TOPOLOGY

LARRY SUSANKA

Abstract. This is an appendix for a book I have (mostly) written on measure theory. It constitutes an introduction to point-set topology, oriented toward some applications to analysis.

It is an essentially self-contained presentation and rather thorough as far as it goes, though it does reference in a few places material of the rest of the text regarding notation, the Integers, the Axiom of Choice and other matters. Most of those references are to Chapter One "Some Preliminaries" which has been included here.

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1. Functions

A **relation**, or more precisely a **binary relation**, is a nonempty subset of some product set $S \times T = \{\, (s,t) \mid s \in S \text{ and } t \in T \,\}$, the set of **ordered pairs** formed from nonempty sets S and T. The **domain** of a relation f consists of the set of first components of any member of f, while the **range** consists of the set of second components, and these sets can be denoted $\mathbf{Domain(f)}$ and $\mathbf{Range(f)}$ respectively.

Relations are used to model many different ideas, but three basic kinds of relations will be of interest over the next few sections. The first of these is the familiar concept of "function."

A **function** f (sometimes also called a **map**) from S to T, described by $f\colon S \to T$, is a relation as above with domain S and range contained in (not necessarily equal to) T and which has the following property: $(a,b) \in f$ and $(a,c) \in f \Rightarrow b = c$.

These properties insure that there is one and only one ordered pair in f having a as first coordinate for every member a of S. Note that the concept of function is domain dependent: the same set of pairs thought of as a subset of $R \times T$, where S is contained in but not equal to R, won't be a function. And there is a certain latitude with regard to T: it can be replaced in the description $f\colon S \to T$ by any set containing $Range(f)$. Often, though, it is T itself under study and how $Range(f)$ sits (or could sit) in T reflects important information about T.

If f is a function, the notation $f(a)$ or f_a is used for $b \in T$ when $(a,b) \in f$. Occasionally a function will be said to **index** its range, and in this case the range is said to be **indexed by** the domain, whose members are called **indices**.

If A is a set, $f(A) = \{\, f(s) \mid s \in A \cap S \,\}$ and $f^{-1}(A) = \{\, s \in S \mid f(s) \in A \,\}$. Both sets can be empty. If $f^{-1}(\{t\})$ contains at most a single member of S for each $t \in T$ we call f **one-to-one** and if $f(S) = T$ we say f is **onto** T.

If f is one-to-one and onto T then f can be used to construct a function $f^{-1}\colon T \to S$ by defining $f^{-1}(t)$, for each $t \in T$, to be that member s of S with $f(s) = t$. This **second** definition of f^{-1} is an abuse of notation that could cause ambiguity in case, for example, both t and $\{t\}$ are members of T.

If f is one-to-one but not onto T then f cannot be used to define f^{-1} as a function from T to S. However f^{-1} would be a function thought of as the set of pairs $\{\, (f(s), s) \mid s \in S \,\}$ in $f(S) \times S$.

The **restriction** of a function $f\colon S \to T$ to a nonempty set $A \subset S$ is denoted $f|_A$ and defined to be $\{\, (a,b) \in f \mid a \in A \,\}$. $f|_A$ is a function with domain A. If g is a function with domain W and $S \subset W$ and $g|_S = f$ then g is called an **extension** of f.

Here are a few more items of notation:

The set of all functions from S to T will be denoted T^S.

If T is the two element set $\{0,1\}$, T^S will sometimes be denoted 2^S.

The set of all subsets of a set S will be denoted $\mathbb{P}(S)$, called the **power set** of S.

If A and X are any sets, the notation $X - A = \{\, x \in X \mid x \notin A \,\}$ is used. $X - A$ is called the **complement of** A **in** X.

If A and X are nonempty sets and $f\colon A \to \mathbb{P}(X)$ the notation $\bigcup_{a\in A} f_a$ denotes $\{\, x \in X \mid x \in f_a \text{ for some } a \in A \,\}$. The notation $\bigcap_{a\in A} f_a$ denotes $\{\, x \in X \mid x \in f_a \text{ for every } a \in A \,\}$.

Sets S and T with similar properties can arise from different sources. Recognizing that two sets are essentially the same in some way often comes through the presentation of a one-to-one function $g\colon S \to T$ that is onto T. When we have this in mind, we will say that S and T are **identified** and that g identifies the element $s \in S$ with the element $g(s) \in T$. The notation $s \leftrightarrow g(s)$ can be used to illustrate such an identification. These identifications can range in utility from a trivial convenience to something more substantial, a shift in context. For instance, on the trivial side, if n is a positive integer, the set $\{1, \ldots, n\}$ can be identified with $\{0, \ldots, n-1\}$ through the function described by $x \leftrightarrow x - 1$. More substantial examples of this vocabulary in action follow.

2^S can be identified with $\mathbb{P}(S)$ via $f \leftrightarrow \{\, a \in S \mid f(a) = 1 \,\}$.

Suppose $S_0, \ldots, S_{n-1}$ are nonempty sets for some integer $n > 2$. Define $S_0 \times S_1 \times S_2$ to be $S_0 \times (S_1 \times S_2)$. More generally, $S_0 \times \cdots \times S_{n-1}$ is given by a recursive definition as $S_0 \times (S_1 \times \cdots \times S_{n-1})$. This last is called the set of all "ordered n-tuples" formed from the S_i in the specified order. Let W denote the set of all functions $f\colon \{0, \ldots, n-1\} \to \bigcup_{i=0}^{n-1} S_i$ having the property that $f(i) \in S_i$ for $i = 0, \ldots, n-1$. Then $S_0 \times \cdots \times S_{n-1}$ can be identified with W by $(a_0, \ldots, a_{n-1}) \leftrightarrow f$ where $f(k) = a_k$ for $k = 0, \ldots, n-1$.

2. Equivalence Relations

Our second use of relations is the standard method used by mathematicians to lump together objects that are manifestly different but which are similar in some way. In this context we focus on the similarities and ignore other properties.

An **equivalence relation** on S is a relation $P \subset S \times S$ that has three properties:

$$
\begin{array}{ll}
(a,a) \in P \ \forall a \in S \quad \text{and} & (\textbf{reflexivity}) \\
(a,b) \in P \Rightarrow (b,a) \in P \quad \text{and} & (\textbf{symmetry}) \\
(a,b) \in P \text{ and } \ (b,c) \in P \Rightarrow (a,c) \in P. & (\textbf{transitivity})
\end{array}
$$

For equivalence relations, the notation $a \sim b$ is usually used when $(a, b) \in P$.

A **partition** of any set S is a set of subsets of S whose union is S and whose **pairwise** (that is, each pair of them) intersections are **void** (that is, the empty set.) Any pair of sets whose intersection is empty is called **disjoint**, and a union of pairwise disjoint sets is called a **disjoint union**.

After presenting an equivalence relation on S, one would typically form, for each a in S, sets $[a] = \{\, b \mid a \sim b \,\}$. These sets are called **equivalence classes** and together form a partition of S denoted $\mathbf{S/P}$ or $\mathbf{S/\!\sim}$. Often any member of an equivalence class will be used to refer to the whole class without comment, and it is the set of classes that is of primary interest.

Alternatively, any partition of a set S could be used to form an equivalence relation on the set, where $a \sim b$ precisely when a and b are in the same partition member.

Most people have seen equivalence relations from grade school. The rational numbers constitute a very important first example.

Let S be the set of all ordered pairs of integers represented as c/d where $d \neq 0$. (For a discussion of the construction of the integers, see Section 5.) We say that $c/d \sim e/f$ if and only if $cf = ed$. It is easy to show this is an equivalence relation. For each c/d in S let $[c/d] = \{\, e/f \in S \mid cf = ed \,\}$. The sets $[c/d]$ form a partition of S. Any ordered pair might be called upon to represent the whole class. That is what is meant by "$2/6 = 4/12$." The collection of these classes is normally referred to as the **rational numbers**, denoted $\mathbb{Q}$.

The operations $[a/b] + [c/d] = [(ad + bc)/(bd)]$ and $[a/b][c/d] = [(ac)/(bd)]$ are **well defined**.

In this context, "well defined" means that the operations, defined here using particular representations of the equivalence classes involved, do not in fact depend on which representative is used. **Statements of this kind, wherever found in the text, require proof.** If not obvious or proved in the text, the reader should supply the proof as an exercise, or simply accept the statement as true. Since all books contain errors, oversights, misstatements or infelicitous phrasing, the former course is the safer.

