = \sum_{n=1}^{\infty} s_n$. We will assume that the limit $\lim_F \sum s_i$ fails to exist. We will then produce a permutation σ for which $\sum_{n=1}^{\infty} s_{\sigma(n)}$ fails to exist, which contradicts our first assumption, and our conclusion must be that (i) holds.

Since $\lim_F \sum s_i$ fails to exist, there is $t > 0$ with the property that whenever $A \in F$ there is a member $B \in F$ with $A \subset B$ and

$$\left\| y - \sum_{n \in B} s_n \right\| > t.$$

However, there does exist an N_1 for which

$$\left\| y - \sum_{i=1}^{n} s_n \right\| < \frac{t}{2} \quad \text{whenever } n \geq N_1.$$

Let $A_1 = \{1, \ldots, N_1\}$ and find $B_1 \in F$ with $A_1 \subset B_1$ and $\left\| y - \sum_{n \in B_1} s_n \right\| > t$.

Having found A_k and B_k where A_k is an interval of positive integers containing A_1 (so automatically $\left\| y - \sum_{n \in A_k} s_n \right\| < t/2$) and $B_k \in F$ and $A_k \subset B_k$ and $\left\| y - \sum_{n \in B_k} s_n \right\| > t$ define $A_{k+1} = \{1, \ldots, \max(B_k)\}$ and select $B_{k+1} \in F$ for which $A_{k+1} \subset B_{k+1}$ and $\left\| y - \sum_{n \in B_{k+1}} s_n \right\| > t$.

The values of the norms in this definition oscillate between "greater than t" and "less than $t/2$" so each A_{k+1} has at least one extra integer not possessed by B_k, and similarly B_{k+1} is strictly larges than A_k.

So the listing of nonempty finite sets of integers

$$A_1, \quad B_1 - A_1, \quad A_2 - B_1, \quad B_2 - A_2, \quad A_3 - B_2, \quad B_3 - A_3, \ \ldots$$

has every positive integer listed somewhere on it, once and only once. Create finite ordered lists of integers using the natural order on each of these sets. Then create permutation σ by appending these lists to each other in order starting with A_1.

We observe that for each k

$$\left| \left\| y - \sum_{i \in A_k} s_i \right\| - \left\| y - \sum_{i \in B_k} s_i \right\| \right| \leq \left\| \sum_{i \in B_k - A_k} s_i \right\|$$

and the left hand expression is at least $\varepsilon/2$ by choice of A_k and B_k.

So let's consider the series $\sum_{i=1}^{\infty} s_{\sigma(i)}$, and in particular the integer n where $\sigma(n)$ is the last member of B_k and integer m where $\sigma(m)$ is the last member of A_k.

$$\left\| \sum_{i=1}^{n} s_{\sigma(i)} - \sum_{i=1}^{m} s_{\sigma(i)} \right\| = \left\| \sum_{i \in B_k - A_k} s_i \right\| > t/2.$$

We conclude that the series $\sum_{i=1}^{\infty} s_{\sigma(i)}$ is not Cauchy and therefore cannot converge. □

11.3. ***Proposition***. *If a series in a Banach space is unconditionally convergent, then the limit of the series is the same for every permutation of terms.*

PROOF. Suppose $\sum_{i=1}^{\infty} s_i$ is unconditionally convergent and σ is a permutation of the positive integers.

By Lemma 11.2 there is a y for which for every $\varepsilon > 0$ there is a set A with

$$\left\| y - \sum_{n \in B} s_n \right\| < \varepsilon \text{ whenever } B \supset A.$$

On the other hand, since $\sum_{i=1}^{\infty} s_{\sigma(i)}$ converges to some w there is an N, which we may choose to be so large that every member of A is of the form $\sigma(i)$ for some $i \leq N$, so that

$$\left\| w - \sum_{n=1}^{N} s_{\sigma(n)} \right\| < \varepsilon.$$

But then we have

$$\| w - y \| \leq \left\| w - \sum_{n=1}^{N} s_{\sigma(n)} \right\| + \left\| y - \sum_{n=1}^{N} s_{\sigma(n)} \right\| < 2\,\varepsilon.$$

We conclude that $w = y$.

The limit is the same (that is, it is y) for *every* permutation. □

A **Schauder basis** for Banach X is a countable *ordered* set of vectors $v_0,\, v_1, \ldots$ for which every member x of X can be written in a *unique* way as

$$x = \sum_{n=0}^{\infty} a^n(x)\, v_n \qquad \text{for certain } a^n(x) \in \mathbb{F}.$$

The convergence of the sequence of partial sums is in norm *for each particular* x. The uniqueness refers to the values of the **coordinate functionals** a^n, which are therefore linear.

We define, for each k and $x \in X$

$$S_k(x) = \sum_{n=0}^{k} a^n(x)\, v_n.$$

The pointwise convergent operators $\mathbf{S_k}$ are called the **partial sum operators** for this Schauder basis.

Uniqueness of coefficients implies that 0 is not among the vectors in a Schauder basis, and in fact the vectors in a Schauder basis must constitute a linearly independent list of vectors.

Other useful facts follow immediately for each x from convergence of the series and uniqueness of coordinate functionals, such as

$$x - \sum_{n=0}^{k} a^n(x)\, v_n = \sum_{n=k+1}^{\infty} a^n(x)\, v_n$$

$$\text{and} \quad \left\| \sum_{i=m}^{n} a^i(x)\, v_i \right\| < \varepsilon \quad \text{for any } \varepsilon > 0 \text{ for sufficiently large } m, n$$

$$\text{and} \quad \lim_{k\to\infty} \left\| \sum_{n=0}^{k} a^n(x)\, v_n \right\| = \|\, x \,\| \quad \text{and} \quad \lim_{k\to\infty} \left\| \sum_{n=k}^{\infty} a^n(x)\, v_n \right\| = 0.$$

When referring to a sequence of vectors in vector space X or X^* the notation

$$\mathbf{v} = (\mathbf{v_n}) \subset X \quad \text{or} \quad \mathbf{a} = (\mathbf{a^n}) \subset X^*$$

will be used. Thus, for Schauder basis as above we have the paired sequences

$$\mathbf{v} \subset X \quad \text{with unique associated coordinate functionals} \quad \mathbf{a} \subset X^*.$$

The coordinate functionals and basis vectors described above satisfy the relation $a^n(v_n) = 1$ for all n and $a^n(v_m) = 0$ whenever $m \neq n$.

Any Banach space with a Schauder basis is separable, so there are Banach spaces without Schauder bases. In fact, there are reflexive separable Banach spaces without a Schauder basis[3]. Still, Banach spaces that have these bases are common in practice. More to the point, "practice" may concentrate on these spaces because they can be more readily understood.

A Schauder basis is called **unconditional** if, for any $x \in X$, the series obtained by any permutation of the terms in the series representation for x in this Schauder basis also converges.

We saw in Proposition 11.3 that if, in fact, *every* permutation of terms for a given convergent series (formed from a Schauder basis or not) for some particular $x \in X$ produces a series that converges to *something*, then they all must converge to x.

Conditional bases (i.e. those that are *not* unconditional) are common. Albiac and Kalton in *Topics in Banach Space Theory* [**?**] develop a number of interesting facts about such things, producing an example of a space with a Schauder basis but with no unconditional basis at all and along the way producing an example of a separable Banach space which is isometrically isomorphic to its bidual, yet is not reflexive. (The evaluation map is not, of course, the isometric isomorphism to which they refer.)

They also show that every Banach space with any Schauder basis at all has one of these conditional bases and provide examples.

For instance if c_0 is the Banach space of sequences that converge to 0 with supremum norm $\|x\| = \sup\{\, x(n) \mid n \in \mathbb{N}\,\}$, define for each k the sequence δ_k by $\delta_k(k) = 1$ and $\delta_k(n) = 0$ if $n \neq k$. Define $f_k = \delta_0 + \cdots + \delta_k$ for each k. The set of these sequences f_k constitute a conditional basis for c_0 with this norm.

Or if $\boldsymbol{\ell^1}$ is the Banach space of sequences x for which $\sum_{i=0}^{\infty} |x(i)|$ converges, with norm $\|x\| = \sum_{i=0}^{\infty} |x(i)|$, the sequence $b_0 = \delta_0$ together with $b_k = \delta_k - \delta_{k-1}$ for $k > 0$ form a conditional Schauder basis.

We refer to Albiac and Kalton [**?**] for further discussion of these matters.

[3]Per Enflo's rather surprising paper *A counterexample to the approximation property in Banach spaces*, Acta Math. 130, 309-317 (1973), provided the first examples. Until then mathematicians (in particular Grothendieck and his followers) had thought it likely, or hoped, there would be none.

A Schauder basis is called **bounded** if there are positive constants A and B for which

$$A \le \|v_n\| \le B \quad \forall n \in \mathbb{N}.$$

If $(\mathbf{v_n})$ is a Schauder basis, so is $(\mathbf{v_n}/\|\mathbf{v_n}\|)$ and the latter Schauder basis is called **normalized.**

In the finite dimensional setting a basis is used to create a coordinate representation of a vector space in $\mathbb{F}^n$, and something similar can be done here.

Let $C_{\mathbf{v}}$ consist of those members c of $\mathbb{F}^{\mathbb{N}}$ for which $K(c) = \sum_{i=0}^{\infty} c^i v_i$ converges. By definition, the sequence of vectors $S_n(K(c)) = \sum_{i=1}^{n} c^i v_i$ converges to $K(c)$.

By uniqueness of the coordinate functionals K is one-to-one, and since every $x \in X$ has a representation of this form K is onto X. It is also easy to show that K is linear (uniqueness again, and linearity of the partial sum operators) and so has linear inverse function.

We will now create a norm on $C_{\mathbf{v}}$ by

$$\| c \|_1 = \sup\{ \|S_n(K(c))\| \mid n \in \mathbb{N} \}.$$

Each convergent sequence is norm bounded so $\| c \|_1 < \infty$ for each c, and homogeneity is immediate as is the fact that $\| c \|_1 = 0$ implies $c = 0$.

If c and d are in $C_{\mathbf{v}}$ then

$$\| c + d \|_1 = \sup\{ \|S_n(K(c)) + S_n(K(d))\| \mid n \in \mathbb{N} \} \le \| c \|_1 + \| d \|_1.$$

So $\| \cdot \|_1$ is in fact a norm on $C_{\mathbf{v}}$.

Further, $K \colon C_{\mathbf{v}} \to X$ is a bounded map, and so continuous, because for any c

$$\|K(c)\| = \left\| \sum_{i=0}^{\infty} c^i v_i \right\| = \lim_{k \to \infty} \|S_k(K(c))\| \le \sup\{ \|S_n(K(c))\| \mid n \in \mathbb{N} \} = \| c \|_1.$$

This implies that the operator norm of K cannot exceed 1, and by examining the specific c for which $K(c) = v_0/\|v_0\|$ we see that $\| c \|_1 = 1$ and $\| K(c) \| = 1$ so the operator norm of K actually attains this upper bound.

We will now show that $C_{\mathbf{v}}$ is complete with norm defined above.

Suppose Θ is a Cauchy sequence in $C_{\mathbf{v}}$.

For each i the sequence of numbers Θ^i is Cauchy too: when $N > M$ and $i > 0$

$$\begin{aligned}
|\Theta^i(N) - \Theta^i(M)| \, \| v_i \| &= \| \left(\Theta^i(N) - \Theta^i(M)\right) \, v_i \| \\
&= \left\| \left(\sum_{k=0}^{i} \left(\Theta^k(N) - \Theta^k(M)\right) \, v_k \right) - \left(\sum_{k=0}^{i-1} \left(\Theta^k(N) - \Theta^k(M)\right) \, v_k \right) \right\| \\
&\le \|\Theta(N) - \Theta(M)\| + \|\Theta(N) - \Theta(M)\|.
\end{aligned}$$

It follows that Θ^i is Cauchy in $\mathbb{F}$, and Θ^0 is Cauchy by an even easier argument. So for each i and some number c^i we have $\lim_{n\to\infty} \Theta^i(n) = c^i$. For now, all we know is that $c \in \mathbb{F}^{\mathbb{N}}$.

We need to show that $\sum_{i=0}^{\infty} c^i v_i$ converges so $c \in C_{\mathbf{v}}$.

And also we must show that $\lim_{n\to\infty} \| c - \Theta(n) \|_1 = 0$.

Since Θ is Cauchy, for each $\varepsilon > 0$ we can find N so that $j > i \geq N$ implies

$$\|\,\Theta(j) - \Theta(i)\,\|_1 = \sup\left\{ \left\| \sum_{t=0}^{n} \big(\,\Theta^t(j) - \Theta^t(i)\,\big)\, v_t \right\| \mid n \in \mathbb{N} \right\} < \varepsilon.$$

For each fixed n and every j larger than i in the set above

$$\begin{aligned} \left\| \sum_{t=0}^{n} \big(\,c^t - \Theta^t(i)\,\big)\, v_t \right\| &= \left\| \sum_{t=0}^{n} \big(\,c^t - \Theta^t(j) + \Theta^t(j) - \Theta^t(i)\,\big)\, v_t \right\| \\ &\leq \left(\sum_{t=0}^{n} |\, c^t - \Theta^t(j)\,|\,\|\, v_t\,\| \right) + \left\| \sum_{t=0}^{n} \big(\,\Theta^t(j) - \Theta^t(i)\,\big)\, v_t \right\| \end{aligned}$$

The first term in the bottom line can be made as small as we like and the second can never be as large as ε.

This means that $\|\,\sum_{t=0}^{n} (\,c^t - \Theta^t(i)\,)\, v_t\,\| \leq \varepsilon$ for *any* n provided $i \geq N$. If we knew at this point that c is in $C_{\mathbf{v}}$ we could then conclude that Θ converged to c and we would be done.

We will finish off by showing this remaining fact.

For any m and n with $m < n$ we have

$$\begin{aligned} \left\| \sum_{t=m+1}^{n} c^t\, v_t \right\| &= \left\| \sum_{t=0}^{n} \big(\,c^t - \Theta^t(N)\,\big)\, v_t - \sum_{t=0}^{m} \big(\,c^t - \Theta^t(N)\,\big)\, v_t + \sum_{t=m+1}^{n} \Theta^t(N)\, v_t \right\| \\ &\leq \left\| \sum_{t=0}^{n} \big(\,c^t - \Theta^t(N)\,\big)\; v_t \right\| + \left\| \sum_{t=0}^{m} \big(\,c^t - \Theta^t(N)\,\big)\; v_t \right\| + \left\| \sum_{t=m+1}^{n} \Theta^t(N)\, v_t \right\|. \end{aligned}$$

For any n and m at all the first two terms cannot exceed ε. And since $\Theta(N)$ is assumed to be in $C_{\mathbf{v}}$, the sequence of partial sums for $\sum_{t=0}^{\infty} \Theta^t(N)\, v_t$ is Cauchy so by requiring n and m to be sufficiently large the last term can be made small as well.

So the sequence of partial sums for $\sum_{t=0}^{\infty} c^t\, v_t$ is Cauchy too, and therefore converges in Banach X.

So $c \in C_{\mathbf{v}}$ and we conclude that $C_{\mathbf{v}}$ is complete.

Since $K\colon C_{\mathbf{v}} \to X$ is a continuous one-to-one function from one Banach space onto another, the open mapping theorem tells us that K has continuous inverse $K^{-1}\colon X \to C_{\mathbf{v}}$. Because the operator norm of K is 1, the operator norm M of K^{-1} cannot be less than 1:

$$\|v\| = \|K \circ K^{-1}(v)\| \leq \|K\|\,\|K^{-1}\|\,\|v\| = \|K^{-1}\|\,\|v\| = M\,\|v\|.$$

The number $M = \|K^{-1}\| \geq 1$ is called the **basis constant for v**. The basis constant does depend on the particular norm used for X, and would differ if a different but equivalent norm was used there.

The function K^{-1} can be used to create an equivalent norm on X. Specifically, define

$$\|x\|_2 = \sup\{\,\|S_n(x)\| \mid n \in \mathbb{N}\,\} = \|K^{-1}(x)\|_1 \text{ for each } x \in X.$$

Since $\|K^{-1}(x)\|_1 \leq M\|x\|$ we have $\|x\| \leq \|x\|_2 \leq M\|x\|$.

Examining the supremum with which $\| \cdot \|_2$ is defined we note for later that $\|S_n(x)\| \leq M\|x\|$ for every x in X: that is to say every partial sum operator is bounded by M, and therefore continuous.

$$\|S_n\| \leq M \quad \forall n \in \mathbb{N}$$

and M is exactly the least upper bound of the operator norms $\|S_n\|$.

Further, the members of the sequence of linear functionals $(\mathbf{a^n})$ are all continuous, because $a^0\, v_0 = S_0$ and $a^n\, v_n = S_n - S_{n-1}$ for $n > 0$. So for Schauder basis $\mathbf{v} \subset X$ we have $\mathbf{a} \subset X'$, not just $\mathbf{a} \subset X^*$.

Actually, we get a bit more out of this: for any $x \in X$

$$\begin{aligned} |a^n(x)|\,\|v_n\| = \|a^n(x)\,v_n\| &= \|S_n(x) - S_{n-1}(x)\| \leq \|S_n(x)\| + \|S_{n-1}(x)\| \\ &\leq M\,\|x\| + M\,\|x\| = 2M\|x\|. \end{aligned}$$

Since $1 = a^n(v_n)$ we conclude that

$$1 \leq \|a^n\|\,\|v_n\| \leq 2M$$

identifying a relationship between the operator norms of the functionals a^n, the magnitudes of the corresponding basis vectors, and the basis constant for the entire basis.

This shows that if the Schauder basis $\mathbf{v}$ for X is bounded, so too are the coordinate functionals:

$$A \leq \|v_n\| \leq B \quad \forall n \in \mathbb{N} \quad \iff \quad \frac{1}{B} \leq \|a^n\| \leq \frac{2M}{A} \quad \forall n \in \mathbb{N}.$$

We will collect all these accumulated related results in the following proposition.

11.4. ***Proposition***. *Suppose X is a Banach space with norm $\|\cdot\|$ and Schauder basis $\mathbf{v}$ and associated coordinate functionals $\mathbf{a}$.*

Let (S_n) denote the sequence of partial sum operators for this Schauder basis.

Let $C_{\mathbf{v}}$ denote the subspace of $\mathbb{F}^{\mathbb{N}}$ consisting of those sequences c for which $\sum_{i=0}^{\infty} c^i v_i$ converges.

Define $K\colon C_{\mathbf{v}} \to X$ by $K(c) = \sum_{i=0}^{\infty} c^i v_i$.

Finally define the function $\|c\|_1 = \sup\{\,\|S_n(K(c))\| \mid n \in \mathbb{N}\,\}$ on $C_{\mathbf{v}}$.

(i) $C_{\mathbf{v}}$ is Banach with norm $\|\cdot\|_1$.

(ii) K is an isomorphism of normed spaces, $\|K\| = 1$ and $\|K^{-1}\| = M \geq 1$.

(iii) Each S_n is continuous and $\|S_n\| \leq M$. So the sequence (S_n) converges to the identity function $I\colon X \to X$ in the strong operator topology.

(iv) $\mathbf{a} \subset X'$.

(v) $1 \leq \|a^n\|\,\|v_n\| \leq 2M$.

(vi) $A \leq \|v_n\| \leq B \quad \forall n \in \mathbb{N} \quad \iff \quad \dfrac{1}{B} \leq \|a^n\| \leq \dfrac{2M}{A} \quad \forall n \in \mathbb{N}.$

Proof. See the remarks above. □

The partial sum operator $S_k = \sum_{n=0}^{k} a^n\, v_n$ is bounded (in fact the whole lot of them is *uniformly* bounded) and linear from X onto the finite dimensional subspace of X spanned by $v_0, \ldots, v_k$. Each operator is finite rank.

S_k is a **projection** onto its image: $S_k \circ S_k = S_k$ and, more generally, if $i \leq j$ then $S_i \circ S_j = S_j \circ S_i = S_i$.

This sequence of projections obviously converges pointwise to the identity operator. However a bit more is true.

Suppose Q has compact closure in X. We saw in Exercise 7.14 that Q must be totally bounded: for each $\varepsilon > 0$ there are members $q_1, \ldots, q_n$ in Q for which $Q \subset \bigcup_{i=1}^{n} (q_i + B_\varepsilon)$ where B_ε is the ball centered at 0 of radius ε.

Select N so large that $j \geq N$ implies $\|q_i - S_j(q_i)\| < \varepsilon$ for each i. If x is any member of K then $x = q_i + y_\varepsilon$ for some $y_\varepsilon \in B_\varepsilon$. But now if $j \geq N$

$$\begin{aligned}\|x - S_j(x)\| &= \|q_i - S_j(q_i) + y_\varepsilon - S_j(y_\varepsilon)\| \\ &\leq \|q_i - S_j(q_i)\| + \|y_\varepsilon\| + \|S_j\|\,\|y_\varepsilon\| \leq \varepsilon + \varepsilon + M\varepsilon.\end{aligned}$$

In other words, the sequence of finite rank operators S_j converges to the identity uniformly on totally bounded subsets of X.

11.5. ***Proposition***. *If Banach space X has a Schauder basis then X has the approximation property. Specifically, the sequence of partial sum operators for this Schauder basis converge to the identity uniformly on totally bounded subsets of X.*

PROOF. The proof is found in the calculations above. □

Suppose $T: W \to X$ is a compact mapping from Banach space W to X. The unit ball B_1 in W is, of course, bounded so $T(B_1)$ has compact closure. If $\varepsilon > 0$ choose N so large that if $j \geq N$ we have $\|x - S_j(x)\| < \varepsilon$ for every $x \in T(B_1)$. Then if $y \in B_1$

$$\|T(y) - S_j \circ T(y)\| = \|(T - S_j \circ T)(y)\| < \varepsilon$$

so the supremum of such numbers over $y \in B_1$ cannot exceed ε. We have shown then that the finite rank mapping $S_j \circ T$ converges uniformly to T.

11.6. ***Proposition***. *If W and X are Banach spaces and if X has a Schauder basis then any compact mapping from W to X has the approximation property. Specifically, if $T \in \boldsymbol{\mathcal{K}}(W, X)$ and (S_j) is the sequence of partial sum operators for the Schauder basis in X then $S_j \circ T$ converges to T in operator norm.*

PROOF. We showed this in the last paragraph. □

For Schauder basis (v_n) of X with associated coordinate functionals (a^n) a natural question involving the symmetry of the situation arises.

Identify (v_n) with members of X'' using the evaluation isometry. Define, for any subset A of any Banach space, $\overline{\mathbf{span}}(\mathbf{A})$ to be the norm closure of the set of finite linear combinations of members of A.

Is $\mathbf{a} = (a^n)$ a Schauder basis of the Banach space $\overline{span}(\mathbf{a}) \subset X'$, using the restriction of operator norm from X', with coordinate functionals (v_n)?

Recall the Banach adjoint operator from Exercise 8.12. We decided there that if $T\colon X \to X$ is continuous so is $T^*\colon X' \to X'$ defined by $(T^*\phi)(x) = \phi(T(x))$, and in fact the operator norms of T and T^* coincide.

That means that, for partial sum operator S_k, the operator $S_k^*\colon X' \to X'$ is continuous, and the supremum of the operator norms $\|S_k^*\|$ is M, the basis constant for (v_n). Actually, M can be defined as the supremum of the numbers $\|S_k\|$, the operator norms of the partial sum operators for the basis.

Note that

$$(S_k^*\phi)(x) = \phi(S_k\, x) = \phi\left(\sum_{i=1}^{k} a^i(x)\, v_i\right) = \sum_{i=1}^{k} \phi(v_i)\, a^i(x) = \left(\sum_{i=1}^{k} v_i(\phi)\, a^i\right)(x).$$

So if (a^n) with coordinate functionals (v_n) turns out to be a Schauder basis for some Banach subspace of X', these operators S_k^* are the partial sum operators for that basis.

An easy calculation shows that if ϕ is any *finite* linear combination of members of (a^n) then for sufficiently large k we have $S_k^*\phi = \phi$.

Define the subset Y of X' to be the set

$$Y = \{\, \phi \in X' \mid \lim_{k\to\infty} \|\, \phi - S_k^*\, \phi\,\| = 0\,\}$$

Y is easily seen to be a vector subspace of X' and contains the finite linear combinations of members of (a^n).

It is also an operator norm closed subspace of X', and so contains $\overline{span}(\mathbf{a})$.

To see this, we suppose ϕ_i is in Y for each $i \in \mathbb{N}$ and $\lim_{i\to\infty} \phi_i = \tau$.

$$\begin{aligned} \|\, \tau - S_k^*\, \tau\,\| &= \|\, \tau - \phi_i + \phi_i - S_k^*\, \phi_i + S_k^*\, \phi_i - S_k^*\, \tau\,\| \\ &\le \|\, \tau - \phi_i\| + \|\phi_i - S_k^*\, \phi_i\| + \|S_k^*\|\, \|\phi_i - \tau\,\| \end{aligned}$$

$\|S_k^*\|$ cannot exceed M, the middle term converges to 0 by assumption, and $\|\, \tau - \phi_i\|$ can be made as small as we like.

Our conclusion must be that $\lim_{k\to\infty} \|\, \tau - S_k^*\, \tau\,\| = 0$ and therefore $\tau \in Y$.

Now suppose τ is a generic member of Y.

Since τ is continuous, if $x = \lim_{n\to\infty} \sum_{j=1}^{n} a^j(x) v_j \in X$ we know

$$\tau(x) = \lim_{n\to\infty} \tau\left(\sum_{j=1}^{n} a^j(x) v_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} a^j(x)\tau(v_j) = \lim_{n\to\infty} (S_n^*\tau)(x).$$

So the series converges to $\tau(x)$ for each $x \in X$. Rephrasing, the sequence of partial sums $S_n^*(\tau)$ converges to τ in the weak* topology. This is not news: the weak* topology is weaker than the norm topology and we have already shown norm convergence.

For $n > k$ and any coefficient sequence $c(i)$

$$S_n^*(\tau)(v_k) = \left(\sum_{i=1}^{n} c(i) a^i\right)(v_k) = c(k)$$

and since this (eventually) constant sequence must converge to $\tau(v_k)$ we have uniqueness of the representation. The coefficient on each a^k *must* be $\tau(v_k)$.

We have a final remark about the basis constant for (a_n) on the Banach subspace $\overline{span}(\mathbf{a})$ of X'. It will be the supremum over all n of the operator norms of the S_n^*, which are defined on all of X', but restricted to $\overline{span}(\mathbf{a})$. Operator norms are defined in terms of suprema, over those τ in the unit ball, of the numbers $\|S_n^*(\tau)\|$. But the unit ball of $\overline{span}(\mathbf{a})$ is a smaller set than the unit ball in X', so the restricted operator norm cannot exceed the unrestricted operator norm. The least upper bound over all n of those unrestricted operator norms is M.

11.7. ***Theorem***. ***The Dual Basis Theorem***

Suppose $\mathbf{v}$ *is a Schauder basis for* X^{Banach} *with associated coordinate functionals* $\mathbf{a} \subset X'$.

(i) Then $\mathbf{a}$ *is a Schauder basis for* $\overline{span}(\mathbf{a})$ *with associated coordinate functionals* $\mathbf{v}$.

(ii) The basis constant for $\mathbf{a}$ *will not exceed that for* $\mathbf{v}$.

(iii) If $\mathbf{v}$ *is bounded so is* $\mathbf{a}$.

(iv) If $\mathbf{v}$ *is unconditional so is* $\mathbf{a}$.

PROOF. We have addressed, in the preceding paragraphs, all the issues of this theorem except for the last. As for the last, none of the conclusions of the paragraphs preceding this theorem are altered if the basis vectors are permuted in any way, assuming that the permuted vectors still constitute a Schauder basis. $\square$

The cleanest case, of course, is when X is reflexive. Then $X' = \overline{span}(\mathbf{a})$ and so we have complete correspondence between Schauder bases with their coefficient sequences for X and those for X', with matching basis constants.

12. Projections and Orthogonality in a Banach Space

Suppose $P\colon V \to V$ is linear and V is a vector space. The function P is called **idempotent, or a projection**, if P^2, defined to be $P \circ P$, is P.

Let K be the kernel of P and M the image of P. Both are subspaces of V.

If P is a projection and $P^2(x) = 0$ then $P(x) = 0$ so $x \in K$. That means $K \cap M = \{0\}$.

Now suppose x is any member of V. Then $P(x - P(x)) = P(x) - P^2(x) = 0$. So $x = P(x) + x - P(x)$ represents x as the sum of a member of M and a member of K. So $V = M \oplus K$.

Every member v of M is of the form $P(x)$ for some $x \in V$. But then $P(v) = P^2(v) = P(x) = v$. So P is the identity on M.

P is called a **projection onto M**.

So any projection on any vector space is the identity function on its image and, of course, the zero function on its kernel, and the domain space is the direct sum of image and kernel.

Given $V = M \oplus K$ for projection P onto M with kernel K define linear function $Q = I - P$, where I is the identity map on V. So K is the image of the map Q, and the kernel of Q is M. So Q is a projection onto K.

For any $x \in V$ we have $x = P(x) + Q(x)$.

12.1. ***Lemma***. *Given any two subspaces K and M of vector space V for which $V = M \oplus K$ there is one and only one projection with image M and kernel K. Projections and corresponding ordered pairs of subspaces for which $V = M \oplus K$ carry the same information.*

PROOF. The argument is given above. □

12.2. ***Lemma***. *Suppose P and W are projections defined on vector space V onto M_P and M_W with kernels K_P and K_W, respectively.*
Suppose further that P and W commute: $PW = WP$.
Then PW is a projection onto $M_{PW} = M_P \cap M_W$ with kernel

$$K_{PW} = span(K_P \cup K_W).$$

PROOF. $(PW)^2 = PWPW = PPWW = PW$ so PW is in fact a projection under the conditions of this lemma.

Also, $PW(V) \subset P(V)$ and $PW(V) = WP(V) \subset W(V)$ so M_{PW} is contained in $M_P \cap M_W$. On the other hand if $x \in M_P \cap M_W$ then $PW(x) = P(x) = x$ so $x \in M_{PW}$ and we conclude $M_{PW} = M_P \cap M_W$.

If $x \in K_W$ then $PW(x) = P(0) = 0$ and if $x \in K_P$ then $PW(x) = WP(x) = W(0) = 0$ so $span(K_P \cup K_W) \subset K_{PW}$.

Finally, suppose $x \in K_{PW}$. So $x = a + b$ where $a \in M_P$, so $P(a) = a$, and $b \in K_P$. But then $0 = PW(x) = WP(x) = W(a)$ so $a \in K_W$. Our conclusion is that $x \in span(K_P \cup K_W)$ and the containment $K_{PW} \subset span(K_P \cup K_W)$ holds too. □

12.3. ***Exercise***. *We will suppose V is a Banach space.*

(i) If the dimension of V is at least 2 and $k \geq 1$ there is a projection defined on V with operator norm k.
(ii) If V is infinite dimensional there is an unbounded projection defined on V.

We will now focus on the case where V is a Banach space.

Suppose A and B are subspaces of V and **both are closed** and $V = A \oplus B$. Following Lorch, *Spectral Theory* [**?**] we define (A, B) to be a **complementary pair**.

The function $P \colon V \to V$ given by $P(x) = a$ whenever $x = a + b$ and $a \in A$ and $b \in B$ is a continuous projection onto A with kernel B.

Continuity follows from the closed graph theorem.

The graph $\gamma(P)$ of P is

$$\gamma(P) = \{\, (x, P(x)) \mid x \in V \,\} = \{\, (a + b, a) \mid a \in A \text{ and } b \in B \,\}.$$

Suppose $x_i = a_i + b_i$ is a sequence converging to a point x in X, where a_i and b_i are drawn from the closed sets A and B, respectively. Suppose also that a_i converges to a. Since A is closed $a \in A$.

Since $b_i = x_i - a_i$ converges to $x - a$ and B is closed we have b_i converging to a point $b \in B$ and $b = x - a$. So $P(x) = a$ which means that the graph of P is closed and therefore P is continuous.

Note that we specifically used the closed nature of both A and B, though we never used the fact that X is complete. This result holds for any normed space.

It is a (perhaps) surprising fact that the direct sum of two complete subspaces might fail to be complete. We can have a situation where A is a normed space with subspaces B and C which are Banach spaces (i.e. complete) and for which $A = B \oplus C$, yet A is not Banach. This can happen even in the nice situation where A is an inner product space so B and C are Hilbert spaces. See Exercise 14.13 for an example.

One of our goals later will be to try to understand the structure of members T of $\mathfrak{CL}_{\mathbb{F}}(V)$, and one way of doing that will be to restrict attention to subspaces of V upon which T is simple, or simpler, and then put the pieces together to re-form T.

A complementary pair (A, B) is said to **reduce** T if $T(A) \subset A$ and $T(B) \subset B$.

If P is the projection described above, the pair will reduce T exactly when $PT = TP$: that is, T commutes with the projection P.

Then $T = T \circ P + T \circ (I - P)$ and T can be thought of as the sum of two functions confined to these simpler domains.

We will be concerned, in later sections devoted to spectral theory, with finding a rich class of projections that commute with T. Our intent will be to write T as a "giant sum" (that is, an integral) of operators that are as simple as possible: scalar multiples of projections if we can, by analogy with diagonalization results from finite-dimensional linear algebra.

We will now consider the concept of orthogonality in a Banach space setting, a concept that is closely related to, but distinct from, the concept of orthogonality we will deal with later in a Hilbert space. Since Hilbert spaces *are* Banach spaces we have a clash of vocabulary which may be clarified by context.

Suppose V is Banach and $u \in V$ and $\phi \in V'$. **Vector v and functional $\boldsymbol{\phi}$ are said to be orthogonal** if $\phi(u) = 0$. The zero functional is orthogonal to every member of V. If u is non-zero and V has dimension exceeding 1 then there is a non-zero member of V' orthogonal to u. And by application of the Hahn-Banach theorem, unless u and v are dependent there is a non-zero member $\phi \in V'$ which is orthogonal to u but *not* orthogonal to v.

We say **nonempty sets $\mathbf{A} \subset \mathbf{V}$ and $\mathbf{B} \subset \mathbf{V'}$ are orthogonal** provided every $\phi \in B$ is orthogonal to every $x \in A$.

Suppose M is any nonempty subset of Banach V. We denote by $\mathbf{M}^{\perp}$ the set of all members of V' which are orthogonal to every member of M.

$M^{\perp}$ is a subspace of V' and easily seen to be closed, called the **orthogonal complement of M**.

12.4. ***Exercise***. *Suppose V is an NLS and M is a closed subspace of V. Verify that the function $\|\cdot\|$ defined on $V/M = \{ v + M \mid x \in V \}$ by $\| v + M \| = \inf\{ \|v+m\| \mid m \in M \}$ is a norm on V/M and V/M is complete with this norm provided V is Banach. This norm is called the **quotient norm** on V/M.*

We suppose now that V is Banach, and make $V'/(M^{\perp})$ into a Banach space in a similar way using operator norm on V'.