$\mathbb{Q}$ has multiplicative and additive identities $[b/b]$ and $[0/b]$ respectively, otherwise known as 1 and 0.

Another example is the usual representation of vectors in the plane as "arrows" with a given length and direction. One takes the point of view that a vector is determined by these two quantities alone and its location is irrelevant. So a vector is really a class of arrows that are alike in these two ways. One refers to the whole class by identifying any member of the class. In the world of vector operations such as vector addition or scalar multiplication the usual representative for a class is the arrow with tail at a specified origin, with coordinate axes centered there. With this choice the coordinates of the tip alone suffice to describe the class, and common vector operations are conveniently calculated.

The concept of equivalency, along with the companion concept of identification, can be seen throughout mathematics.

3. Order Relations

In this section we try to extract the essence of the idea of "less than" as thought of in the following three examples:

> 3 is said to be "less than" 7 on the number line because it is to the left when one represents the real numbers ordered as a line in the usual way.

> Consider a desk covered with many layers of paper. We might say one piece of paper is "less than or equal to" another if its distance to the table top is equal or less than the other: an ordering by "height above the table."
>
> We think of one set as "bigger than or equal to" another if it contains the other. Sets can be said to be **ordered by containment**, a very important example.

The relations we use to model these ideas are called **order relations**.

A **preorder** on a set S is a relation $P \subset S \times S$ that has the reflexivity and transitivity properties: $((a,a) \in P \ \forall a \in S)$ and $((a,b) \in P$ and $(b,c) \in P \Rightarrow (a,c) \in P)$.

The notation $a \leq b$ will be used to indicate that $(a,b) \in P$, while $a < b$ will indicate that $(a,b) \in P$ but $(b,a) \notin P$.

Suppose $B \subset S$. b is called an **upper bound** for B (in S if the specificity is warranted) if $b \in S$ and $a \leq b \ \forall a \in B$. If B has an upper bound, B is called **bounded above**. If, further, $c \in S$ and $a \leq c \ \forall a \in B \Rightarrow b \leq c$ then b is called a **least upper bound** for B. If b is a unique least upper bound for B (that is, the only one) then b is also called the **supremum** of B, denoted $\sup B$ or $\sup(B)$. The supremum of a set $\{a,b\}$, if it exists, is denoted $a \vee b$. An element b of S is called **maximal** if $c \in S$ and $b \leq c \Rightarrow c \leq b$.

A function $b\colon J \to S$ is called **bounded above** if its range is bounded above. The **supremum of a function** b is denoted $\sup(b)$ or $\bigvee_{\alpha\in J} b(\alpha)$ and is defined to be $\sup\{\, b(\alpha) \mid \alpha \in J \,\}$ whenever the supremum exists. In the case of $J = \mathbb{N}$, the nonnegative integers, a function $b\colon \mathbb{N} \to S$ is called a **sequence** in S. In this case the notation $\bigvee_{i=0}^{\infty} b(i)$ can be used in place of $\bigvee_{i\in\mathbb{N}} b(i)$. When $J = \{k, k+1, \ldots, n\}$ we may write $\bigvee_{i=k}^{n} b(i)$ rather than $\bigvee_{\alpha\in J} b(\alpha)$.

The definitions of $\geq$, $>$, **lower bound, bounded below, greatest lower bound, infimum,** $\inf B$, $\inf(B)$, $a \wedge b$, **minimal**, $\inf(b)$, $\bigwedge_{\alpha\in J} b(\alpha)$, $\bigwedge_{i=0}^{\infty} b(i)$ and $\bigwedge_{i=k}^{n} b(i)$ are the obvious adaptations of the list of definitions above with ordered pairs (that is, inequalities) reversed.

A set or function as above is called **bounded** if it is bounded both above and below. Otherwise, it is called **unbounded.**

Since functions are defined to *be* sets there is potential for ambiguity in the definitions just given involving functions, which focus on the order properties in the range alone. The ordered pairs in a function will not usually have an order specified for them so this will rarely be an issue.

The preorder P is called a **partial order** if, in addition to reflexivity and transitivity, we have:

$$a \leq b \text{ and } b \leq a \Rightarrow a = b.$$

A partial order can be created from any preorder on a set S by the following process: Let $a \sim b$ if and only if $a \leq b$ and $b \leq a$. Consider the set of equivalence classes $S/\!\sim$ generated by this equivalence relation. We will say $[a] \leq [b]$ precisely when $a \leq b$. This relation is well defined (that is, it does not depend on the representatives of the classes used to define it) and a partial order on $S/\!\sim$.

Note that if P is a partial order, least upper bounds and greatest lower bounds are unique if they exist. Also, in this case, if b is maximal, $a \in S$ and $b \leq a$ then $b = a$. A similar result holds if b is minimal.

If P is a preorder and $A \subset S$, A is called a **chain** if whenever a and b are in A then either $a \leq b$ or $b \leq a$.

If P is a preorder and for each pair a and b in S one can find c in S such that $a \leq c$ and $b \leq c$ we say that P **directs** S, and S with P is called a **directed set**.

```
          Preorder
        /    |    \
  Chain  Partial  Directed
     \   /   |   \   /
     Total  Tree  Lattice
         \   |   /
           Well
```

P, or sometimes S with P, is called a **lattice** if it is a partial order for which each pair of elements has a greatest lower bound and a least upper bound.

If S with partial order P is itself a chain, P is called a **total order**. In some contexts a total order is also called a **linear order**.

A total order on S which has the property that every nonempty subset of S contains a minimal element is called a **well order**.

For brevity, one often refers to S as "**ordered by** P," P is said to "**order** S" and S is said to "**have the order** P." This vocabulary is extended to the various types of orders. Sometimes, when this will not cause ambiguity, the set S will be said to be ordered and the specific order P will be understood to exist without explicit mention. As an example of this vocabulary in use we have the following interesting result:

Every subset of well ordered S is itself well ordered with the order **inherited** from S.

We define, for each α in preordered S, the sets $\mathbf{I}_\alpha$ and $\mathbf{T}_\alpha$ to be $\{\, \beta \in S \mid \beta < \alpha \,\}$ and $\{\, \beta \in S \mid \beta \geq \alpha \,\}$, respectively. They are called **initial** and **terminal segments** in S, respectively. Note that the T_α are distinct for different values of α in partially ordered S, but the I_α need not be distinct. They will be distinct if S is totally ordered.

If S is partially ordered and $\alpha, \beta \in S$ and $\beta > \alpha$ we say that β is **a successor** to α and α is **a predecessor** to β. If, further, there is no other member of S between α and β we say that β is **an immediate successor** to α and α is **an immediate predecessor** to β.

Suppose S is a generic well ordered set. Unless there is a compelling reason to deviate, it will be standard practice to denote the first element of S as 0 and the second member by 1. For $\alpha \in S$, we will use $\alpha + 1$ to denote the least successor to α. Unless α is the supremum of S, this immediate successor to α will always exist in well ordered S. $\alpha + 1$ is called **the successor** to α and α is called **the predecessor** to $\alpha + 1$. The vocabulary recognizes the fact that there can be at most one immediate successor or immediate predecessor in any totally ordered set. Any member of S **except the first** that cannot be written as $\alpha + 1$ for some α in S is called a **limit member** of S. A limit member has predecessors (many of them) but no *immediate* predecessor.

If S is partially ordered and if, for each $\alpha \in S$, the initial segment I_α is well ordered with the order inherited from S we call S with this order a **tree.** A **branch** of a tree is a nonempty subset B of S which is a chain with the induced order and which is maximal in the following sense: for each $s \in S$, either $s \in B$ or $\{s\} \cup B$ is not a chain.

A **root** of a tree S is a member r of S for which $S = T_r$. A tree S is called **rooted** if it has a unique root.

3.1**.** ***Exercise.*** *(i) Is it true that the intersection of two or more (distinct) branches in a rooted tree is an initial segment?*

(ii) If S is a tree and $r \in S$ then T_r is a rooted tree.

If S is well ordered and A is any union or any intersection of initial segments then A is either all of S or itself an initial segment. To see this, let α be the least member of S, if any, that is not in A. Then A is I_α. Similarly, if A is any union or intersection of terminal segments in well ordered S, then A is itself a terminal segment or void.

3.2**.** ***Exercise.*** *Suppose S is a well ordered set. Prove:*

$S = I_\alpha \cup T_\alpha$ *for every* α *in* S $\qquad I_0 = \varnothing \qquad T_0 = S$

T_α *is never empty* $\qquad I_{\alpha+1} = I_\alpha \cup \{\alpha\}$ *whenever* $\alpha + 1$ *is defined.*

A function $f\colon A \to B$ between two preordered sets is called **nondecreasing** if $f(\alpha) \le f(\beta)$ whenever $\alpha \le \beta$. f is called **increasing** if $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$. Neither condition implies that f is one-to-one. However if the order on A is a total order, the second condition does imply that f is one-to-one.

f is called **nonincreasing** if $f(\alpha) \ge f(\beta)$ whenever $\alpha \le \beta$. f is called **decreasing** if $f(\alpha) > f(\beta)$ whenever $\alpha < \beta$.

f is called **monotone** if it is nonincreasing or nondecreasing. If f is nondecreasing and f^{-1} exists and is nondecreasing, f is called an **order isomorphism** and A and B are said to be **order isomorphic**.

3.3**.** ***Exercise.*** *(i) In the definition of order isomorphism, if A is totally ordered, the requirement that f^{-1} be nondecreasing is redundant.*

(ii) Suppose $f\colon A \to B$ is nonincreasing, where A is totally ordered and B is well ordered. Then f is eventually constant: that is, there is an $a \in A$ for which $c \ge a$ implies $f(c) = f(a)$.

(iii) If S is well ordered, S is order isomorphic to $\{\, I_\alpha \mid \alpha \in S \,\}$ ordered by containment.

(iv) If S is partially ordered, S is order isomorphic to $\{\, \{\alpha\} \cup I_\alpha \mid \alpha \in S \,\}$ ordered by containment.

(v) If S is partially ordered, S is order isomorphic to $\{\, T_\alpha \mid \alpha \in S \,\}$ ordered by ***reverse containment****: that is, $T_\alpha < T_\beta$ if $T_\alpha \ne T_\beta$ and $T_\alpha \supset T_\beta$.*

Parts (iii) through (v) of the exercise show that any partial order—and well orders in particular—can be thought of as containment orders on families of subsets of S in several ways.

4. The Integers

We sketch in some detail the recognition of a set we will identify with the natural numbers, as you have come to know them from ordinary counting and grade-school arithmetic.

To get things started, the following assumption, to be accepted without proof, is required.

Axiom of the Empty Set:

> There exists a set, denoted $\varnothing$, which has no elements.

Without this (or some similar) assumption, we cannot conclude that there are any sets whatsoever! That would mean all our discussions about sets have been about nothing, a situation tailor-made for irony if ever there was one. We accept this axiom.

If X is a set, for now we will let X^* be the set $\{X\} \cup X$.

We let $0 = \varnothing,\ 1 = 0^*,\ 2 = 1^*,\ 3 = 2^*$ and so forth. Another way of writing this is: $0 = \varnothing,\ 1 = \{0\},\ 2 = \{0, 1\},\ 3 = \{0, 1, 2\}$ and so on.

To complete the definition of the natural numbers it is necessary to invoke another axiom of set theory, called the Axiom of Infinity. Essentially this axiom asserts that there exists an infinite set.

Axiom of Infinity:

> There exists a set A with $\varnothing \in A$ and such that whenever X is a set and $X \in A$ then $X^* \in A$.

Note that the intersection of any pair of sets of the type whose existence is guaranteed by this axiom is also of this type.

Let A be one of these sets. The **natural numbers**, denoted $\mathbb{N}$, consist of the intersection of all subsets S of A for which $\varnothing \in S$ and such that whenever X is a set and $X \in S$ then $X^* \in S$. In light of the last observation, $\mathbb{N}$ does not depend on the specific choice of A, only that there is at least one such set.

It is only because of the Axiom of Infinity that we know that $\mathbb{N}$, which we might have carelessly denoted $\{\, 0, 1, 2, 3, \dots \}$, is actually a set, and therefore eligible to participate in the various set operations.

The empty set is said to **have 0 elements**. If S is a nonempty set and n is a **positive integer** (that is, $n \in \mathbb{N}$ and $n \neq 0$) we say **S has n elements** if there is a one-to-one and onto function $f : S \to n$. We say S is **finite** if it has n elements for some $n \in \mathbb{N}$. S is called **infinite** if it is not finite. We will discuss this concept again in Sections **??** and **??**.

The natural numbers are ordered by containment. Henceforth, if $n \in \mathbb{N}$ we will use $\mathbf{n+1}$ in preference to n^*.

The definition of $\mathbb{N}$ is just what we need to create **Proof by Induction.** If we have some property P which is either true or false for members of $\mathbb{N}$, let $S = \{\, n \in \mathbb{N} \mid P \text{ is true for } n \,\}$. If $0 \in S$ and if $n \in S$ implies $n + 1 \in S$ then S is a set of the kind whose existence is asserted in the Axiom of Infinity. Since $\mathbb{N}$ is the intersection of all such sets, $S = \mathbb{N}$. In other words, we can conclude that P is true for every member of $\mathbb{N}$.

As an exercise using induction, show that every member of $\mathbb{N}$ except 0 can be written as $n+1$ for some $n \in \mathbb{N}$.

Then show that $S = \{\, n \in \mathbb{N} \mid n \subset k \text{ or } k \subset n \text{ for all } k \in \mathbb{N} \,\}$ actually equals $\mathbb{N}$. This implies that $\mathbb{N}$ with containment order is totally ordered.

Define B to be the set:

$$\{\, n \in \mathbb{N} \mid \text{ if } k \in S \subset \mathbb{N} \text{ for some } k \subset n \text{ then } S \text{ contains a least member.} \,\}$$

Obviously $\varnothing \in B$ and it is not hard to show that if $n \in B$ then $n+1 \in B$. We conclude that $B = \mathbb{N}$ so $\mathbb{N}$ is well ordered by containment order.

We can use this to show that a set C cannot have both m elements and n elements for natural numbers $n \neq m$. Let S consist of those members s of $\mathbb{N}$ for which there is a member n of $\mathbb{N}$, $n \neq s$, and a set C which has both s elements and n elements. Obviously $\varnothing \notin S$, so every member of S has the form $k+1$ for some natural number k. Should S be nonempty, it would contain a least member, and this leads easily to a contradiction. We conclude S is empty, and there is at most a single natural number n for which the statement "C has n elements" is true.

Note that each positive integer n is, itself, a well ordered set which has n elements. Any natural number n is, in fact, the initial segment I_n in $\mathbb{N}$.

$\mathbb{N}$ is an infinite set. To see this, suppose $h\colon \mathbb{N} \to k+1$ is one-to-one and onto. Then $h(m) = k$ for some $m \in \mathbb{N}$. Define $f\colon \mathbb{N} \to \mathbb{N}$ by $f(a) = a$ when $a < m$ and $f(a) = a+1$ if $a \geq m$. But then $h \circ f\colon \mathbb{N} \to k$ is one-to-one and onto. Since it is obvious that there can be no one-to-one function $h\colon \mathbb{N} \to 1$, the result follows.

Our next steps, left to the energetic reader, are to define negative integers and then the **integers**, denoted $\mathbb{Z}$, comprised of the natural numbers and the negative integers. The order on $\mathbb{N}$ is used to form a total order on $\mathbb{Z}$. The usual arithmetic operations are then defined and shown to obey the commonly listed algebraic properties.

Addition and multiplication of integers can be defined now, but demonstrating that they have the usual properties such as commutativity, the distributive law and so on using the tools we have to this point is a lengthy series of applications of induction. Connecting these operations to the order relation on the integers also requires more than a bit of work.

For instance, if a, b and m are positive integers and $m = ab$ then both $a \leq m$ and $b \leq m$. And if $a > 1$ then $b < m$. But what does it take, exactly, to prove that? Laying out every last detail about basic integer arithmetic is a big project, falling under the heading of number theory and logic. Gottlob Frege and Bertrand Russell are among the luminaries who broke teeth on it. You might wish to review Exercises **??** and **??** and the more sophisticated techniques assembled in Appendix **??**. In this work we will simply assume various "obvious" facts about integers.

With the integers in hand, one can define the rational numbers, $\mathbb{Q}$, as suggested in Section 2.

4.1. ***Exercise.*** *Till now we have had no specific well ordered sets (other than $\mathbb{N}$ and its initial segments) with which to work. Now we can create examples.*

(i) Let $\mathcal{S}$ denote the set $\left\{ m + \frac{n}{n+1} \mid n, m \in \mathbb{N} \right\}$ with the usual order from $\mathbb{Q}$. Show that $\mathcal{S}$ is well ordered.