(i) If $f \in V'$ and $g \in M^{\perp}$ then $(f+g)(m) = f(m) \forall m \in M$. So the restriction of f to M is the same as the restriction of $f+g$ to M. Both correspond to the same member of M'.

(ii) If $f+M^{\perp} \in V'/(M^{\perp})$ define $A(f+M^{\perp})$ to be f restricted to M, a member of M'. By (i), $A\colon V'/(M^{\perp}) \to M'$ is well defined, and by Hahn-Banach this map is onto M'. Show that A is linear and an isometry.

(iii) Suppose $\phi \in M^{\perp}$. If $x + M = y + M \in V/M$ then x and y differ by a member of M so $\phi(x-y) = 0$ and then $\phi(x) = \phi(y)$. Define $B\colon M^{\perp} \to (V/M)'$ by $B(\phi)(x+M) = \phi(x)$. You must verify that B is well defined and linear and also that each $B(\phi)$ is continuous.

(iv) If $\Psi \in (V/M)'$ and $v \in V$ define $\phi(v) = \Psi(v+M)$. If $v \in M$ then $\phi(v) = 0$ so $\phi \in M^{\perp}$ if it is continuous. And by definition then $B(\phi) = \Psi$. So B is onto $(V/M)'$. Show that each ϕ is continuous and B is an isometry.

12.5. ***Exercise***. *Suppose $T\colon \mathcal{D}_T \to W$ is a closed operator where $\mathcal{D}_T \subset V$ and both V and W are Banach. Provide $V/Ker(T)$ with quotient norm.*

Define $\widetilde{T}\colon \mathcal{D}_T/Ker(T) \to W$ by

$$\widetilde{T}(\, x + Ker(T)\,) = T(x) \quad \textit{for } x \in \mathcal{D}_T.$$

Then $\widetilde{T}$ is closed and invertible and $\mathcal{R}_T = \mathcal{R}_{\widetilde{T}}$.

The set $\left(M^{\perp}\right)^{\perp} = M^{\perp\perp}$ is in V''. If V is reflexive, the second dual of V is (identified with) V, so in that case we may, and will, consider $M^{\perp\perp}$ to be a subset of V.

Suppose V is reflexive and M is a closed subspace of V. If $x \in M$ then, of course, $x \in M^{\perp\perp}$. But if $x \notin M$ then by Hahn-Banach there is a continuous functional ϕ that is 0 on every vector in closed M but non-zero on x so $x \notin M^{\perp\perp}$. In other words, if M is a closed subspace of reflexive Banach V then $M = M^{\perp\perp}$.

Suppose now that $T \in \mathcal{CL}_{\mathbb{F}}(V)$. The Banach adjoint of T is $T^*\colon V' \to V'$ defined by $T^*(\phi) = \phi \circ T$. We saw earlier that the adjoint operator is operator norm continuous and an isometry onto its range in $\mathcal{CL}_{\mathbb{F}}(V')$, which will be all of $\mathcal{CL}_{\mathbb{F}}(V')$ if V is reflexive.

The kernel of T^* consists of those continuous functionals that are zero on all of the image space $T(V)$ so $Ker(T^*) = T(V)^{\perp} = \overline{T(V)}^{\perp}$, the last equality established by continuity of members of V'.

Now suppose V is a reflexive Banach space.

We define $N = Ker(T)$ and $N^* = Ker(T^*)$ and $M = \overline{T(V)}$ and $M^* = \overline{T^*(V')}$.

We are going to establish some relationships among these subspaces.

First, we already have (even in spaces that are not reflexive) that $N^* = M^\perp$.

Since in reflexive spaces $T^{**} = T$ this gives also $N = N^{**} = (M^*)^\perp$.

If $\phi = \tau \circ T$ for some $\tau \in V'$ then $\phi(x) = 0$ for every $x \in N$ so $\phi \in N^\perp$ and so (again by continuity of members of V') we find that $M^* = \overline{T^*(V')} \subset N^\perp$.

On the other hand, suppose $\phi \in N^\perp$ but ϕ is *not* in M^*. Since M^* is closed there must be a member x of V'' (identified with reflexive V) for which

$$x(T^*(V')) = \{\, \tau(T(x)) \mid \tau \in V' \,\} = \{0\} \quad \text{but} \quad x(\phi) = \phi(x) \neq 0.$$

Since $\phi \in N^\perp$ we cannot, therefore, have $x \in N$. So $T(x) \neq 0$ and there must be a member $\tau \in V'$ which is non-zero on $T(x)$. But this contradicts the assumption that $\tau(T(x)) = 0$ for any $\tau \in V'$.

Therefore $N^\perp - M^*$ is empty and we conclude that $M^* = N^\perp$.

Using, again, the fact that $T^{**} = T$ in our context, we also get $M = (N^*)^\perp$.

12.6. ***Lemma***. *Suppose V is a reflexive Banach space and $T \in \mathfrak{CL}_{\mathbb{F}}(V)$. Define $N = Ker(T)$ and $N^* = Ker(T^*)$ and $M = \overline{T(V)}$ and $M^* = \overline{T^*(V')}$.*

Then:

$$N^* = M^\perp \quad \textit{and} \quad N^\perp = M^* \quad \textit{and} \quad (N^*)^\perp = M \quad \textit{and} \quad (M^*)^\perp = N.$$

Proof. The proof can be found in the preceding discussion. □

Getting back to projections, suppose P is the projection on Banach V for complementary pair (M, N) as above. By definition both the image $M = P(V)$ and $N = Ker(P)$ are closed and $V = M \oplus N$, and this implies P is continuous.

Under these conditions the Banach adjoint $P^* \colon V' \to V'$ is also a continuous projection.

$$(P^*)^2(\phi) = \phi \circ P^2 = \phi \circ P = P^*(\phi)$$

shows that it is idempotent.

And $P^*(V') = M^* = N^\perp$ and $Ker(P^*) = N^* = M^\perp$ by the remarks from above, so $M^* \cap N^* = \{0\}$ and $V' = M^* \oplus N^*$. This means P^* has closed range and kernel and is, therefore, continuous too.

The following result is Edgar Lorch's 1939 generalization to "power bounded" operators on reflexive Banach spaces of a result proved initially in 1931 by von Neumann for contractions on a Hilbert space, and we adapt the following proof from Lorch [**?**].

12.7. ***Theorem***. ***The Mean Ergodic Theorem***
Suppose V is a reflexive Banach space and $T \in \mathfrak{CL}_{\mathbb{F}}(V)$ and $\|T^n\| \le k$ for some fixed $k \ge 1$ and all integers $n \ge 0$, with T^0 defined to be the identity operator I on V. Define for each integer $n \ge 1$ the operator

$$P_n = \frac{1}{n}\left(I + T + T^2 + \cdots + T^{n-1}\right).$$

Let A consist of those members $g \in V$ for which $g = Tg$. A is the kernel of $I - T$. Let B be the closure of $(I - T)(V)$, the image of $I - T$.

Then (A, B) is a complementary pair, and P_n converges to the projection P for this complementary pair in the strong operator topology. The operator norm of P will not exceed k.

PROOF. If $f \in A$ then $f = Tf$ and generally $f = T^n f$ and so $P_n f = f$ for all n. So for members of A, we have the constant sequence $P_n f$ converging, obviously, to f.

On the other hand, suppose $f \in B$, the closure of $(I - T)(V)$. So there is a member g of V for which $\|f - (I - T)g\| < \varepsilon$. Let $h = f - (I - T)g$. Then

$$\begin{aligned} P_n f &= \frac{1}{n}\left(f + Tf + \cdots + T^{n-1}f\right) \\ &= \frac{1}{n}\Big((h + (I-T)g) + T(h + (I-T)g) + \cdots + T^{n-1}(h + (I-T)g)\Big) \\ &= \frac{1}{n}\left(h + Th + T^2 h + \cdots + T^{n-1}h + g - T^n g\right). \end{aligned}$$

Noting that $\|h\| < \varepsilon$ and $\|T^n\| \le k$ for all n, the triangle inequality gives

$$\begin{aligned} \|P_n f\| &\le \frac{1}{n}\left(\|h\| + \|T\|\,\|h\| + \cdots + \|T^{n-1}\|\,\|h\| + \|g\| + \|T^n\|\,\|g\|\right) \\ &\le \frac{\varepsilon}{n} + \frac{\varepsilon}{n}\,k(n-1) + \frac{1+k}{n}\,\|g\| \;\le\; k\varepsilon + \frac{1+k}{n}\,\|g\|. \end{aligned}$$

The last term converges to 0 for each g, and g can be chosen to make the previous term as small as we want. We conclude that $P_n f$ converges to the zero vector for each $f \in B$.

It is now obvious that the intersection of A and B is the zero vector, so the sum $A + B$ is direct and $P_n f$ converges for each f in $A \oplus B$ to $Pf \in A$ where P is the projection for the complementary pair (A, B), defined on $A \oplus B$.

Applying the triangle inequality to $P_n = \frac{1}{n}\left(I + T + T^2 + \cdots + T^{n-1}\right)$ we observe that the operator norm of P_n cannot exceed k for any $n \ge 1$.

If f is any member of $A \oplus B$ then for any $\varepsilon > 0$ and sufficiently large n we have $\|Pf\| \le \|P_n f\| + \varepsilon \le k\,\|f\| + \varepsilon$.

We conclude that the operator norm of P on $A \oplus B$ cannot exceed k.

Suppose $f_n = a_n + b_n$ is a Cauchy sequence in $A \oplus B$, with each $a_n \in A$ and $b_n \in B$, converging to a limit f in Banach V. Then

$$\|\, a_n - a_m \,\| = \|\, P(f_n) - P(f_m) \,\| \le \|\, P \,\|\, \|\, f_n - f_m \,\|$$

so the sequence a_n is Cauchy, converging to a in closed A. But then $f_n - a_n = b_n$ is Cauchy converging to some b in closed B, and it follows that $f = a + b \in A \oplus B$. So $A \oplus B$ is closed in V.

It only remains to show that $A \oplus B = V$. We have not used, yet, the reflexiveness condition on V, and here is where that assumption is needed.

We first note that the norms of the adjoint powers $\|(T^*)^n\| = \|(T^n)^*\| = \|T^n\|$ are also bounded by the same k for each $n \ge 0$ so the convergence procedure can be carried out in just the same way as before, except this time using T^* and $I^* - T^*$ and P_n^*.

By the preceding lemma $B^{\perp}$ is the kernel A^* of $(I-T)^* = I^* - T^*$ and $A^{\perp}$ is B^*, the closure of the range of $I^* - T^*$.

That means for each $\phi \in A^*$ we have $P_n^*(\phi) = \phi$ and so this constant sequence, just as before, converges. For $\phi \in B^*$ we have $P_n^*(\phi)$ converging to the zero functional. As before, $A^* \cap B^* = \{0\}$.

Now suppose that $x \in V - A \oplus B$. Since $A \oplus B$ is closed there is a functional $\phi \in V'$ with $\phi(A \oplus B) = \{0\}$ but $\phi(x) \neq 0$.

But then $\phi \in A^{\perp}$ and $\phi \in B^{\perp}$ so

$$0 \neq \phi \in B^{\perp} \cap A^{\perp} = A^* \cap B^* = \{0\}.$$

This contradiction allows us to conclude $V - A \oplus B = \varnothing$ so $V = A \oplus B$. $\square$

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13. Inner Product Spaces

A **sesquilinear form** on an $\mathbb{F}$-vector space V is a map $S\colon V \times V \to \mathbb{F}$ for which, for all $\alpha \in \mathbb{F}$ and $x, y, z \in V$

(i) $S(\alpha x + y,\ z) = \alpha S(x, z) + S(y, z)$ and
(ii) $S(z,\ \alpha x + y) = \overline{\alpha} S(z, x) + S(z, y)$.

Item (i) is called **linearity in the first factor** and item (ii) is called **conjugate linearity** in the second factor. If the field is real, an inner product is a covariant tensor of order 2. But if the field is complex it is not quite multilinear: in Latin sesqui means "one and a half."

A **pre-inner product** is a sesquilinear form S for which, for all $x, y \in V$

(iii) $S(x, x) \geq 0$ and
(iv) $S(x, y) = \overline{S(y, x)}$.

Item (iii) is called **positivity**, while item (iv) is called **symmetry**.

A pre-inner product is called an **inner product** provided

(v) $S(x, x) = 0$ implies $x = 0$.

The additional feature is called **nondegeneracy**.

V together with a specified inner product is called an **inner product space**. It is customary to use a notation such as $\langle \mathbf{x}, \mathbf{y} \rangle$ rather than $S(x, y)$ for inner products, with different inner products distinguished by subscripts if necessary. The inner product itself, if it must be referred to without arguments, would be denoted $\langle \cdot\,, \cdot \rangle$.

The **norm associated with an inner product** is given by

$$\| v \| = \sqrt{\langle v\,, v \rangle}.$$

The fact that this actually *is* a norm, that it has the properties required of a norm, are immediate.

So an inner product space is also a normed linear space and, through the norm, a metric space. An inner product space is given the topology generated by this metric.

The norm from an inner product has two additional properties: for any vectors v and w

$$|\langle u\,, v \rangle| \leq \|v\|\,\|w\|$$

and also

$$2\,\|u\|^2 + 2\,\|v\|^2 = \| u + v \|^2 + \| u - v \|^2.$$

The first of these is the **Schwarz inequality**. Or the **Cauchy-Schwarz inequality**. Or the **Bunyakovsky-Cauchy-Schwarz inequality**. It depends on who you talk to. The second is called the **parallelogram law.**

We will prove the **BCS** inequality.

First, if either u or v is the zero vector the result is obvious, so presume neither are 0. We note that if

$$w = u - \frac{\langle u\,, v \rangle}{\langle v\,, v \rangle} v \quad \text{then } \langle w\,, v \rangle = 0.$$

We write $u = w + \frac{\langle\, u\,,\, v\,\rangle}{\langle\, v\,,\, v\,\rangle} v$ and expand $\|\, u\,\|^2 = \langle\, u\,,\, u\,\rangle$ and the inequality we are looking for falls out.

It is not hard to show from the BCS inequality that the inner product is continuous on $V \times V$ where the product space has the product topology. Note for comparison Exercise 9.7.

Norms can come from various sources but only norms that come from (or *could* have come from) an inner product satisfy the parallelogram law. And an inner product can be built from any norm satisfying the parallelogram law.

If $\mathbb{F} = \mathbb{R}$ then a norm satisfying the parallelogram law comes from the inner product given by

$$\langle\, u\,,\, v\,\rangle = \frac{\|\, u + v\,\|^2 - \|\, u - v\,\|^2}{4}.$$

If $\mathbb{F} = \mathbb{C}$ then a norm satisfying the parallelogram law comes from the inner product given by

$$\langle\, u\,,\, v\,\rangle = \frac{\|\, u + v\,\|^2 - \|\, u - v\,\|^2}{4} + i\,\frac{\|\, u + i\,v\,\|^2 - \|\, u - i\,v\,\|^2}{4}.$$

This last equality is called the **polarization identity** and comes in handy now and then.

13.1. ***Lemma***. *Suppose V is a complex normed linear space with norm $\|\cdot\|$ that satisfies the Parallelogram Law. Define the real inner product $\langle\, x, y\,\rangle_R$ on V as shown above. Then define the function $\langle\,\cdot,\cdot\,\rangle_C$ by*

$$\langle\, x, y\,\rangle_C = \langle x, y\rangle_R + i\langle x, iy\rangle_R.$$

$\langle\,\cdot,\cdot\,\rangle_C$ is a complex inner product on V and $\langle\, x, x\,\rangle_C = \|x\|^2$ for each $x \in V$.

PROOF. Show that $i\langle\, x, y\,\rangle_C = \langle\, ix, y\,\rangle_C$ and $\langle\, y, x\,\rangle_C = \overline{\langle\, x, y\,\rangle_C}$. The rest of the required properties are immediate. □

The ordinary **dot product** on $\mathbb{R}^3$ is the inner product everyone knows about. Here $\langle\, u\,,\, v\,\rangle = u \cdot v$.

It is important to note that the properties of dot product in $\mathbb{R}^n$ not only let us define **length of a vector**, as

$$\|v\| = \sqrt{v \cdot v}$$

but also lets us define **angle between two vectors** by

$$u \cdot v = \|u\|\|v\|\cos(\theta).$$

Angle is defined in an analogous way for *any* real inner product: it is given by the equation

$$\langle\, u\,,\, v\,\rangle = \|u\|\|v\|\cos(\theta)$$

and the BCS inequality guarantees that θ can be defined by this equation. That it behaves as an angle should, that it corresponds to our intuition provided by our experience with Euclidean geometry, is a different matter. You might satisfy yourself about this by looking at the two dimensional space $\mathbb{R}^2$ and the lengths of the edges of the triangle in this space with corners at 0, u and v.

The notion of angle is the additional element the inner product supplies beyond the homogeneous and translation invariant "length" idea given by the norm: in particular, a notion of perpendicularity. Perpendicularity can be defined in *any* inner product space, real or complex.

Two vectors u and v are said to be **orthogonal** if $\langle u , v \rangle = 0$. If both vectors are nonzero and they are orthogonal we sometimes call them **perpendicular.** A set of vectors is called **orthonormal** if each vector in the set has norm 1 and any two vectors from the set are orthogonal.

We have previously discussed (Section 12) the concept of orthogonality in a Banach space. There it referred to vectors and continuous functionals. ϕ and v were said to be orthogonal if $\phi(v) = 0$. Our usage here is different, referring to a relationship between vectors. We will see in Section 16 that in a Hilbert space the concepts are very close to coincident.

14. Geometry in Hilbert Spaces

A **Hilbert Space** is a complete inner product space, and therefore a type of Banach space. We will assemble some basic facts about these special spaces.

A vector v is called **orthogonal to a vector w or orthogonal to a nonempty set of vectors B** if v is orthogonal to, respectively, w or every vector in B. **Two sets of vectors are called orthogonal** if every vector in one set is orthogonal to every vector in the other set.

If B is a nonempty set of vectors in an inner product space, $\mathbf{B}^\perp$ denotes the set of all members of the space orthogonal to every member of B. It is called the **orthogonal complement of B**. "Orthogonal complement" may be shortened to "**orthocomplement.**"

14.1. ***Theorem***. *If M is any subset of a Hilbert space then $M^\perp$ is a closed subspace.*

PROOF. That $M^\perp$ is a subspace follows immediately by linearity properties of the inner product. Closure follows by continuity: suppose sequence (x_i) in $M^\perp$ is Cauchy and converges to y. Then for each i we have $0 = \langle x_i, m \rangle$ for any $m \in M$. Since the inner product is continuous $\langle y, m \rangle = 0$ too, for each $m \in M$. That means $y \in M^\perp$. □

14.2. ***Lemma***. *If M is a nonempty closed convex subset of Hilbert space $\mathcal{H}$ and $x \notin M$ then there is a unique member y of M closest to x.*

PROOF. Consider the number

$$\alpha = \inf\{\, \|x - m\| \mid m \in M \,\}.$$

Suppose (m_i) is a sequence of members of M for which $\|x - m_i\| \to \alpha$. Note that if $\|x - m_i\| < \alpha + \varepsilon$ and $\|x - m_j\| < \alpha + \varepsilon$ then for any $t \in [0, 1]$

$$\begin{aligned}\alpha &\le \|x - (tm_i + (1-t)m_j)\| = \|t(x - m_i) + (1-t)(x - m_j)\| \\ &\le t\,\|x - m_i\| + (1-t)\|x - m_j\| < t(\alpha + \varepsilon) + (1-t)(\alpha + \varepsilon) = \alpha + \varepsilon.\end{aligned}$$

In other words, the whole line between m_i and m_j is in convex M and similarly close to x.

By the parallelogram law

$$\begin{aligned} 2\|x-m_i\|^2+2\|x-m_j\|^2 &= \|x-m_i+x-m_j\|^2+\|m_j-m_i\|^2 \\ &= 4\left\|x-\frac{m_i+m_j}{2}\right\|^2+\|m_j-m_i\|^2. \end{aligned}$$

The first expression approaches $4\alpha^2$ and the last is near $4\alpha^2+\|m_j-m_i\|^2$, which means that (m_i) is Cauchy and so converges to some y in closed M. By continuity of the norm, $\|x-y\|=\alpha$.

As for uniqueness, no condition was made on the sequence (m_i) other than it be drawn from M and $\|x-m_i\|\to\alpha$. If there were another point $z\in M$ with $\|x-z\|=\alpha$ we could create a new sequence (w_j) with $w_j=m_j$ with j odd and $w_j=z$ for j even. This new sequence is also Cauchy, and has subsequences converging to both y and z, hence $y=z$. □

14.3. ***Lemma***. *If M is a closed subspace of a Hilbert space $\mathcal{H}$ and $M\neq\mathcal{H}$ then $M^\perp\neq\{0\}$.*

Proof. Pick a member x of $\mathcal{H}-M$. Since M is convex, by the last lemma there is a unique point w in M closest to x.

Now suppose m is any member of M for which $\langle x-w,m\rangle\neq 0$. Replacing m by the appropriate multiple, we may assume that $\langle x-w,m\rangle=1$. Suppose a is an arbitrary *real* number. Then $w+am$ is also in M, and by the minimality condition on w we have

$$\begin{aligned} \langle x-w,x-w\rangle \leq& \langle x-w-am,x-w-am\rangle \\ =&\langle x-w,x-w\rangle+\langle x-w,-am\rangle+\langle -am,x-w\rangle+\langle am,am\rangle \\ =&\langle x-w,x-w\rangle-a\langle x-w,m\rangle-a\langle m,x-w\rangle+a^2\langle m,m\rangle \\ =&\langle x-w,x-w\rangle-2a+a^2\langle m,m\rangle. \end{aligned}$$

This means that for *any* positive real number a we have $\frac{2}{a}\leq\langle m,m\rangle$, an obvious contradiction.

Our conclusion must be that $\langle x-w,m\rangle=0$ for any member m of M, and so $x-w$ is a nonzero member of $M^\perp$. □

14.4. ***Lemma***. *If M is a closed subspace of a Hilbert space $\mathcal{H}$ then $\mathcal{H}=M\oplus M^\perp$. It follows that $M^{\perp\perp}=M$.*

Proof. Suppose $x\in\overline{M\oplus M^\perp}$. So there is a sequence (y_i) in $M\oplus M^\perp$ that converges to x. Then there exists unique sequences $(a_i)\in M$ and $(b_i)\in M^\perp$ so that $y_i=a_i+b_i$. The sequence (y_i) is Cauchy. So for any $\varepsilon>0$ and any correspondingly large subscripts i and j we have

$$\begin{aligned} \varepsilon >& \|\, a_i+b_i-a_j-b_j\,\|^2=\langle\, a_i+b_i-a_j-b_j\ ,\ a_i+b_i-a_j-b_j\,\rangle \\ =&\langle\,(a_i-a_j)+(b_i-b_j)\ ,\ (a_i-a_j)+(b_i-b_j)\,\rangle \\ =&\langle\, a_i-a_j\ ,\ a_i-a_j\,\rangle+\langle\, b_i-b_j\ ,\ b_i-b_j\,\rangle \\ =&\|\, a_i-a_j\,\|^2+\|\, b_i-b_j\,\|^2. \end{aligned}$$

which implies that both $\| a_i - a_j \|$ and $\| b_i - b_j \|$ can be made arbitrarily small by requiring i and j to be large enough. That means both sequences are Cauchy in their respective closed subspaces. So $a_i \to c \in M$ and $b_i \to d \in M^\perp$ and $x = c + d \in M \oplus M^\perp$. We find, therefore, that $M \oplus M^\perp$ itself is closed.

By Lemma 14.3, if $\mathcal{H} \neq M \oplus M^\perp$ then there would be a nonzero member of $\left(M \oplus M^\perp\right)^\perp$ and this vector would necessarily be orthogonal to both M and $M^\perp$. A contradiction follows easily. $\square$

14.5. ***Lemma***. *Suppose B and C are nonempty subsets of a Hilbert space $\mathcal{H}$.*

$$B \subset C^\perp \quad \Longrightarrow \quad \overline{span}(B) \subset \overline{span}(C)^\perp.$$

$$\textit{Also,} \qquad \left(\overline{B}\right)^\perp = B^\perp.$$

$B^{\perp\perp}$ is the smallest closed subspace containing B.

Proof. Using linearity and continuity of inner product, show in turn that $B \subset span(C)^\perp$, then $B \subset \overline{span}(C)^\perp$, then $span(B) \subset \overline{span}(C)^\perp$ and finally $\overline{span}(B) \subset \overline{span}(C)^\perp$.

The second and third lines are left entirely to the reader. $\square$

An **orthonormal basis** for an inner product space is a maximal orthonormal subset of that space: that is, a set of unit vectors, each one orthogonal to all other members of the set, and contained in no larger set of this kind. It is an easy exercise to verify that any orthonormal subset constitutes a linearly independent set of vectors, so its cardinality cannot exceed the Hamel dimension of the inner product space.

14.6. ***Lemma***. *The closure of the span of an orthonormal basis B of a Hilbert space $\mathcal{H}$ is all of $\mathcal{H}$.*

Proof. Let $\mathcal{H}_B = \overline{span}(B)$. Since $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_B^\perp$, if $\mathcal{H}_B^\perp$ is anything but $\{0\}$ we could add a normalized nonzero member of $\mathcal{H}_B^\perp$ to B, increasing the size of maximal orthonormal B, a contradiction. So $\mathcal{H}_B = \mathcal{H}$. $\square$

14.7. ***Lemma***. *If B and C are disjoint and nonempty and $B \cup C = A$ for orthonormal basis A of a Hilbert space $\mathcal{H}$, we define*

$$\mathcal{H}_B = \overline{span}(B) \quad \textit{and} \quad \mathcal{H}_C = \overline{span}(C).$$

$$\textit{Then} \quad \mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_C.$$

We find that $C^\perp = \mathcal{H}_B$ and $B^\perp = \mathcal{H}_C$. So we could rewrite the line above as

$$\mathcal{H} = C^\perp \oplus B^\perp.$$

Proof. Any member v of $\mathcal{H}_B \cap \mathcal{H}_C$ is the limit of two sequences (b_i) and (c_i) where each b_i is a finite linear combination of members of B, and each c_i is a finite linear combination of members of C. Since $\langle b_i, c_i \rangle = 0$ for all i, continuity of inner product requires that $v = 0$ and the sum $\mathcal{H}_B + \mathcal{H}_C$ is direct.

Now by an argument identical to Lemma 14.4 we find that this direct sum is closed. If it is not all of $\mathcal{H}$ we could produce, by Lemma 14.3, a unit vector orthogonal to every vector in $B \cup C = A$, which could then be used to create an orthonormal subset of $\mathcal{H}$ larger than A. This is a contradiction. The necessary conclusion for the first equality of the lemma follows.

The second observation of the lemma now follows from Lemma 14.5. □

14.8. ***Lemma***. *If B and C are disjoint and $B \cup C = A$ for orthonormal basis A of a Hilbert space $\mathcal{H}$ and if $\langle x, b \rangle = 0$ for all $b \in B$ then $x \in \overline{span}(C)$.*

Proof. Easy. □

14.9. ***Theorem***. *Every nontrivial Hilbert space has an orthonormal basis. In fact, if A is an orthonormal subset of a Hilbert space, there is an orthonormal basis containing A.*

Proof. If A is given, let $\mathcal{S}$ denote the set of all orthonormal subsets of Hilbert space $\mathcal{H}$ containing orthonormal subset A. $\mathcal{S}$ is not empty: A itself is in $\mathcal{S}$.

If A is not specified, define it to be $\{v/\|v\|\}$ where v is any nonzero member of $\mathcal{H}$ and proceed as above.

The union of any chain in $\mathcal{S}$ (containment order) is also orthonormal and contains A, and so constitutes an upper bound for the chain. So by Zorn's Lemma there is a maximal member of $\mathcal{S}$: an orthonormal set not contained in any other. This is the orthonormal basis we sought. □

14.10. ***Proposition***. *The cardinalities of any two orthonormal bases of a Hilbert space coincide.*

Proof. Suppose A and B are two orthonormal bases of Hilbert space $\mathcal{H}$. If the cardinality of either is finite, results from finite dimensional linear algebra imply these cardinalities are equal. So suppose both are infinite and the cardinality of A is not larger than that of B. Every $a \in A$ is the limit of finite linear combinations of members of B. Let B_a denote all the members of B involved in one sequence of this kind, for each $a \in A$. The set B_a is countable, so if C is the union of all these B_a the cardinality of C does not exceed that of A. If C does not contain some $b \in B$ then $b \in C^{\perp} = span(C)^{\perp} = \overline{span}(C)^{\perp}$. But $A \subset \overline{span}(C)$ so $b \in A^{\perp} = \{0\}$, a contradiction. So $C = B$ and the cardinality of B cannot exceed that of A: that is, the two cardinalities are equal. □

The cardinality of any orthonormal basis for a Hilbert space is called the **Hilbert dimension** of the space. The content of the last proposition is (essentially) that Hilbert dimension is well defined.

14.11. ***Proposition***.
Suppose $\mathcal{H}$ and $\mathcal{M}$ are Hilbert spaces and $\Psi \colon \mathcal{H} \to \mathcal{M}$ is linear.

(i) If Ψ is an isometry then Ψ takes any orthonormal basis of $\mathcal{H}$ to an orthonormal basis of $\Psi(\mathcal{H})$, which is closed in $\mathcal{M}$ and therefore itself a Hilbert space.

(ii) On the other hand, suppose Ψ is continuous and linear and B is an orthonormal basis of $\mathcal{H}$. If $\Psi(b)$ is a unit vector for each $b \in B$ and $\psi(b)$ is orthogonal to $\Psi(c)$ whenever $c \neq b$ where both c and b are in B then Ψ is an isometry.

Proof. (i) Suppose Ψ is an isometry. Isometries must take vectors of length 1 to vectors of length 1, so we need to check that an isometry takes orthogonal vectors to orthogonal vectors. Suppose $a, b \in \mathcal{H}$ are orthogonal. Expand $\langle\, a-b,\, a-b\,\rangle = \langle\, \Psi(a)-\Psi(b),\, \Psi(a)-\Psi(b)\,\rangle$ to find that the real part of $\langle\, \Psi(a),\, \Psi(b)\,\rangle$ is 0. If the field is $\mathbb{C}$ expand $\langle\, a-i\,b,\, a-i\,b\,\rangle = \langle\, \Psi(a)-i\,\Psi(b),\, \Psi(a)-i\,\Psi(b)\,\rangle$ to find that the complex part of $\langle\, \Psi(a),\, \Psi(b)\,\rangle$ is 0 too. So an isometry does take an orthonormal basis to an orthonormal set in the range space.

Any Cauchy sequence $\Psi(x_n)$ in $\Psi(\mathcal{H})$ corresponds to a Cauchy sequence x_n in $\mathcal{H}$ which converges to some $x \in \mathcal{H}$. Since isometries are continuous $\Psi(x_n)$ converges to $\Psi(x)$ so $\Psi(\mathcal{H})$ is closed and therefore itself a Hilbert space, a subspace of $\mathcal{M}$. If B is an orthonormal basis for $\mathcal{H}$ and $\Psi(B)$ is *not* an orthonormal basis for $\Psi(\mathcal{H})$ then there must be a member $\Psi(x)$ of $\Psi(B)^\perp$ in $\Psi(\mathcal{H})$. This vector is orthogonal to every member of $\Psi(B)$ so (by an argument identical to the one above) x is orthogonal to every member of B which implies $x = 0$. So $\Psi(B)^\perp = \{\,0\,\}$ and so $\Psi(B)$ is an orthonormal basis for $\Psi(\mathcal{H})$.

(ii) Now suppose Ψ is continuous and the other conditions of (ii) apply. If Ψ is *not* an isometry there there is an x for which $\|\,x\,\| \neq \|\,\Psi(x)\,\|$. But then there would be a sequence of *finite* linear combinations y_n of members of B for which $y_n \to x$ but $\|\,y_n\,\| = \|\,\Psi(y_n)\,\|$ does not converge to $\|\,\Psi(x)\,\|$, which contradicts continuity of Ψ. So Ψ is an isometry. □

14.12. ***Exercise***. *Suppose S is any countable subset of Hilbert space $\mathcal{H}$ and suppose S contains more than the zero vector. Adapt the usual Gram-Schmidt process to deduce that there is an orthonormal set B for which $span(S) = span(B)$. The orthonormal set B might be finite or, at most, countably infinite. B can be extended to an orthonormal basis A of $\mathcal{H}$ by adding, if necessary, elements in an orthonormal basis $C = A - B$ for $S^\perp = B^\perp$. Then we would have*

$$\mathcal{H} = C^\perp \oplus B^\perp = \mathcal{H}_B \oplus \mathcal{H}_C.$$

A closed subspace S of any Banach space X is called **complemented** if there is a closed subspace T of X for which $X = S \oplus T$. Earlier (see page 74) we referred to (S, T) as a complementary pair. When in possession of such a pair, the projection onto S with kernel T is continuous.

If S is complemented as above and if $F\colon X \to Y$ is continuous and linear where Y is also Banach then S and T are also Banach spaces and F induces continuous linear maps $F|_S\colon S \to Y$ and $F|_T\colon T \to Y$ by restriction. The operator norms of $F|_S$ and $F|_T$ cannot exceed that of F.

On the other hand, suppose $Q\colon S \to Y$ and $R\colon T \to Y$ are continuous. Let P be the projection onto S with kernel T. So $I - P$ is the projection onto T with kernel S. We can define function $F\colon X \to Y$ by

$$F(x) = Q \circ P(x) + R \circ (I - P)(x) = Q(s) + R(t)$$

where s and t are the unique members of S and T, respectively, for which $x = s + t$.

F is continuous, with operator norm not exceeding $\|Q\|\,\|P\| + \|R\|\,\|I - P\|$.

Any finite dimensional subspace is complemented, but Lindenstrauss and Tzafriri[4] proved that any Banach space for which *every* closed subspace is complemented has an equivalent norm with which it is Hilbert.

Philips[5] showed that in the Banach space $\boldsymbol{\ell^\infty}$ of bounded sequences with supremum norm the closed subspace c_0 of sequences that converge to 0 is not complemented, so there are common examples that do not have this nice property.

If you let A be a Hamel basis of c_0 and add members B of $\boldsymbol{\ell^\infty} - c_0$ to create a Hamel basis for $\boldsymbol{\ell^\infty}$ we have $\boldsymbol{\ell^\infty} = span(A) \oplus span(B)$. But $span(A) = c_0$ is closed and, according to this result of Philips, $span(B)$ is not. So the Banach space $\boldsymbol{\ell^\infty}$ is the direct sum of two subspaces, one of which is closed and the other is not closed.

But every closed subspace of a Hilbert space is complemented. If S is a closed (nontrivial) subspace of Hilbert $\mathcal{H}$ then S is, itself, a Hilbert space with restriction inner product. Find an orthonormal basis B of S, add additional vectors C to create an orthonormal basis for $\mathcal{H}$, and then $S = \mathcal{H}_B$ and $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_C$.

Note that we actually showed more: every closed subspace is complemented by an *orthogonal* closed subspace, and $\mathcal{H}_C$ is called the **orthocomplement** of $\mathcal{H}_B$.