(ii) For any $f \in \mathbb{N}^{\mathbb{N}}$ let

$$Support(f) = \{ n \in \mathbb{N} \mid f(n) \neq 0 \}.$$

Define $\mathcal{F}$ to be those members of $\mathbb{N}^{\mathbb{N}}$ for which $Support(f)$ is a finite set. For $f \in \mathcal{F}$ let m_f denote the greatest member of $Support(f)$.

*We will define an order $\leq_R$ on $\mathcal{F}$ called **reverse lexicographic order.***

*Declare $f \leq_R f$. Suppose $f, g \in \mathcal{F}$ and $f \neq g$. Let j be the **last** integer for which $f(j) \neq g(j)$. Declare $f \leq_R g$ if $f(j) < g(j)$.*

Show that $\mathcal{F}$ is well ordered with $\leq_R$. (hint: First show transitivity and conclude that $\leq_R$ is a total order. With that in hand, suppose H is a subset of $\mathcal{F}$ with at least two members. Let n_1 denote the least m_f of any $f \in H$. Let

$$H_1 = \{ f \in H \mid m_f = n_1 \} \quad \textit{and} \quad G_1 = \{ g \in H_1 \mid g(n_1) \leq f(n_1) \ \forall f \in H_1 \}.$$

If G_1 contains a single member we stop: this member is the minimal member of H. If G_1 contains more than one member, let n_2 denote the smallest integer for which there is some $g \in G_1$ with $g(n_2) \neq 0$ but $g(k) = 0$ for all k with $n_2 < k < n_1$. Possibly, $n_2 = n_1 - 1$. Now let

$$G_2 = \{ g \in G_1 \mid g(n_2) \leq f(n_2) \ \forall f \in G_1 \}.$$

If G_2 contains a single element it is the least member of H. If G_2 contains more than a single member we can continue, creating by this procedure a strictly decreasing list $n_1, n_2, \ldots$ in $\mathbb{N}$. Such a list cannot be infinite in any well ordered set. It must terminate at some least n_k, and the sole member of G_k is the minimal member of H.)

What is the role of the "finite support" condition for members of $\mathcal{F}$?

(iii) Define $\mathcal{F}_1$ to be those members f of $\mathcal{F}$ with $m_f = 0$ or 1. This set inherits the reverse lexicographic well order from $\mathcal{F}$. How is this order on $\mathcal{F}_1$ related to that on $\mathcal{S}$ from part (i)?

*(iv) We will define a different order $\leq_L$ on $\mathcal{F}$ called **lexicographic order.** Declare $f \leq_L f$. Suppose $f, g \in \mathcal{F}$ and $f \neq g$. Let j be the **first** integer for which $f(j) \neq g(j)$. Declare $f \leq_L g$ if $f(j) < g(j)$ and $g \leq_L f$ otherwise. Though $\mathcal{F}$ is totally ordered with $\leq_L$, it is not well ordered.*

The difference between $\leq_R$ and $\leq_L$ boils down to the following fact. In a well ordered set it is impossible to have a strictly decreasing sequence, but it is certainly possible to have a strictly increasing one.

(v) Suppose A and B are disjoint well ordered sets. Create an order on $A \cup B$ corresponding to "elements of A all follow any element of B," while retaining the given orders on A and B. Show that this order is a well order.

*(vi) Sometimes it will be convenient in certain arguments to have a well ordered set with a last member. In general, well ordered sets might not **have** a last member. Suppose C is well ordered with first element a_1 and more than one element. Give*

$A = C - \{a_1\}$ the inherited well order and let $B = \{a_1\}$. Using (v) create a well order on C that ***does*** *have a last member.*

(vii) Suppose A and B are well ordered sets. Create a well order on a ***subset*** *of A^B analogous to the reverse lexicographic order $\leq_R$ we created for $\mathcal{F}$ in part (ii).*

5. The Real Numbers

We will now make one of the common definitions of the real numbers and discuss some important properties of this set. The following construction is due to Dedekind.

Let $\mathbb{Q}^+$ be the set of nonnegative rational numbers. We define $\mathbb{R}^+ \subset \mathbb{P}(\mathbb{Q}^+)$ to consist of exactly those sets A of nonnegative rational numbers with the following three properties:

(i) A has no largest member and

(ii) $q \in A \Rightarrow p \in A \; \forall p \in \mathbb{Q}^+$ with $p \leq q$ and

(iii) $A \neq \mathbb{Q}^+$.

$\mathbb{R}^+$ is (obviously) nonempty and called the set of **nonnegative real numbers**. Sometimes a nonnegative real number, created this way, is also called a **Dedekind cut.**

If r and s are nonnegative real numbers, we say $r < s$ if $r \neq s$ and $r \subset s$.

This relation is a total order on $\mathbb{R}^+$ but it is not a well order. In fact no explicit well order of the real numbers is known.

If r and s are nonempty (that is, "**positive**") members of $\mathbb{R}^+$ and $t \in \mathbb{R}^+$ we define binary operations "+" and "·" by:

$$t + \varnothing = t \quad \text{and} \quad r + s = \{\, u \in \mathbb{Q}^+ \mid u < q + p \text{ for some } q \in r \text{ and } p \in s \,\},$$
$$t \cdot \varnothing = \varnothing \quad \text{and} \quad r \cdot s = \{\, u \in \mathbb{Q}^+ \mid u < q \cdot p \text{ for some } q \in r \text{ and } p \in s \,\}.$$

It is an exercise to show that $r + s$ and $r \cdot s$ are nonnegative real numbers and the operations satisfy the commonly listed properties of addition and multiplication with multiplicative identity given by $\{\, [a/b] \in \mathbb{Q}^+ \mid 0 < a < b \,\}$ and additive identity $\varnothing$, which will henceforth be denoted 1 and 0, respectively. Multiplicative inverses exist for positive real numbers.

Note that this is the third usage for the symbol 1 in this section. $1 \in \mathbb{N}$ was defined to be $\{\varnothing\}$ and $1 \in \mathbb{Q}^+$ was defined as a set of ordered pairs $\{\, a/a \mid a \in \mathbb{Z} \text{ and } a \neq 0 \,\}$. We unify these disparate definitions by identifying $n \in \mathbb{N}$ with $[n/1] \in \mathbb{Q}$, and $q \in \mathbb{Q}^+$ with $\{\, p \in \mathbb{Q}^+ \mid p < q \,\} \in \mathbb{R}^+$.

Let S be any nonempty set of nonnegative real numbers. If S has an upper bound in $\mathbb{R}^+$, we can show that $\bigcup_{A \in S} A \in \mathbb{R}^+$. In fact it is the supremum of S.

Let S be any nonempty set of nonnegative real numbers. $\bigcap_{A \in S} A$ might actually contain a largest rational. If it does not, then $\bigcap_{A \in S} A \in \mathbb{R}^+$ and is the infimum of S. If it does contain a largest rational, remove that rational from the intersection. The result is now in $\mathbb{R}^+$ and is the infimum of S.

If $r\colon \mathbb{N} \to \mathbb{R}^+$ is a nondecreasing sequence of nonnegative real numbers that is bounded above we let

$$\lim_{\mathbf{n}\to\infty} \mathbf{r_n} = \sup\{\, r_n \mid n \in \mathbb{N}\,\}$$

If $r\colon \mathbb{N} \to \mathbb{R}^+$ is a nonincreasing sequence of nonnegative real numbers let

$$\lim_{\mathbf{n}\to\infty} \mathbf{r_n} = \inf\{\, r_n \mid n \in \mathbb{N}\,\}$$

In either case, this number is called the **limit** of the corresponding sequence.

At this point it is an exercise to extend all of the above to a definition of the **negative real numbers** and then to the **real numbers**—consisting of both nonnegative and negative real numbers. Extend the total order on the nonnegative real numbers to the real numbers. Then define **multiplication, division, addition** and **subtraction** for these numbers, and show they have the familiar properties. Define **absolute value**. Define **limits of bounded monotone sequences** of real numbers.

Henceforth we let $\mathbb{R}$ denote the real numbers.

Define **intervals** $[a, b),\ (a, b),\ (a, b],\ (-\infty, b),\ (-\infty, b],\ (a,\infty),\ [a,\infty)$ and $[a, b]$ for real numbers a and b with $a \leq b$. The **standard topology on** $\mathbb{R}$ is that formed from a basis consisting of all intervals (a, b) with $a, b \in \mathbb{Q}$.

If $r\colon \mathbb{N} \to \mathbb{R}$ is a bounded sequence we define

$$\begin{aligned}\limsup(\mathbf{r}) &= \lim_{n\to\infty}(\sup\{\, r_k \mid k \in \mathbb{N} \text{ and } k > n\,\}) \quad \text{and}\\ \liminf(\mathbf{r}) &= \lim_{n\to\infty}(\inf\{\, r_k \mid k \in \mathbb{N} \text{ and } k > n\,\})\end{aligned}$$

Since the supremum and infimum above are being taken over smaller and smaller sets, the sequences whose limits are referred to are monotone and the limits are defined.

When these limits are equal we refer to their common value as the **limit of the sequence** r and denote this number by $\lim_{\mathbf{n}\to\infty} \mathbf{r_n}$. When the limit exists and is L we say the **sequence converges** or, when specificity is required, **converges to** L.

In applications, it is common for sequence values r_n to be defined only for n in a terminal segment of $\mathbb{N}$. Limits, if they exist, depend only on the value of r on any terminal segment. So when considering limits, we might define r_n values in any way that is convenient or not at all for n in any particular initial segment of $\mathbb{N}$.

Show that $||a|-|b|| \leq |a-b| \leq |a|+|b|$. This is known as the **triangle inequality**.

Show that the limit of a sequence r exists and is a number L exactly when the limit of the sequence $|r - L|$ exists and is 0.

Two sequences r and s are called **equivalent** if $\lim_{n\to\infty} |r_n - s_n| = 0$. The exercises above can be used to show that equivalent sequences converge or not together, and if they converge it is to the same limit.

A sequence r is called a **Cauchy sequence** if $\lim_{n\to\infty}(\sup\{\, |r_n - r_k| \ \mid k > n\,\})$ exists and is 0.

It is a fact that a sequence of real numbers converges precisely when it is Cauchy, and the definition of equivalent sequences from above forms an equivalence relation on the set of convergent sequences.

These last concepts can be used in an alternative construction of the real numbers. One examines the set of all Cauchy sequences of rational numbers, and partitions that set using the equivalence relation for sequences defined above. This does involve the creation of a preliminary definition of limit, but only for rational sequences that converge to 0. The set of these classes constitute the real numbers in this formulation.

There is a more general concept of limit that pops up sometimes. This is where the indexing set is a more general directed set and not necessarily $\mathbb{N}$, and we might as well define it here.

If J is a directed set, a function $r\colon J \to Y$ is called a **net** in Y. A net is a generalization of the idea of a sequence.

Now suppose $r\colon J \to \mathbb{R}$ is a net in $\mathbb{R}$ and $L \in \mathbb{R}$. We call L the **limit of the net** r and write $r_\alpha \xrightarrow{\alpha} L$ if and only if $\forall \varepsilon > 0\ \exists \alpha \in J$ so that $\alpha \leq \beta \Rightarrow |r_\beta - L| < \varepsilon$.

Limits of nets in $\mathbb{R}$, when they exist, depend only on the values of the net on any particular terminal segment of J. So when considering these limits, we are free to modify or define the r_α values in any way that is convenient or not at all for α outside of any terminal segment of J. Also, it is possible that J has a supremum, $\sigma = sup(J)$. In that case the limit is simply the number r_σ.

When the limit of a net in $\mathbb{R}$ exists we say the **net converges** or, when specificity is required, **converges to** L.

A net in $\mathbb{R}$ has at most one limit.

In case $J = \mathbb{N}$, show that $\lim_{n\to\infty} r_n$ exists and equals L if and only if r converges as a net and $r_n \xrightarrow{n} L$.

Suppose D is a nonempty subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Make D into a directed set by $a \preccurlyeq b$ if and only if $|c - a| \geq |c - b|$. Now suppose that $D \subset A \subset \mathbb{R}$ and $f\colon A \to \mathbb{R}$. The function $f|_D$ is a net in $\mathbb{R}$ which might converge.

In case $c \notin D$ and D contains a set of the form $\{\, x \in \mathbb{R} \mid x \neq c \text{ and } |x - c| < \xi \,\}$ for some $\xi > 0$ and if $f|_D$ converges, a limit of this net is denoted $\lim_{x\to c} f(x)$.

If there is any D satisfying the conditions above then the existence of $\lim_{x\to c} f(x)$ and its unique value do not depend on the particular D (satisfying the specified conditions) used in its definition, and the set D will usually not be mentioned explicitly.

The various properties of $\mathbb{R}$, such as the total order on $\mathbb{R}$ and the existence of suprema and infima of bounded subsets of $\mathbb{R}$, have numerous consequences of importance here. The reader should recall, prove, look up or accept the following miscellaneous facts about the real numbers. The various topological concepts can be found in Appendix **??**.

5.1. ***Exercise.*** *(i) Suppose* $f\colon (a,b) \to \mathbb{R}$ *and* $c \in (a_1, b_1) \subset (a,b)$. $\lim_{x\to c} f(x)$ *exists and equals L if and only if* $\lim_{n\to\infty} f(x_n)$ *exists and equals L for every sequence* $x\colon \mathbb{N} \to (a_1, c) \cup (c, b_1)$ *with* $\lim_{n\to\infty} x_n = c$.

(ii) If $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *both exist and equal* L *and* M *respectively, then* $\lim_{x\to c}(f(x)+g(x))$ *and* $\lim_{x\to c}(f(x)g(x))$ *exist and* $\lim_{x\to c}(f(x)+g(x)) = L+M$ *and* $\lim_{x\to c}(f(x)g(x)) = LM$*. If* $M \neq 0$ *then* $\lim_{x\to c}(1/g(x))$ *exists and equals* $1/M$*.*

(iii) $f\colon (a,b) \to \mathbb{R}$ *is continuous (with respect to the subspace topology on* (a,b)*) if and only if* $\lim_{x\to c} f(x)$ *exists and equals* $f(c)$ *for all* $c \in (a,b)$*.*

(iv) Constant functions are continuous, and the product and sum of continuous functions with common domain are continuous.

(v) If $f\colon (a,b) \to (c,d)$ *and* $g\colon (c,d) \to \mathbb{R}$ *are continuous then so is* $g \circ f$*.*

(vi) The function $f\colon (0,\infty) \to (0,\infty)$ *defined by* $f(x) = x^2$ *is one-to-one and onto* $(0,\infty)$*. Its inverse function is denoted* $f^{-1}(x) = \sqrt{x}$*. These functions are continuous and nondecreasing on their respective domains.*

(vii) The function $g\colon (0,\infty) \to (0,\infty)$ *defined by* $g(x) = 1/x$ *is one-to-one and onto* $(0,\infty)$*. It is its own inverse function. It is continuous and nonincreasing.*

(viii) $A \subset \mathbb{R}$ *is compact if and only if* A *is closed and bounded. This is the* ***Heine-Borel Theorem****.*

(ix) Suppose $f\colon (a,b) \to \mathbb{R}$ *is continuous and* $[a_1,b_1] \subset (a,b)$*. Let* $B = \{\, f(x) \mid x \in [a_1,b_1]\,\}$*. Suppose* $\inf(B) \leq L \leq \sup B$*. Then* $\exists c \in [a_1,b_1]$ *with* $f(c) = L$*. This is called the* ***Intermediate Value Theorem****.*

(x) Suppose $f\colon (a,b) \to \mathbb{R}$ *is continuous. If* K *is a compact subset of* (a,b) *then* $f(K)$ *is compact. If* J *is a subinterval of* (a,b) *then* $f(J)$ *is an interval.*

(xi) If $f\colon (a,b) \to (c,d)$ *is one-to-one and onto and continuous then the inverse function* $f^{-1}\colon (c,d) \to (a,b)$ *is continuous.*

(xii) If $f\colon (a,b) \to \mathbb{R}$ *is continuous, the values of* f *on* $\mathbb{Q} \cap (a,b)$ *determine the values of* f *on all of* (a,b)*.*