14.13. ***Exercise***. *Consider the Hilbert space $\boldsymbol{\ell^2}$ and the product Hilbert space $X = \boldsymbol{\ell^2} \times \boldsymbol{\ell^2}$ with inner product produced by norm $\|(u,v)\|^2 = \|u\|^2 + \|v\|^2$. Consider the operator $F\colon \boldsymbol{\ell^2} \to \boldsymbol{\ell^2}$ given by*

$$F(v_1, v_2, v_3, \dots) = (v_1/1, v_2/2, v_3/3, \dots).$$

Note that $F(\boldsymbol{\ell^2})$ is dense in $\boldsymbol{\ell^2}$ but is not all of $\boldsymbol{\ell^2}$.

Also F is bounded and therefore $\gamma(F) = \{\, (v, F(v)) \mid v \in \boldsymbol{\ell^2} \,\}$, the graph of F, is a closed subspace of X.

Consider also the graph of the zero operator on $\boldsymbol{\ell^2}$, $\gamma(0) = \boldsymbol{\ell^2} \times \{\, 0 \,\}$. It too is closed in X.

These two subspaces share only the vector $(0,0) \in X$ and are, themselves, Hilbert spaces. Consider now the direct sum $Y = \gamma(0) \oplus \gamma(F)$.

Show $\{\, 0 \,\} \times F(\boldsymbol{\ell^2}) \subset Y$. Since there are members of $\boldsymbol{\ell^2}$ which are limits points of $F(\boldsymbol{\ell^2})$, but not members of $F(\boldsymbol{\ell^2})$, decide that Y is not closed in X.

Conclusion: ***Even in a Hilbert space, the direct sum of two closed infinite dimensional subspaces might fail to be complete.***

This cannot happen when the direct summands involved are orthogonal. Could this happen if one of the spaces is finite dimensional?

[4]Lindenstrauss, J. and Tzafriri, L. *On the Complemented Subspaces Problem* Israel J. Math. 9 263-269 (1971)

[5]Phillips, R. S. *On Linear Transformations* Trans. Amer. Math. Soc. 48 516-541 (1940)

15. Using an Orthonormal Basis

Suppose (b_i) is a countable **orthonormal sequence** of vectors (no repeats, each vector normal and orthogonal to all the others) in an inner product space and consider the sequence $y_k = \sum_{i=1}^{k} a^i b_i$ for certain numbers a^i defined for $i > 0$.

We are going to find a condition on the a^i which is equivalent to the Cauchy condition on the sequence (y_k).

In our context, (y_k) is Cauchy precisely when, for any $\varepsilon > 0$ there is an integer $k > 0$ so that whenever $j > i > k$ we have $\| y_j - y_i \|^2 < \varepsilon$. Expanding, the square norm on the left equals $\sum_{k=i+1}^{j} \left|a^k\right|^2$. We record the important implication in the following result.

15.1. ***Lemma***. *If (b_i) is a countable orthonormal sequence of vectors (no repeats) in an inner product space and (a^i) is a sequence of numbers, the sequence (y_k) defined by $y_k = \sum_{i=1}^{k} a^i b_i$ is Cauchy precisely when the series $\sum_{i=1}^{\infty} \left|a^i\right|^2$ converges.*

Proof. Examine the discussion above. □

15.2. ***Theorem***. ***The Riesz-Fisher Theorem: Hilbert Space Version***

*If A is an orthonormal basis of Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ only countably many of the numbers $\langle x, p \rangle$ for $p \in A$ are nonzero. These numbers are called the **Fourier coefficients** for x in basis A.*

There is a countable sequence of distinct vectors (b_i) from A and, for this sequence, a unique sequence of constants (a^i) with $x = \sum_{i=0}^{\infty} a^i b_i$. In fact, $a^i = \langle x, b_i \rangle$ for each i and this list contains all the nonzero Fourier coefficients for x in basis A.

The convergence of the sequence of partial sums is absolute in the sense that any re-ordering of the terms of the series also converges to x, and $\|x\|^2 = \sum_{i=0}^{\infty} |a^i|^2$.

Proof. Since convergence in $\mathcal{H}$ is given by a metric, a member x of $\overline{span}(A)$ is the limit of a sequence (s_i) of *finite* linear combinations of members of A. Only a countable number of members of A are involved in any of these linear combinations for this specific x.

Let (b_i) denote a sequence formed from all these members of A without repeats. Define B to be the set of these b_i, a subset of A. Since $\langle s_i, w \rangle = 0$ whenever $w \in C = A - B$, by continuity of inner product we have $\langle x, w \rangle = 0$ whenever $w \in C$.

Each sum $s_i = \sum_{j=1}^{\infty} \gamma_i^j b_j$ is in $span(B)$ and so only finitely many of its terms are nonzero. Examining $\langle s_i, b_j \rangle = \gamma_i^j$, linearity and continuity of inner product implies that for each j the sequence $\left(\gamma_i^j\right)$ converges to a number a^j and, in fact, a_j must be $\langle x, b_j \rangle$.

Consider the series $\sum_{j=1}^{\infty} a^j b_j$ with a^j as above with partial sums $y_k = \sum_{j=1}^{k} a^j b_j$.

Expanding $0 \le \langle\ x - \sum_{j=1}^{k} a^j b_j\ ,\ x - \sum_{j=1}^{k} a^j b_j\ \rangle$ yields $\sum_{j=1}^{k} \left|a^j\right|^2 \le \|x\|^2$ which means that $\sum_{j=1}^{\infty} \left|a^j\right|^2$ converges which means (Lemma 15.1) that (y_k) is Cauchy in complete $\mathcal{H}$, and therefore converges to some vector $z = \sum_{j=1}^{\infty} a^j b_j$.

By continuity of inner product, $\langle\, z, w\,\rangle = 0$ for all $w \in C$, and $\langle\, x - z, w\,\rangle = 0$ for all $w \in B$ by definition of z. So (Lemma 14.8) $x - z = 0$: that is, $x = \sum_{j=1}^{\infty} a^j b_j$.

The remaining comments of the Theorem are immediate. □

The reader is invited to recall and reflect upon the relationship between this theorem and Theorem **??**, another result commonly referred to as "the" Riesz-Fisher Theorem, which states that $\boldsymbol{L^p}$ is a complete metric space for $0 < p \le \infty$.

In any case, it appears that orthonormal bases may be good for something.

15.3. ***Theorem***. *Suppose B and C are disjoint subsets of orthonormal basis A of Hilbert space $\mathcal{H}$ and $B \cup C = A$. Suppose x and y are in $\mathcal{H}$.*

The numbers $\langle\, x, w\,\rangle$ are nonzero for only countably many $w \in A$.

The number $\langle\, x, y\,\rangle$ can be calculated using the absolutely convergent series[6]

$$\sum_{w\in A} \langle\, x, w\,\rangle \langle\, w, y\,\rangle = \langle\, x, y\,\rangle. \qquad \textbf{\textit{(Parseval's Identity)}}$$

The series $\sum_{w\in B} |\langle\, x, w\,\rangle|^2$ converges and

$$\sum_{w\in B} |\langle\, x, w\,\rangle|^2 \le \|x\|^2. \qquad \textbf{\textit{(Bessel's Inequality)}}$$

The series $\sum_{w\in A} |\langle\, x, w\,\rangle|^2$ converges and

$$\sum_{w\in A} |\langle\, x, w\,\rangle|^2 = \|x\|^2. \qquad \textbf{\textit{(Plancherel's Identity)}}$$

Further, the series $\sum_{w\in B}\langle\, x, w\,\rangle\, w$ and $\sum_{w\in C}\langle\, x, w\,\rangle\, w$ converge to vectors x_B and x_C, respectively.

Finally, $x = x_B + x_C$ and $\|x\|^2 = \|x_B\|^2 + \|x_C\|^2$. ***(Pythagorean Law)***

Proof. We will show that the series $\sum_{w\in A}\langle\, x, w\,\rangle \langle\, w, y\,\rangle$ converges absolutely to $\langle\, x, y\,\rangle$. The remaining parts of this theorem follow immediately from this fact or Theorem 15.2.

Let (b_j) denote the sequence composed (without repeats) of all the members w of A where either $\langle\, w, y\,\rangle \ne 0$ or $\langle\, x, w\,\rangle \ne 0$.

By continuity of inner product, we know that the sequence

$$\left\langle \sum_{i=1}^{k} \langle\, x, b_i\,\rangle\, b_i\,,\ \sum_{i=1}^{k} \langle\, y, b_i\,\rangle\, b_i \right\rangle = \sum_{i=1}^{k} \langle\, x, b_i\,\rangle \langle\, b_i, y\,\rangle$$

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[6] The two relations listed here as Parseval's Identity and Plancherel's Identity are ascribed to Marc-Antoine Parseval (1755-1836) and Michel Plancherel (1885-1967). These men proved results involving the particular case of trigonometric series. Investigating the literature, there seems to be little doubt in anyone's mind which result should be attributed to which man. Different minds, however, disagree.

converges to $\langle x, y \rangle$. It remains to justify absolute convergence of these numbers. Modify x to a new vector $\tilde{x} = \sum_{i=1}^{k} \alpha_i \langle x, b_i \rangle b_i$, where each α_i is a complex number of norm 1 and for which each $\langle \tilde{x}, b_i \rangle \langle b_i, y \rangle = \alpha_i \langle x, b_i \rangle \langle b_i, y \rangle \geq 0$. The magnitude of each coefficient is unchanged by this, and the sum of positive terms

$$\sum_{i=1}^{k} \langle \tilde{x}, b_i \rangle \langle b_i, y \rangle = \sum_{i=1}^{k} | \langle x, b_i \rangle \langle b_i, y \rangle |$$

converges to the number $\langle \tilde{x}, y \rangle$. □

For any orthonormal set B in Hilbert space $\mathcal{H}$ we now officially define $\mathcal{H}_B$ to be

$$\mathcal{H}_B = \overline{span}(B).$$

$\mathcal{H}_B$ is a closed subspace and so is, itself, a Hilbert space. Note also that any orthonormal set B, if it is not already an orthonormal basis, can be extended by adding vectors from the orthogonal orthonormal set C, so that $B \cup C$ is an orthonormal basis for $\mathcal{H}$. Then $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_C = C^\perp \oplus B^\perp$.

We define for any $x \in \mathcal{H}$ the **orthogonal projection vector** onto $\mathcal{H}_B$ to be

$$\mathbf{x_B} = \sum_{w \in B} \langle x, w \rangle \, w.$$

We saw above that this sum has only countably many nonzero terms, converges, and that $\|x_B\| \leq \|x\|$ for any $x \in \mathcal{H}$.

Finally, we define the **orthogonal projection function**

$$\mathbf{P_B} \colon \mathcal{H} \to \mathcal{H} \quad \text{by} \quad P_B(x) = x_B \text{ for any } x \in \mathcal{H}.$$

Orthogonal projection functions may be referred to as **orthoprojections.**

15.4. ***Corollary***. *The orthogonal projection function P_B defined above is linear and continuous, with $\|P_B\| = 1$.*

PROOF. Follows immediately from the remarks above. □

15.5. ***Lemma***. *Suppose J and K are closed subspaces of Hilbert space $\mathcal{H}$ and the Hilbert dimension of J is less than the Hilbert dimension of K. Then there is a nonzero vector in $J^\perp \cap K$.*

PROOF. If the dimension of J is the finite number n let $P \colon \mathcal{H} \to \mathcal{H}$ denote the orthogonal projection onto J. There are at least $n+1$ linearly independent vectors in K, and without loss we assume K is the span of these vectors, and so has dimension $n+1$. If P restricted to K had no kernel the images in J of the $n+1$ basis vectors of K would be linearly independent, impossible since J has dimension n. And any member of the kernel of P is in $J^\perp$.

We now suppose that J has infinite dimension. Let A be an orthonormal basis of J and B an orthonormal basis of K. For each $a \in A$ the set B_a consisting of those $b \in B$ for which $\langle a, b \rangle \neq 0$ is countable. So the cardinality of $\bigcup_{a \in A} B_a$ is no more than that of A. So there is a member of B not in this union, and this member of B is in $J^\perp$. □

15.6. ***Exercise***. *Suppose $\mathcal{H}$ is a Hilbert space with countable orthonormal basis $\mathbf{v} = (v_n)$. Define a^i to be the functional $\langle \cdot, v_i \rangle$ for each i. Then $\mathbf{v}$ is a Schauder basis for $\mathcal{H}$ with associated coordinate functionals $\mathbf{a} = (a^n)$.*

$\mathbf{v}$ is an unconditional bounded Schauder basis with basis constant 1. Also $\|v_n\| = 1 = \|a^n\|$ for all n. Further, $\mathbf{a}$ forms a Schauder basis on $\mathcal{H}'$ with associated coordinate functionals $\mathbf{v}$, and $\mathbf{a}$ also has basis constant 1.

The isomorphism $K \colon C_{\mathbf{v}} \to \mathcal{H}$ of Proposition 11.4 is an isometry onto $\mathcal{H}$, so $\mathcal{H}$ is isometrically isomorphic to the subspace $C_{\mathbf{v}}$ of $\mathbb{F}^{\mathbb{N}}$ which consists of all those sequences c for which $\sqrt{\sum_{i=0}^{\infty}(c(i))^2} < \infty$, with this expression as norm on $C_{\mathbf{v}}$.

There is really just one Hilbert space with a countable orthonormal basis, and if you want to visualize that Hilbert space as the space of square summable sequences you will not be led astray.

We will now consider issues of compactness again.

If E is any totally bounded set in Hilbert space $\mathcal{H}$ (for instance a compact set) then by Lemma 5.20 E has a countable dense subset S. By the result in Exercise 14.12 there is a countable orthonormal basis B for $\overline{span}(E) = \overline{span}(S)$ which can be extended to an orthonormal basis A of all of $\mathcal{H}$ by adding vectors C as in that exercise.

Then $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_C$ where $\mathcal{H}_B$ and $\mathcal{H}_C$ are each, themselves, Hilbert spaces with restriction inner product and B, with an appropriate well order, is an unconditional Schauder basis for $\mathcal{H}_B$.

We know that the identity map restricted to $\mathcal{H}_B$, the function $I_B \colon \mathcal{H}_B \to \mathcal{H}_B$, can be approximated pointwise by the finite rank partial sum operators for the basis B, and the approximation is uniform on the compact set $\overline{E}$.

And since $\mathcal{H}_B$ and $\mathcal{H}_C$ are both closed in $\mathcal{H}$ these finite rank operators on $\mathcal{H}_B$ can be extended to continuous finite rank mapping on all of $\mathcal{H}$: define each to be the zero map on the $\mathcal{H}_C$ direct summand.

15.7. ***Proposition***. *Every Hilbert space $\mathcal{H}$ has the approximation property: for each compact set in $\mathcal{H}$ there is a sequence of finite rank continuous operators that converges to the identity uniformly on that compact set.*

Proof. Suppose E is any totally bounded subset of $\mathcal{H}$. The sequence of partial sum operators for a Schauder basis for $\overline{span}(E)$ converges to the identity of $\overline{span}(E)$, and the convergence is uniform on E. This can be carried out for any totally bounded set. □

15.8. ***Proposition***. *If W is a Banach spaces and $\mathcal{H}$ is a Hilbert space and if $T \in \boldsymbol{\mathcal{K}}(W, \mathcal{H})$ then T has the approximation property: that is, T is the operator norm limit of finite rank operators.*

Proof. The range of T is $\overline{span}(T(B))$ where B is the unit ball in W. The set $T(B)$ has compact closure, so the remarks of Proposition 15.7 apply with $E = T(B)$. There is a sequence of orthonormal vectors in $\mathcal{H}$ and sequence of partial sum operators (S_j) for this sequence for which $S_j \circ T$ converges to T in operator norm, as in Proposition 11.6. □

16. Riesz Representation on a Hilbert Space

We confine our attention in this section to functionals defined on a Hilbert space $\mathcal{H}$ or on vector subspaces of $\mathcal{H}$.

We presume A is an orthonormal basis for $\mathcal{H}$ and B and C are disjoint with $A = B \cup C$.

16.1. ***Lemma***. *Each member ϕ of $\mathcal{H}^*$ can be used to create unique members ϕ_B of $\mathcal{H}_B^*$ and ϕ_C of $\mathcal{H}_C^*$ by restriction. If ϕ is continuous both ϕ_B and ϕ_C will be continuous.*

Conversely, any pair of members ϕ_B of $\mathcal{H}_B^*$ and ϕ_C of $\mathcal{H}_C^*$ can be used to create a member ϕ of $\mathcal{H}^*$ by

$$\phi(x) = \phi_B(x_B) + \phi_C(x_C) \quad \textit{for } x = x_B + x_C \in \mathcal{H}$$

where x_B and x_C are the projections of x onto the orthogonal subspaces $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively.

ϕ is continuous exactly when both ϕ_B and ϕ_C are continuous.

Proof. Easy. □

16.2. ***Lemma***. *Each bounded member ϕ of $span(B)^*$ (i.e. a member of $span(B)'$) can be extended in one and only one way to a member of $\mathcal{H}_B'$.*

Proof. By the Hahn-Banach theorem, ϕ can be extended while preserving the norm bound on ϕ: that is, extended to a continuous function on $\mathcal{H}_B$. Because *span*(B) is dense in $\mathcal{H}_B$, values on limit points are determined as limits of values on members of *span*(B). □

Suppose $\phi \in \mathcal{H}^*$ and B consists of those members of a of A for which $\phi(a) \neq 0$. For each $b \in B$ let $d_b = k_b b$ where k_b is a complex number of norm 1 for which $\phi(d_b) > 0$.

If B is uncountable then there is a positive number α for which $\phi(d_b) > \alpha$ for infinitely many $b \in B$. Let (h_i) denote a sequence of distinct members of B with this property, and define

$$w_i = \frac{1}{\beta}\sum_{j=1}^{i}\frac{h_j}{j} \quad \textit{where } \beta \text{ is the constant } \sqrt{\sum_{i=1}^{\infty} 1/n^2}.$$

For each i, $\langle\, w_i, w_i\rangle = \frac{1}{\beta^2}\sum_{j=1}^{i}\frac{1}{j^2} < 1$.

But $\phi(w_i) = \frac{1}{\beta}\sum_{j=1}^{i}\frac{\phi(h_j)}{j} > \frac{\alpha}{\beta}\sum_{j=1}^{i}\frac{1}{j}$. Since that sum can become unboundedly large, ϕ is not continuous. We enshrine this result as:

16.3. ***Lemma***. *If $\phi \in \mathcal{H}'$ then ϕ is nonzero on only countably many members of orthonormal basis A.*

Proof. The argument is given above. □

This result concerning continuity is not a sufficient condition, however. A functional for which B given above is countable could still fail to be continuous. Let (b_i) be a sequence formed from all the members of B without repeats.

Consider the sequence of vectors (y_k) given by $y_k = \sum_{i=1}^{k} \overline{\phi(b_i)}\, b_i$ where $\overline{\phi(b_i)}$ denotes the complex conjugate of $\phi(b_i)$.

So $\|y_k\|^2 = \sum_{i=1}^{k} |\phi(b_i)|^2$.

The sequence of vectors is Cauchy and will converge to some vector w exactly when the series $\sum_{i=1}^{\infty} |\phi(b_i)|^2$ converges.

Now suppose that in spite of the countability condition on B the functional ϕ is unbounded. Then for any constant $\alpha > 0$ we could find x in $span(B)$ with norm 1 for which $|\phi(x)| > \alpha$. The vector x can be written as $x = \sum_{i=1}^{k} a^i b_i$ for some finite k. Then we have, by linearity of ϕ and the BCS inequality,

$$\begin{aligned} |\phi(x)| &= \left| \phi\left(\sum_{i=1}^{k} a^i b_i \right) \right| = \left| \sum_{i=1}^{k} a^i \phi(b_i) \right| \\ &= \left| \left\langle \sum_{i=1}^{k} a^i b_i\,,\, \sum_{i=1}^{k} \overline{\phi(b_i)} b_i \right\rangle \right| = |\,\langle\, x, y_k \rangle\,| \leq \|\, x \,\|\, \|\, y_k \,\| \\ &\leq \sqrt{\sum_{i=1}^{\infty} |\phi(b_i)|^2}. \end{aligned}$$

We conclude that $\sum_{i=1}^{\infty} |\phi(b_i)|^2$ must fail to converge, and so too must (y_k).

On the other hand, suppose that $\sum_{i=1}^{\infty} |\phi(b_i)|^2$ does converge. So for each $x \in span(B)$ of norm 1 the same calculation has $|\phi(x)| \leq \sqrt{\sum_{i=1}^{\infty} |\phi(b_i)|^2}$ so ϕ is bounded on $span(B)$, and therefore on its closure $\mathcal{H}_B$. ϕ is continuous there, and zero on $\mathcal{H}_B^{\perp}$, so it is continuous on all of $\mathcal{H}$.

And ϕ agrees with the continuous function $\langle\,\cdot,\, w\,\rangle$ on the dense subset $span(B)$ of $\mathcal{H}_B$ and therefore on all of $\mathcal{H}$. We have just proven the following critical result.

16.4. ***Theorem***. *Suppose A is an orthonormal basis of Hilbert space $\mathcal{H}$. $\phi \in \mathcal{H}^*$ is continuous exactly when $\sum_{a \in A} |\phi(a)|^2$ converges, which will occur exactly when $\sum_{a \in A} \overline{\phi(a)}\, a$ converges.*

That happens if and only if there is a vector $w \in \mathcal{H}$ for which

$$\phi(\cdot) = \langle\, \cdot\,,\, w\,\rangle. \quad \textbf{\textit{(The Riesz Representation Theorem)}}$$

Nondegeneracy of inner product then requires uniqueness of w.

In case of continuity, $w = \sum_{a \in A} \overline{\phi(a)}\, a$ and so $\|\phi\| = \|w\|$.

When $\mathcal{H}'$ has operator norm, the association of $\phi \in \mathcal{H}'$ with $w \in \mathcal{H}$ is a conjugate-linear isometry onto $\mathcal{H}$.

Proof. The argument is found in the discussions above. □

Let $\mathcal{R}\colon \mathcal{H}' \to \mathcal{H}$ be the conjugate linear isometry sending ϕ to w as above. And let $\mathcal{Z}\colon \mathcal{H} \to \mathcal{H}'$ be its inverse function: $\mathcal{Z}(w) = \langle \cdot, w \rangle$. So $\mathcal{Z}$ is also a conjugate linear isometry.

16.5. ***Theorem***. *The operator norm on $\mathcal{H}'$ satisfies the parallelogram law and so $\mathcal{H}'$ is itself a Hilbert space, and so then is $\mathcal{H}''$. The composition of two onto conjugate linear isometries is a linear isometry onto $\mathcal{H}''$. If $w \in \mathcal{H}$ is associated with member Ψ of $\mathcal{H}''$ under this linear isometry then $\Psi(\phi) = \phi(w)$ for all $\phi \in \mathcal{H}'$: in other words, the composition of these isometries is the evaluation map, $E\colon \mathcal{H} \to \mathcal{H}''$, and is onto $\mathcal{H}''$.*

$$\mathcal{Z}_1\colon \mathcal{H} \to \mathcal{H}', \qquad \mathcal{Z}_2\colon \mathcal{H}' \to \mathcal{H}'', \qquad \mathcal{Z}_2 \circ \mathcal{Z}_1 = E \;:\; \mathcal{H} \to \mathcal{H}''.$$

We conclude that any Hilbert space is reflexive.

PROOF. The argument follows from the last theorem. □

16.6. ***Corollary***. *Suppose $\mathcal{H}$ is a Hilbert space and x_n is a sequence in $\mathcal{H}$.*

(i) If x_n is norm bounded then x_n has a weakly convergent subsequence .

(ii) If x_n is weakly convergent to x and $\|x_n\|$ converges to $\|x\|$ then x_n converges to x in norm.

PROOF. In Exercise 8.11 we saw that (i) holds in any reflexive normed space, and by the last theorem we know any Hilbert space is reflexive.

On the other hand, suppose x_n is weakly convergent to x and also $\|x_n\|$ converges to $\|x\|$. Then

$$\langle x - x_n,\, x - x_n \rangle = \langle x,\, x \rangle + \langle x_n,\, x_n \rangle - \langle x,\, x_n \rangle - \langle x_n,\, x \rangle$$

converges to 0 and (ii) follows. □

16.7. ***Exercise***. *(i) If (b_i) is an orthonormal sequence of vectors in a Hilbert space $\mathcal{H}$ and $v \in \mathcal{H}$ and c is any positive number. Then $v + c\, b_i$ converges weakly to v but $\|\, v + c\, b_i \,\|^2$ converges to $\|\, v \,\|^2 + c^2$.*

(ii) If sequence (w_i) converges weakly to v then $\limsup \|\, w_i \,\| \geq \|\, v \,\|$. (hint: $\langle w_i,\, v \rangle \leq \|\, w_i \,\|\, \|\, v \,\|$.)

16.8. ***Exercise***. *(i) Adapt Exercise 9.7 to apply to sesquilinear forms.*

(ii) Define $\phi(x,y) = \langle A(x),\, B(y) \rangle$ for linear $A, B\colon \mathcal{H} \to \mathcal{H}$. If ϕ is continuous in each coordinate separately, which will happen (for instance) if both A and B are continuous, then ϕ is continuous.

(iii) Suppose A is continuous. If $\mathcal{R}_B$ is the range of B there is unique linear $C\colon \mathcal{R}_B \to \mathcal{H}$ for which

$$\phi(x,y) = \langle A(x),\, B(y) \rangle = \langle x,\, C(B(y)) \rangle \quad \forall x, y \in \mathcal{H}.$$

16.9. ***Exercise***. *By Hölder's inequality if x and y are in $\boldsymbol{\ell^2}$ we know that $xy \in \boldsymbol{\ell^1}$. We now suppose that x is a fixed sequence. Does the condition $xy \in \boldsymbol{\ell^1}$ whenever $y \in \boldsymbol{\ell^2}$ actually imply $x \in \boldsymbol{\ell^2}$? (hint: For any sequence x define $G_n \colon \boldsymbol{\ell^2} \to \boldsymbol{\ell^1}$ by*

$$(G_n y)(k) = x(k)\, y(k) \text{ for } k \le n \text{ and } (G_n y)(k) = 0 \text{ otherwise.}$$

Each G_n is bounded from $\boldsymbol{\ell^2}$ to $\boldsymbol{\ell^1}$ with their usual norms. If x has the stated property then for every $y \in \boldsymbol{\ell^2}$ the sequence $G_n(y)$ converges to a limit in $\boldsymbol{\ell^1}$. The uniform boundedness principle as described in Exercise 9.5 implies that the set of operator norms $\{\, \|G_n\| \mid n > 0 \,\}$ is bounded by some constant M, and this means

$$\sum_{k=1}^{\infty} |\, x(k)\, y(k) \,| \le M \|\, y \,\|_2.$$

The functional $\psi \colon \boldsymbol{\ell^2} \to \boldsymbol{\ell^2}$ given by $\psi(y) = \sum_{k=1}^{\infty} x(k)\, y(k)$ is therefore continuous and any such functional on the Hilbert space $\boldsymbol{\ell^2}$ is given as inner product against a member of $\boldsymbol{\ell^2}$, which obviously must be x.)

17. The Hermitian Adjoint and Normal Operators

The **Hermitian adjoint** of any continuous operator F on Hilbert space $\mathcal{H}$ is the unique operator $\mathbf{F}^\dagger$ for which

$$\langle\, F(x), y \,\rangle = \langle\, x, F^\dagger(y) \,\rangle \text{ for all } x, y \in \mathcal{H}.$$

The Riesz representation theorem applied to the continuous and linear function $\langle\, F(\cdot), y \,\rangle$ assures us that a vector $F^\dagger(y)$ exists for each y, and properties of inner product then assure us the resulting function is uniquely defined and linear.

Note that if G is also a continuous operator then $(FG)^\dagger = G^\dagger F^\dagger$.

The Hermitian adjoint operator is a combination of the conjugate linear isometries provided in Theorem 16.4 with the Banach adjoint operator *. Specifically,

$$\mathcal{Z} \colon \mathcal{H} \to \mathcal{H}', \qquad ^* \;:\; \mathfrak{C}\boldsymbol{\mathcal{L}}\,(\mathcal{H}) \to \mathfrak{C}\boldsymbol{\mathcal{L}}\,(\mathcal{H}'), \qquad \mathcal{R} \colon \mathcal{H}' \to \mathcal{H}.$$

So for $F \in \mathfrak{C}\boldsymbol{\mathcal{L}}\,(\mathcal{H})$ we have $F^\dagger = \mathcal{R} \circ F^* \circ \mathcal{Z}$.

Since F^* is continuous and $\|F^*\| = \|F\|$ and both $\mathcal{Z}$ and $\mathcal{R}$ are conjugate linear isometries onto their ranges we conclude that $F^\dagger$ is continuous and

$$\left\|F^\dagger\right\| = \|F\| = \|F^*\|.$$

If α is a scalar and F and G are continuous operators then

$$(\, \alpha\, F + G\,)^\dagger = \overline{\alpha}\, F^\dagger + G^\dagger.$$

It is worth comparing the following result to Lemma 12.6. That lemma involves Banach adjoints and their associated subspaces. In fact most results about Hermitian adjoints translate to/from similar results about Banach adjoints using the $\mathcal{Z}$ and $\mathcal{R}$ isometries.

17.1. ***Lemma***. *If $F \in \mathcal{CL}(\mathcal{H})$ then $F = F^{\dagger\dagger}$ and*

$$Ker(F^\dagger)^\perp = \overline{F(\mathcal{H})} \quad \text{and} \quad Ker(F)^\perp = \overline{F^\dagger(\mathcal{H})}.$$

$$Ker(F^\dagger) = \overline{F(\mathcal{H})}^\perp \quad \text{and} \quad Ker(F) = \overline{F^\dagger(\mathcal{H})}^\perp.$$

Proof. Adapting Proposition 8.16 (or by direct proof) and using the fact that any Hilbert space is reflexive, we have $F = F^{\dagger\dagger}$.

Since $\langle F(x), y \rangle = \langle x, F^\dagger(y) \rangle$ it is obvious that $F(\mathcal{H}) \subset Ker(F^\dagger)^\perp$, and since $Ker(F^\dagger)^\perp$ is closed we have also that $\overline{F(\mathcal{H})} \subset Ker(F^\dagger)^\perp$.

On the other hand, suppose $y \in F(\mathcal{H})^\perp$. Then $\langle F(x), y \rangle = 0 = \langle x, F^\dagger(y) \rangle$ for all x, which means $y \in Ker(F^\dagger)$. So $F(\mathcal{H})^\perp \subset Ker(F^\dagger)$.

Recalling Lemma 14.5 we have $Ker(F^\dagger)^\perp \subset F(\mathcal{H})^{\perp\perp} = \overline{F(\mathcal{H})}$ and therefore $Ker(F^\dagger)^\perp = \overline{F(\mathcal{H})}$.

The remaining set equalities are left as an easy exercise. □

17.2. ***Exercise***. *Adapt Exercises 8.15 and 8.18 to show that the Hermitian adjoint* $\;^\dagger : \mathcal{CL}(\mathcal{H}) \to \mathcal{CL}(\mathcal{H})$ *is continuous when $\mathcal{CL}(\mathcal{H})$ is given either operator norm or weak operator topology. In the following example we show the adjoint operator need not be continuous when $\mathcal{CL}(\mathcal{H})$ has strong operator topology.*

Define $H_{x,y} \colon \mathcal{H} \to \mathcal{H}$ by $H_{x,y}(w) = \langle w, x \rangle y$.

Note that

$$\langle H_{x,y}(z), w \rangle = \langle \langle z, x \rangle y, w \rangle = \langle z, x \rangle \langle y, w \rangle$$

while

$$\langle z, H_{y,x}(w) \rangle = \langle z, \langle w, y \rangle x \rangle = \langle y, w \rangle \langle z, x \rangle.$$

In other words, $H_{y,x}$ and $H_{x,y}$ are Hermitian adjoints to each other.

Now suppose (b_n) is a countable orthonormal sequence in $\mathcal{H}$. Defining, for each x and n, numbers $\zeta_n = \langle x, b_n \rangle$, Bessel's Inequality implies that the numerical sequence (ζ_n) converges to 0.

Let $T_n = H_{b_n, b_1}$. So $T_n(x) = \langle x, b_n \rangle b_1 = \zeta_n b_1$. So the sequence (T_n) converges to 0 in the strong operator topology.

But $T_n^\dagger = H_{b_1, b_n}$ and for given x, $T_n^\dagger(x) = \langle x, b_1 \rangle b_n = \zeta_1 b_n$. This sequence of vectors will not generally converge in norm (never, unless ζ_1 happens to be 0) so this sequence $(T_n^\dagger)$ of Hermitian adjoints does not converge in strong operator topology.

So Hermitian adjoint is not continuous when $\mathcal{H}$ is infinite dimensional and $\mathcal{CL}(\mathcal{H})$ has strong operator topology.

17.3. ***Exercise***. *Adapt Proposition 8.20 or work directly to show that the Hermitian adjoint $F^\dagger$ of a continuous operator F on a Hilbert space is compact or has finite rank exactly when F is compact or, respectively, has finite rank.*

An operator $F \in \mathcal{CL}(\mathcal{H})$ is called **normal** if it commutes with its adjoint.

$$F\, F^\dagger = F^\dagger\, F.$$

In the finite dimensional setting, an operator is normal if the matrix that represents it in a basis commutes with the matrix of its adjoint. The adjoint of a matrix is its conjugate transpose, and a matrix that commutes with its adjoint called **normal**. The adjoint of a matrix which represents an operator is the matrix of the adjoint operator. Normal matrices are precisely those that can be diagonalized (see Exercise 17.8) using matrix of transition whose adjoint and inverse coincide (i.e. $U^\dagger U = I$) and we will be concerned with analogous results in this infinite dimensional setting.

17.4. ***Lemma***. *Suppose F is a normal operator on Hilbert space $\mathcal{H}$.*

$$\|\, F(x)\,\| = \sqrt{\langle\, F(x), F(x)\,\rangle} = \sqrt{\langle\, F^\dagger(x), F^\dagger(x)\,\rangle} = \|\, F^\dagger(x)\,\|.$$

In particular, the kernels of F and $F^\dagger$ coincide.

$$\textit{Also}\quad \forall x, y \in \mathcal{H} \qquad \langle\, F(x), F(y)\,\rangle = \langle\, F^\dagger(x), F^\dagger(y)\,\rangle.$$

So $F(x)$ and $F(y)$ are orthogonal exactly when $F^\dagger(x)$ and $F^\dagger(y)$ are, and F is an isometry exactly when $F^\dagger$ is an isometry.

Proof.

$$\langle\, F(x), F(y)\,\rangle = \langle\, x, F^\dagger\, F(y)\,\rangle = \langle\, x, F\, F^\dagger(y)\,\rangle = \langle\, F^\dagger(x), F^\dagger(y)\,\rangle.$$

□

See Corollary 18.4 for the converse implication, which holds when $\mathcal{H}$ is a complex Hilbert space.

Not only do the kernels of a normal operator and its adjoint coincide, so too do the images of these operators.

17.5. ***Lemma***. *Suppose F is a normal operator on Hilbert space $\mathcal{H}$. Then $F(\mathcal{H}) = F^\dagger(\mathcal{H})$.*

Proof. Define linear D on $F^\dagger(\mathcal{H})$ by $Dz = Fy$ whenever $z = F^\dagger y$. If a second vector w satisfies $z = F^\dagger w$ then $w - y \in Ker(F^\dagger) = Ker(F)$ so $Fw = Fy$ and we conclude that D is well defined on $F^\dagger(\mathcal{H})$.

Suppose $z \in F^\dagger(\mathcal{H})$ and $\|z\| = 1$. So z is of the form $F^\dagger\left(y/\left\|F^\dagger y\right\|\right)$.

$$\text{Then}\quad \|Dz\| = \left\|F\left(\frac{y}{\|F^\dagger y\|}\right)\right\| = \frac{\|Fy\|}{\|F^\dagger y\|} = 1.$$

So D is an isometry from $F^\dagger(\mathcal{H})$ onto $F(\mathcal{H})$ and therefore can be extended (Exercise 5.13) in a unique way to a continuous operator (an isometry, actually) from $\overline{F^\dagger(\mathcal{H})}$ onto $\overline{F(\mathcal{H})}$. Every closed subspace of a Hilbert space is complemented so D can be further extended to a continuous operator defined on all of $\mathcal{H}$, an isometry when restricted to $\overline{F^\dagger(\mathcal{H})}$ and the zero operator on $\overline{F^\dagger(\mathcal{H})}^\perp$. And using this extended D we have $DF^\dagger = F$ on all of $\mathcal{H}$. We have factorized F as the composition of two continuous operators.

So $FD^\dagger = F^\dagger$ and therefore $F^\dagger(\mathcal{H}) \subset F(\mathcal{H})$. The reverse inclusion follows by an identical argument, switching F and $F^\dagger$. □

17.6. ***Corollary***. *Suppose F is a normal operator on Hilbert space $\mathcal{H}$.*

$Ker(F) = Ker\left(F^\dagger\right)$ *and* $F^\dagger(\mathcal{H}) = F(\mathcal{H})$.

$Ker(F)^\perp = \overline{F(\mathcal{H})} = Ker\left(F^\dagger\right)^\perp = \overline{F^\dagger(\mathcal{H})}$.

So the image of F is dense in $\mathcal{H}$ exactly when the kernel of F is trivial.

Proof. This collects earlier results. □

Actually, we can get a little more out of the proof of Lemma 17.5. By the corollary above we see that the images of F and $F^\dagger$ coincide when F is normal, as do their kernels in any case.

And if $X = Ker(F)$ and $Y = F(\mathcal{H})$ then X is closed and $\overline{Y} = X^\perp$ and $\overline{Y}^\perp = X$.

Both F and $F^\dagger$ are one-to-one when restricted to $\overline{Y}$ and $F\left(\overline{Y}\right) = F^\dagger\left(\overline{Y}\right) = Y$. So the complementary pair $\left(X, \overline{Y}\right)$ reduces both F and $F^\dagger$.

Note that if I is the identity operator and F is normal then $F - \lambda I$ is also normal. Any nonzero vector in the kernel of $F - \lambda I$ is also in the kernel of $(F - \lambda I)^\dagger = F^\dagger - \overline{\lambda} I$, and conversely. Note that any member of this kernel must be in Y. So eigenvectors for eigenvalue λ for normal F are exactly the eigenvectors for eigenvalue $\overline{\lambda}$ for $F^\dagger$.

This implies that eigenvectors for distinct eigenvalues are orthogonal: if x is an eigenvector for normal F for eigenvalue λ and y is an eigenvector for F for eigenvalue α with $\lambda \neq \alpha$ then

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle F(x), y \rangle = \langle x, F^\dagger(y) \rangle = \langle x, \overline{\alpha}\, y \rangle = \alpha \langle x, y \rangle$$

which implies that $\langle x, y \rangle = 0$.

Since normal F is one-to-one when restricted to $\overline{Y}$, it follows that for any $n > 1$ the operator F^n is one-to-one when restricted to $\overline{Y}$ as well: if $a, b \in \overline{Y}$ and $F^n(a) = F^n(b)$ then $F\left(F^{n-1}(a) - F^{n-1}(b)\right) = 0$ so $F^{n-1}(a) = F^{n-1}(b)$. We find, after $n-1$ such steps, that $F(a) = F(b)$ so $a = b$.

And we have $Ker\left(F^n\right) = Ker(F)$ for any $n > 1$. To see this, suppose $n > 1$ and $F^n(x) = F(F^{n-1}(x)) = 0$. That means $F^{n-1}(x) \in Y \cap X = \{\,0\,\}$ so $x \in Ker\left(F^{n-1}\right)$ and the proof follows by induction.

So the complementary pair $\left(X, \overline{Y}\right)$ reduces both F^n and $\left(F^\dagger\right)^n = (F^n)^\dagger$ for $n \geq 1$, and these iterates of F are also normal.

Applying this to the normal operator $F - \lambda I$ we see that the kernel of $(F - \lambda I)^n$ is the same as the kernel of $F - \lambda I$ for each λ and each $n \geq 1$. So for a normal operator F and each eigenvalue λ, the generalized eigenspaces (the set of vectors a for which $(F - \lambda I)^n(a) = 0$ for *some* n) are not larger than the eigenspaces.

Now suppose $x \in \overline{Y}$ and $F(F(x)) = \lambda F(x)$ for some λ where $x \neq 0$. Then $F(F(x) - \lambda x) = 0$ so $F(x) - \lambda x \in \overline{Y} \cap X = \{\,0\,\}$ so x is, itself, an eigenvector for eigenvalue λ.

This means that the action of F cannot turn a non-eigenvector in $\overline{Y}$ into an eigenvector. Specifically, if $\mathbf{Eig}_\lambda$ is the eigenspace for eigenvalue λ then $\left(Eig_\lambda, Eig_\lambda^\perp\right)$ is a complementary pair of subspaces that reduces F, and also reduces any iterate

F^n and their adjoints, where orthogonal complementation is taken in either $\overline{Y}$ or all of $\mathcal{H}$.

In the proof of Lemma 17.5 we construct an isometry D, and we may deduce that there is an isometry E, from $\overline{Y}$ onto $\overline{Y}$, for which

$$DF^\dagger = F \qquad \text{and} \qquad EF = F^\dagger.$$

In that proof we extend D to all of $\mathcal{H}$ by defining it to be 0 on X, which we discovered along the way was $\overline{Y}^\perp$, extending by linearity to the direct sum $\mathcal{H} = X \oplus \overline{Y}$.

With that definition $DEF = F$ and we conclude that DE is the identity on Y and an isometry on $\overline{Y}$ so DE is the orthogonal projection onto $\overline{Y}$.

Any linear map such as D that is an isometry when restricted to the orthogonal complement of its kernel is called a **partial isometry**, and we have seen they can come in handy. Orthogonal projections provide examples, but are not the only ones: except in special cases D itself will not be an orthogonal projection.

Leaving that for now (obviously partial isometries have properties we have not explored) we could have, instead, defined D (and E too) to be the identity on X rather than 0, and after extending by linearity both D and E would be isometries of $\mathcal{H}$ onto $\mathcal{H}$.

Using the new definition, the complementary pair $\left(X, \overline{Y}\right)$ reduces D and E and their adjoints too. These adjoints are also isometries, both when restricted as operators from $\overline{Y}$ onto $\overline{Y}$ and as operators from $\mathcal{H}$ onto $\mathcal{H}$.

Now let's focus on F, D and E, and their adjoints, as operators from $\overline{Y}$ to $\overline{Y}$. All six operators are one-to-one and D and E and their adjoints are isometries onto $\overline{Y}$. The restricted operators F and $F^\dagger$ are onto Y, which might (or might not) equal $\overline{Y}$.

Since $DEF = F$ the map DE and the identity operator I agree on the dense subset Y of $\overline{Y}$ and both D and E are the identity on X, so $DE = I$ on all of $\mathcal{H}$.

Since D is an isometry, for all $x, y \in \mathcal{H}$ we have

$$\langle\, x,\, y \,\rangle = \langle\, D(x), D(y) \,\rangle = \langle\, x, D^\dagger D(y) \,\rangle$$

which implies that $D^\dagger D(y) = y$ for all $y \in \mathcal{H}$: that is, $D^\dagger\, D = I$.

This means that $D^\dagger = D^{-1}$, a fact that holds for any invertible isometry, not just this one. So $D^\dagger = E$.

We have then

$$DF^\dagger = F \quad \text{and} \quad DF = FD \quad \text{and} \quad DF^\dagger = F^\dagger D$$

and these identities hold not just on $\overline{Y}$ but on all of $\mathcal{H}$.

We have accumulated a rather lengthy list of important results in this discussion, which we record in the following proposition.

17.7. ***Proposition***. *Suppose F is a normal operator on Hilbert space $\mathcal{H}$.*

(i) $\overline{F(\mathcal{H})}^{\perp} = Ker(F)$ *and* $\left(Ker(F), \overline{F(\mathcal{H})} \right)$ *is a complementary pair that reduces both F and $F^\dagger$, and both F and $F^\dagger$ are one-to-one when restricted to $\overline{F(\mathcal{H})}$.*

*(ii) There is an invertible isometry D (so $D^{-1} = D^\dagger$: these are called **unitary operators** and we will have more to say about them later) which is the identity on $Ker(F)$ and for which $DF^\dagger = F$ and which commutes with both F and $F^\dagger$.*

(iii) Eigenvectors for F for different eigenvalues are orthogonal.

(iv) For each complex number λ, the operator $F - \lambda I$ is normal, so it and its adjoint have the same kernel. So eigenvalues for F are the conjugates of eigenvalues for $F^\dagger$, and the eigenspace for F with eigenvalue λ is the same as the eigenspace for $F^\dagger$ for eigenvalue $\overline{\lambda}$.

(v) $Ker(F) = Ker\left(F^n\right)$ *and* $F^n(\mathcal{H}) = F^n\left(\overline{F(\mathcal{H})}\right)$ *is dense in $F(\mathcal{H})$ for any $n \geq 1$ and any normal F. This implies that the kernel of $F - \lambda I$ is the same as the kernel of $(F - \lambda I)^n$ for each $n \geq 1$ so the generalized eigenvectors for F and λ are simply the members of the eigenspace Eig_λ.*

(vi) For each λ the complementary pair $\left(Eig_\lambda, Eig_\lambda^\perp\right)$ reduces F, $F^\dagger$ and all their powers. This includes the case of $\lambda = 0$, for which $\left(Eig_0, Eig_0^\perp\right) = \left(Ker(F), \overline{F(\mathcal{H})} \right)$.

Proof. The proof is taken from the preceding remarks. $\square$

17.8. ***Exercise***. *If N is a normal operator on a finite dimensional Hilbert space then there is an invertible isometry U (these are called unitary operators, and $U^{-1} = U^\dagger$ for any such) so that the matrix of $U^{-1} \circ N \circ U$ is diagonal.*

18. Bounded Self-Adjoint Operators

In this section all Hilbert spaces will have complex field. Not only do the given proofs fail, most of the results we discuss here are actually false for real Hilbert spaces.

An operator $F \in \mathfrak{CL}(\mathcal{H})$ is called **self-adjoint** when $F = F^\dagger$: that is,

$$\langle F(x), y \rangle = \langle x, F(y) \rangle \quad \forall x, y \in \mathcal{H}.$$

Of course, self-adjoint operators are normal. For self-adjoint operators the image of F and the kernel of F are orthogonal.

A second, on its face weaker, condition

$$\langle F(x), x \rangle = \langle x, F(x) \rangle \text{ for all } x \in \mathcal{H}$$

is actually equivalent to, or implies, self-adjointness of F on complex Hilbert spaces.

To see this, suppose the second condition holds. Then

$$\begin{aligned}
0 =& \langle\, F(x+y), x+y\,\rangle - \langle\, x+y, F(x+y)\,\rangle \\
=& (\langle\, F(x), x\,\rangle + \langle\, F(y), x\,\rangle + \langle\, F(x), y\,\rangle + \langle\, F(y), y\,\rangle) \\
& \quad - (\langle\, x, F(x)\,\rangle + \langle\, y, F(x)\,\rangle + \langle\, x, F(y)\,\rangle + \langle\, y, F(y)\,\rangle) \\
=& \langle\, F(y), x\,\rangle + \langle\, F(x), y\,\rangle - \langle\, y, F(x)\,\rangle - \langle\, x, F(y)\,\rangle.
\end{aligned}$$

That means

$$\langle\, F(y), x\,\rangle - \langle\, y, F(x)\,\rangle = \langle\, x, F(y)\,\rangle - \langle\, F(x), y\,\rangle.$$

Since the two sides are conjugates they must be real. But even more, if they were nonzero and $\mathcal{H}$ a complex Hilbert space (as our spaces usually will be) the vector y could be replaced by αy for any complex α and the right side would be multiplied by $\overline{\alpha}$, while the left would acquire a factor of α. That means both sides must actually be zero.

18.1. ***Lemma***. *A continuous operator F on a complex Hilbert space $\mathcal{H}$ is self-adjoint if and only if*

$$\langle\, F(x), x\,\rangle = \langle\, x, F(x)\,\rangle \textit{ for all } x \in \mathcal{H}.$$

Rephrasing, the condition is that $\langle\, F(x), x\,\rangle$ is real for every $x \in \mathcal{H}$.

Proof. This is the content of the last calculation. □

The following result provides, for self-adjoint operators on a complex Hilbert space, an alternative to the usual method of calculating the operator norm.

18.2. ***Lemma***. *Suppose F is a continuous and self-adjoint operator on complex Hilbert space $\mathcal{H}$.*

$$\textit{Then } \|F\| = \sup\{\,\langle\, F(x),\, x\,\rangle \mid \|x\| \le 1\,\}.$$

Proof. Let $N = \sup\{\,\langle\, F(x),\, x\,\rangle \mid \|x\| \le 1\,\}$. For each x we have

$$\langle\, F(x),\, x\,\rangle \le \|F\|\,\|x\|^2 \quad \text{so } N \le \|F\|.$$

We need to show the reverse inequality to prove this lemma.

Expand

$$\frac{\langle\, F(x+y),\, x+y\,\rangle - \langle\, F(x-y),\, x-y\,\rangle}{4} - i\,\frac{\langle\, F(x+i\,y),\, x+i\,y\,\rangle - \langle\, F(x-i\,y),\, x-i\,y\,\rangle}{4}$$

to verify that it simplifies to $\langle\, F(x), y\,\rangle$. Since F is self-adjoint all four inner products in this expression are real so the second fraction is the complex part of $\langle\, F(x),\, y\,\rangle$. Replace y by $w = \alpha\, y$ where α is a complex number of norm 1 chosen so that $\langle\, F(x),\, w\,\rangle$ is real. Then the complex part of the expression above is 0, and the magnitude of w is still the same as that of y. We have:

$$\langle\, F(x),\, w\,\rangle = \frac{\langle\, F(x+w),\, x+w\,\rangle - \langle\, F(x-w),\, x-w\,\rangle}{4}.$$

Squaring and applying the definition of N and the parallelogram law we have

$$\begin{aligned}
|\langle F(x),\, w\rangle|^2 &= \frac{|\langle F(x+w),\, x+w\rangle - \langle F(x-w),\, x-w\rangle|^2}{16} \\
&\leq \frac{(N\|x+w\|^2 + N\|x-w\|^2)^2}{16} \\
&\leq \frac{N^2}{4}\left(\frac{\|x+w\|^2 + \|x-w\|^2}{2}\right)^2 \\
&= \frac{N^2}{4}(\|x\|^2 + \|w\|^2)^2.
\end{aligned}$$

If we choose x of norm 1, the maximum value of $\langle F(x),\, w\rangle$ among *other* vectors w of norm 1 occurs when w is a norm one complex multiple of $F(x)$, so in that case

$$|\langle F(x),\, w\rangle|^2 = \left|\left\langle F(x),\, \frac{F(x)}{\|F(x)\|}\right\rangle\right|^2 \leq \frac{N^2}{4}(4) = N^2.$$

We conclude that $\|F\|$ cannot exceed N and the lemma is proved. □

This result is false on real Hilbert spaces, even in dimension 2. For instance, if F represents a rotation by angle α in the plane, the operator norm of F is 1 but the supremum indicated in the last lemma is $|\cos(\alpha)|$. This also provides a counterexample to the statement of Lemma 18.1 for real spaces.

18.3. ***Corollary***. *If F is a continuous operator on a complex Hilbert space $\mathcal{H}$ we have $\|F^\dagger F\| = \|F\|^2$. So in the self-adjoint case, $\|F^2\| = \|F\|^2$.*

Proof. If we apply Lemma 18.2 to the self-adjoint operator $F^\dagger F$ we find

$$\begin{aligned}
\left\|F^\dagger F\right\| = {}& \sup\left\{\left|\left\langle F^\dagger F(x),\, x\right\rangle\right| \;\middle|\; \|x\| = 1\right\} \\
&= \sup\{|\langle F(x),\, F(x)\rangle| \mid \|x\| = 1\} = \|F\|^2.
\end{aligned}$$

□

The following is the converse of Lemma 17.4 for complex Hilbert spaces.

18.4. ***Corollary***. *If F is a continuous operator on complex Hilbert space $\mathcal{H}$ and $\|F(x)\| = \|F^\dagger(x)\|$ for all x then F is normal.*

Proof. If $\langle F(x), F(x)\rangle = \langle F^\dagger(x), F^\dagger(x)\rangle$ then $\langle F^\dagger F(x), x\rangle = \langle FF^\dagger(x), x\rangle$. So the assumption of the corollary is that

$$\left\langle \left(F^\dagger F - FF^\dagger\right) x, x\right\rangle = 0 \quad \forall x.$$

$F^\dagger F - FF^\dagger$ is self-adjoint, so $\left\|F^\dagger F - FF^\dagger\right\| = 0$ and $F^\dagger F = FF^\dagger$. □

18.5. ***Corollary***. *If $F \in \mathcal{CL}(\mathcal{H})$ is normal on complex $\mathcal{H}$ then $\|F^n\| = \|F\|^n$ for all $n \geq 1$.*

Proof. Suppose D is self-adjoint. By Corollary 18.3 we know that $\left\|D^2\right\| = \|D\|^2$ and $\left\|D^4\right\| = \left\|D^2\right\|^2 = \|D\|^4$ and then, generally, $\left\|D^{2^n}\right\| = \|D\|^{2^n}$. So if $2^n \geq j \geq 0$ we have

$$\|D\|^{2^n} = \left\|D^{2^n}\right\| = \left\|D^{2^n-j}D^j\right\| \leq \left\|D^{2^n-j}\right\| \left\|D^j\right\| \leq \|D\|^{2^n}$$

so equality holds throughout and $\left\|D^j\right\| = \|D\|^j$. By choosing n appropriately we have this identity for any $j \geq 0$ and self-adjoint D.

Suppose F is normal. $D = F^\dagger F$ is self-adjoint and normality implies that, for $n \geq 1$, $D^n = \left(F^\dagger\right)^n F^n$. But then

$$\begin{aligned}\|F\|^{2n} &= \left\|F^\dagger F\right\|^n = \left\|\left(F^\dagger F\right)^n\right\| \\ &= \left\|\left(F^\dagger\right)^n F^n\right\| \leq \left\|\left(F^\dagger\right)^n\right\| \|F^n\| = \|F^n\|^2 \leq \|F\|^{2n}.\end{aligned}$$

So equality holds throughout and we have $\|F^n\| = \|F\|^n$ for all n and any normal F, not just those which are self-adjoint. □

As a final note in this section, we provide a means of partially ordering the self-adjoint operators on a complex Hilbert space.

A self-adjoint operator F is called **positive** if $\langle\, F(x), x\,\rangle \geq 0$ for every x.

Self-adjoint F is called **positive definite** if $\langle\, F(x), x\,\rangle > 0$ whenever $x \neq 0$.

If F is self-adjoint then $F + \|\,F\,\|\, I$ is a positive operator, where I is the identity operator, which allows us to transfer many facts true for positive operators to the merely self-adjoint case.

We **partially order the self-adjoint operators on** $\mathcal{H}$ by $\mathbf{F} \leq \mathbf{G}$ if and only if $G - F$ is a positive operator.

18.6. ***Lemma***. *Any operator F on a complex Hilbert space that satisfies the condition $\langle\, F(x), x\,\rangle \geq 0$ for every x is automatically self-adjoint. So these operators are positive.*

Proof. If F satisfies the specified inner product condition then, of course, $\langle\, F(x), x\,\rangle$ is always real and Lemma 18.1 applies to tells us that F is self-adjoint. □

Note that for any operator F, the products $F^\dagger F$ and $F\,F^\dagger$ are both positive operators.

18.7. ***Exercise***. *If $T \in \mathfrak{CL}(\mathcal{H})$ then $T^\dagger + T$ and $i\,(T^\dagger - T)$ are self adjoint. So*

$$T = \frac{1}{2}\left(T^\dagger + T\right) + i\,\left[\frac{-i}{2}\left(T^\dagger - T\right)\right]$$

represents T in the form $T = A + i\,B$ where A and B are self-adjoint operators. If $C + i\,D$ is any other such representation, $A - C = i\,(\,D - B\,)$. But the right side is not self-adjoint unless $D = B$, while the left is self-adjoint. In other words, $A = C$ and $B = D$; this type of representation is unique.

$T^\dagger = A - i\,B$ so $T\,T^\dagger = A^2 + i\,(BA - AB) + B^2$ and it follows that $T\,T^\dagger = T^\dagger T$ (i.e. T is normal) when and only when A commutes with B.

The analogy associating normal operators T with complex numbers and self-adjoint operators A, B with real numbers in

$$T = A + i\,B \qquad T^{\dagger} = A - i\,B$$

is compelling.

19. C* and Other Algebras

Suppose given an algebra A; that is, a vector space equipped with an associative multiplication. Matrices with matrix multiplication, or the bounded operators on a Hilbert space with composition come to mind.

If A has a norm and $\|\,F\,G\,\| \leq \|\,F\,\|\,\|\,G\,\|$ for all F and G in A then A is called a **normed algebra**. If A has a multiplicative identity I and $\|\,I\,\| = 1$ the normed algebra is called **unital**. If A is complete with this norm it is called a **Banach algebra.**

We have seen that $\mathfrak{C}\boldsymbol{\mathcal{L}}(\mathcal{H})$ is an example of a unital Banach algebra, non-commutative except in trivial cases.

Any function ${}^{\diamond}\colon A \to A$ is called an **involution** on A if

$$F^{\diamond\diamond} = F \text{ and } (a\,F + b\,G)^{\diamond} = \overline{a}\,F^{\diamond} + \overline{b}\,G^{\diamond} \text{ and } (F\,G)^{\diamond} = G^{\diamond}\,F^{\diamond}$$

for every $F, G \in A$ and numbers a, b. Normally, the field of numbers will be $\mathbb{C}$.

If it possesses an involution, A is called an **involution algebra** or, synonymously, a $*$**-algebra.**

An algebra homomorphism $\phi\colon A \to B$ between two $*$-algebras is called a $*$**-homomorphism** if $\phi\,(F^{\diamond}) = \phi(F)^{*}$ for all F in A, where ${}^{\diamond}$ denotes the involution on A and * is the involution on B.

The Hermitian adjoint is an involution on $\mathfrak{C}\boldsymbol{\mathcal{L}}(\mathcal{H})$, and a primary example is when $\mathcal{H} = \boldsymbol{L^2}(\mu)$ for a σ-finite measure μ.

If an involution ${}^{\diamond}$ on the Banach algebra A satisfies $\|\,F^{\diamond}F\,\| = \|\,F\,\|\,\|\,F^{\diamond}\,\|$ for all $F \in A$ then A is called a **C*-algebra**.

We just proved above that for a complex Hilbert space $\mathcal{H}$, the algebra of operators $\mathfrak{C}\boldsymbol{\mathcal{L}}(\mathcal{H})$ with Hermitian adjoint satisfies $\|\,F^{\dagger}F\,\| = \|\,F\,\|^2$ for $F \in \mathfrak{C}\boldsymbol{\mathcal{L}}(\mathcal{H})$, and since $\|\,F\,\| = \|\,F^{\dagger}\,\|$ for these F we find that $\mathfrak{C}\boldsymbol{\mathcal{L}}(\mathcal{H})$ is a unitary C^*-algebra.

It is also (fairly) obvious that the continuous complex valued functions on a fixed compact space with supremum norm and (pointwise) complex conjugation is a commutative and unitary C*-algebra. With complex conjugation and pointwise a.e. multiplication, $\boldsymbol{L^{\infty}}(\mu)$ is a C*-algebra for σ-finite measure μ, where the essentially finite measurable functions have essential supremum norm.

Sub-algebras of any $*$-algebra which are closed under the operation of taking adjoints are also $*$-algebras, though completeness or the presence of a unit may not be preserved.

Finally, any unitary $*$-algebra A which is a sub-algebra of some $\mathcal{CL}(\mathcal{H})$ and which is weak operator closed[7] is called a **von Neuman algebra** or, synonymously, a $\mathbf{W^*}$**-algebra.**

20. Orthogonal Projection and Unitary Operators

Suppose V is an inner product space and P is a projection onto subspace M with kernel K. The pair (M, K) is called a complementary pair when both M and K are closed subspaces. Recall the material on *orthogonal* projections (page 91) where the two subspaces M and K were orthogonal, and in that case we had the Pythagorean Law $\|x\|^2 = \|P(x)\|^2 + \|Q(x)\|^2$ for all x, where $Q = I - P$.

There, we had $\langle P(x), Q(x) \rangle = 0$ for all x, and that is sufficient to ensure that the Pythagorean Law holds for these paired projections, as can be seen by expanding $\|x\|^2 = \|P(x) + Q(x)\|^2$.

Wherever linear P might come from, projection or not, if it satisfies the Pythagorean Law for some function Q then P must be continuous, and in fact $\|P\| \leq 1$.

This will certainly happen if M and K are orthogonal. In that case, continuity of P has it that K and $M = K^{\perp}$ are closed, and there are disjoint orthonormal bases B of M and C for K for which P is the orthogonal projection P_B and Q is the orthogonal projection P_C.

20.1. ***Lemma***. *Suppose $\mathcal{H}$ is a Hilbert Space and M is a closed subspace of $\mathcal{H}$. The orthogonal projections onto M and $M^{\perp}$ are continuous and can be realized as projections P_B and P_C as in Corollary 15.4, where B is an orthonormal basis for M, C is an orthonormal basis for $M^{\perp}$ and $B \cup C$ is an orthonormal basis for $\mathcal{H}$.*

Proof. See the discussion preceding this lemma. $\square$

Let P_B be an orthogonal projection onto $\overline{span}(B)$ for orthonormal set B, and suppose C is an orthonormal basis for $B^{\perp}$ and denote $P_B(x) = x_B$ and $x_C = x - x_B$.

For any $x = x_B + x_C$ and $y = y_B + y_C$ represented this way we have

$$\langle P_B(x), y \rangle = \langle x_B, y_B + y_C \rangle = \langle x_B, y_B \rangle = \langle x_B + x_C, y_B \rangle = \langle x, P_B(y) \rangle.$$

So any orthogonal projection P_B is self-adjoint.

On the other hand, suppose a projection P is *not* orthogonal. In other words, there are vectors x in $M = P(\mathcal{H})$ and y in $K = Ker(P)$ for which $\langle x, y \rangle \neq 0$. Then we have

$$\langle P(x), y \rangle = \langle x, y \rangle \neq 0 = \langle x, 0 \rangle = \langle x, P(y) \rangle.$$

So P is not self-adjoint.

20.2. ***Lemma***. *A projection on a Hilbert space is orthogonal if and only if it is self-adjoint.*

[7] In this context A is weak operator closed if, whenever $S \in \mathcal{CL}(\mathcal{H})$ and T_ν is a net in A for which $\langle h, (T_\nu - S)(g) \rangle \longrightarrow 0$ for all $h, g \in \mathcal{H}$, then S must be in A too.

PROOF. The argument is in the last remarks. □

Even in two dimensions, two orthogonal projections need not commute, nor will their product (that is, composition) be a projection in general. Still, we have the following result.

20.3. ***Lemma***. *Suppose P and W are orthogonal projections onto subspaces M_P and M_W, with nullspaces K_P and K_W, respectively.*

(i) If P and Q commute their composition is the orthogonal projection onto the intersection of their ranges.

(ii) $P+W$ is an orthogonal projection exactly when $WP = 0 = PW$. In that case, the sum $M_P + M_W$ is direct and $P+W$ is a projection onto $M_P \oplus M_W$.

(iii) $P \leq W$ exactly when $PW = P$. This happens exactly when $M_P \subset M_W$.

(iv) $W - P$ is an orthogonal projection exactly when $M_P \subset M_W$. In that case it is a projection onto $K_P \cap M_W$.

PROOF. The proof is left for the reader. □

20.4. ***Proposition***. *Suppose Q_1 and Q_2 are two orthogonal projections on Hilbert space $\mathcal{H}$ with associated complementary pairs (A_1, B_1) and (A_2, B_2), respectively.*

Let $T = Q_1 \circ Q_2$ and $B = \overline{(I-T)(\mathcal{H})}$ and $A = A_1 \cap A_2$.

Then $B = A^{\perp}$ and (A, B) is a complementary pair for $\mathcal{H}$ and T^n converges in the strong operator topology to the projection P for this complementary pair.

PROOF. The operator norm of T^n cannot exceed 1 for any n and every Hilbert space is reflexive so the Mean Ergodic Theorem, Theorem 12.7, applies to T.

Suppose $Tg = Q_1 \circ Q_2(g) = g$. Then g is in the range of Q_1 so $g = a_1 = a_2 + b_2$ where $a_1 \in A_1$ and $a_2 \in A_2$ and $b_2 \in B_2$.

Then $\|a_1\|^2 = \|a_2\|^2 + \|b_2\|^2$ so $\|a_1\| \leq \|a_2\|$.

But $Q_2(a_1) = a_2$ and $\|Q_2\| \leq 1$ so $\|a_2\| \leq \|a_1\|$.

We conclude that $\|b_2\| = 0$ so $a_1 = a_2$ and $g \in A_1 \cap A_2$.

According to the Mean Ergodic Theorem

$$P_n = \frac{1}{n}\left(I + T + T^2 + \cdots + T^{n-1}\right)$$

converges in strong operator topology to P, the projection for the complementary pair (A, B).

Now suppose $a \in A = A_1 \cap A_2$ and recall that Q_1 and Q_2 are self-adjoint. Then for every $w \in \mathcal{H}$ we have

$$\begin{aligned}\langle a, (I-T)w \rangle &= \langle a, w \rangle - \langle a, Q_1 \circ Q_2(w) \rangle \\ &= \langle a, w \rangle - \langle Q_2 \circ Q_1(a), w \rangle = \langle a, w \rangle - \langle a, w \rangle = 0.\end{aligned}$$

So every member of A is orthogonal to every member of $(I-T)(\mathcal{H})$, and it follows that $B = A^{\perp}$. That means P is an orthogonal projection.

$T^n(a) = P_n(a) = Pa = a$ for every $a \in A$. It remains to show than $T^n(b)$ converges to $Pb = 0$ for every $b \in B$.

Suppose $b \in B$. Then $T(b) = a + c$ for certain $a \in A$ and $c \in B$. Note that T and P_k commute for each k.

$$P_k T(b) = a + P_k(c) = T P_k(b).$$

The vector $P_k(c)$ converges to 0, and also $P_k(b)$ converges to 0. That means $a = 0$ and so $T(b) \in B$. So the complementary pair (A, B) reduces T and this implies (A, B) reduces T^n and P_k for every n and k.

Given this reduction, we may presume without loss that $B = \mathcal{H}$ and $A = \{0\}$, and we make this presumption now. With this assumption, P_n converges strongly to the zero operator, and we are trying to show that T^n does too.

Consider $S = Q_2 T$.

Then S is self-adjoint, operator norm bounded by 1, $S^n = Q_2 T^n$ and (by an argument very similar to the above) the range of $I - S$ is dense in $\mathcal{H}$. Also

$$R_n = \frac{1}{n}\left(I + S + S^2 + \cdots + S^{n-1}\right) = Q_2 P_n - \frac{1}{n}(Q_2 - I)$$

converges strongly to the zero operator and each R_n is self-adjoint.

....unfinished $\square$

Recall that a linear isometry on $\mathcal{H}$ is a member Ψ of $\mathfrak{CL}(\mathcal{H})$ for which

$$\|\Psi(v)\| = \|v\| \quad \text{for all } v \in \mathcal{H}.$$

If Ψ is a linear isometry, expanding $\langle \Psi(x+y), \Psi(x+y) \rangle = \langle x+y, x+y \rangle$ and noting that both sides are real we see that

$$\langle \Psi(x), \Psi(y) \rangle + \langle \Psi(y), \Psi(x) \rangle = \langle x, y \rangle + \langle y, x \rangle.$$

So the real part of $\langle \Psi(x), \Psi(y) \rangle$ equals the real part of $\langle x, y \rangle$. In case $\mathcal{H}$ is a complex Hilbert space, replacing y by $i\,y$ we see that the complex parts are equal too, so we find that in any Hilbert space, isometries preserve the inner product of *any* pair of vectors, not just those inner products of the form $\langle x, x \rangle$ used to define the norm.

$$\langle \Psi(x), \Psi(y) \rangle = \langle x, y \rangle \quad \forall x, y \in \mathcal{H}.$$

A linear isometry on $\mathcal{H}$, therefore, sends an orthonormal basis of $\mathcal{H}$ to an orthonormal basis of its range. It is an isometric isomorphism onto its image, and in the finite dimensional case must be invertible.

Invertible or not, we saw earlier that in the case of isometries, for $x, y \in \mathcal{H}$

$$\langle x, y \rangle = \langle \Psi(x), \Psi(y) \rangle = \langle x, \Psi^\dagger \Psi(y) \rangle$$

which implies that $\Psi^\dagger \Psi(y) = y$ for all $y \in \mathcal{H}$: that is, $\Psi^\dagger\, \Psi = I$, the identity operator on $\mathcal{H}$.

Recall that if it *is* invertible, an isometry on a Hilbert space is called a unitary operator, and the set of these will be denoted $\mathfrak{U}(\mathcal{H})$.

Unitary operators are normal and, in fact, if Ψ is unitary, $\Psi\,\Psi^\dagger = \Psi^\dagger\,\Psi = I$. In this case $\Psi^\dagger = \Psi^{-1}$ is also an isometry. Conversely, any bounded operator Ψ on

$\mathcal{H}$ which is *onto* $\mathcal{H}$ for which $\Psi^\dagger \Psi = I$ must be an invertible isometry and hence unitary. So a normal isometry is unitary.

20.5. ***Lemma***. *If $\Psi\colon \mathcal{H} \to \mathcal{H}$ is an isometry it has closed range.*
So if that range is dense then Ψ is onto $\mathcal{H}$, unitary, and $\Psi^\dagger = \Psi^{-1}$.