(xiii) If a *is a sequence of real numbers we define a new sequence* S*, called the* ***sequence of partial sums of*** $\mathbf{a}$*, by* $S_n = \sum_{k=0}^{n} a_k$*. A sequence formed this way is called a* ***series****. Sometimes* S *converges. When it does its limit is denoted* $\sum_{k=0}^{\infty} a_k$ *and the* ***series is said to converge.*** *If* S_n *does not converge it is said to* ***diverge****. If* $\sum_{k=0}^{\infty} |a_k|$ *exists the series* S *is said to* ***converge absolutely*** *and it is a fact that if a series converges absolutely then it converges. If the series converges but does* ***not*** *converge absolutely we say that the series converges* ***conditionally.***

When discussing the existence of the limit $\sum_{k=0}^{\infty} a_k$*, we often say that the symbol* $\sum_{k=0}^{\infty} a_k$ *itself converges, diverges or converges absolutely or conditionally.*

(xiv) Suppose $\sum_{k=0}^{\infty} a_k$ *converges absolutely, and* b *is a real valued sequence. Define for each* $k \in \mathbb{N}$ *the number* $c_k = \sum_{i=0}^{k} a_{k-i} b_i$*.*

If $\sum_{k=0}^{\infty} b_k$ *converges then so too does* $\sum_{k=0}^{\infty} c_k$ *and*

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_{k-i} b_i\right).$$

(xv) The series $E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ *converges absolutely for each real* x*. The limit is denoted* e^x*. The function* $\mathbf{Exp}\colon \mathbb{R} \to (0,\infty)$ *defined by* $\mathbf{Exp(x)} = e^x$ *is*

one-to-one and onto $(0,\infty)$. Its inverse is denoted **Ln**. *For each real x and y, $e^{x+y} = e^x e^y$. Exp and Ln are continuous and nondecreasing on their respective domains.*

(xvi) The series $\sum_{k=0}^{n} x^k$ converges absolutely to $\frac{1}{1-x}$ for each $x \in (-1,1)$.

(xvii) The series $S_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$ converge absolutely for each real x. Their limits are denoted **Sin(x)** *and* **Cos(x)**, *respectively, and the functions formed from these values are called the* **Sine** *and* **Cosine** *functions. They are continuous.*

(xviii) If a and b are real valued sequences define $\Delta a_n = a_{n+1} - a_n$ and $\Delta b_n = b_{n+1} - b_n$ for each $n \in \mathbb{N}$. Then for $0 \le m < n$

$$\sum_{k=m}^{n} a_k \, \Delta b_k = a_{n+1} b_{n+1} - a_m b_m - \sum_{k=m}^{n} b_{k+1} \, \Delta a_k$$

which is called the ***summation by parts formula*** *for series. In case the sequence ab converges, the left sequence of partial sums converges exactly when the right sequence of partial sums does.*

(xix) Suppose a and c are real valued sequences and we want to discover facts about the convergence of $S_n = \sum_{i=0}^{n} a_i c_i$. We define $b_k = \sum_{i=0}^{k} c_i$. Then $\Delta b_{n-1} = c_n$. The following equality of partial sums is called ***Abel's transformation*** *and is useful in several common applications.*

$$S_n = \sum_{k=0}^{n} a_k \, c_k = a_0 \, c_0 + \sum_{k=1}^{n} a_k \, \Delta b_{k-1} = a_{n+1} b_n - \sum_{k=0}^{n} b_k \, \Delta a_k .$$

(xx) If the sequence a/b converges to a nonzero constant L then the series $\sum_{k=0}^{\infty} a_k$ converges exactly when $\sum_{k=0}^{\infty} b_k$ converges.

(xxi) Suppose a is a sequence of non-zero numbers. Then $lim_{n\to\infty} |a_{n+1}| \,/\, |a_n|$ exists exactly when $lim_{n\to\infty} |a_n|^{1/n}$ exists. In case this common limit exists define R to be the reciprocal of the limit (if the limit is 0 let $R = \infty$.) For real x the series $\sum_{k=0}^{\infty} a_k x^k$ is called a ***power series*** *and R is called the* ***radius of convergence*** *of the series. This power series converges whenever $|x| < R$.*

(xxii) Suppose a is a sequence of non-zero numbers and $L = |a_n|^{1/n}$. If $L = 0$ the power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all x. If $L = \infty$ the power series converges only for $x = 0$. Otherwise, let $R = 1/L$. The power series converges absolutely if $|x| < R$ and diverges if $|x| > R$. This result is called the ***Cauchy-Hadamard Theorem.***

Finally, we get to the issue of specific common representations of real numbers.

If p is an integer bigger than 1, we can represent any real number between 0 and 1 as $\sum_{k=1}^{\infty} \frac{a_k}{p^k}$ where the sequence a consists of integers with $0 \le a_n < p$ for all n.

This representation is not quite unique as stated.

Sequences a that terminate, for some n, with $a_n \ne 0$ and $a_k = 0$ for all $k > n$ and exactly one sequence b with $b_k = p - 1$ for all $k > n$ generate series for the same real number.

However this is the only duplication in the representation, so uniqueness is acquired by forbidding all representations that use sequences b that terminate in $b_k = p - 1$ for all $k > n$ for some n.

With this convention, any real number can be represented uniquely (for each p and some $k \geq 0$) as

$$\pm \left(\sum_{n=0}^{k} a_{-n} p^n + \sum_{n=1}^{\infty} \frac{a_n}{p^n} \right) \qquad \text{where}$$

(i) $\quad 0 \leq a_j < p$ for all $j \geq -k$ and

(ii) $\quad a_{-k} \neq 0$ unless $k = 0$ and

(iii) $\quad$ the sequence does not terminate with $a_j = p - 1$ for all $j > m$ for any m.

The case of $p = 10$ corresponds to the ordinary **decimal representation** of numbers, while $p = 2$ and $p = 3$ generate the **binary or dyadic and ternary representations**.

We will take one further step in the progression $\varnothing \to 1 \to \mathbb{N} \to \mathbb{Q} \to \mathbb{R}$. The **complex numbers**, denoted $\mathbb{C}$, consist of the set of all ordered pairs of real numbers with operations of addition and multiplication given by $(a, b) + (c, d) = (a+c, b+d)$ and $(a, b) * (c, d) = (ac-bd, ad+bc)$. An alternative way of representing an ordered pair of real numbers (a, b) thought of as a complex number is using the symbol $a + bi$. If $z = a + bi$ is a complex number, a is called its **real part**, and b is called its **imaginary part**. This is purely a notational device: we are associating real number a with ordered pair $(a, 0)$ and bi with $(0, b)$. $\overline{\mathbf{z}} = a - bi$ is called the **conjugate of z**. The **magnitude of z** is $\sqrt{a^2 + b^2}$ and denoted $|\mathbf{z}|$. Note that if $z \neq (0, 0)$ then $(1, 0) = z \left(\frac{\overline{z}}{|z|^2} \right)$. The map that associates x in $\mathbb{R}$ with $(x, 0)$ in $\mathbb{C}$ preserves the arithmetic operations on $\mathbb{R}$ and sends the multiplicative identity there to the multiplicative identity in $\mathbb{C}$, so the range of this map can (and will) be identified with $\mathbb{R}$.

5.2. ***Exercise.*** *A sequence of complex numbers $z_n = a_n + b_n i$ converges to complex number $w = x + yi$ exactly when both $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = y$. This happens exactly when the real sequence $|\, w - z_n \,|$ converges to 0. Adapt Exercise 5.1 wherever necessary to handle series of complex numbers. Then define the series $E_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ and show it converges absolutely for every complex z and define e^z to be the limit. If w and z are complex show that $e^{z+w} = e^z e^w$ and if $z = a + bi$ then*

$$e^z = e^a e^{bi} = e^a \left(Cos(b) + i\, Sin(b) \right).$$

Delete the negative x axis and the origin from the complex plane and define an inverse to a piece of the exponential function there. If you want this logarithm to be continuous, what choices do you have? Could you delete another half-line terminating at the origin and define another logarithm with this domain?

6. An Axiomatic Characterization of $\mathbb{R}$

Let A be any field containing a copy $\mathbb{N}_A$ of $\mathbb{N}$ and a copy $\mathbb{Q}_A$ of $\mathbb{Q}$. By this we mean that $\mathbb{N}_A$ and $\mathbb{Q}_A$ are subrings of A containing the identity of A and which are ring isomorphic to $\mathbb{N}$ and $\mathbb{Q}$ respectively. If A contains a copy of $\mathbb{N}$ in this sense then, since it is a field, it must contain a copy of $\mathbb{Q}$, and these subrings are unique if they exist for a given ring A. We are simply positing here that A have characteristic 0.