PROOF. The range of any isometry with complete domain must be closed, so if the range is dense in $\mathcal{H}$ it must be all of $\mathcal{H}$. For the rest, adapt the remarks above. □

If Ψ is an isometry but *not* invertible, then $\Psi\,\Psi^\dagger \neq \Psi^\dagger\,\Psi = I$ so Ψ is not normal. But at least

$$\langle\, x,\, \Psi\Psi^\dagger(y)\,\rangle = \langle\, \Psi^\dagger(x), \Psi^\dagger(y)\,\rangle = \langle\, \Psi\Psi^\dagger(x), \Psi\Psi^\dagger(y)\,\rangle = \langle\, x, \Psi\Psi^\dagger\Psi\Psi^\dagger(y)\,\rangle$$

for all x and y, so $\Psi\Psi^\dagger$ is a projection, obviously orthogonal.

20.6. ***Exercise***. *The unitary operators $\boldsymbol{\mathfrak{U}}(\mathcal{H})$ form a group under composition. An operator is unitary exactly when it sends any (and hence every) orthonormal basis of $\mathcal{H}$, in a one-to-one fashion, to an orthonormal basis of $\mathcal{H}$.*

Operators $S\colon \mathcal{D}_S \to \mathcal{R}_S$ and $T\colon \mathcal{D}_T \to \mathcal{R}_T$, where domains and ranges are all subspaces of a given Hilbert space $\mathcal{H}$, are called **unitarily equivalent** if there is a member $U \in \boldsymbol{\mathfrak{U}}(\mathcal{H})$ for which

$$S = U^\dagger T\, U = U^{-1} T\, U.$$

The relationship is obviously reciprocal. If $\mathcal{D}_S = \mathcal{H}$ for unitarily equivalent S and T then also $\mathcal{D}_T = \mathcal{H}$, and in that case the same unitary operator U can be used to show that $S^\dagger$ and $T^\dagger$ are unitarily equivalent.

20.7. ***Exercise***. *Unitarily equivalent operators are normal or self-adjoint together.*

21. Unbounded Operators

Until now, our concentration has been on bounded (i.e. continuous) operators. But many important operators, differential operators among them, are unbounded, defined on a dense subset of a Hilbert space but not on the whole space.

We discussed a few properties of such operators—closed operators— in Section 10 and we continue that discussion here.

A primary user-group of this material, physicists thinking about quantum mechanics, employs a slightly different notation from that used by most mathematicians, and that can lead to communication problems. For instance an inner product $\langle\, x\,,\, y\,\rangle$ is denoted $\langle\, y \,|\, x\,\rangle$, so to these physicists an inner product is conjugate linear in the first "slot" and linear in the second. Also, it is pretty universal for physicists to employ λ^* rather than $\overline{\lambda}$ to denote complex conjugation of the number λ.

Using this vocabulary a vector w is indicated by what is termed a **ket**, $|\, \mathbf{w}\, \rangle = w$, while a functional corresponding to inner product against vector v is called a **bra**, $\langle\, \mathbf{v}\, | = \langle\, \cdot\,,\, v\, \rangle$. Thus we have a **bracket**, a number, given by

$$\langle\, v\, |\; |\, w\, \rangle = \langle\, v\, |\, w\, \rangle = \langle\, w\,,\, v\, \rangle.$$

The map that identifies $\mathcal{H}$ with its dual $\mathcal{H}'$ is conjugate linear. Thus

$$\langle\, f + \lambda\, g\, | = \langle\, f\, | + \lambda^* \langle\, g\, |.$$

Bras and kets are frequently transformed into each other, and it is this conjugate linear identification that does the job.

If Ψ is an operator, the symbol $\langle\, v\, |\, \Psi$ denotes the functional defined by

$$\langle\, v\, |\, \Psi\, (\, |\, w\, \rangle\,) = \langle\, \Psi(w)\,,\, v \rangle$$

and this number is usually denoted $\langle\, v\, |\, \Psi\, |\, w\, \rangle$. The notation is linear in the "spots" occupied by Ψ and w (individually) and conjugate linear in the location of v.

$|\, v\, \rangle\, \langle\, w\, |$ is intended to represent a linear operator mapping onto the span of v related to a projection. Specifically,

$$|\, v\, \rangle\, \langle\, w\, |\, (\, |\, u\, \rangle\,) = \langle\, w\, |\, u\, \rangle\, |\, v\, \rangle = \langle\, u\,,\, w\, \rangle\, v$$

and if $w = v$ and a unit vector this is the projection of u onto the span of v.

If you see symbols rather than vectors inside a bra or a ket, the intent is to label instances of a vector or functional. A mathematician might use subscripts or superscripts on a generic vector symbol for this. For instance, if you see an indexed ket $|\, i, j\, \rangle$ the intent is that there are vectors $v_{i,j}$ for which $|\, i, j\, \rangle = |\, v_{i,j}\, \rangle = v_{i,j}$. Or if λ is an eigenvalue of an operator the ket $|\, \lambda\, \rangle$ would denote an eigenvector for λ.

Apparently context and habit helps the expert user keep this kind of shorthand straight, but it is a *common* source of confusion for the beginner.

A function T is often defined as a set of ordered pairs where each first component is associated with exactly one ordered pair in this set. The set of first components is the domain of the function, and the set consists of ordered pairs of the form $(x, T(x))$. In other words, with this definition a function actually *is* its graph $\gamma(T)$. The difference is that $\gamma(T)$ has subspace topology from a product space, while T itself has no topology or other structures associated with it. We can (and will) endow it with a topology of our choosing for special purposes. It is rarely relevant to draw this distinction.

For each such T we will let $\mathcal{D}_T$ denote the domain of T and indicate the image $T(\mathcal{D}_T)$ of T by $\mathcal{R}_T$. Unless otherwise noted, whenever we use this notation for domain and range we presume T to be linear on vector space $\mathcal{D}_T$.

Two functions are equal if *the two domains and all function values coincide.*

For two functions S and T we have a partial ordering given by containment. So $S \subset T$ provided that T is an extension of S to larger domain. In particular, $\mathcal{D}_S \subset \mathcal{D}_T$ and $S(x) = T(x)$ for every $x \in \mathcal{D}_S$.

Generally, we define $S+T$ in the obvious way, with domain $\mathcal{D}_{S+T} = \mathcal{D}_S \cap \mathcal{D}_T$. And $S \circ T$ is defined with domain $\mathcal{D}_{S\circ T} = \{\, x \in \mathcal{D}_T \mid T(x) \in \mathcal{D}_S\, \}$.

When T is one-to-one we define T^{-1} for the members $\mathcal{D}_{T^{-1}} = \mathcal{R}_T$ of $\mathcal{H}$ by $T^{-1}(T(x)) = x$. Recall from Exercise 10.2 that if T is closed, so is T^{-1}.

Because of domain confusion we use the word "commute" cautiously when applied to unbounded operators.

Note that $T \circ S$ has domain consisting of those $x \in \mathcal{D}_S$ for which $S(x) \in \mathcal{D}_T$, while $S \circ T$ has domain consisting of those $x \in \mathcal{D}_T$ for which $T(x) \in \mathcal{D}_S$. These domains will often be different, and even when they are equal their common domain could easily contain just the zero vector. It is unclear how useful a relation such as $S \circ T = T \circ S$ might be. If T is one-to-one, and so possesses an inverse function $T^{-1} \colon \mathcal{R}_T \to \mathcal{D}_T$ the compositions $T \circ T^{-1}$ and $T^{-1} \circ T$ are both the identity, but possibly on different subsets of $\mathcal{H}$. These maps cannot be said to commute in general.

With these issues in mind, we define a relation between two operators B and T that recovers part of what we use "commutativity" for, but with restrictions, and which is *not* symmetric.

We say **B commutes with T** (in that order) only when $\mathcal{D}_T \cup \mathcal{R}_T \subset \mathcal{D}_B$ and $B(T(x)) = T(B(x))$ for every $x \in \mathcal{D}_T$. It is implied by the existence of the right-hand side that $B(x) \in \mathcal{D}_T$ whenever $x \in \mathcal{D}_T$.

Equivalently, $\mathcal{D}_{B \circ T} = \mathcal{D}_T$ and $B \circ T \subset T \circ B$. A common case is when B is defined on all of $\mathcal{H}$, in which case the first condition (in both forms of the definition) is superfluous.

Note that if B commutes with T and T commutes with B then our definition puts strong conditions on domains and ranges. Specifically, $\mathcal{D}_B = \mathcal{D}_T$ and $\mathcal{R}_B \cup \mathcal{R}_T \subset \mathcal{D}_T$.

And whenever $\mathcal{D}_B = \mathcal{D}_T$, if B commutes with T then we do have symmetry: T commutes with B also.

Recall now the definition of a closed operator, and the results of Section 10.

In our context, and using our identification of a function *as* its graph, T is closed when T is a closed subset of $\mathcal{H} \times \mathcal{H}$ with its natural inner product. This means that whenever sequence x_i, $i \in \mathbb{N}$, in $\mathcal{H}$ converges to a point $a \in \mathcal{H}$ and provided $T(x_i)$ converges to a point $b \in \mathcal{H}$ then $a \in \mathcal{D}_T$ and $T(a) = b$.

This condition is implied by continuity, but does not require continuity. The Closed Graph Theorem, Proposition 9.8, states that when $f \colon X \to Y$ is linear and X and Y are Banach then f closed implies f continuous. But in our case that theorem applies only when $\mathcal{D}_T$ is a closed subset of the Hilbert space $\mathcal{H}$. This is allowed, but not the case of primary concern to us.

T is closed when T and its topological closure $\overline{T}$ in $\mathcal{H} \times \mathcal{H}$ coincide, and T is called closeable if $\overline{T}$ is a function. We saw in Corollary 10.3 that this will happen exactly when $(0, y) \in \overline{T}$ implies $y = 0$. In terms of $\mathcal{D}_T$, this condition means that whenever x_i converges to 0 in $\mathcal{D}_T$ then either $T(x_i)$ converges to 0 or $T(x_i)$ fails to converge at all.

In particular examples, a closed operator S is usually not given directly, but instead is the closure $\overline{T} = S$ of an operator T defined on smaller domain. $\mathcal{D}_T$ is then called a **core** of S.

21.1. ***Exercise***. *(i) If it is to be a core of closed Hilbert space operator S, it is necessary that subspace X of $\mathcal{D}_S$ be dense in $\mathcal{D}_S$. If S is bounded this is also sufficient.*

(ii) Assuming that there actually is a closeable unbounded operator with domain dense in $\mathcal{H}$ (we discuss several in later sections) show that the boundedness condition on S in (i) cannot be removed. (hint: In Exercise 10.8 (ii) we saw that no unbounded closed operator T can be defined on all of $\mathcal{H}$. That means there is a vector x for which $T(y_i)$ fails to converge for every sequence y_i in $\mathcal{D}_T$ that converges to x. So let $\mathcal{D}_S = \mathbb{F}\,x \oplus \mathcal{D}_T$ and define S on $\mathcal{D}_S$ to be an extension of T.)

(iii) Suppose T is closeable and $R \subset \overline{T}$. Then $\mathcal{D}_R$ is a core of $\overline{T}$ if every member x of $\mathcal{D}_T$ is a limit of some sequence y_i in $\mathcal{D}_R$ for which $R(y_i)$ converges.

(iv) If closed S is bounded below and has closed range $\mathcal{R}_S$, and if $S(X)$ is dense in $\mathcal{R}_S$ for dense subspace X of $\mathcal{D}_S$ then X is a core of S.

For unbounded T, if $\overline{T}$ is a function then $\mathcal{D}_{\overline{T}}$ contains every member a of $\mathcal{H}$ for which there is *any* sequence x_i in $\mathcal{D}_T$ converging to a with the property that $T(x_i)$ is also convergent. If this limit is b then $\overline{T}(a) = b$.

Suppose S is any extension of $\overline{T}$, and suppose $(a, b) \in S$.

If $a \in \overline{\mathcal{D}_T}$ then since $\mathcal{D}_T$ is dense in $\overline{\mathcal{D}_T}$ there is a sequence x_i from $\mathcal{D}_T$ with x_i converging to a. If $(a, b) \notin \overline{T}$ then it must be that $S(x_i) = T(x_i)$ fails to converge, and in fact there is no sequence y_i in $\mathcal{D}_T$ converging to a for which $T(y_i)$ converges to *anything*. So $b = S(a)$ has no connection to the values of T. And if $a \notin \overline{\mathcal{D}_T}$ then this is even more obviously true.

Extensions of T beyond $\overline{T}$ can be made "at random" (subject to linearity) but these extensions, even if they might be good for *something*, cannot be said to have anything to do with T. On the other hand, every point of $\overline{T}$ not already in T is connected to the values of T by a continuity condition at that new domain member, and there is only one possible way of doing this.

So $\overline{T}$ is the smallest closed extension of T, and the only one whose values are all connected by a continuity condition to the values of T itself.

It is important to recognize that if T is closeable there is no reason why either $\mathcal{D}_{\overline{T}}$ or $\mathcal{R}_{\overline{T}}$ should be closed. Generally, the domain will not be closed and the range will often not be closed.

In the following, we will consider functions defined on vector subspaces of a Hilbert space $\mathcal{H}$. We have certain requirements on a function T in the remainder of this section, and *if any these requirements are not met for some T we will be explicit about that.*

- $T \colon \mathcal{D}_T \to \mathcal{H}$ is linear on vector subspace $\mathcal{D}_T$ of Hilbert space $\mathcal{H}$.
- We ***require*** $\mathcal{D}_T$ to be ***dense*** in $\mathcal{H}$.
- We assume the field to be the complex numbers.
- We specifically ***do not*** require T to be bounded.

Generally, an operator satisfying these conditions is called an **unbounded operator on $\mathcal{H}$**.

It is an awkward phrase, as we do not exclude the possibility that such an operator is bounded, but it may not be, and many of the operators from important applications will not be. Also, the operator will not generally be defined *on* $\mathcal{H}$, but only on a proper dense subset.

For each $y \in \mathcal{H}$ define the function $A_y(\cdot) = \langle\, T(\cdot)\,,\, y\,\rangle \colon \mathcal{D}_T \to \mathbb{C}$.

A_y is linear on $\mathcal{D}_T$ and *if* it is bounded on $\mathcal{D}_T$ corresponds to inner product against a unique member $w_y \in \mathcal{H}$: viz.

$$A_y(\cdot) = \langle\, \cdot\,,\, w_y\,\rangle.$$

Since $\mathcal{D}_T$ is assumed dense in $\mathcal{H}$, w_y is the unique member of $\mathcal{H}$ that "works" for this y, and this uniqueness is one reason to require $\mathcal{D}_T$ to be dense in $\mathcal{H}$.

We define $T^\dagger(y) = w_y$ whenever A_y is a bounded functional on $\mathcal{D}_T$.

So $T^\dagger$ has its own domain, consisting of all those y for which the functional $A_y(\cdot) = \langle\, T(\cdot)\,,\, y\,\rangle$ is bounded.

The function $\mathbf{T}^\dagger \colon \mathcal{D}_{T^\dagger} \to \mathcal{H}$ defined by

$$\langle\, x\,,\, T^\dagger(y)\,\rangle = \langle\, T(x)\,,\, y\,\rangle \quad \forall x \in \mathcal{D}_T,\ y \in \mathcal{D}_{T^\dagger}$$

is called **the adjoint** of T. **Without further conditions we might not have $\mathcal{D}_{T^\dagger}$ dense in $\mathcal{H}$, and will provide an example illustrating this later.**

If S extends T (that is, if $T \subset S$) there is a relationship between $T^\dagger$ and $S^\dagger$.

21.2. ***Lemma***. *If S and T are unbounded operators on $\mathcal{H}$ and $T \subset S$ then $S^\dagger \subset T^\dagger$.*

Proof. Since $\mathcal{D}_T \subset \mathcal{D}_S$ and S agrees with T on $\mathcal{D}_T$, for each $y \in \mathcal{H}$ it is harder for $A_y(x) = \langle\, T(x)\,,\, y\,\rangle\ \forall x \in \mathcal{D}_T$ to be bounded than for $B_y(x) = \langle\, T(x)\,,\, y\,\rangle\ \forall x \in \mathcal{D}_S$ to be bounded. So $\mathcal{D}_{S^\dagger} \subset \mathcal{D}_{T^\dagger}$.

Now suppose $y \in \mathcal{D}_{S^\dagger}$. Then

$$\langle\, x\,,\, S^\dagger(y)\,\rangle = \langle\, S(x)\,,\, y\,\rangle \quad \forall x \in \mathcal{D}_S.$$

But $\mathcal{D}_T \subset \mathcal{D}_S$ and S agrees with T on $\mathcal{D}_T$ so

$$\langle\, x\,,\, S^\dagger(y)\,\rangle = \langle\, T(x)\,,\, y\,\rangle = A_y(x) \quad \forall x \in \mathcal{D}_T.$$

Since $T^\dagger(y)$ is the unique member of $\mathcal{H}$ that represents A_y on $\mathcal{D}_T$, we find $S^\dagger(y) = T^\dagger(y)$ for $y \in \mathcal{D}_{S^\dagger}$. □

21.3. ***Lemma***. *$T^\dagger$ is a closed operator for any unbounded operator T.*

Proof. Suppose x_i is a sequence in $\mathcal{D}_{T^\dagger}$ and x_i converges to a and $T^\dagger(x_i)$ converges to b. So

$$\langle\, x\,,\, T^\dagger(x_i)\,\rangle = \langle\, T(x)\,,\, x_i\,\rangle \quad \forall x \in \mathcal{D}_T.$$

The left side converges to $\langle\, x\,,\, b\,\rangle$ and the right side to $\langle\, T(x)\,,\, a\,\rangle$ for all $x \in \mathcal{D}_T$.

So we have: $\qquad \langle\, x\,,\, b\,\rangle = \langle\, T(x)\,,\, a\,\rangle \quad \forall x \in \mathcal{D}_T.$

That means $|\,\langle\, T(x)\,,\, a\,\rangle\,| \le \|b\|\,\|x\|$ for all $x \in \mathcal{D}_T$ so $a \in \mathcal{D}_{T^\dagger}$. And the uniqueness condition mentioned earlier implies that $T^\dagger(a) = b$. □

21.4. ***Corollary***. *$T^\dagger$ is continuous exactly when $\mathcal{D}_{T^\dagger}$ is closed.*

Proof. By the last lemma $T^\dagger$ is closed so if $\mathcal{D}_{T^\dagger}$ is closed the closed graph theorem tells us that $T^\dagger$ is continuous.

On the other hand, suppose $T^\dagger$ is continuous and x_n is a sequence in $\mathcal{D}_{T^\dagger}$ converging to a point x. The sequence is bounded and $\| T^\dagger \| < \infty$ so there is a constant K for which $\| T^\dagger(x_n) \| < K$ for all n.

For each $w \in \mathcal{D}_T$ we find $| \langle T(w) , x_n \rangle | = | \langle w , T^\dagger(x_n) \rangle | \leq K \| w \|$. So by continuity of inner product we have $| \langle T(w) , x \rangle | \leq K \| w \|$ as well, which puts $x \in \mathcal{D}_{T^\dagger}$. □

21.5. ***Corollary***. *If $T^\dagger$ is bounded below (i.e. $T^\dagger$ has continuous inverse) then $\mathcal{R}_{T^\dagger}$ is closed.*

Proof. Suppose $T^\dagger(x_n)$ is Cauchy in $\mathcal{R}_{T^\dagger}$, converging to point $y \in \mathcal{H}$. Since $T^\dagger$ is bounded below x_n is Cauchy too, converging to some $x \in \mathcal{H}$.

Since $T^\dagger$ is closed $T^\dagger(x) = y \in \mathcal{R}_{T^\dagger}$. □

We define $\boldsymbol{J} \colon \boldsymbol{\mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}}$ by

$$J(x, y) = (-y, x).$$

Note that J is an isometry and $J^2 = -I$ where I is the identity operator on $\mathcal{H} \times \mathcal{H}$ with product inner product

$$\langle (a, b) , (x, y) \rangle = \langle a , x \rangle + \langle b , y \rangle$$

Because J is an isometry on a complete space, if $A \subset \mathcal{H} \times \mathcal{H}$ is closed then so is $J(A)$. Obviously, two pairs (x, y) and (a, b) are orthogonal in $\mathcal{H} \times \mathcal{H}$ exactly when $J(x, y)$ and $J(a, b)$ are orthogonal.

If T is any linear operator and c a nonzero number, the operator cT is normally regarded as the operator that sends x to $cT(x)$ and so the function cT will contain the point $(x, cT(x))$ for each $x \in \mathcal{D}_T$. But when T is regarded as a subset of $\mathcal{H} \times \mathcal{H}$, cT means the set of all $(cx, cT(x))$ for $x \in \mathcal{D}_T$, and by linearity this is just T.

Keep an eye out for this: in the equation $J^2(T) = -T = T$ it is this second meaning that is intended. $J^2(x, y) = (-x, -y)$ for each $(x, y) \in T$ but the *sets* $J^2(T)$ and T coincide when T is linear.

21.6. ***Lemma***. *When T is a closed operator.*

Then $J(T^\dagger) = T^\perp$ and $\mathcal{H} \times \mathcal{H} = T \oplus T^\perp = T \oplus J(T^\dagger)$

where these direct sums are orthogonal direct sums.

Proof. Suppose $(y, T^\dagger(y)) \in T^\dagger$. Then for every $x \in \mathcal{D}_T$ we find

$$0 = \langle x , T^\dagger(y) \rangle \; - \; \langle T(x) , y \rangle = \langle J(x, T(x)) , (y, T^\dagger(y)) \rangle.$$

So every member of $T^\dagger$ is orthogonal to every member $J(T)$ and, of course, we then have every member of $J(T^\dagger)$ orthogonal to every member of T as well.

On the other hand, suppose $(a, b) \in T^\perp$. Then for every $x \in \mathcal{D}_T$ we have

$$0 = \langle x , a \rangle + \langle T(x) , b \rangle$$

which implies that $\langle\, T(\cdot)\,,\, b\,\rangle$ is bounded on $\mathcal{D}_T$ so $b \in \mathcal{D}_{T^\dagger}$. By the uniqueness condition implied by the density of $\mathcal{D}_T$ we then have $a = -T^\dagger(b)$ and $(a, b) \in J(T^\dagger)$.

Our conclusion here is that $J(T^\dagger) = T^\perp$, and the remaining statements of the lemma are immediate. $\square$

21.7. ***Proposition***. *If T is a closeable operator then $\overline{T}^\dagger = T^\dagger$ and $\mathcal{D}_{T^\dagger}$ is dense in $\mathcal{H}$.*

PROOF. Since $\overline{T}$ extends T we know that $\mathcal{D}_{\overline{T}^\dagger} \subset \mathcal{D}_{T^\dagger}$ and that $\overline{T}^\dagger$ agrees with $T^\dagger$ on $\mathcal{D}_{\overline{T}^\dagger}$. We will now show that $\mathcal{D}_{T^\dagger} - \mathcal{D}_{\overline{T}^\dagger}$ must be empty, so that $\overline{T}^\dagger = T^\dagger$.

Suppose $y \in \mathcal{D}_{T^\dagger} - \mathcal{D}_{\overline{T}^\dagger}$.

The operator norm of $\langle\, \overline{T}(\cdot)\,,\, y\,\rangle = \langle\, T(\cdot)\,,\, y\,\rangle$ is bounded by some number k on $\mathcal{D}_T$ but $\langle\, \overline{T}(\cdot)\,,\, y\,\rangle$ is unbounded on $\mathcal{D}_{\overline{T}}$.

So there must be a point $(x, \overline{T}(x)) \in \overline{T}$ for which x is in the unit sphere in $\mathcal{D}_{\overline{T}}$ and $|\,\langle\, \overline{T}(x)\,,\, y\,\rangle\,| > k + 1$.

This x cannot be in $\mathcal{D}_T$, but there is a sequence x_i in the unit sphere in $\mathcal{D}_T$ converging to x and for which $T(x_i)$ converges to $\overline{T}(x)$. But then

$$|\,\langle\, T(x_i)\,,\, y\,\rangle\,| \le k < k + 1 < |\,\langle\, \overline{T}(x)\,,\, y\,\rangle\,|$$

contradicting continuity of inner product.

So no such y can exist and therefore $\overline{T}^\dagger = T^\dagger$.

We will now show that in case T is closeable that $\mathcal{D}_{T^\dagger} = \mathcal{D}_{\overline{T}^\dagger}$ is dense in $\mathcal{H}$.

We will suppose, without loss, that T itself is closed: that is, T is a closed subspace of the Hilbert space $\mathcal{H} \times \mathcal{H}$.

We now suppose that $a \in (\mathcal{D}_{T^\dagger})^\perp$. Our goal here is to conclude that $a = 0$ so $(\mathcal{D}_{T^\dagger})^\perp = \{\, 0\,\}$ and we could conclude that $\mathcal{D}_{T^\dagger}$ is dense.

We do know that for every $y \in \mathcal{D}_{T^\dagger}$ we have

$$0 = \langle\, -T^\dagger(y)\,,\, 0\,\rangle + \langle\, a\,,\, y\,\rangle = \langle\, (0, a)\,,\, (-T^\dagger(y), y)\,\rangle = \langle\, (0, a)\,,\, J(y, T^\dagger(y))\,\rangle.$$

That means $(0, a) \in T = \big(J(T^\dagger)\big)^\perp$ by Lemma 21.6 (remember, we are assuming here that $T = \overline{T}$) and since T is a linear function we must have $a = 0$ as required. $\square$

21.8. ***Proposition***. *For unbounded operator T if $\mathcal{D}_{T^\dagger}$ is dense then T is closeable and $\overline{T} = T^{\dagger\dagger}$.*

PROOF. Suppose $(0, b) \in \overline{T}$. So there is a sequence x_i of members of $\mathcal{D}_T$ converging to 0 for which $T(x_i)$ converges to b. Then for each $y \in \mathcal{D}_{T^\dagger}$ we have

$$\langle\, T(x_i)\,,\, y\,\rangle = \langle\, x_i\,,\, T^\dagger(y)\,\rangle$$

and the right side converges to 0. If $\mathcal{D}_{T^\dagger}$ is dense, continuity of inner product requires b to be 0 and so $\overline{T}$ is a function and T is closeable.

Recall the notation in the proof of Lemma 21.6 and Proposition 21.7.

If $\mathcal{D}_{T^\dagger}$ is dense

$$\mathcal{H} \times \mathcal{H} = T^\dagger \oplus (T^\dagger)^\perp = T^\dagger \oplus J(T^{\dagger\dagger}) = \big(J(\overline{T})\big)^\perp \oplus J(T^{\dagger\dagger})$$

where all direct sums are orthogonal direct sums.

So $(\overline{T})^\perp = \big(T^{\dagger\dagger}\big)^\perp$ and it follows that $\overline{T} = T^{\dagger\dagger}$. □

21.9. ***Exercise***. *(i)* $(\mathcal{R}_T)^\perp = Ker\,\big(T^\dagger\big)$.

(ii) $(\mathcal{R}_{T^\dagger})^\perp \supset Ker\,(T)$. *And if* T *is closed then* $(\mathcal{R}_{T^\dagger})^\perp = Ker\,(T)$.

21.10. ***Corollary***. *Suppose* T *is closeable.*
T is one-to-one if and only if $\mathcal{R}_{T^\dagger}$ is dense in $\mathcal{H}$.

Proof. Rephrase Exercise 21.9. □

21.11. ***Corollary***. *If $\mathcal{D}_T = \mathcal{H}$ and $\mathcal{D}_{T^\dagger}$ is dense in $\mathcal{H}$ then T is continuous.*

Proof. If $\mathcal{D}_{T^\dagger}$ is dense then $\overline{T} = T^{\dagger\dagger}$. Since T is already defined on $\mathcal{H}$ we have $\overline{T} = T$ so T is closed. Now Corollary 21.4 tells us that $T = T^{\dagger\dagger}$ is continuous. □

21.12. ***Lemma***. *Suppose S is one-to-one and $\mathcal{R}_S$ is dense in $\mathcal{H}$.*
So both S and S^{-1} are unbounded operators.
Then $S^\dagger$ is one-to-one and $\big(S^\dagger\big)^{-1} = \big(S^{-1}\big)^\dagger$.

Proof. By Exercise 21.9 (i) $Ker(S^\dagger)$ is trivial so $S^\dagger$ is one-to-one.

$$\begin{aligned}
(a,b) \in \big(S^\dagger\big)^{-1} &\iff (b,a) \in S^\dagger \iff S^\dagger(b) = a \\
&\iff \forall x \in \mathcal{D}_S \quad \langle\, S(x)\,,\, b\rangle = \langle\, x\,,\, a\rangle \\
&\iff \forall y \in \mathcal{R}_S \quad \langle\, y\,,\, b\rangle = \langle\, S^{-1}(y)\,,\, a\rangle \\
&\iff b = \big(S^{-1}\big)^\dagger(a) \iff (a,b) \in \big(S^{-1}\big)^\dagger.
\end{aligned}$$

□

21.13. ***Lemma***. *$\mathcal{R}_{S^\dagger} = \mathcal{H}$ if and only if S is bounded below.*

Proof. Assume $\mathcal{R}_{S^\dagger} = \mathcal{H}$ but S is not bounded below.

Select sequence x_n in the domain of S with $\|\, x_n\,\| \to \infty$ but $S(x_n) \to 0$. Consider the family of continuous functionals ϕ_n given by $\phi_n(w) = \langle\, w\,,\, x_n\,\rangle$ for each n. Note that $\|\,\phi_n\,\| = \|\, x_n\,\| \to \infty$.

Since $S^\dagger$ is onto $\mathcal{H}$ every $w \in \mathcal{H}$ is of the form $S^\dagger(y)$ for some $y \in \mathcal{D}_{S^\dagger}$. Then

$$\phi_n(w) = \langle\, w\,,\, x_n\,\rangle = \langle\, S^\dagger(y)\,,\, x_n\,\rangle = \langle\, y\,,\, S(x_n)\,\rangle \to 0.$$

So the set $\{\,\phi_n(w) \mid n \in \mathbb{N}\,\}$ is bounded for every $w \in \mathcal{H}$. So by the Banach-Steinhaus Theorem there is a constant M for which $\|\,\phi_n\,\| \le M$ for every n. This contradicts our earlier observation about these operator norms. So S is bounded below.

We now suppose S is bounded below.

Then $S^{-1}\colon \mathcal{R}_S \to \mathcal{D}_S$ exists and is continuous.

For every $w \in \mathcal{H}$ the functional $\phi_w(\cdot) = \langle\, S^{-1}(\cdot)\,,\, w\,\rangle$ is continuous on $\mathcal{R}_S$ and so can be extended to a continuous functional on all of $\mathcal{H}$. Any such can be represented as inner product against a member z of $\mathcal{H}$, so

$$\langle\, S^{-1}(x)\,,\, w\,\rangle = \langle\, x\,,\, z\,\rangle \quad \forall x \in \mathcal{R}_S.$$

So if $x = S(y)$ we have

$$\langle\, y\,,\, w\,\rangle = \langle\, S(y)\,,\, z\,\rangle \quad \forall y \in \mathcal{D}_S.$$

That means $z \in \mathcal{D}_{S^\dagger}$ and $S^\dagger(z) = w$. So arbitrarily chosen w is in $\mathcal{R}_{S^\dagger}$. □

21.14. ***Corollary***. *Suppose S is closed.*
$\mathcal{R}_S = \mathcal{H}$ if and only if $S^\dagger$ is bounded below.

Proof. Since S is closed, $S^{\dagger\dagger} = S$ and we can apply the last lemma, replacing S with $S^\dagger$. □

21.15. ***Theorem***. *Suppose S is closed.*
$\mathcal{R}_S = \mathcal{H}$ and S is bounded below if and only if
$\mathcal{R}_{S^\dagger} = \mathcal{H}$ and $S^\dagger$ is bounded below.

Proof. This follows from Lemma 21.13 and its corollary. □

22. Spectrum and Resolvent

For unbounded operator T on $\mathcal{H}$ and complex number α, if $T - \alpha\, I$ has trivial kernel then it has an inverse function called the **resolvent for T and $\boldsymbol{\alpha}$**

$$\mathbf{R}_{\boldsymbol{\alpha}}(\mathbf{T}) = (\,T - \alpha\, I\,)^{-1}\colon (T - \alpha\, I)(\mathcal{D}_T) \to \mathcal{D}_T.$$

$T - \alpha\, I$ is bounded below if and only if $R_\alpha(T)$ is bounded on its domain, which is equivalent to continuity of this inverse map. In particular,

$$\|\,(T - \alpha\, I)(x)\,\| \geq c\,\|\,x\,\|\ \forall x \in \mathcal{D}_T \text{ if and only if } \|\,R_\alpha(y)\,\| \leq \frac{1}{c}\,\|\,y\,\|\ \forall y \in \mathcal{R}_{T-\alpha\, I},$$

where in this statement c is a positive constant.

If $R_\alpha(T)$ is continuous with dense domain α is called a **regular value for T**. The set of all regular values is called the **resolvent set**, $\boldsymbol{\rho}(\mathbf{T})$, and the set of complex numbers *not* in the resolvent set is called the **spectrum**, $\boldsymbol{\sigma}(\mathbf{T})$.

Complex numbers can wind up in the spectrum, potentially, for three reasons.

It might be that α is an eigenvalue for T, so $T - \alpha\, I$ has no inverse at all. The collection of eigenvalues is called the **point spectrum.** The point spectrum is denoted $\boldsymbol{\sigma}_{\mathbf{p}}(\mathbf{T})$. There is no distinction made among members of the point spectrum for which $T - \alpha\, I$ has dense range and those for which the range is not dense.[8]

[8]Different authors chop the spectrum up in various ways and with various names, and we don't propose to analyze all the different vocabularies. Descriptors for pieces of the spectrum include: discrete, pure point, peripheral, essential, absolutely continuous, singular and compression, in addition to our vocabulary. The reader of a given text must winkle out the usage in context.

If α is *not* an eigenvalue and $T-\alpha I$ has dense range but $T-\alpha I$ is *not* bounded below, then $R_\alpha(T)$ still exists as an unbounded operator and these α comprise the **continuous spectrum** denoted $\boldsymbol{\sigma_c}(\mathbf{T})$.

And members α of the spectrum which are *not* eigenvalues but for which $T-\alpha I$ does *not* have dense range, whether or not $T-\alpha I$ is bounded below, comprise the **residual spectrum** denoted $\boldsymbol{\sigma_r}(\mathbf{T})$.

As a final piece of vocabulary we gather together all of σ_p and σ_c and possibly some of the members of σ_r to form the **approximate point spectrum** denoted $\boldsymbol{\sigma_{ap}}(\mathbf{T})$. This set consists of those α for which there is a sequence x_n of unit vectors in $\mathcal{D}_T$ with $\|\,(T-\alpha I)(x_n)\,\| \to 0$. It may be convenient to find a sequence x_n of vectors in $\mathcal{D}_T$ for which $\|\,x_n\,\| \to \infty$ and for which $\|\,(T-\alpha I)(x_n)\,\| \to 0$, or a sequence for which there exists $\varepsilon > 0$ with $\limsup \|x_n\| \geq \varepsilon$ and $\|\,(T-\alpha I)(x_n)\,\| \to 0$. Either condition can, equivalently, serve to define the approximate point spectrum.

22.1. ***Lemma***. *$\alpha \in \sigma_{ap}(T)$ if and only if there is a sequence x_n of unit vectors in $\mathcal{D}_T$ for which*

$$\langle\, T(x_n) - \alpha\, x_n\,,\, x_n\,\rangle \to 0 \textit{ and } \langle\, T(x_n)\,,\, x_n\,\rangle \to \alpha \textit{ and } \langle\, T(x_n)\,,\, T(x_n)\,\rangle \to \alpha\,\overline{\alpha}.$$

PROOF. Suppose $\alpha \in \sigma_{ap}$. Then there is a sequence x_n of unit vectors in $\mathcal{D}_T$ with $\|\,T(x_n) - \alpha\, x_n\,\| \to 0$. But then

$$|\,\langle\, T(x_n) - \alpha\, x_n\,,\, x_n\,\rangle\,| \leq \|\,T(x_n) - \alpha\, x_n\,\|\,\|\,x_n\,\| = \|\,T(x_n) - \alpha\, x_n\,\| \to 0.$$

So $\langle\, T(x_n)\,,\, x_n\,\rangle - \langle\, \alpha\, x_n\,,\, x_n\,\rangle = \langle\, T(x_n)\,,\, x_n\,\rangle - \alpha \to 0.$

Expand $\langle\, T(x_n) - \alpha\, x_n\,,\, T(x_n) - \alpha\, x_n\,\rangle$ to obtain the remaining limits and the converse implication. $\square$

$\sigma_{ap}(T)$ consists of exactly those α for which $T-\alpha I$ is *not* bounded below.

So $\sigma_r(T) - \sigma_{ap}(T)$ corresponds to those complex numbers α for which $T-\alpha I$ *is* bounded below but for which $\mathcal{R}_{T-\alpha I}$ *is not* dense.

If T is fixed during a discussion, repeated reference to it might be suppressed to clean up the notation. So the resolvent R_α exists and is bounded on the resolvent set ρ, and the spectrum $\sigma = \mathbb{C} - \rho$ can be written as:

the ***disjoint*** **union** $\quad \sigma = \sigma_p \cup \sigma_c \cup \sigma_r \quad$ and as the **union** $\quad \sigma = \sigma_{ap} \cup \sigma_r$.

22.2. ***Lemma***. *If T is an unbounded operator and $T \subset S \subset \overline{T}$ then*

$$\sigma_r\,(S) \subset \sigma_r(T) \textit{ and } \sigma_{ap}\,(S) = \sigma_{ap}(T) \textit{ so } \sigma\,(S) \subset \sigma(T).$$

While S could have more eigenvalues than T, it acquires them from $\sigma_c(T)$ or $\sigma_{ap}(T) \cap \sigma_r(T)$.

$$\textit{Also} \quad \sigma_c\,(S) - \sigma_c\,(T) \subset \sigma_{ap}(T) \cap \sigma_r(T).$$

Any member of the continuous spectrum for S which is not in $\sigma_c\,(T)$ must have come from the part of $\sigma_r\,(T)$ in $\sigma_{ap}(T)$.

PROOF. If $(S - \alpha I)(\mathcal{D}_S)$ is not dense then its subset $(T - \alpha I)(\mathcal{D}_T)$ cannot be, so clearly $\sigma_r(S) \subset \sigma_r(T)$.

It is also obvious that $\sigma_{ap}(T) \subset \sigma_{ap}(S)$, since any sequence of unit vectors x_n drawn from $\mathcal{D}_T$ for which $(T - \alpha I)(x_n) \to 0$ will serve as well to place $\alpha \in \sigma_{ap}(S)$.

Now suppose x_n is a sequence in $\mathcal{D}_S$ for which $(S - \alpha I)(x_n) \to 0$. Since $\mathcal{D}_S \subset \mathcal{D}_{\overline{T}}$ for each n there is a unit vector y_n from $\mathcal{D}_T$ for which both $\| x_n - y_n \| < \frac{1}{n}$ and $\| S(x_n) - T(y_n) \| < \frac{1}{n}$. But then

$$\begin{aligned} 0 \le | \, \| (S - \alpha I)(x_n) \| - \| (T - \alpha I)(y_n) \| \\ \le \| S(x_n) - \alpha x_n - T(y_n) + \alpha y_n \| \\ \le \| S(x_n) - T(y_n) \| + \alpha \| y_n - x_n \| \to 0. \end{aligned}$$

By assumption $(S - \alpha I)(x_n) \to 0$ so $(T - \alpha I)(y_n) \to 0$ as well so $\alpha \in \sigma_{ap}(T)$.

The final containment of the lemma is now clear. □

22.3. ***Lemma***. *Suppose T is closed and operator $T - \alpha I$ is bounded below for some number α. This is equivalent to continuity of the resolvent function for T and α.*
Then the range of $T - \alpha I$ is closed. So if it is dense in $\mathcal{H}$ it must be all of $\mathcal{H}$.

PROOF. Suppose $(T - \alpha I)(x_n) \to y$. Since $T - \alpha I$ is bounded below x_n is Cauchy and so converges to some $x \in \mathcal{H}$. That means $T(x_n) = \alpha x_n + y$ is a sequence in $\mathcal{R}_T$ which converges to $\alpha x + y$, and since T is closed $x \in \mathcal{D}_T$ and $T(x) = \alpha x + y$. So $(T - \alpha I)(x) = y$ is in the range of $T - \alpha I$. □

22.4. ***Exercise***. *For operator T the residual spectrum can be decomposed as*

$$\sigma_r = (\sigma_r - \sigma_{ap}) \cup (\sigma_r \cap \sigma_{ap}).$$

(i) If $\alpha \in \sigma_r - \sigma_{ap}$, the resolvent R_α is continuous. If $\alpha \in \sigma_r \cap \sigma_{ap}$ the resolvent is not continuous.

(ii) Whenever $\alpha \in \sigma_r$ the domain of the resolvent R_α, $\mathcal{R}_{T-\alpha I}$, is not dense in $\mathcal{H}$. In particular

$$(\mathcal{R}_{T-\alpha I})^{\perp} = Ker\left(T^\dagger - \overline{\alpha} I\right) \ne \{0\}.$$

$\overline{\mathcal{R}_{T-\alpha I}}$ is the orthogonal complement of the eigenspace of $T^\dagger$ for eigenvalue $\overline{\alpha}$, and this eigenspace is nontrivial.

(iii) If T is closed and $\alpha \in \sigma_r - \sigma_{ap}$ we have $\mathcal{R}_{T-\alpha I}$ closed (Lemma 22.3) and not all of $\mathcal{H}$. And $T^\dagger - \overline{\alpha} I$ is closed so $Ker\left(T^\dagger - \overline{\alpha} I\right)$ is closed and, here, nontrivial.

$$\mathcal{H} = \mathcal{R}_{T-\alpha I} \oplus Ker\left(T^\dagger - \overline{\alpha} I\right).$$

Suppose for a moment that T is bounded and defined on all of $\mathcal{H}$. So $T^\dagger$ is also bounded and, we saw that $\left\|T^\dagger\right\| = \|T\|$. The last exercise applies to bounded as well as unbounded operators so in this case $\mathcal{H} = \mathcal{R}_{T-\alpha I} \oplus Ker\left(T^\dagger - \overline{\alpha} I\right)$ implies $|\alpha| \le \|T\|$ whenever $Ker\left(T^\dagger - \overline{\alpha} I\right) \ne \{0\}$.

And if $(T - \alpha I)(x_n) \to 0$ for sequence x_n of unit vectors then we must have $|\alpha| \le \|T\|$ here too.

We have the following result, applicable to **bounded operators only:**

22.5. ***Proposition***. *Suppose T is a bounded operator defined on all of $\mathcal{H}$. The entire spectrum of T is contained on or inside the circle centered at 0 of radius $\|T\|$. Rephrasing, the region outside this circle is in the resolvent set for T.*

Proof. The remarks of the previous paragraphs deal with this. □

Stepping back to the more general setting of unbounded operators, we have:

22.6. ***Lemma***. *If T is an unbounded operator and $T - \alpha I$ is bounded below for some number α then $T - \beta I$ is bounded below for all β in some disk of positive radius around α.*

Specifically, if $\| R_\alpha(T) \| \leq k$ and $| \beta - \alpha | < \frac{1}{k}$ then

$$\| R_\beta(T) \| \leq \left(\frac{1}{k} - | \beta - \alpha | \right)^{-1}.$$

Proof. If $\| (T - \alpha I)(x) \| \geq \varepsilon \| x \|$ for some positive ε and number α and operator T, and if $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ then

$$\begin{aligned} \| (T - \alpha I - \eta I)(x) \| &\geq \| (T - \alpha I)(x) \| - \| \eta x \| \\ &\geq \varepsilon \| x \| - |\eta| \| x \| = (\varepsilon - |\eta|) \| x \|. \end{aligned}$$

□

This lemma tells us that complex numbers sufficiently near a member of the resolvent set are themselves in the resolvent set *provided that the domain of the corresponding resolvent function is dense.*

22.7. ***Lemma***. *For unbounded operator T, if $\rho = \rho(T) \neq \varnothing$ then T is closed.*

Proof. Suppose ρ is nonempty and $\alpha \in \rho$. Suppose further that $(x_n, T(x_n))$ is convergent to a point (a, b) in $\mathcal{H} \times \mathcal{H}$. Then R_α is continuous so

$$\begin{aligned} R_\alpha(b - \alpha a) &= \lim_{n\to\infty} R_\alpha(T(x_n) - \alpha x_n) \\ &= \lim_{n\to\infty} (T - \alpha I)^{-1} (T - \alpha I)(x_n) = \lim_{n\to\infty} x_n = a. \end{aligned}$$

Since the range of R_α is $\mathcal{D}_T$, this means $a \in \mathcal{D}_T$. And now

$$b - \alpha a = (T - \alpha I)(T - \alpha I)^{-1}(b - \alpha a) = (T - \alpha I)(a) = T(a) - \alpha a$$

So we find that $T(a) = b$ and therefore T is closed. □

Suppose α is in the resolvent set for operator T. We have just found that R_α has domain $\mathcal{H}$. We know that R_β is bounded for each β near to α but we don't (yet) know if the domain of a nearby R_β is dense—and therefore *also* $\mathcal{H}$.

If $\| R_\alpha \| = k$ define for complex β with $| \beta - \alpha | < \frac{1}{k}$ the expression

$$S_\beta = \sum_{n=0}^{\infty} (\beta - \alpha)^n R_\alpha^{n+1}.$$

The sequence of partial sums $P_j = \sum_{n=0}^{j} (\beta - \alpha)^n R_\alpha^{n+1}$ is Cauchy in operator norm and the continuous operators defined on all of $\mathcal{H}$ form a Banach space so the series does converge to a bounded operator defined, as is R_α, on all of $\mathcal{H}$.

$$\begin{aligned}(T-\beta I)P_j &= (T-\alpha I-(\beta-\alpha)I)P_j = (T-\alpha I)P_j-(\beta-\alpha)P_j\\ &= (T-\alpha I)\left(\sum_{n=0}^{j}(\beta-\alpha)^n R_\alpha^{n+1}\right)-(\beta-\alpha)\left(\sum_{n=0}^{j}(\beta-\alpha)^n R_\alpha^{n+1}\right)\\ &= I+\left(\sum_{n=1}^{j}(\beta-\alpha)^n R_\alpha^{n}\right)-\left(\sum_{n=0}^{j}(\beta-\alpha)^{n+1} R_\alpha^{n+1}\right)\\ &= I-(\beta-\alpha)^{j+1}R_\alpha^{j+1}.\end{aligned}$$

The last term converges to 0 in operator norm and we conclude that $(T-\beta I)P_j(x)$ converges to x for every $x \in \mathcal{H}$. Since the range of $T-\beta I$ is closed (we are assuming here that $\rho(T)$ is nonempty so T is closed) that range must be all of $\mathcal{H}$. So the domain of R_β is all of $\mathcal{H}$.

A similar argument shows that $P_j(T-\beta I)(x) \to x$ for each $x \in \mathcal{D}_T$, and it also converges to $S_\beta(T-\beta I)(x)$.

Putting these two together yields $S_\beta = R_\beta$, a series representation for the resolvent function for β in a neighborhood of any α in the resolvent set.

22.8. ***Proposition***. *(i) The resolvent set for any unbounded operator T is open.*

(ii) The spectrum of any unbounded operator T is closed.

(iii) The domain of R_α is $\mathcal{H}$ (not just dense in $\mathcal{H}$) for every $\alpha \in \rho(T)$.

(iv) If $\alpha \in \rho(T)$ and $\| R_\alpha \| = k$ and $\beta \in \mathbb{C}$ with $|\beta-\alpha| < \frac{1}{k}$ then $\beta \in \rho(T)$ and the following series representation is convergent in operator norm:

$$R_\beta = \sum_{n=0}^{\infty}(\beta-\alpha)^n R_\alpha^{n+1}.$$

Proof. If $\rho(T) = \varnothing$ there is nothing to prove. The case $\rho(T) \neq \varnothing$ is dealt with in the preceding remarks. $\square$

When the resolvent set for an operator T is nonempty, there are several identities involving the resolvent that are useful and we develop them now.

22.9. ***Proposition***. *Suppose S and T are unbounded operators with nonempty resolvent sets. Suppose $\alpha, \beta \in \rho(T)$ and $\gamma \in \rho(T) \cap \rho(S)$.*

(i) Operator V commutes with T if and only if $R_\alpha(T)$ commutes with V.

(ii) $R_\alpha(T) - R_\beta(T) = (\alpha-\beta)\, R_\alpha(T)\, R_\beta(T)$.

(iii) $R_\gamma(T) - R_\gamma(S) = R_\gamma(T)\,(S-T)\,R_\gamma(S)$ when $\mathcal{D}_S \subset \mathcal{D}_T$.

Proof. (i) Suppose V commutes with T. Then both $\mathcal{D}_T$ and $\mathcal{R}_T$ are in $\mathcal{D}_V$ and for every $x \in \mathcal{D}_T$ we have $V(T(x)) = T(V(x))$. It is implied by the existence of the right-hand side that $V(x) \in \mathcal{D}_T$ whenever $x \in \mathcal{D}_T$.

We need to show that this implies that R_α commutes with V. Since $\mathcal{D}_{R_\alpha} = \mathcal{H}$ we have both domain and range of V in the domain of R_α. We need to show that $R_\alpha(V(x)) = V(R_\alpha(x))$ for all $x \in \mathcal{D}_V$. We do have the right side of this equation defined, since the range of R_α is $\mathcal{D}_T$ which is contained in $\mathcal{D}_V$ by assumption. Since $T - \alpha I$ is onto $\mathcal{H}$ there is a $y \in \mathcal{D}_T$ for which $T(y) - \alpha y = x$. Then

$$\begin{aligned} V(R_\alpha(x)) &= V(R_\alpha(T(y) - \alpha y)) = V(y) = R_\alpha(T - \alpha I)V(y) \\ &= R_\alpha(T(V(y)) - \alpha V(y)) = R_\alpha(V(T(y)) - V(\alpha y)) = R_\alpha(V(x)). \end{aligned}$$

So we have shown that if V commutes with T then $R_\alpha(T)$ commutes with V.

For the converse implication, suppose $R_\alpha(V(x)) \neq V(R_\alpha(x))$ for some $x \in \mathcal{D}_V$. Revisiting the last calculation above in reverse order, it must fail somewhere. There are only three places in the chain of equalities that this might occur. First, it may be that $T(y) \notin \mathcal{D}_V$, so V does not commute with T because $\mathcal{R}_T \not\subset \mathcal{D}_V$. But if $T(y) \in \mathcal{D}_V$, it might be that $T(V(y)) \neq V(T(y))$, and again V does not commute with T. Finally, it could be that $R_\alpha(x) \notin \mathcal{D}_V$. But the range of R_α is $\mathcal{D}_T$ so $\mathcal{D}_T \not\subset \mathcal{D}_V$ and, again, V does not commute with T.

(ii) $(\,T - \beta I\,)R_\beta(T)$ is the identity on $\mathcal{H}$ while $R_\alpha(T)(\,T - \alpha I\,)$ is the identity on $\mathcal{D}_T$.

$$\begin{aligned} R_\alpha(T) - R_\beta(T) &= R_\alpha(T)(\,T - \beta I\,)R_\beta(T) - R_\alpha(T)(\,T - \alpha I\,)R_\beta(T) \\ &= R_\alpha(T)(\,T - \beta I - T + \alpha I\,)R_\beta(T) = (\alpha - \beta)\, R_\alpha(T)\, R_\beta(T). \end{aligned}$$

(iii) Similarly to the calculation above, and using $\mathcal{D}_S \subset \mathcal{D}_T$, we have

$$\begin{aligned} R_\gamma(T) - R_\gamma(S) &= R_\gamma(T)(\,S - \gamma I\,)R_\gamma(S) - R_\gamma(T)(\,T - \gamma I\,)R_\gamma(S) \\ &= R_\gamma(T)(\,S - \gamma I - T + \gamma I\,)R_\gamma(S) = R_\gamma(T)(S - T)R_\gamma(S). \end{aligned}$$

□

Item (ii) of this proposition is generally referred to as **the First Resolvent Identity** while (iii) is called **the Second Resolvent Identity.** The First Resolvent Identity implies that $R_\alpha(T)$ and $R_\beta(T)$ commute whenever $\alpha, \beta \in \rho(T)$.

23. The Spectrum of Unitary or Normal Operators

Suppose $U \colon \mathcal{H} \to \mathcal{H}$ is unitary: that is, an isometry onto $\mathcal{H}$. For such operators we have seen that the inverse isometry U^{-1} coincides with the adjoint $U^\dagger$.

Suppose λ is complex and $|\,\lambda\,| \neq 1$.

The function $U - \lambda I$ is easily seen to have trivial kernel, and the adjoint $(\,U - \lambda I\,)^\dagger = U^\dagger - \overline{\lambda} I = U^{-1} - \overline{\lambda} I$ also has trivial kernel.

That means the range of $U - \lambda I$ (and also that of $U^\dagger - \lambda I$) is dense in $\mathcal{H}$.

Suppose $U - \lambda I$ *is not* bounded below. Then there is a sequence x_n of unit vectors for which $\|\,U(x_n) - \lambda x_n\,\| \to 0$. By the BCS inequality both

$$\langle\, U(x_n) - \lambda x_n\,,\, U(x_n)\,\rangle \to 0 \text{ and } \langle\, U(x_n) - \lambda x_n\,,\, \lambda x_n\,\rangle \to 0.$$

But then

$$\langle\, U(x_n) - \lambda x_n\,,\, U(x_n)\,\rangle = 1 - \langle\, \lambda x_n\,,\, U(x_n)\,\rangle \to 0$$

so $\langle\, U(x_n)\,,\, \lambda x_n\,\rangle \to 1$.

And also

$$\langle\, U(x_n) - \lambda\, x_n\,,\, \lambda\, x_n\,\rangle = \langle\, U(x_n)\,,\, \lambda\, x_n\,\rangle - \langle\, \lambda\, x_n\,,\, \lambda\, x_n\,\rangle \to 0$$

so $\langle\, U(x_n)\,,\, \lambda\, x_n\,\rangle \to \lambda\,\overline{\lambda} \neq 1$.

This contradiction implies that $U - \lambda\, I$ *is* bounded below, and this applies to its adjoint $U^{\dagger} - \overline{\lambda}\, I$ as well. So by Corollary 21.14 we have $\mathcal{R}_{U-\lambda\, I} = \mathcal{R}_{U^{\dagger}-\overline{\lambda}\, I} = \mathcal{H}$.

The point is that the resolvent R_{λ} exists and is continuous with domain $\mathcal{H}$ whenever $|\,\lambda\,| \neq 1$.

We knew from Proposition 22.5 that no point of the spectrum could be strictly *outside* the unit circle $\mathbb{S}^1$ of $\mathbb{C}$. We now know that points in the spectrum cannot be strictly inside this circle either.

So the spectrum is confined to the unit circle itself.

If b_n is an enumeration of a countable orthonormal basis of $\mathcal{H}$ and r_n is any countable set of real numbers then the function

$$U\colon \mathcal{H} \to \mathcal{H} \text{ given by } U\left(\sum_{n=1}^{\infty} x_n\, b_n\right) = \sum_{n=1}^{\infty} e^{i\, r_n}\, x_n\, b_n$$

is easily seen to be an isometry onto $\mathcal{H}$, and each $e^{i\, r_n}$ is an eigenvalue of U. By choosing the r_n properly, we see that the spectrum of a unitary operator on an infinite dimensional separable Hilbert space can be any closed subset of the unit circle in $\mathbb{C}$.

If $\mathcal{H}$ has an uncountable orthonormal basis A, that basis can be written as a disjoint union of countable subsets $S_{\alpha},\ \alpha \in B$ for an uncountable index set B, and if we define U as above on each $\mathcal{H}_{S_{\alpha}}$ and extend by linearity to $\mathcal{H}$ we have the same result for Hilbert spaces of uncountable dimension too.

23.1. ***Proposition***. *If U is a unitary operator on complex Hilbert space $\mathcal{H}$ then the spectrum of U is a compact subset of the unit circle in $\mathbb{C}$. Any compact subset of the unit circle in $\mathbb{C}$ is the spectrum of some unitary operator on any infinite dimensional Hilbert space.*

Proof. The proof can be found above. □

For normal operator N and any complex λ the operator $N - \lambda\, I$ commutes with its adjoint $N^{\dagger} - \overline{\lambda}\, I$ (i.e. these operators are also normal) so they have the same range and the same kernel. This shared kernel is the orthogonal complement of shared range.

It follows that λ is an eigenvalue for N exactly when $\overline{\lambda}$ is an eigenvalue for $N^{\dagger}$, and the respective eigenspaces coincide.

Suppose that x_n is a sequence of unit vectors in $\mathcal{H}$ and $\|\,(N - \lambda\, I)(x_n)\,\| \to 0$. In other words, λ is in the approximate point spectrum of N. So

$$\langle\,(N - \lambda\, I)(x_n)\,,\,(N - \lambda\, I)(x_n)\,\rangle = \langle\,(N^{\dagger} - \overline{\lambda}\, I)(x_n)\,,\,(N^{\dagger} - \overline{\lambda}\, I)(x_n)\,\rangle \to 0$$

which means $\overline{\lambda} \in \sigma_{ap}\left(N^{\dagger}\right)$.

Moreover, for any normal operator the closure of its range is the orthogonal complement of its kernel. So if the range $\mathcal{R}_{N-\lambda\, I}$ is not dense in $\mathcal{H}$ then λ must be an eigenvalue for N.

So normal operators, including unitary operators, have no residual spectrum.

This leads to the following proposition for normal operators, which applies to unitary operators as a special case.

23.2. ***Proposition***. *Suppose operator N is normal on complex Hilbert $\mathcal{H}$.*

(i) Every point strictly outside the circle centered at 0 *of radius $\|N\|$ is in the resolvent set for N.*

(ii) $\sigma_r(N) = \varnothing$.

(iii) $\lambda \in \sigma_p(N)$ *exactly when* $\overline{\lambda} \in \sigma_p\left(N^\dagger\right)$. *For these* λ,
$Ker(N - \lambda I) = Ker\left(N^\dagger - \overline{\lambda} I\right) \neq \{0\}$ *and* $\mathcal{R}_{N-\lambda I} = \mathcal{R}_{N^\dagger - \overline{\lambda} I}$
and $\overline{\mathcal{R}_{N-\lambda I}} = Ker(N - \lambda I)^\perp \neq \mathcal{H}$.

(iv) $\lambda \in \sigma_c(N)$ *exactly when* $\overline{\lambda} \in \sigma_c\left(N^\dagger\right)$.

Proof. The proof is contained in the remarks above and Proposition 22.5. □

23.3. ***Corollary***. *Suppose operator N is normal on complex Hilbert $\mathcal{H}$.*

(i) $\sigma(N) = \sigma_p(N) \cup \sigma_c(N)$.

(ii) $\mathcal{R}_{N-\lambda I} = \mathcal{H}$ *for every* $\lambda \in \sigma_c(N)$.

Proof. □

We have shown that the spectrum of any linear operator, bounded or not, is closed. We *have not* shown it to be nonempty. That it is nonempty for *bounded* operators is true, but the proof uses techniques we will not discuss till a bit later. See Proposition **??**.

24. Symmetric Operators

In this section we work with operators defined on a dense vector subspace of a complex Hilbert space $\mathcal{H}$ unless explicitly stated to the contrary.

We call an unbounded operator T as above **symmetric** if $T \subset T^\dagger$: in other words, if $T^\dagger$ is an extension of T. Linear T is symmetric exactly when $T^\dagger(x)$ is defined for every $x \in \mathcal{D}_T$ and $T(x) = T^\dagger(x)$.

So T symmetric means that

$$\langle T(x)\,,\, y\,\rangle = \langle\, x\,,\, T(y)\,\rangle \quad \forall\, x, y \in \mathcal{D}_T.$$

The argument of Lemma 18.1 needs no modification for unbounded operators.

$$T \text{ is symmetric} \Leftrightarrow \langle T(x)\,,\, x\,\rangle = \langle\, x\,,\, T(x)\,\rangle \quad \forall\, x \in \mathcal{D}_T$$

and then of course we have the equivalent condition

$$T \text{ is symmetric} \Leftrightarrow \langle T(x)\,,\, x\,\rangle \ \text{ is real} \quad \forall\, x \in \mathcal{D}_T.$$

24.1. ***Proposition***. *If T is symmetric, the approximate point spectrum σ_{ap} of T is contained in* $\mathbb{R}$.

So if α is non-real and in the spectrum of T then the resolvent function R_α is continuous but fails to have dense domain: $\mathcal{R}_{T-\alpha I}$ cannot be dense in $\mathcal{H}$ for any such α.

Proof. Suppose $\alpha \in \sigma_{ap}$. Then there is a sequence x_n of unit vectors in $\mathcal{D}_T$ with $\| T(x_n) - \alpha\, x_n \| \to 0$. By Lemma 22.1 we have $\langle T(x_n)\,,\, x_n \rangle \to \alpha$.

Symmetry now requires α to be real. $\square$

See Proposition 24.4 for the next step in this direction.

Since $T^\dagger$ is always closed and $T^\dagger$ extends symmetric T, every symmetric operator is closeable.

If symmetric T is bounded on $\mathcal{D}_T$ then it could be extended in a unique way to a continuous operator $\overline{T}$ defined on all of $\mathcal{H}$. In that case, since $T^\dagger$ is closed we have $T^\dagger = \overline{T}$ and then $\overline{T} = T^{\dagger\dagger} = \overline{T}^\dagger$ so $\overline{T}$ is its own adjoint. This is the ordinary adjoint for bounded operators, and the symmetry condition just means that this unique extension of T to all of $\mathcal{H}$ is self-adjoint.

Bounded or not, the domain of $T^\dagger$ for symmetric T will often be larger than the domain of T.

24.2. ***Lemma***. *(i) A symmetric operator is always closeable.*
(ii) If T is symmetric and $\mathcal{D}_{\overline{T}} = \mathcal{H}$ (so $\overline{T} = T^\dagger$) then T is bounded.

Proof. See the preceding remarks for (i). Case (ii) follows Corollary 21.11 or from the preceding remarks and the closed graph theorem directly. $\square$

Item (ii) of Lemma 24.2 is called **The Hellinger-Toeplitz Theorem.**

So if we know that symmetric T is *not* bounded, then we know that the domain of $T^\dagger$ *cannot* be all of $\mathcal{H}$: the Hellinger-Toeplitz Theorem tells us there is no getting away (in cases of crucial importance) from consideration of operators defined on only part of $\mathcal{H}$.

24.3. ***Lemma***. *Suppose S is symmetric and one-to-one.*
If $\mathcal{R}_S = \mathcal{D}_{S^{-1}}$ is dense in $\mathcal{H}$ then S^{-1} is symmetric too.

Proof. The domain of S^{-1} is $\mathcal{R}_S$.

Suppose $g = S(x)$ and $f = S(y)$ are in $\mathcal{D}_{S^{-1}}$. Then

$$\langle S^{-1}(g)\,,\, f \rangle = \langle x\,,\, S(y) \rangle = \langle S(x)\,,\, y \rangle = \langle g\,,\, S^{-1}(f) \rangle.$$

$\square$

A symmetric T *might* have the same domain as $T^\dagger$, *and it is only to these unbounded operators that some of our most important theorems apply.*

If $T = T^\dagger$ we say that the unbounded operator T is **self-adjoint**. Self-adjoint operators are, of course, closed. And if a self-adjoint operator is bounded its domain *must* be all of $\mathcal{H}$.

If a self-adjoint operator (like any symmetric operator) has any eigenvalues at all they must be real.

24.4. ***Proposition***. *Suppose T is self-adjoint.*
If $\mathcal{R}_{T-\alpha I} = (T - \alpha I)(\mathcal{D}_T)$ is not dense in $\mathcal{H}$
then α is a real eigenvalue of T.
Self-adjoint operators have no residual spectrum.

PROOF. Suppose $A = \mathcal{R}_{T-\alpha I}$ is not dense in $\mathcal{H}$. Select nonzero $x \in A^{\perp}$.

So $\forall w \in \mathcal{D}_T$ we have (since T is self-adjoint)

$$0 = \langle (T - \alpha I)(w), x \rangle = \langle T(w), x \rangle - \langle \alpha w, x \rangle.$$

So $\langle T(w), x \rangle = \langle \alpha w, x \rangle$ for all $\forall w \in \mathcal{D}_T$, so the functional $\langle T(\cdot), x \rangle$ is bounded on $\mathcal{D}_T$. That means x is in the domain of $T^{\dagger}$, assumed to be equal to T. We can now carry the calculation above a step further:

$$0 = \langle (T - \alpha I)(w), x \rangle = \langle T(w), x \rangle - \langle \alpha w, x \rangle = \langle w, T(x) \rangle - \langle w, \overline{\alpha} x \rangle.$$

Since $\mathcal{D}_T$ is dense in $\mathcal{H}$ this means $T(x) = \overline{\alpha} x$. But T can have only real eigenvalues so α is real. □

24.5. ***Corollary***. *Suppose T is self-adjoint.*
The spectrum σ of T is contained in $\mathbb{R}$ and
$\sigma = \sigma_p \cup \sigma_c$.

PROOF. By Proposition 24.1 the resolvent function is continuous off the real numbers for every symmetric T. For self-adjoint operators, which are closed, Proposition 24.4 and Lemma 22.3 then imply that the domains of these resolvent functions are not only dense but actually all of $\mathcal{H}$. So every non-real complex number is in the resolvent set.

By Proposition 24.1 the approximate point spectrum is contained in $\mathbb{R}$ and by Proposition 24.4 there is no residual spectrum, so there is no part of σ_{ap} outside $\sigma_p \cup \sigma_c$. □

Suppose S and T are symmetric operators and $\overline{S} \subset S^{\dagger} \subset \overline{T} \subset T^{\dagger}$. Then

$$T^{\dagger\dagger} = \overline{T} \subset T^{\dagger} \subset S^{\dagger\dagger} = \overline{S} \subset S^{\dagger}.$$

In other words $\overline{T} = \overline{S} = S^{\dagger}$ and $\overline{S}$ is, therefore, self-adjoint.

We can draw conclusions from this.

First, there could be no proper self-adjoint extension of $S^{\dagger}$ for symmetric S.

More, $S^{\dagger}$ can not even have a nontrivial *symmetric* extension (that is, a symmetric extension bigger than itself) and if $S^{\dagger}$ is symmetric it must be self-adjoint.

Looking at this from the "T standpoint," if T is *any* symmetric operator, $\overline{T}$ cannot be a nontrivial extension of the adjoint of *any* symmetric operator.

So not only can there be no nontrivial "chains" of self-adjoint operators, there cannot even be nontrivial containment chains of *symmetric* operators, where the adjoint of one is contained in the closure of the next.

This implies, for instance, that different self-adjoint extensions of symmetric S, should there be more than one, cannot be compatible in the sense that neither could be an extension of the other.

On the other hand, suppose $S \subset \overline{T} \subset T^\dagger$. Then $T^{\dagger\dagger} = \overline{T} \subset T^\dagger \subset S^\dagger$ so

$$\overline{S} \subset \overline{T} \subset T^\dagger \subset S^\dagger.$$

We find that if S is a restriction of *any* symmetric operator, then S is symmetric too and any possible self-adjoint extension of S must lie between $\overline{S}$ and $S^\dagger$.

In the hunt for $\overline{T}$ for which $\overline{T} = T^\dagger$, increasing the size of $\mathcal{D}_{\overline{T}}$ decreases the size of $\mathcal{D}_{T^\dagger}$ *and there might be no way to "meet in the middle."* Some symmetric operators have no self-adjoint extension at all. Later, we examine a condition that will guarantee such extensions.

We say that unbounded T is **essentially self-adjoint** if $\overline{T} = T^\dagger$. By the discussion above this implies that $\overline{T}$ is the only self-adjoint extension of T. And there can there be no nontrivial self-adjoint restrictions of $\overline{T}$.

In the physics literature, the term **Hermitian operator** may refer to any of the operators we describe as symmetric, self-adjoint and essentially self-adjoint, depending on the predilections of the author. Busy physicists sometimes prefer not to think about the distinction unless absolutely necessary.

24.6. ***Exercise***. *Two unitarily equivalent unbounded operators are both self-adjoint or both not.*

25. Counterexamples: Self-Adjointness

Here are a few very simple examples/counterexamples of the phenomena under consideration.

First we examine ***an operator that is not closeable*** and ***an adjoint without dense domain***.

Define T on $\mathcal{D}_T = C([0,1])$ by $T(\psi) = \psi(0)$, the constant function. So T is an unbounded operator on the Hilbert space $\mathcal{H} = \mathcal{L}^2([0,1])$.

Strictly speaking, of course, we are saying that members of $\mathcal{D}_T$ are those equivalence classes of measurable functions on $[0,1]$ that differ from a continuous function on a null set. There is at most one continuous function in any such class. T uses that member in its definition and returns the class of the relevant constant function.

$g \in \mathcal{D}_{T^\dagger}$ provided

$$\langle\, T(\psi)\,,\, g\,\rangle = \int_0^1 \overline{g(x)}\, \psi(0)\, dx = \psi(0) \int_0^1 \overline{g(x)}\, dx$$

is a bounded operator. But there is no limit to how large $\psi(0)$ can be among ψ of norm 1 in the Hilbert space, so we must have $\int_0^1 \overline{g(x)}\, dx = 0$. In other words, $\mathcal{D}_{T^\dagger} = 1^\perp$ which *is not* dense in $\mathcal{H}$.

And then for $g \in 1^\perp$ we have

$$\langle\, \psi\,,\, T^\dagger(g), \rangle = \langle\, T(\psi)\,,\, g\,\rangle = \psi(0) \int_0^1 \overline{g(x)}\, dx = 0.$$

So $T^\dagger$ is the zero operator on $1^\perp$.