If this field A is equipped with a linear order $<$ satisfying

(*i*) $\quad x + y > 0$ whenever $x, y > 0$

(*ii*) $\quad xy > 0$ whenever $x, y > 0$

(*iii*) $\quad x + z > y + z$ whenever $x > y$

we call A an **ordered field**. Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.

If both x and $-x$ were positive (i.e. greater than 0) we would have by (i) that $0 > 0$, contradicting the assumption that our order is a linear order. So if $x \neq 0$ at most one of x or $-x$ is positive.

If $-1 > 0$ then (ii) implies $(-1)(-1) = 1 > 0$, contradicting (i). So $-1 < 0$. Using this in (iii) yields $0+1 > -1+1$ so $1 > 0$. Repeated application of (iii) yields $n + 1 > n > 0$ for all positive integers n. In particular, we can never have $n = 0$. So the ordered field properties **imply** that the underlying field has characteristic 0: we do not need to assume it.

An ordered field has the quality of **Dedekind completeness** or **DKC** provided that each subset which is bounded above has a least upper bound.

The real numbers as we have built them constitute a Dedekind complete ordered field. Every property of the real numbers used in analysis follows from just a few properties: those which define a field, the properties defining a linear order, (i), (ii), (iii) and DKC.

An ordered field has the **Archimedean order property** or **AOP** if, for each $x, y \in A$ with $0 < x < y$ there is an $n \in \mathbb{N}_A$ so that $y < nx$.

These properties have consequences, a few of which are explored below.

6.1**.** ***Exercise.*** *We will presume that A is an ordered field as above.*

(i) $x > 0$ *exactly when* $-x < 0$. *(ii)* $x < y$ *exactly when* $0 < y - x$.

(iii) $0 < x$ *and* $0 < y < z$ *implies* $0 < xy < xz$.

(iv) $x < 0$ *and* $y < 0$ *implies* $xy > 0$. *(v)* $x < 0$ *and* $y > 0$ *implies* $xy < 0$.

(vi) $x > 0$ *exactly when* $\frac{1}{x} > 0$. *(vii)* $0 < x < y$ *exactly when* $0 < \frac{1}{y} < \frac{1}{x}$.

6.2**.** ***Exercise.*** *Suppose A is an ordered field as above.*

Define $|0| = 0$. *If x is nonzero in A, either* $x > 0$ *or* $-x > 0$ *but not both. Define* $|x|$ *to be* x *or* $-x$*, chosen so that* $|x| > 0$.

(i) For each $x, y \in A$*, show that* $|xy| = |x||y|$.

(ii) For each $x, y \in A$*, show that* $|x + y| \leq |x| + |y|$.

6.3. ***Exercise.*** *Suppose A is an ordered field as above.*

(i) AOP is equivalent to each of the following three conditions:
For each $x > 0$ there is an $n \in \mathbb{N}_A$ so that $\frac{1}{n} < x$.
For each $x > 0$ there is an $n \in \mathbb{N}_A$ so that $x < n$.
For each $x > 0$ there is a unique $n \in \mathbb{N}_A$ for which $n < x \leq n + 1$.

(ii) DKC implies AOP. (hint: If A does not have AOP then there is a y with $0 < 1 < y$ but $y \geq m$ for all $m \in \mathbb{N}_A$. So $\mathbb{N}_A$ is bounded above. If A had DKC there would be a least upper bound $p \in A$ for $\mathbb{N}_A$. Show that $p - 1$ must also be an upper bound for $\mathbb{N}_A$, a contradiction. So A cannot have DKC.)

(iii) DKC implies that sets in A which are bounded below have infima.

(iv) $\mathbb{Q}$ has AOP but not DKC so AOP does not imply DKC.

6.4. ***Exercise.*** *Suppose A is an ordered field as above with AOP.*

(i) If B is a subset of A for which $s = \sup B$ exists then for each integer $n > 0$ there is a member $t \in B$ with $s - t < \frac{1}{n}$.

(ii) If $r \in A$ then $r = \sup\{\, t \in \mathbb{Q}_A \mid t \leq r \,\}$.

6.5. ***Exercise.*** *If A is a Dedekind complete ordered field then there is a ring isomorphism between A and $\mathbb{R}$. This ring isomorphism is unique, and is an order isomorphism.*

We have a collection of properties, axioms if you will, satisfied by the real numbers. These axioms are (some of) the axioms of ordinary set theory plus those axioms associated with a Dedekind complete ordered field. The real numbers as we have created them constitute a realization or **model** of the axioms of a Dedekind complete ordered field "inside" ordinary set theory. We have shown by our construction that these axioms are consistent (if the axioms of set theory are consistent) and that was an important finding.

However neither the "Dedekind cut" construction of the real numbers nor the "Cauchy sequence" construction correspond in a compelling way to our simple intuition about real numbers as, for example, "points on a line."

In fact, all properties of the real numbers important to analysts **follow from the axioms mentioned above, not from the details of construction employed in forming our particular realization.**

It is these axioms which capture some of our intuition about real numbers, not any particular construction. The last exercise guarantees that if someone produces a different realization of these axioms, their underlying object shares all essential features with ours. We are free, when that is convenient, to remember the axioms and forget as irrelevant their particular embodiment.

Finally, it is worth noting that the usual identification of the real numbers with **all** the points on a line is not set in stone. It is not implied by the ancient concept of a line, nor by the standard practices of the inventors of calculus who routinely employed "infinitesimals," since replaced by limits. Practitioners of **nonstandard analysis** use a larger ordered field called the **hyperreal numbers** ${}^*\mathbb{R}$ in place of

$\mathbb{R}$. The hyperreal numbers contain positive numbers smaller than any real number, and limit-taking is replaced by hyperreal arithmetic.

The main technical challenges involved in transferring nonstandard results to the standard world were overcome by Abraham Robinson, the creator of this subject, in 1960.

Though conceptually attractive, it is currently unclear if this approach offers net advantages over standard analytic technique.

7. $[-\infty, \infty]^X$ AND $\mathbb{R}^X$

$[-\infty, \infty]$ is called the set of **extended real numbers** and defined to be $\mathbb{R} \cup \{-\infty, \infty\}$, where members of $\mathbb{R}$ have their usual properties and $\pm\infty$ are not real numbers and have the order, addition and multiplication properties that would seem reasonable for "infinitely large" numbers.

For example, $-\infty \leq a \leq \infty \ \forall a \in [-\infty, \infty]$. If $a > 0$ we define $a \cdot \infty = \infty$ $a \cdot (-\infty) = -\infty$ and $a + \infty = \infty$ and $(-a) \cdot \infty = -\infty$ and $(-a) + (-\infty) = -\infty$. We also define $-\infty \cdot 0 = \infty \cdot 0 = 0$. However $-\infty + \infty$ is not defined. The symbols $\pm\infty$ have no multiplicative inverses.

Every set in $[-\infty, \infty]$ is bounded above and below by ∞ and $-\infty$ respectively. We abuse vocabulary and declare a subset of $[-\infty, \infty]$ to be **bounded above or below** if it is bounded by a real number in the specified sense. With this usage, the bounded sets in $[-\infty, \infty]$ and $\mathbb{R}$ are the same.

$[-\infty, \infty]$ is a compact topological space, where neighborhoods of a point in $\mathbb{R}$ are sets containing an open interval around that point, neighborhoods of $-\infty$ are those sets containing an interval of the form $[-\infty, a)$, and neighborhoods of ∞ are those sets containing an interval of the form $(a, \infty]$.

Suppose J is a directed set, such as $\mathbb{N}$. Recall that for each $j \in J$ the symbol T_j denotes the terminal segment of J consisting of those members n of J for which $n \geq j$.

If $r\colon J \to [-\infty, \infty]$ is a net, each $r\left(T_j\right)$ is a set of extended real numbers, and any such has both supremum and infimum in $[-\infty, \infty]$. So, for example, both $u(j) = \sup r\left(T_j\right)$ and $l(j) = \inf r\left(T_j\right)$ are defined for each $j \in J$ and so form extended real valued nets l and u defined on J. u and l are are monotone: u is nonincreasing while l is nondecreasing.

The reader should investigate the modifications to the definition of limits of real valued sequences needed to make sense out of notation such as

$$\limsup(r) = L \quad \text{or} \quad \liminf(r) = L \quad \text{or} \quad r_\alpha \xrightarrow{\alpha} L$$

when L is an extended real number and r is a net in $[-\infty, \infty]$. In particular it must be verified that this definition agrees with the previously defined limit for a real valued net, when the former limit existed.