Next, we look at an example of ***an operator that is symmetric but not essentially self-adjoint.***

Consider the operator $S = i\,\frac{d}{dx}$ defined on $\mathcal{D}_S$, which consists of those members $f \in C^1([0,1])$ with $f(0) = f(1) = 0$. So S is an unbounded operator on the Hilbert space $\mathcal{H} = \mathcal{L}^2([0,1])$. S is symmetric.

$$\langle\, S(g)\,,\, f\,\rangle = \langle\, g\,,\, S(f)\,\rangle \quad \forall f, g \in \mathcal{D}_S$$

as shown by the calculation

$$\begin{aligned}\int_0^1 (i\,g'(x))\,\overline{f(x)}\,dx &= i\,g(x)\,\overline{f(x)}\Big|_0^1 - \int_0^1 i\,g(x)\,\overline{f'(x)}\,dx \\ &= g(1)\overline{f(1)} - g(0)\overline{f(0)} \;+\; \int_0^1 g(x)\,\overline{(\,i\,f'(x)\,)}\,dx.\end{aligned}$$

Let E_t consist of those members f of $C^1([0,1])$ for which $f(1) = e^{it}f(0)$ for a fixed real t. And let A_t be the operator with the same formula as S but defined on E_t. For these functions

$$g(1)\overline{f(1)} - g(0)\overline{f(0)} = e^{it}g(0)e^{-it}\overline{f(0)} - g(0)\overline{f(0)} = 0$$

so the A_t are symmetric by the same calculation used above for S and are extensions of S. Note that the A_t are incompatible except for values of t for which they are equal: they cannot have a shared self-adjoint extension.

The next example features ***a symmetric operator whose adjoint is not symmetric, and therefore (of course) not self-adjoint.***

The set $C = C_c^{\infty}(\,(0,1)\,)$ consists of those infinitely differentiable functions on $(0,1)$ which are 0 off some compact subinterval of $(0,1)$.

C may be considered as a dense subset of $\mathcal{L}^2([0,1])$.

We define operator T_C by the formula $-\frac{d^2}{dx^2}$ applied to members of C.

Suppose f and g are in C. Then since members of C and all their derivatives are 0 on the boundary of $[0,1]$

$$\begin{aligned}\langle\, T_C(f)\,,\, g\,\rangle = \int_0^1 -f''(x)\,\overline{g(x)}\,dx &= \int_0^1 f'(x)\,\overline{g'(x)}\,dx \\ = \int_0^1 -f(x)\,\overline{g''(x)}\,dx &= \langle\, f\,,\, T_C(g)\,\rangle.\end{aligned}$$

So T_C is symmetric.

For any $f \in C$ we have

$$\langle\, T_C(f)\,,\, 1\,\rangle = \int_0^1 -f''(x)\,dx = -f'(x)\Big|_0^1 = -f'(1) + f'(0) = 0.$$

So this functional is (very) bounded and $1 \in \mathcal{D}_{T_C^{\dagger}}$ and $T_C^{\dagger}(1) = 0$.

We have discovered that $(1,0) \in T_C^{\dagger}$.

Note that for any twice differentiable f and $x \in [0,1]$ we have

$$\left|\int_0^x \int_0^t f''(s)\,ds\,dt\right| = \left|\int_0^x f'(t) - f'(0)\,dt\right| = |f(x) - x\,f'(0) - f(0)|\,.$$

In case $f \in C$ we have then

$$\begin{aligned} |\, f(x) \,| = & \left| \int_0^x \int_0^t f''(s)\, ds\, dt \right| \le \int_0^x \int_0^t |\, f''(s) \,|\; ds\, dt \le \int_0^1 \int_0^1 |\, f''(s) \,|\; ds\, dt \\ = & \int_0^1 |\, f''(s) \,|\; ds \le \sqrt{\int_0^1 |\, f''(s) \,|^2\; ds} = \|\, T_C(f) \,\| \end{aligned}$$

where the last inequality is Jensen's inequality applied to the squaring function.

So if f_n is a sequence in C for which $\|\, T_C(f_n) \,\|$ converges to 0 then the sequence f_n *cannot* converge to 1, and that means $(1, 0) \notin \overline{T_C}$.

The domain of the closed operator $T_C^\dagger$ is therefore strictly larger than the domain of $T_C^{\dagger\dagger} = \overline{T_C}$ so $T_C^\dagger$ is *not* symmetric.

And so $\overline{T_C}$, which is symmetric, is *not* self-adjoint.

26. The Cayley and Inverse Cayley Transforms

The next few results rely only on **the symmetry condition** $\langle\, T(x)\,,\, y \,\rangle = \langle\, x\,,\, T(y) \,\rangle$ for members $x, y \in \mathcal{D}_T$ and not the density of $\mathcal{D}_T$ in $\mathcal{H}$, so we will assume T possesses the former property and not necessarily the latter.

If T has the symmetry condition it can have no complex eigenvalues. So if complex number α is not real the equation $T(g) = \alpha\, g$ has no solution except $g = 0$. This implies that $T - \alpha\, I$ is one-to-one from its domain $\mathcal{D}_T$ onto the set $\mathcal{R}_{T-\alpha\, I} = (T - \alpha\, I)\,(\mathcal{D}_T)$, where I is the identity operator on $\mathcal{H}$.

So the resolvent function $R_\alpha = (\, T - \alpha\, I \,)^{-1}$ is defined on the set $\mathcal{R}_{T-\alpha\, I}$.

A quick calculation shows that, since T has the symmetry condition, for any complex α and $x, y \in \mathcal{D}_T$

$$\langle\, (\, T - \alpha\, I \,)(x)\,,\, y \,\rangle = \langle\, x\,,\, (\, T - \overline{\alpha}\, I \,)(y) \,\rangle.$$

For T with the symmetry condition and non-real number α define C_α by

$$C_\alpha = (\, T - \overline{\alpha}\, I \,)(\, T - \alpha\, I \,)^{-1}$$

on the vector subspace $\mathcal{D}_{C_\alpha} = (\, T - \alpha\, I \,)(\mathcal{D}_T) = \mathcal{R}_{T-\alpha\, I}$ of $\mathcal{H}$.

The image of C_α is $\mathcal{R}_{C_\alpha} = C_\alpha\,(\mathcal{D}_{C_\alpha}) = (\, T - \overline{\alpha}\, I \,)(\mathcal{D}_T) = \mathcal{R}_{T-\overline{\alpha}\, I}$.

Note that $C_{\overline{\alpha}} \circ C_\alpha$ is the identity map on $\mathcal{D}_{C_\alpha}$.

It may be that $\mathcal{D}_{C_\alpha}$ fails to be dense in $\mathcal{H}$, which happens for symmetric T exactly when the number α is in the residual spectrum σ_r.

But dense domain or not, C_α is an isometry on its domain, as verified in the following calculation.

If $f = (\, T - \alpha\, I \,)(u)$ and $g = (\, T - \alpha\, I \,)(v)$ for $u, v \in \mathcal{D}_T$ expand and compare the right sides (using the symmetry condition) to show equality of the left sides.

$$\begin{aligned} \langle\, C_\alpha(f)\,,\, C_\alpha(g) \,\rangle = &\ \langle\, (\, T - \overline{\alpha}\, I \,)(u)\,,\, (\, T - \overline{\alpha}\, I \,)(v) \,\rangle \\ \langle\, f\,,\, g \,\rangle = &\ \langle\, (\, T - \alpha\, I \,)(u)\,,\, (\, T - \alpha\, I \,)(v) \,\rangle. \end{aligned}$$

The case of $\alpha = -i$ is singled out and called the **Cayley transform** of T with the symmetry condition. The Cayley transform

$$\mathbf{C(T)} = (\,T - i\,I\,)\,(\,T + i\,I\,)^{-1}$$

of T is an isometry from $(\,T + i\,I\,)(\mathcal{D}_T) = \mathcal{R}_{T+i\,I}$ onto $(\,T - i\,I\,)(\mathcal{D}_T) = \mathcal{R}_{T-i\,I}$.

Suppose $(\,T+i\,I\,)(x)$ is a generic member of the domain of $C(T)$. So this vector is also in the domain of $C(T) - I$, and

$$\begin{aligned}(\,C(T) - I\,)(\,T + i\,I\,)(x) &= C(T)(\,T + i\,I\,)(x) - (\,T + i\,I\,)(x) \\ &= (\,T - i\,I\,)(x) - (\,T + i\,I\,)(x) = -2\,i\,x.\end{aligned}$$

So the range of $C(T) - I$ is $\mathcal{D}_T$, and since $T - i\,I$ is one-to-one so is $C(T) - I$.

The Cayley transform of an operator as above is not just any isometry. ***It is an isometry that moves every domain member except the zero vector.***

The domain of $(C(T)-I)^{-1}$ is $\mathcal{D}_T$, and $(C(T)-I)^{-1}$ sends x to $\frac{i}{2}\,(\,T+i\,I\,)(x)$.

Now we calculate

$$\begin{aligned}(\,C(T) + I\,)(\,T + i\,I\,)(x) &= C(T)(\,T + i\,I\,)(x) + (\,T + i\,I\,)(x) \\ &= (\,T - i\,I\,)(x) + (\,T + i\,I\,)(x) = 2\,T(x).\end{aligned}$$

Putting these two calculations together yields

$$-i\,(\,C(T) + I\,)(C(T) - I)^{-1}(x) = T(x) \quad \forall x \in \mathcal{D}_T.$$

The operation K defined by

$$\mathbf{K(A)} = -i\,(\,A + I\,)(A - I)^{-1}$$

is called the **inverse Cayley transform**, and *it is defined for any isometry A which moves every domain member except 0*, required so that $A - I$ is one-to-one and $(A - I)^{-1}$ is defined.

Let's suppose $A\colon \mathcal{D}_A \to \mathcal{R}_A$ is any isometry of this type on $\mathcal{H}$.

Let $T = K(A)$. So $T\colon \mathcal{R}_{A-I} \to \mathcal{R}_{A+I}$.

If $x, y \in \mathcal{R}_{A-I}$ then $x = (A - I)(a)$ and $y = (A - I)(b)$ for certain $a, b \in \mathcal{D}_A$. But then

$$\begin{aligned}\langle\, T(x)\,,\, y\,\rangle &= \langle\, -i\,(A + I)(a)\,,\,(A - I)(b)\,\rangle \\ &= \;-\,i\;(\,\langle\, A(a)\,,\,A(b)\,\rangle - \langle\, A(a)\,,\,b\,\rangle + \langle\, a\,,\,A(b)\,\rangle - \langle\, a\,,\,b\,\rangle\,)\end{aligned}$$

and because A is an isometry this is $i\,(\langle\, A(a)\,,\,b\,\rangle - \langle\, a\,,\,A(b)\,\rangle\,)$.

Similarly,

$$\begin{aligned}\langle\, x\,,\,T(y)\,\rangle &= \langle\;(A - I)(a)\,,\,-i\,(A + I)(b)\,\rangle \\ &= i\;(\,\langle\, A(a)\,,\,A(b)\,\rangle + \langle\, A(a)\,,\,b\,\rangle - \langle\, a\,,\,A(b)\,\rangle - \langle\, a\,,\,b\,\rangle\,)\end{aligned}$$

and this simplifies to the same number, so T has the symmetry condition.

If $T = K(A)$ is densely defined we now know that T is symmetric, though none of the calculations involving these transforms so far used this density. Dense domain or not, the symmetry condition implies that $T + i\,I$ and $T - i\,I$ are one-to-one on their common domain $\mathcal{D}_T = \mathcal{R}_{A-I}$.

Now suppose $x \in \mathcal{D}_A$ so $(A-I)(x)$ is a generic member of $\mathcal{D}_{K(A)}$.

$$\text{Then} \quad (K(A)+iI)(A-I)(x) = K(A)(A-I)(x) + i\,(A-I)(x)$$
$$= -i\,(A+I)(x) + i\,(A-I)(x) = -2i\,x.$$

So $K(A)+iI$ is onto $\mathcal{D}_A$ and $x = \frac{i}{2}\,(K(A)+iI)(A-I)(x)$. Now we have

$$\begin{aligned} C(K(A))(x) &= (K(A)-iI)(K(A)+iI)^{-1}(x) \\ &= (K(A)-iI)(K(A)+iI)^{-1}\frac{i}{2}\,(K(A)+iI)(A-I)(x) \\ &= \frac{i}{2}\,(K(A)-iI)(A-I)(x) = \frac{i}{2}\,(-i\,(A+I)(x) - i\,(A-I)(x)) \\ &= A(x). \end{aligned}$$

Let's recapitulate.

1 is not an eigenvalue of any isometry produced by applying the Cayley transform to any operator that satisfies the symmetry condition; that is, the isometry thereby produced moves every member of its domain except 0. The inverse Cayley transform can be applied to any isometry from one subspace of $\mathcal{H}$ to another, so long as 1 is not an eigenvalue of this isometry. The operator produced satisfies the symmetry condition, and will be a symmetric operator if it has dense domain.

26.1. ***Proposition***. *If T has the symmetry condition then the Cayley transform $C(T)$ of T is an isometry with domain $\mathcal{D}_{C(T)} = (T+iI)(\mathcal{D}_T) = \mathcal{R}_{T+iI}$ onto range $\mathcal{R}_{C(T)} = (T-iI)(\mathcal{D}_T) = \mathcal{R}_{T-iI}$. This isometry moves every domain member except 0, and $K(C(T)) = T$.*

If A is any isometry extending $C(T)$ (which must, perforce, move every domain member except 0: see Exercise 27.4) then the inverse Cayley transform $K(A)$ of A is an extension of T with the symmetry condition and $C(K(A)) = A$.

Proof. Examine the remarks above. □

26.2. ***Lemma***. *Suppose T is symmetric.*
T is closed exactly when $\mathcal{R}_{T+iI}$ is closed.
So T is closed exactly when its Cayley transform has closed domain.

Proof. For $x \in \mathcal{D}_T$ note that

$$\begin{aligned} &\langle (T+iI)(x), (T+iI)(x) \rangle \\ &\qquad = \langle T(x), T(x) \rangle + i\,\langle x, T(x) \rangle - i\,\langle T(x), x \rangle + i(-i)\,\langle x, x \rangle. \end{aligned}$$

By symmetry of T the middle terms cancel and we have, for every $x \in \mathcal{D}_T$,

$$\|\,(T+iI)(x)\,\|^2 = \|\,T(x)\,\|^2 + \|\,x\,\|^2.$$

Also by symmetry, $T+iI$ is one-to-one. So the map that sends $(T+iI)(x) \in \mathcal{R}_{T+iI}$ to $(x, T(x)) \in T$ is an isometry onto T with norm induced from $\mathcal{H} \times \mathcal{H}$. The two sets are closed or not together. □

26.3. ***Corollary***. *Suppose T is symmetric and λ is*
a complex number with nonzero imaginary part.
T is closed exactly when $\mathcal{R}_{T+\lambda I}$ is closed.

Proof. Suppose T is unbounded and $\lambda = r + s\,i$ is a complex number with nonzero imaginary part s. Then T is closed and symmetric exactly when $\frac{1}{s}\,T + \frac{r}{s}\,I$ is closed and symmetric, which happens exactly when $\mathcal{R}_{\frac{1}{s}\,T+\frac{r}{s}\,I+i\,I} = \mathcal{R}_{T+\lambda\,I}$ is closed by Lemma 26.2. $\square$

26.4. ***Lemma***. *If T is symmetric then* $\overline{\mathcal{R}_{T-i\,I}} = \mathcal{R}_{\overline{T}-i\,I}$.

Proof. Expanding $\|\,(T-i\,I)(f)\,\|^2$ shows that $\|\,(T-i\,I)(f)\,\| \geq \|\,f\,\|\ \forall f \in \mathcal{D}_T$. That is, $T - i\,I$ is bounded below by 1. So if $(T - i\,I)(h_n) = T(h_n) - i\,h_n \in (T - i\,I)(\mathcal{D}_T)$ for $h_n \in \mathcal{D}_T$ is a Cauchy sequence then h_n must be Cauchy too and this implies $T(h_n)$ is Cauchy so $(h_n, T(h_n))$ converges to a point $(f, \overline{T}(f)) \in \overline{T}$. So $(T - i\,I)(h_n)$ converges to $(\overline{T} - i\,I)(f) \in (\overline{T} - i\,I)(\mathcal{D}_{\overline{T}})$, and we find that $\overline{(T - i\,I)(\mathcal{D}_T)} \subset (\overline{T} - i\,I)(\mathcal{D}_{\overline{T}})$.

The inclusion $(\overline{T} - i\,I)(\mathcal{D}_{\overline{T}}) \subset \overline{(T - i\,I)(\mathcal{D}_T)}$ is similar but easier. $\square$

26.5. ***Corollary***. *Suppose T is symmetric and λ is a complex number with nonzero imaginary part.*

Then $\overline{\mathcal{R}_{T-\lambda\,I}} = \mathcal{R}_{\overline{T}-\lambda\,I}$.

Proof. Suppose T is symmetric and $\lambda = r + s\,i$ is a complex number with nonzero imaginary part s. Then $\frac{1}{s}\,T - \frac{r}{s}\,I$ is symmetric. By Lemma 26.4 we have

$$\overline{\mathcal{R}_{T-\lambda\,I}} = \overline{\mathcal{R}_{\frac{1}{s}\,T-\frac{r}{s}\,I-i\,I}} = \mathcal{R}_{\overline{\frac{1}{s}\,T-\frac{r}{s}\,I}-i\,I} = \mathcal{R}_{\overline{T}-\lambda\,I}.$$

$\square$

26.6. ***Lemma***. ***Test for Self-Adjointness***

Symmetric T is self-adjoint exactly when its Cayley transform $C(T)$ is unitary.

Proof. If T is self-adjoint then it is closed. By Lemma 26.2 so are $\mathcal{R}_{T+i\,I}$ and $\mathcal{R}_{T-i\,I}$. Also, $\mathcal{R}^{\perp}_{T+i\,I} = Ker(T^{\dagger} - i\,I) = Ker(T - i\,I) = \{\,0\,\}$ since symmetric operators cannot have non-real eigenvalues. So $\mathcal{R}_{T+i\,I}$ is dense in $\mathcal{H}$ and so *is* $\mathcal{H}$. $\mathcal{R}_{T-i\,I}$ is found to be $\mathcal{H}$ by similar means. Since $C(T)$ is an isometry onto $\mathcal{H}$ it is unitary.

On the other hand suppose $C(T)$ is unitary. That means $\mathcal{R}_{T+i\,I} = \mathcal{R}_{T-i\,I} = \mathcal{H}$. Suppose that $x \in \mathcal{D}_T$ and $y \in \mathcal{D}_{T^\dagger}$ and select $w \in \mathcal{D}_T$ so that $(T - i\,I)(w) = (T^{\dagger} - i\,I)(y)$. Now we have

$$\begin{aligned}\langle\,(T + i\,I)(x)\,,\,y\,\rangle &= \langle\,x\,,\,(T^{\dagger} - i\,I)(y)\,\rangle = \langle\,x\,,\,(T - i\,I)(w)\,\rangle \\ &= \langle\,(T^{\dagger} + i\,I)(x)\,,\,w\,\rangle = \langle\,(T + i\,I)(x)\,,\,w\,\rangle.\end{aligned}$$

Since $\mathcal{R}_{T+i\,I} = \mathcal{H}$ this means $y = w$, so $y \in \mathcal{D}_T$; so $\mathcal{D}_{T^\dagger} \subset \mathcal{D}_T$ and $T = T^{\dagger}$. $\square$

Suppose T is self-adjoint with associated unitary operator A on $\mathcal{H}$. Then

$$A = C(T) = (T - i\,I)(T + i\,I)^{-1} \colon \mathcal{R}_{T+i\,I} = \mathcal{H} \to \mathcal{R}_{T-i\,I} = \mathcal{H}$$

while T itself is given by

$$T = K(A) = -i\,(A + I)(A - I)^{-1} \colon \mathcal{R}_{A-I} \to \mathcal{R}_{A+I}.$$

Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{C}$. Then if $\lambda = a + bi \in \mathbb{S}^1 - \{1\}$ and $p \in \mathbb{R}$ define

$$m(\lambda) = -i\,\frac{\lambda+1}{\lambda-1} = \frac{b}{a-1} \quad \text{and} \quad g(p) = \frac{p-i}{p+i} = \frac{p^2-1}{p^2+1} - \frac{2p}{p^2+1}\,i.$$

Obviously m and g are related to the inverse Cayley and Cayley transforms, respectively. And $m(g(p)) = p$ and $g(m(\lambda)) = \lambda$.

Suppose p is an eigenvalue for T with eigenvector x and define the vector y to be the nonzero multiple $y = (T + i\,I)\,x = (p+i)\,x$. Then

$$A(y) = (T - i\,I)(T + i\,I)^{-1}y = (T - i\,I)\,x = (p-i)\,x = \frac{p-i}{p+i}\,y.$$

In other words the vector y (and also, of course, x) is an eigenvector for A for eigenvalue $g(p)$.

If λ is an eigenvalue for A with eigenvector x then, since $\lambda \neq 1$ for Cayley transforms, define the vector y to be the nonzero multiple $y = (A - I)\,x = (\lambda - 1)\,x$. Then

$$T(y) = -i\,(A+I)(A-I)^{-1}y = -i\,(A+I)\,x = (\lambda+1)\,x = -i\,\frac{\lambda+1}{\lambda-1}\,y.$$

In other words the vectors y and x are eigenvectors for T for eigenvalue $m(\lambda)$. We see, for example, that $Ker(T)$ is the eigenspace for A for eigenvalue -1.

Similar results hold for elements of the continuous spectrum of these operators.

Suppose x_n is a sequence of unit vectors for which $(T - p\,I)(x_n) \to 0$ where the x_n are *not* eigenvectors for p and T. The real number p still might be an eigenvalue, but if not it is in the continuous spectrum of T. In any case, $T(x_n) = p\,x_n + z_n$ where $z_n \to 0$ and this sequence demonstrates that $T - p\,I$ is *not* bounded below so the resolvent function for T at p is *not* continuous.

Since $T + i\,I$ is bounded below (its inverse is continuous) it follows that there is a positive number a for which $a \le \|y_n\|$, where $y_n = (T + i\,I)\,x_n = p\,x_n + i\,x_n + z_n = (p+i)\,x_n + z_n$ for all n.

$$\begin{aligned} A\,y_n &= (T - i\,I)\,x_n = T\,x_n - i\,x_n = (p\,x_n + z_n) - i\,x_n = (p-i)\,x_n + z_n \\ &= (p-i)\,\frac{y_n - z_n}{p+i} + z_n = \frac{p-i}{p+i}\,y_n - \frac{p-i}{p+i}\,z_n + z_n. \end{aligned}$$

This implies that $\left(A - \frac{p-i}{p+i}\,I\right) y_n \to 0$ and since $\|y_n\|$ is bounded below by positive a for every n

$$(A - g(p)\,I)\,\frac{y_n}{\|y_n\|} \to 0.$$

So the sequence $y_n/\|y_n\|$ of unit vectors demonstrates that $A - g(p)\,I$ is *not* bounded below so the resolvent function for A at $g(p)$ is *not* continuous.

Now suppose $\lambda \neq 1$ and x_n is a sequence of unit vectors for which $(A - \lambda\,I)x_n \to 0$. So $A\,x_n = \lambda\,x_n + z_n$ where $z_n \to 0$. Define, for each n, the vector y_n by

$$y_n = (A - I)x_n = \lambda\,x_n - x_n + z + n = (\lambda - 1)x_n + z_n.$$

Since $\lambda \neq 1$ we may assume (by shifting x_n to start later in the sequence, if necessary, so that z_n is small enough) that $\|y_n\| \geq a$ for all n and some positive a.

$$\begin{aligned} T\,y_n &= -\,i\,(A+I)\,x_n = -i\,A\,x_n - i\,x_n = -i\,(\lambda\,x_n + z_n) - i\,x_n = -i\,(\lambda+1)\,x_n - i\,z_n \\ &= -\,i\,(\lambda+1)\,\frac{y_n - z_n}{\lambda - 1} - i\,z_n = -i\,\frac{\lambda+1}{\lambda-1}\,y_n + i\,\frac{\lambda+1}{\lambda-1}\,z_n - i\,z_n. \end{aligned}$$

As above, we have $\left(T + i\,\frac{\lambda+1}{\lambda-1}\,I\right)y_n = (T - m(\lambda)\,I)\,y_n \to 0$. So the resolvent function for T fails to be continuous at $m(\lambda)$.

It remains to consider the possibility that $1 \in \sigma(A)$. It is not possible for 1 to be an eigenvalue for A, but it may be part of the continuous spectrum. If it is, select sequence x_n of unit vectors demonstrating this: $y_n = (A - I)x_n \to 0$. But then

$$T\,y_n = -i\,(A+I)\,x_n = -i\,A\,x_n - i\,x_n = -i\,y_n - 2i\,x_n.$$

Since $y_n \to 0$ the magnitude of the far right side converges to 2, which shows that T is unbounded.

On the other hand, if we assume T to be unbounded, with unit vectors x_n and $\|T(x_n)\| \to \infty$ chosen so that $\|T(x_n)\| > 2$ for all n. Define $y_n = (T + i\,I)\,x_n$ as before. In this case $\|y_n\| \to \infty$. Let $w_n = y_n/\|y_n\|$.

$$(A - I)\,w_n = \frac{A\,y_n - y_n}{\|y_n\|} = \frac{(T - i\,I)\,x_n - (T + i\,I)\,x_n}{\|y_n\|} = \frac{-2i\,x_n}{\|y_n\|} \to 0.$$

So 1 is in the continuous spectrum of A. We note that since the spectrum of A is closed, if 1 is *not* in the spectrum of A there must be an "interval" on the unit circle in C around 1 disjoint from that spectrum, and the points of the spectrum nearest to 1 provide upper and lower bounds (through the function m) of the interval in $\mathbb{R}$ within which the spectrum of T is contained. In that case the domain of the resolvent function $\mathcal{R}_1(A)$ is all of $\mathcal{H}$, and that is the domain of T.

Let's recapitulate some of the facts we have accumulated.

26.7. ***Theorem***. *Suppose T is self-adjoint with Cayley transform A. Then $\sigma(T) \subset \mathbb{R}$ and $\sigma(A) \subset \mathbb{S}^1$.*

Eigenvectors for T and A coincide, with eigenvalues related by the functions g and m above. Generally, $g(\,\sigma(T)\,) = \sigma(A) - \{\,1\,\}$ and $m(\,\sigma(A) - \{\,1\,\}\,) = \sigma(T)$.

1 may be in the continuous spectrum of A but 1 cannot be an eigenvalue for A. T is bounded exactly when $1 \notin \sigma(A)$.

If T is bounded and λ_- and λ_+ are those points in the spectrum of A nearest to 1 with non-positive and non-negative complex parts, respectively, then the spectrum of T is contained in the interval $[m(\lambda_+),\, m(\lambda_-)]$.

Proof. Everything but the last remark has been dealt with in the comments preceding the theorem. Consider $m(a + b\,i) = \frac{b}{1-a}$ defined on $\mathbb{S}^1 - \{\,1\,\}$. m can be split into two functions: the part with negative $b = -\sqrt{1-a^2}$ given by

$$f\colon [-1, 1) \to (-\infty, 0], \quad f(a) = -\sqrt{\frac{1+a}{1-a}}\,, \qquad f' \text{ is negative on } (-1, 1)$$

and the part with positive $b = \sqrt{1-a^2}$ given by

$$h\colon [-1,1) \to [0,\infty), \quad h(a) = \sqrt{\frac{1+a}{1-a}}\,, \qquad h' \text{ is positive on } (-1,1).$$

The comment about the spectrum of T is now immediate. □

27. When is an Operator Essentially Self-Adjoint?

We will now delve into various means by which we can show a symmetric operator to be self-adjoint, or essentially self-adjoint.

We deal with the easiest case first.

27.1. ***Proposition***. ***Test for Essential Self-Adjointness.***
Suppose T is a symmetric operator and there is a real number α for which $\mathcal{R}_{\overline{T}-\alpha I} = \mathcal{H}$. Then $\overline{T}$ is self-adjoint.

PROOF. Our goal below is to show that $\mathcal{D}_{T^\dagger} \subset \mathcal{D}_{\overline{T}}$ and, since the other containment is implied by the symmetry of $\overline{T}$, we will have $\overline{T} = T^\dagger$.

Assume the conditions on $\overline{T}$ and α. If $y \in \mathcal{D}_{T^\dagger}$ then since $\mathcal{R}_{\overline{T}-\alpha I} = \mathcal{H}$ we can find $w \in \mathcal{D}_{\overline{T}}$ for which $(\overline{T} - \alpha I)(w) = (T^\dagger - \alpha I)(y)$. Then $\forall x \in \mathcal{D}_{\overline{T}}$ we have

$$\begin{aligned}\langle\, (\overline{T} - \alpha I)(x)\,,\, y\,\rangle &= \langle\, x\,,\, (T^\dagger - \alpha I)(y)\,\rangle \\ &= \langle\, x\,,\, (\overline{T} - \alpha I)(w)\,\rangle = \langle\, (\overline{T} - \alpha I)(x)\,,\, w\,\rangle.\end{aligned}$$

Since $\overline{T} - \alpha I$ is onto $\mathcal{H}$ this means $y = w$: that is, $y \in \mathcal{D}_{\overline{T}}$. □

We define the **deficiency subspaces $\boldsymbol{\mathcal{D}_-}$** and **$\boldsymbol{\mathcal{D}_+}$** by

$$\begin{aligned}\mathcal{D}_+ = \ (\,\mathcal{R}_{T-i\,I}\,)^\perp &= Ker\left(T^\dagger + i\,I\right) \\ \text{and}\ \ \mathcal{D}_- = (\,\mathcal{R}_{T+i\,I}\,)^\perp &= Ker\left(T^\dagger - i\,I\right).\end{aligned}$$

We will eschew a cumbersome but more precise reference to T, as in $\mathcal{D}_-(T)$ and $\mathcal{D}_+(T)$, whenever possible.

The **deficiency indices $\boldsymbol{n_-}$** and **$\boldsymbol{n_+}$** are the Hilbert dimensions of the respective deficiency subspaces.

If T is closed and symmetric then the following are orthogonal direct sums:

$$\mathcal{H} = \mathcal{R}_{T+i\,I} \oplus \mathcal{D}_- \quad \text{and} \quad \mathcal{H} = \mathcal{R}_{T-i\,I} \oplus \mathcal{D}_+.$$

The Cayley transform $C(T)\colon \mathcal{R}_{T+i\,I} \to \mathcal{R}_{T-i\,I}$ is an isometry onto $\mathcal{R}_{T-i\,I}$ which moves every nonzero member of its domain, so an isometry from $\mathcal{D}_-$ onto $\mathcal{D}_+$ which moves every nonzero member of $\mathcal{D}_-$, if there are any, could be combined with $C(T)$ to produce a unitary operator U on $\mathcal{H}$ extending $C(T)$. Then $K(U)$ is a self-adjoint extension of T.

27.2. ***Theorem***. ***Test for the Existence of a Self-Adjoint Extension.*** *If T is symmetric then T has a self-adjoint extension if and only if the deficiency indices n_+ and n_- of T are equal. If $\mathcal{D}_+ = \mathcal{D}_- = \{ 0 \}$ then $\overline{T}$ itself is self-adjoint. Otherwise, the distinct self-adjoint extensions of T correspond to those isometries from $\mathcal{D}_-$ onto $\mathcal{D}_+$ which leave no point of $\mathcal{D}_-$ except 0 unmoved. The association is through the inverse Cayley transform.*

Proof. The argument follows from the preceding discussions. □

A **conjugation** on a Hilbert space $\mathcal{H}$ is a conjugate linear isometry whose square is the identity. The obvious example is complex conjugation applied to a function space such as $\mathcal{L}^2([0,1])$.

By an application of the polarization identity, we see that whenever Q is a conjugation

$$\langle x, y \rangle = \langle Q(y), Q(x) \rangle.$$

27.3. ***Theorem***. ***Test for the Existence of a Self-Adjoint Extension.*** *Suppose T is symmetric on complex Hilbert space $\mathcal{H}$ and there is a conjugation Q on $\mathcal{H}$ for which $Q(\mathcal{D}_T) \subset \mathcal{D}_T$ and with the property that Q commutes with T: that is, $T \circ Q(x) = Q \circ T(x) \, \forall x \in \mathcal{D}_T$. Then T has a self-adjoint extension.*

Proof. Since $\mathcal{D}_T = Q(\, Q(\mathcal{D}_T)\,)$ and $Q(\mathcal{D}_T) \subset \mathcal{D}(T)$ we have $Q(\mathcal{D}_T) = \mathcal{D}_T$.

If x is any member of $(\mathcal{R}_{T-i\,I})^{\perp} = \mathcal{D}_+$ and y is any member of $\mathcal{D}_T$ then

$$0 = \langle\, (T - i\,I)(y),\, x \,\rangle = \langle\, Q(x), Q \circ (\,(T - i\,I)(y)\,)\,\rangle = \langle\, Q(x),\, (T + i\,I)(Q(y))\,\rangle$$

Coupled with the remark of the last line we have $Q(x) \in (\mathcal{R}_{T+i\,I})^{\perp} = \mathcal{D}_-$.

Then Q restricted to $\mathcal{D}_+$ is an invertible conjugate linear isometry onto $\mathcal{D}_-$, and so the deficiency indices for T coincide. □

27.4. ***Exercise***. *If $g \in \mathcal{D}_- \cap \mathcal{D}_+$ then $g \in (\mathcal{D}_T)^{\perp} \cap (\mathcal{R}_T)^{\perp}$. So if T is symmetric (i.e. it satisfies the symmetry condition and has dense domain) any isometry of $\mathcal{D}_-$ onto $\mathcal{D}_+$ whatsoever satisfies the condition in Theorem 27.2, since there are no shared points to leave unmoved except 0.*

Test for Essential Self-Adjointness *So symmetric T is essentially self-adjoint if and only if the deficiency indices are both 0. If the indices are both 1 there is a "circle" of extensions of T: if a and b are unit vectors and $\mathcal{D}_- = \mathbb{C}\,a$ and $\mathcal{D}_+ = \mathbb{C}\,b$ then the linear map sending a to $e^{i\,t}b$ is an isometry for each fixed real t. If the indices are equal but exceed 1 the collection of isometries of $\mathcal{D}_-$ onto $\mathcal{D}_+$ is larger, and each one corresponds to a distinct self-adjoint extension of T.*

We now reprise some of the discussions specialized above for Cayley transforms to get (in some ways) a slightly more general result.

27.5. ***Theorem***. ***Test for Essential Self-Adjointness***
If T is symmetric and both $\mathcal{D}_{C_\alpha}$ and $\mathcal{D}_{C_{\overline{\alpha}}}$ are dense in $\mathcal{H}$ for some non-real complex number α then $\overline{T}$ is the unique self-adjoint extension of T.

Proof. Suppose first that symmetric T is closed and both domains are dense as indicated in the statement of this theorem.

Suppose also that $f_i = (T - \alpha I)(g_i) = T(g_i) - \alpha g_i$ is a Cauchy sequence in $\mathcal{D}_{C_\alpha}$ where g_i is a sequence in $\mathcal{D}_T$. Since C_α is an isometry, $C_\alpha(f_i) = (T - \overline{\alpha} I)(g_i) = T(g_i) - \overline{\alpha} g_i$ is also Cauchy. So the difference sequence

$$f_i - C_\alpha(f_i) = (T(g_i) - \alpha g_i) \;-\; (T(g_i) - \overline{\alpha} g_i) = (\overline{\alpha} - \alpha) g_i$$

is Cauchy and it follows that both g_i and $T(g_i)$ are Cauchy, and therefore converge to limits a and b, respectively, in $\mathcal{H}$.

Using the assumption that T is closed we conclude that $a \in \mathcal{D}_T$ and $b = T(a)$. Therefore $f_i = T(g_i) - \alpha g_i$ converges to $T(a) - \alpha a \in \mathcal{D}_{C_\alpha}$ and so $\mathcal{D}_{C_\alpha}$ is closed. Our assumption then that $\mathcal{D}_{C_\alpha}$ is dense implies that $\mathcal{D}_{C_\alpha} = \mathcal{H}$ and, similarly, $\mathcal{D}_{C_{\overline{\alpha}}} = \mathcal{H}$.

Now suppose that $f \in \mathcal{D}_{T^\dagger}$.

By definition of adjoint, $\left(T^\dagger - \alpha I\right)(f)$ is the unique member of $\mathcal{H}$ for which

$$\langle f , (T - \overline{\alpha} I)(h) \rangle = \langle \left(T^\dagger - \alpha I\right)(f) , h \rangle \qquad \forall h \in \mathcal{D}_T.$$

Because $(T - \alpha I)(\mathcal{D}_T) = \mathcal{D}_{C_\alpha} = \mathcal{H}$ there is a member g of $\mathcal{D}_T$ for which

$$(T - \alpha I)(g) = (T^\dagger - \alpha I)(f).$$

But then for every $h \in \mathcal{D}_T$

$$\begin{aligned}\langle f , (T - \overline{\alpha} I)(h) \rangle &= \langle \left(T^\dagger - \alpha I\right)(f) , h \rangle \\ &= \langle (T - \alpha I)(g) , h \rangle = \langle g , (T - \overline{\alpha} I)(h) \rangle\end{aligned}$$

where the last equality follows from symmetry of T.

Since $(T - \overline{\alpha} I)(\mathcal{D}_T) = \mathcal{D}_{C_{\overline{\alpha}}} = \mathcal{H}$ this means $f = g$. But $g \in \mathcal{D}_T$ by assumption. So $\mathcal{D}_{T^\dagger} \subset \mathcal{D}_T$. By the symmetry of T we know $\mathcal{D}_T \subset \mathcal{D}_{T^\dagger}$. So $T = T^\dagger$. Since T is closed $T^{\dagger\dagger} = T = T^\dagger$.

Now we remove the assumption that T is closed. Then $T \subset \overline{T} \subset T^\dagger$. The domain of $\overline{T}$ contains that of T so the domain of C_α calculated for $\overline{T}$ contains that of the similar transform found using T for each non-real α, so the density condition of this theorem for T implies the density condition for $\overline{T}$.

So $\overline{T} = T^\dagger$. Since $\overline{T}$ is the smallest possible extension of T which *could* be self-adjoint, and since there can be no proper self-adjoint extensions of a known self-adjoint extension, $\overline{T}$ is the unique self-adjoint extension of T. $\square$

27.6. ***Exercise***. *(i)* ***Test for Essential Self-Adjointness.*** *Review Exercise 21.9 and prove that if T is symmetric and there is any non-real complex number α such that neither α nor $\overline{\alpha}$ is an eigenvalue of $T^\dagger$ then T is essentially self-adjoint and $\overline{T}$ is self-adjoint. (hint: $Ker(T^\dagger - \overline{\alpha} I) = (\mathcal{D}_{C_\alpha})^\perp$.)*

(ii) If T is symmetric and there is a single non-real α for which neither α nor $\overline{\alpha}$ is an eigenvalue for $T^\dagger$ then no non-real β is an eigenvalue for $T^\dagger$.

(iii) If symmetric T fails to be essentially self-adjoint, at least one of β or $\overline{\beta}$ is an eigenvalue of $T^\dagger$ for every non-real β, and therefore at least one of β or $\overline{\beta}$ is in the residual spectrum of T.

(iv) ***Test for Essential Self-Adjointness.*** *Show that if T is symmetric, $\overline{T}$ is self-adjoint if and only if for every Cauchy sequence x_n of unit vectors in $\mathcal{D}_T$ for which $T(x_n)$ is also Cauchy, neither $(T+i\,I)(x_n)$ nor $(T-i\,I)(x_n)$ converges to 0.*

This is equivalent to the following: For symmetric T, $\overline{T}$ is self-adjoint if and only if for any Cauchy sequence x_n of vectors in $\mathcal{D}_T$ for which $T(x_n)$ is also Cauchy, if either $(T+i\,I)(x_n)\to 0$ or $(T-i\,I)(x_n)\to 0$ then $x_n\to 0$.

(v) In Lemma 22.6 we determined that a (possibly small) disk around every member of the resolvent set is also in the resolvent set. If λ is real and in the resolvent set then there will be some non-real α for which both α and $\overline{\alpha}$ are in the resolvent set. So we have the following:

Test for Essential Self-Adjointness. *If T is symmetric, $\overline{T}$ is self-adjoint if there is a real number λ in the resolvent set.*

27.7. ***Exercise***. *Show that if T is a closed symmetric operator then the spectrum of T must be one of the following four possibilities. a) The whole complex plane. b) The closed upper half-plane. c) The closed lower half-plane. d) A closed subset of $\mathbb{R}$. In case d) the operator is self-adjoint.*

We are now in a position to efficiently deal with a case that is very common, possibly even the most important case, in applications.

27.8. ***Theorem***. ***Test for Essential Self-Adjointness.***
Suppose T has the symmetry condition and $\mathcal{H}$ has an orthonormal basis B of eigenvectors of T, with eigenvalue λ_b for each $b\in B$.
Then the domain of T is dense and $\overline{T}$ is self-adjoint and $\sigma\left(\overline{T}\right)=\overline{\{\,\lambda_b\mid b\in B\,\}}$.

Proof. $\mathcal{D}_T$ must contain the dense set $span(B)$, the finite linear combinations of members of B, so T is symmetric and so too then is $\overline{T}$, defined on $\mathcal{D}_{\overline{T}}$. The orthonormal basis B is contained in $\mathcal{R}_{\overline{T}+i\,I}$ and $\mathcal{R}_{\overline{T}-i\,I}$, which are closed and dense and hence equal $\mathcal{H}$. By Theorem 27.5 $\overline{T}$ is self-adjoint, and we know that $\overline{T}$ must be contained in any possible self-adjoint extension of T so T is essentially self-adjoint.

Let L denote $\overline{\{\,\lambda_b\mid b\in B\,\}}$. The eigenvalues of T are all eigenvalues of $\overline{T}$ and the spectrum of any operator is closed, so $L\subset\sigma(\overline{T})$.

Suppose a is any number not in L. Suppose the distance from a to L is d. Given generic $x=\sum_{b\in B}a_b\,b$ define

$$F(x)=\sum_{b\in B}\frac{a_b}{\lambda_b-a}\,b.$$

The denominators in this formula are never smaller than d, so the operator norm of F does not exceed $1/d$. F is one-to-one and its range contains every member of B, so is dense in $\mathcal{H}$.

$$(T-a\,I)\circ F(b)=b=F\circ(T(b)-a\,b)\quad\forall b\in B.$$

So F is the resolvent $R_a(\overline{T})$, continuous and densely defined. So $a\notin\sigma\left(\overline{T}\right)$. □

We note here that the eigenvalues of a self-adjoint operator on an infinite dimensional Hilbert space could be dense in any subset (or all of) $\mathbb{R}$, so the spectrum could be any closed subset of $\mathbb{R}$.

28. Extending a Cayley Transform

Suppose T is closed and symmetric. We saw in Exercise 27.4 that the deficiency subspaces

$$\begin{aligned} \mathcal{D}_+ = \; & (\,\mathcal{R}_{T-i\,I}\,)^{\perp} = Ker\left(T^{\dagger} + i\,I\,\right) \\ & \text{and} \;\; \mathcal{D}_- = (\,\mathcal{R}_{T+i\,I}\,)^{\perp} = Ker\left(T^{\dagger} - i\,I\,\right). \end{aligned}$$

do not intersect. And of course we also have

$$\mathcal{H} = \mathcal{R}_{T+i\,I} \oplus \mathcal{D}_+ = \mathcal{R}_{T-i\,I} \oplus \mathcal{D}_- \quad \text{(orthogonal sums).}$$

Suppose that $y = a + b \in \mathcal{D}_T$ where $a \in \mathcal{D}_+$ and $b \in \mathcal{D}_-$.

$$\left(T^{\dagger} - i\,I\,\right)(y) = T^{\dagger}a - i\,a \quad \text{and} \quad \left(T^{\dagger} + i\,I\,\right)(y) = T^{\dagger}b + i\,b.$$

Adding the two left sides gives $2\,T(y)$ and the two right sides add to $T(y) + i\,(a+b)$ So $T(y) = i\,y$ which implies $y = 0$ and then $a = b = 0$ as well.

So the following sum is verified to be direct: $\mathcal{D}_T \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$ and all three summands are contained in $\mathcal{D}_{T^{\dagger}}$.

Now suppose z is a generic member of $\mathcal{D}_{T^{\dagger}}$.

Pick member $y \in \mathcal{D}_T$ and $a \in \mathcal{D}_+$ so that $\left(T^{\dagger} - i\,I\right)(z) = (T - i\,I)(y) + a$.

Note that $2i\,a = \left(T^{\dagger} + i\,I - 2i\,I\right)(a) = \left(T^{\dagger} - i\,I\right)(a)$ so we have, finally,

$$\left(T^{\dagger} - i\,I\right)\left(z - y + \frac{i}{2}\,a\right) = 0$$

So there is a $w \in \mathcal{D}_-$ with $w = z - y + \frac{i}{2}\,a$ and then $z = w + y - \frac{i}{2}\,a$.

This means that

$$\mathcal{D}_{T^{\dagger}} = \mathcal{D}_T \oplus \mathcal{D}_+ \oplus \mathcal{D}_-.$$

There is no reason to expect that these direct summands are orthogonal with respect to the usual inner product. However

$$\langle\, g\,,\,h\,\rangle_G = \langle\, g\,,\,h\,\rangle + \langle\, T^{\dagger}(g)\,,\,T^{\dagger}(h)\,\rangle \quad \forall g, h \in \mathcal{D}_{T^{\dagger}}$$

is an inner product on $\mathcal{D}_{T^{\dagger}}$ called the graph inner product. A sequence x_n is Cauchy with graph inner product exactly when both x_n and $T(x_n)$ are Cauchy with the usual inner product. Since $T^{\dagger}$ is closed, graph inner product makes $\mathcal{D}_{T^{\dagger}}$ into a Hilbert space.

With our assumptions (recall T is closed and symmetric) the three direct summands indicated above are graph closed.

And they are graph orthogonal. To see this suppose $x \in \mathcal{D}_T$, $y \in \mathcal{D}_+ = Ker(\,T^{\dagger} + i\,I\,)$ and $w \in \mathcal{D}_- = Ker(\,T^{\dagger} - i\,I\,)$.

$$\begin{aligned} \langle\, x\,,\,y\,\rangle_G &= \langle\, x\,,\,y\,\rangle + \langle\, T^{\dagger}(x)\,,\,T^{\dagger}(y)\,\rangle \\ &= \langle\, x\,,\,y\,\rangle + \langle\, T(x)\,,\,-i\,y\,\rangle = \langle\, x\,,\,y\,\rangle + \langle\, x\,,\,-i\,T^{\dagger}(y)\,\rangle \\ &= \langle\, x\,,\,y\,\rangle + \langle\, x\,,\,-i\,(-i)\,y\,\rangle = 0. \end{aligned}$$

An identical argument shows $\langle\, x\,,\, w\,\rangle_G = 0$, and

$$\begin{aligned}\langle\, w\,,\, y\,\rangle_G &= \langle\, w\,,\, y\,\rangle + \langle\, T^\dagger(w)\,,\, T^\dagger(y)\,\rangle\\ &= \langle\, w\,,\, y\,\rangle + \langle\, i\,w\,,\, -i\,y\,\rangle = \langle\, w\,,\, y\,\rangle - \langle\, w\,,\, y\,\rangle = 0.\end{aligned}$$

Let's examine these decompositions more closely in conjunction with extensions of the Cayley transform of T. Below it is convenient to use the more precise $\mathcal{D}_-(T)$ and $\mathcal{D}_+(T)$ for deficiency subspaces of T.

$$C(T)\colon \mathcal{R}_{T+i\,I} \to \mathcal{R}_{T-i\,I}$$

$$\mathcal{H} = \mathcal{D}_-(T) \oplus \mathcal{R}_{T+i\,I} = \mathcal{D}_+(T) \oplus \mathcal{R}_{T-i\,I} \quad \text{(orthogonal)}$$

$$\mathcal{D}_T \subset \mathcal{D}_{T^\dagger} = \mathcal{D}_T \oplus \mathcal{D}_+(T) \oplus \mathcal{D}_-(T) \quad \text{(graph orthogonal)}$$

$C(T)$ sends a vector of the form $T(f) + i\,f$ for $f \in \mathcal{D}_T$ to $T(f) - i\,f$.

One can visualize the creation of any isometry from a subspace of $\mathcal{D}_-(T)$ onto a subspace of $\mathcal{D}_+(T)$ one step at a time by sending an orthonormal basis element a of $\mathcal{D}_-(T)$ to an orthonormal basis element b of $\mathcal{D}_+(T)$ and extending $C(T)$ to an isometry $A\colon \mathcal{R}_{T+i\,I} \oplus \mathbb{C}\,a \to \mathcal{R}_{T-i\,I} \oplus \mathbb{C}\,b$ by linearity.

A sends a vector of the form $T(f) + i\,f + k\,a$ to $T(f) - i\,f + k\,b$.

The inverse Cayley transform $S = K(A) = -i\,(A+I)(A-I)^{-1}$ will be a symmetric extension of T with domain $\mathcal{R}_{A-I}$. A generic member of this domain is of the form

$$\begin{aligned}(A-I)(T(f) + i\,f + k\,a) &= T(f) - i\,f + k\,b - T(f) - i\,f - k\,a\\ &= -\,2i\,f + k(b-a) \quad \text{for } f \in \mathcal{D}_T,\ k \in \mathbb{C}.\end{aligned}$$

So extension $S = -i\,(A+I)(A-I)^{-1}$ acts on this domain member by

$$\begin{aligned}-2i\,f + k(b-a) &\longrightarrow T(f) + i\,f + k\,a\\ &\longrightarrow T(f) - i\,f + k\,b + T(f) + i\,f + k\,a = 2\,T(f) + k\,(b+a)\\ &\longrightarrow\ -\,2i\,T(f) - ik\,(b+a).\end{aligned}$$

Letting $g = -2i\,f$ we have $S(\,g + k\,(b-a)\,) = T(g) - ik\,(b+a)$.

So $\mathcal{D}_S = \mathcal{D}_T \oplus \mathbb{C}\,(b-a)$ and the range of S is $\mathcal{R}_S = \mathcal{R}_T \oplus \mathbb{C}\,(b+a)$

Each deficiency subspace $\mathcal{D}_+(S)$ and $\mathcal{D}_-(S)$ will lose one orthonormal basis vector, b and a respectively. So

$$\mathcal{D}_S \subset \mathcal{D}_{S^\dagger} = \mathcal{D}_S \oplus \mathcal{D}_+(S) \oplus \mathcal{D}_-(S)$$

has the right side losing two basis elements and gaining one, $b-a$. The left side gains one, $b-a$. So the "gap" between them is diminished by two basis elements compared to $\mathcal{D}_T \subset \mathcal{D}_{T^\dagger}$.

S is "closer" to being self-adjoint than was T, and

$$T \subset S \subset S^\dagger \subset T^\dagger.$$

29. Examples: Self-Adjointness

Here is an example of ***incompatible essentially self-adjoint extensions of a symmetric operator.***

Let D consist of the infinitely differentiable functions f defined on $[0,1]$ (one-sided derivatives at the endpoints) with Dirichlet boundary conditions: $f(0) = f(1) = 0$. Let N consist of the infinitely differentiable functions f defined on $[0,1]$ with Neumann boundary conditions: $f'(0) = f'(1) = 0$. Finally, let M consist of the infinitely differentiable functions f defined on $[0,1]$ with mixed boundary conditions: $f(0) = f'(1) = 0$.

Define operators T_D and T_N on D and N respectively by the same formula

$$T_D(f) = -f'' \ \forall f \in D, \qquad T_N(f) = -f'' \ \forall f \in N \qquad T_M(f) = -f'' \ \forall f \in M.$$

As in the example on page 128, all three domains may be considered as a dense subset of $\mathcal{L}^2([0,1])$, and with that assumption these operators extend T_C, which was shown to be *not* essentially self-adjoint by explicitly producing a point on the graph of $T_C^\dagger$ which was not in $\overline{T_C}$.

$$\begin{aligned}\langle\, T_D(f)\,,\, g\,\rangle &= \int_0^1 -f''(x)\,\overline{g(x)}\,dx = -f'(x)\overline{g(x)}\Big|_0^1 + \int_0^1 f'(x)\,\overline{g'(x)}\,dx \\ &= \, -f'(x)\overline{g(x)}\Big|_0^1 + f(x)\overline{g'(x)}\Big|_0^1 - \int_0^1 f(x)\,\overline{g''(x)}\,dx = \ \langle\, f\,,\, T_D(g)\,\rangle.\end{aligned}$$

So T_D is symmetric, and the same calculation reveals that both T_N and T_M are symmetric as well.

Let's examine $(T_D + i\,I)(\mathcal{D}_{T_D})$. For integer n we find that $(T_D + i\,I)(\,\sin(n\pi x)\,) = (n^2\pi^2 + i)\sin(n\pi x)$ and the finite linear combinations of functions of this type form a dense subset of D, and hence of $\mathcal{L}^2([0,1])$ itself. Also finite linear combinations of functions of the form $(T_D - i\,I)(\,\sin(n\pi x)\,) = (n^2\pi^2 - i)\sin(n\pi x)$ are dense. Therefore $\overline{T_D}$ is self-adjoint.

Similar statements involving $(T_N + i\,I)(\,\cos(n\pi x)\,) = (n^2\pi^2 + i)\cos(n\pi x)$ and $(T_N - i\,I)(\,\cos(n\pi x)\,) = (n^2\pi^2 - i)\cos(n\pi x)$ allow us to conclude that T_N is also essentially self-adjoint.

$$(T_M + i\,I)\left(\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)\right) = \left(\left(n+\frac{1}{2}\right)^2\pi^2 + i\right)\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)$$

and

$$(T_M - i\,I)\left(\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)\right) = \left(\left(n+\frac{1}{2}\right)^2\pi^2 - i\right)\sin\left(\left(n+\frac{1}{2}\right)\pi x\right).$$

So T_M is essentially self-adjoint too, subject to verification that the span of the relevant functions form a dense set in M.

Appealing to Exercise 27.6 and referring again to the example on page 128 we can also conclude that for every non-real α *at least one of* α *or* $\overline{\alpha}$ is an eigenvalue of $T_C^\dagger$ but *neither* α *nor* $\overline{\alpha}$ is an eigenvalue of $T_N^\dagger$, $T_D^\dagger$ or $T_M^\dagger$.

And **we have found an example for which different self-adjoint extensions have different spectra.** In this case the spectra of T_D and T_N consists of the numbers $n^2\pi^2$ for integer n (we appeal here to Theorem 27.8) while the spectrum of T_M consists of the numbers $\left(n+\frac{1}{2}\right)^2\pi^2$.

29.1. ***Lemma***. *If S is symmetric and $T \subset S$ and $(T + i\,I)(\mathcal{D}_T) = (S + i\,I)(\mathcal{D}_S)$ then $T = S$.*

Proof. Suppose $(S+i\,I)(f)$ for $f \in \mathcal{D}_S$ is some member of the range of $S+i\,I$. Since the ranges of $S + i\,I$ and $T + i\,I$ are equal, there must be a $g \in \mathcal{D}_T$ for which $(T + i\,I)(g) = (S + i\,I)(f)$. But S extends T so $(S + i\,I)(g) = (S + i\,I)(f)$. Since S is symmetric $S + i\,I$ is one-to-one so $f = g$. That means $\mathcal{D}_S \subset \mathcal{D}_T$. □

Finally, we look at an example of ***a symmetric operator with no self-adjoint extension at all.***

We will let our Hilbert space be $\boldsymbol{\ell^2}$, the space of square summable sequences with orthonormal basis e_n given by $e_n(j) = 1$ if $j = n$ and $e_n(j) = 0$ otherwise, for $n \geq 0$.

Define $s_n = e_n - e_{n+1}$ for $n \geq 0$. The set of these s_n is linearly independent, and if $f \in \boldsymbol{\ell^2}$ and $\langle\, f,\, s_n \,\rangle = 0$ for all n it is pretty easy to show that f is the zero sequence.

So the vector space formed from the finite linear combinations of these s_n, which we will denote $\mathcal{D}_T$, is dense in $\boldsymbol{\ell^2}$.

We define T on $\mathcal{D}_T$ by $T(s_n) = i\,e_n + i\,e_{n+1}$, extending by linearity.

A quick calculation verifies that $\langle\, T(s_n),\, s_m \,\rangle$ and $\langle\, s_n,\, T(s_m) \,\rangle$ are equal, and in fact both inner products are i if $m = n+1$ and $-i$ if $m = n-1$ and 0 otherwise.

So T is symmetric.

$(T + i\,I)(s_n) = 2\,i\,e_n$ so the range of $T + i\,I$ is dense. So (Lemma 26.4) we find that $(\overline{T} + i\,I)(\mathcal{D}_{\overline{T}}) = \boldsymbol{\ell^2}$.

However $(T - i\,I)(s_n) = 2\,i\,e_{n+1}$ which means that $e_1 \in (\,(T - i\,I)(\mathcal{D}_T)\,)^{\perp}$. i is in the residual spectrum $\sigma_r(T)$ of T.

So $e_1 \in \left(\,(\overline{T} - i\,I)(\mathcal{D}_T)\,\right)^{\perp}$, by Lemma 26.4. So $(\overline{T} - i\,I)(\mathcal{D}_T)$ is not dense in $\boldsymbol{\ell^2}$ so $\overline{T}$ is not self-adjoint. Also, Theorem 27.2 tells us that this operator has no self-adjoint extension.

If S is symmetric closed extension of T then the range of $S+i\,I$ would necessarily contain $(\overline{T} + i\,I)(\mathcal{D}_{\overline{T}}) = \boldsymbol{\ell^2}$, so $(S + i\,I)(\mathcal{D}_{\overline{T}}) = (\overline{T} + i\,I)(\mathcal{D}_{\overline{T}})$. By Lemma 29.1 we then have $S = \overline{T}$: in other words, $\overline{T}$ cannot even have a nontrivial *symmetric* extension. That fact could also be deduced by appeal to Theorem 27.2.

30. The Friedrichs Extension

We now consider a condition for which a symmetric operator does have at least one self-adjoint extension.

We say operator T is **semi-bounded** (from below) if there is a real constant c for which

$$\langle\, T(g)\,,\, g\,\rangle \geq c\,\langle\, g\,,\, g\,\rangle \quad \forall g \in \mathcal{D}_T$$

and **positive** if c can be chosen to be 0 and **positive definite** if it is positive and $\langle\, T(g)\,,\, g\,\rangle = 0$ only when $g = 0$. If $c > 0$ we will call T **strongly positive**. We will refer to the greatest such c as **the lower semi-bound for T**.

$\langle\, T(g)\,,\, g\,\rangle$ is assumed to be real in this definition for all g, so semi-bounded operators are symmetric. That means the approximate point spectrum is a subset of the real numbers. But even more, the approximate point spectrum must be in $[c, \infty)$. To see this we select $\alpha \in \sigma_{ap}$. Then there is a sequence x_n of unit vectors in $\mathcal{D}_T$ with $\langle\, T(x_n)\,,\, x_n\,\rangle \to \alpha$. But by the semi-bound condition $\langle\, T(x_n)\,,\, x_n\,\rangle \geq c$ for each n and so $\alpha \in [c, \infty)$.

Note the distinction between semi-bounded and bounded below: T is bounded below if there is a *positive* c for which

$$\langle\, T(g)\,,\, T(g)\,\rangle \geq c\,\langle\, g\,,\, g\,\rangle \quad \forall g \in \mathcal{D}_T.$$

This latter condition is used for different purposes than the one we deal with now; for instance, T has bounded inverse exactly when it is bounded below.

In any event, if T is *semi-bounded* with constant c then $\langle\, T(g) - c\,g\,,\, g\,\rangle \geq 0$ and in fact if I is the identity operator,

$$\langle\, T(g) - c\,g + g\,,\, g\,\rangle = \langle\, (\,T - (c-1)\,I\,)(g)\,,\, g\,\rangle \geq \langle\, g\,,\, g\,\rangle \quad \forall g \in \mathcal{D}_T.$$

For this reason, many facts involving semi-bounded operators with any lower semi-bound are easy consequences of similar facts about positive operators, or semi-bounded operators with lower semi-bound 1.

In an attempt to find a self-adjoint extension for symmetric S we are looking for operator T with $\overline{S} \subset T = T^\dagger \subset S^\dagger$.

$\overline{S}$ is one possibility, and the domain of $\overline{S}$ can be described using a modified inner product, the graph inner product, on $\mathcal{D}_S$.

$$\langle\, g\,,\, h\,\rangle_G = \langle\, g\,,\, h\,\rangle + \langle\, S(g)\,,\, S(h)\,\rangle \quad \forall g, h \in \mathcal{D}_S.$$

Any Cauchy sequence in $\mathcal{D}_S$ using this inner product—we will call such sequences G-Cauchy—is also Cauchy with the Hilbert space inner product, and will converge in both senses to the same limit in $\mathcal{D}_{\overline{S}}$. Conversely, every member of $\mathcal{D}_{\overline{S}}$ is the limit of a G-Cauchy sequence. And the values of $\overline{S}$ on such a limit are the limits of the S-values on any G-Cauchy sequence in $\mathcal{D}_S$ converging to that limit vector.

Limits of G-Cauchy sequences not already in $\mathcal{D}_S$ are exactly the members of $\mathcal{D}_{\overline{S}}$ not already in $\mathcal{D}_S$. In other words, $\mathcal{D}_{\overline{S}}$ is the G-completion of $\mathcal{D}_S$.

In many cases $\overline{S}$ is not self-adjoint, and a self-adjoint extension of S, if one exists, will have larger domain. Our goal below is to find such an extension in a special case commonly found in applications.

Suppose now that S is not just symmetric but also semi-bounded with constant 1. Any operator with positive lower semi-bound must be one-to-one, and that is the case here.

First, we define the Friedrichs inner product and norm on $\mathcal{D}_S$ as follows.

$$\langle\, g\,,\, h\,\rangle_F = \langle\, S(g)\,,\, h\,\rangle \quad \text{and} \quad \|\, g\,\|_F = \sqrt{\langle\, g\,,\, g\,\rangle_F} \qquad \forall g, h \in \mathcal{D}_S.$$

Both the positive lower semi-bound constant and symmetry are needed to show that this is, indeed, an inner product on $\mathcal{D}_S$.

Now suppose a sequence g_n is in $\mathcal{D}_S$. Using the lower semi-bound constant and the BCS inequality,

$$\begin{aligned}\|\, g_n - g_m\,\|^2 \le \|\, g_n - g_m\,\|_F^2 &= \langle\, S(g_n - g_m)\,,\, g_n - g_m\,\rangle \\ &\le \|\, S(g_n - g_m)\,\|\,\|\, g_n - g_m\,\|.\end{aligned}$$

If the sequence g_n is G-Cauchy, both g_n and $S(g_n)$ are Cauchy. So the right side of this inequality can be made small by choosing m and n large enough. Then $\|\, g_n - g_m\,\|_F$ can also be made small by choosing m and n large enough. In other words, a G-Cauchy sequence is also F-Cauchy.

Looking at the left side of the inequality, we see that every F-Cauchy sequence is Cauchy in $\mathcal{H}$, and therefore converges in both senses to the same member of $\mathcal{H}$.

So if $\mathcal{K}$ is the F-completion of $\mathcal{D}_S$ we have

$$\mathcal{D}_S \subset \mathcal{D}_{\overline{S}} \subset \mathcal{K} \subset \overline{\mathcal{D}_S} = \mathcal{H}.$$

We extend the Friedrichs inner product to all of $\mathcal{K}$ by continuity. In particular, we note that since $\langle\, g\,,\, g\,\rangle_F = \langle\, S(g)\,,\, g\,\rangle \ge \langle\, g\,,\, g\,\rangle$ for all $g \in \mathcal{D}_S$ we also have $\langle\, g\,,\, g\,\rangle_F \ge \langle\, g\,,\, g\,\rangle$ for all $g \in \mathcal{K}$.

Then $\mathcal{K}$ is itself a Hilbert space with Fredrichs inner product, and by definition $\mathcal{D}_S$ is F-dense in $\mathcal{K}$.

For each $y \in \mathcal{H}$ the linear function

$$A_y(\cdot) = \langle\, \cdot\,,\, y\,\rangle \colon \mathcal{D}_S \to \mathbb{C}$$

has F-operator norm bound $\|\, y\,\|$, as calculated below:

$$|\, A_y(x)\,| = |\,\langle\, x\,,\, y\,\rangle\,| \le \|\, x\,\|\,\|\, y\,\| \le \|\, x\,\|_F\,\|\, y\,\|.$$

Since $\mathcal{D}_S$ is F-dense in $\mathcal{K}$, A_y corresponds to F-inner product against a unique member $w_y \in \mathcal{K}$: $A_y(\cdot) = \langle\, \cdot\,,\, w_y\,\rangle_F$, and w_y has F-norm not exceeding $\|\, y\,\|$.

We define $B(y) = w_y$ for every $y \in \mathcal{H}$. Then $B\colon \mathcal{H} \to \mathcal{K}$ is easily seen to be linear; $B(y)$ is the unique member of $\mathcal{K}$ for which

$$\langle\, x\,,\, B(y)\,\rangle_F = \langle\, x\,,\, y\,\rangle \quad \forall x \in \mathcal{D}_S.$$

We will now accumulate some properties of B.

If $x \in \mathcal{K}$ and $y \in \mathcal{H}$ and $B(y) = 0$ then

$$\langle\, x\,,\, 0\,\rangle = \langle\, x\,,\, 0\,\rangle_F = \langle\, x\,,\, B(y)\,\rangle_F = \langle\, x\,,\, y\,\rangle$$

and the density of $\mathcal{K}$ in $\mathcal{H}$ implies $y = 0$. So B is one-to-one.

B is symmetric with respect to the original inner product, as seen below. For any $x, y \in \mathcal{H}$ we have

$$\langle\, B(x)\,,\, y\,\rangle = \langle\, B(x)\,,\, B(y)\,\rangle_F = \overline{\langle\, B(y)\,,\, B(x)\,\rangle_F} = \overline{\langle\, B(y)\,,\, x\,\rangle} = \langle\, x\,,\, B(y)\,\rangle.$$

So B is self-adjoint and, by the Hellinger-Toeplitz theorem, continuous.

Also B is positive because $\forall x \in \mathcal{H}$

$$\langle\, B(x)\,,\, x\,\rangle = \langle\, B(x)\,,\, B(x)\,\rangle_F \geq \langle\, B(x)\,,\, B(x)\,\rangle \geq 0.$$

Since $B = B^\dagger$ and $(\mathcal{R}_B)^\perp = Ker\left(B^\dagger\right) = Ker\,(B) = \{\,0\,\}$ we know that the subset $\mathcal{R}_B$ of $\mathcal{K}$ is dense in $\mathcal{H}$. So B^{-1}, which we will henceforth denote by Q, is a symmetric operator $Q\colon \mathcal{R}_B \to \mathcal{H}$ and is onto $\mathcal{H}$.

Note that if $x, y \in \mathcal{D}_S$ then

$$\langle\, x\,,\, y\,\rangle_F = \langle\, S(x)\,,\, y\,\rangle = \langle\, x\,,\, S(y)\,\rangle = \langle\, x\,,\, B \circ S(y)\,\rangle_F.$$

But then for each $y \in \mathcal{D}_S$, $\langle\, x\,,\, y - B \circ S(y)\,\rangle_F = 0\ \forall x \in \mathcal{D}_S$.

Since $\mathcal{D}_S$ is F-dense in $\mathcal{K}$ we have $y = B \circ S(y)\ \forall y \in \mathcal{D}_S$.

There are a couple of interesting conclusions to be drawn from this. First, all of $\mathcal{D}_S$ is in $\mathcal{R}_B = \mathcal{D}_Q$. And, second, $Q(y) = S(y)\ \forall y \in \mathcal{D}_S$.

We now know that Q is a symmetric extension of S, and since B is a closed operator, so too is its inverse Q.

Suppose $x = B(y)$ is a generic member of the domain of Q.

$$\begin{aligned}\langle\, Q(x)\,,\, x\,\rangle &= \langle\, x\,,\, Q(x)\,\rangle = \langle\, B(y)\,,\, y\,\rangle = \langle\, B(y)\,,\, B(y)\,\rangle_F \\ &\geq \langle\, B(y)\,,\, B(y)\,\rangle = \langle\, x\,,\, x\,\rangle.\end{aligned}$$

So Q is semi-bounded with constant 1, the same lower semi-bound as S.

Let $\psi\colon \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H}$ be the map that switches coordinates, $\psi(x, y) = (y, x)$. So ψ sends any one-to-one operator to its inverse, and in particular $\psi(B) = Q$ and $\psi(Q) = B$.

Recall the notation and result of Lemma 21.6 and observe that $J \circ \psi = -\psi \circ J$.

As subsets of $\mathcal{H} \times \mathcal{H}$ we have (since $B = B^\dagger$ and Q is closed)

$$\begin{aligned}Q = \ -Q &= -\psi(B) = -\psi(\,(\,J(\,B^\dagger\,)\,)^\perp\,) = -\psi(\,(\,J(\,B\,)\,)^\perp\,) \\ &= (\,J(\,Q\,)\,)^\perp = Q^\dagger.\end{aligned}$$

So Q is self-adjoint.

30.1. ***Theorem***. *A symmetric and semi-bounded operator has a self-adjoint extension that is also semi-bounded with the same lower semi-bound constant.*

PROOF. Suppose T is symmetric and semibounded with lower semi-bound constant c. Then $S = T - (c-1)\,I$ is *also* symmetric with lower semi-bound 1. According to the remarks before the theorem, there is a self-adjoint extension Q of S with lower semi-bound constant 1. And then $Q + (c-1)\,I$ is a self-adjoint extension of T with lower semi-bound constant c. $\square$

The semi-bounded self-adjoint operator of this theorem is called the **Friedrichs extension of T**.

Several additional sections (including the final result on the spectral theorem) will be included in the final version of these notes

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