For sets X and Y, recall that Y^X is the set of functions from X to Y. When Y has a partial order, there is a partial order induced on Y^X given by $f \leq g \Leftrightarrow f(a) \leq g(a) \ \forall a \in X$. This is called the **pointwise order** on Y^X.

Infima and suprema of indexed sets of functions, such as $\{\, f_\alpha \mid \alpha \in J \,\} \subset Y^X$, are themselves members of Y^X whose values on each $x \in X$ are indicated by:

$$\left(\bigvee_{\alpha \in J} f_\alpha \right)(x) = \bigvee_{\alpha \in J} f_\alpha(x) \quad \text{and} \quad \left(\bigwedge_{\alpha \in J} f_\alpha \right)(x) = \bigwedge_{\alpha \in J} f_\alpha(x)$$

provided, of course, that the "pointwise" infima and suprema exist in Y for every $x \in X$.

Note that these definitions depend on the existence of limits *in* Y. If $Y \subset W$, an infimum or supremum might exist in W^X but not in Y^X.

There is a notational issue that should be observed here. If $f_\alpha \in Y^X$, we already have a definition for the infimum and supremum of a function f_α, namely:

$$\bigvee_{x \in X} f_\alpha(x) = \sup\{\, f_\alpha(x) \mid x \in X \,\} \quad \text{and} \quad \bigwedge_{x \in X} f_\alpha(x) = \inf\{\, f_\alpha(x) \mid x \in X \,\}$$

Confusion can arise when there are functions with multiple arguments or if multiple infima and suprema are being calculated if care is not taken in specifying order and arguments. Consider, for example:

$$\bigvee_{\alpha \in J} \left(\bigwedge_{x \in X} f_\alpha(x) \right) \quad \text{and} \quad \bigwedge_{x \in X} \left(\bigvee_{\alpha \in J} f_\alpha \right)(x).$$

There is no reason to think the two will be equal.

Note that Y is a lattice $\Leftrightarrow$ Y^X is a lattice.

More generally, infima and suprema always exist in Y^X precisely when such always exist in Y. These always exist if $Y = [-\infty, \infty]$ but not if $Y = \mathbb{R}$.

Suppose $f \colon J \to [-\infty, \infty]^X$, where J is a directed set. We say f **converges pointwise** to a function $\widehat{f}$ provided $f_\alpha(b) \xrightarrow{\alpha} \widehat{f}(b)\ \forall b \in X$. To describe this situation and to assert the existence of such a limit we will write $f_\alpha \xrightarrow{\alpha} \widehat{f}$ or, when $J = \mathbb{N}$, we often write $\lim_{n \to \infty} f_n = \widehat{f}$.

For f and $g \in [-\infty, \infty]^X$, we define $f \cdot g$ by $(f \cdot g)(a) = f(a)g(a)$ and $f + g$ by $(f+g)(a) = f(a) + g(a)\ \forall a \in X$. These are called **pointwise multiplication and addition**.

The multiplication and addition defined above are commutative, and the functions that are constantly one and zero are the multiplicative and additive identities, respectively. $[-\infty, \infty]^X$ is not a real vector space, but only because addition is not defined for all pairs of functions.

For any set X define $\chi \colon \mathbb{P}(X) \to 2^X$ by $\chi_A(a) = \begin{cases} 0 & \text{if } a \notin A; \\ 1 & \text{if } a \in A. \end{cases}$

The map χ is an order isomorphism.

Each χ_A is called a **step** or **characteristic function** and finite real linear combinations of these are called **simple functions**.

Note that $\chi_A \vee \chi_B = \chi_{A \cup B}$, $\chi_A \wedge \chi_B = \chi_{A \cap B} = \chi_A \cdot \chi_B$, $\chi_{A-B} = \chi_A - \chi_{A \cap B}$ and $|\chi_A - \chi_B| = \chi_{(A-B) \cup (B-A)}$.

When $\mathbb{G} \subset \mathbb{P}(X)$, we will use $\mathbf{\mathcal{S}}(\mathbb{G})$ to denote the set of simple functions constructed from the sets in $\mathbb{G}$.

A function that has constant range value t on its whole domain will sometimes be denoted t, with this usage (and the domain) taken from context. Thus, for example, χ_X is sometimes denoted by 1 and $0\chi_X$ by 0, in yet another use of each of those symbols.

When $\mathbf{H}$ is a subset of $[-\infty, \infty]^X$, we will use $\mathbf{\mathcal{B}}(\mathbf{H})$ to denote the bounded members of $\mathbf{H}$; $f \in \mathbf{\mathcal{B}}(\mathbf{H}) \Leftrightarrow f \in \mathbf{H}$ and $\exists a \in \mathbb{R}$ with $0 \leq a < \infty$ and $-a \leq f \leq a$.

If X is a topological space, let $\mathbf{\mathcal{C}}(X)$ denote the continuous functions from X to $\mathbb{R}$.

$\mathbb{R}^X$, $\mathbf{\mathcal{B}}(\mathbb{R}^X)$ and, when X has a topology, $\mathbf{\mathcal{C}}(X)$, are all **vector lattices**: real vector spaces and lattices. They are also **commutative rings with multiplicative identity** χ_X.

7.1**. *Exercise.*** $\mathbf{\mathcal{S}}(\mathbb{G})$ *is obviously a (possibly empty) vector space. Give conditions on* $\mathbb{G}$ *under which* $\mathbf{\mathcal{S}}(\mathbb{G})$ *is a vector lattice and a commutative ring with multiplicative identity* χ_X.

8. The Axiom of Choice

In this section we will discuss an axiom of set theory, the Axiom of Choice.

Every human language has grammar and vocabulary, and people communicate by arranging the objects of the language in patterns. We imagine that our communications evoke similar, or at least related, mental states in others. We also use these patterns to elicit mental states in our "future selves," as reminder of past imaginings so that we can start at a higher level in an ongoing project and not have to recreate each concept from scratch should we return to a task. It is apparent that our brains are built to do this.

But words are all defined in terms of each other. Ultimate meaning, if there is any to be found, is derived from pointing out the window at instances in the world, or from introspection. Very often ambiguity or multiple meaning of a phrase is the point of a given communication, and provides the richness and subtlety characteristic of poetry, for instance, or the beguiling power of political speech.

Set theory is a language mathematicians have invented to encode mathematics. But unlike most human languages, this language does everything possible to avoid blended meaning, to expose the logical structure of statements and keep the vocabulary of undefined terms to an absolute minimum. Many mathematicians believe what they do is "art." But ambiguity and internal discord is not part of our particular esthetic ensemble.

Most mathematicians believe that, though set theory may be unfinished, it serves its purpose well. Virtually all mathematical structures can be successfully modeled in set theory, to the extent that most mathematicians never think of any other way of speaking or writing.

Together, the collection of axioms (which, along with logical conventions defines the language) normally used by most mathematicians is called the **Zermelo-Fraenkel Axioms**, or simply **ZF** and the set theory that arises from these axioms is called **Zermelo-Fraenkel Set Theory**. You saw explicit mention of two axioms from ZF, the Axiom of Infinity and the Axiom of the Empty Set, in Section 5. We have used others without mention on almost every page. For example we have formed power sets.

> **The Axiom of the Power Set** For any set A there is a set $\mathbb{P}(A)$ consisting of all, and only, the subsets of A.

Asserting the existence of a set with this feature is a dramatic and "non-constructive" thing to do, *particularly* when the underlying set is infinite. We are not told how to create this set. We just have a means of recognizing if a set we have in hand is a member of this power set, or not.

And where, exactly, did that first infinite set come from? The Axiom of Infinity brings it into existence, out of nothing, simply because mathematicians *want infinite sets* and this seems to be a logically consistent way to produce them.

There is another extremely useful—and arguably even less constructive—axiom which we discuss now.

We will present and presume to be true, wherever convenient, the four equivalent and useful statements below, one of which is called the Axiom of Choice. This axiom is frequently abbreviated to **AC**. The collection of the axioms of standard set theory plus this axiom is frequently denoted **ZFC**.

The discussions regarding equivalence of the Axiom of Choice and the other three statements, and the history associated with them, is a fascinating story which deserves study by every serious student of mathematics.

> **The Axiom of Choice**: If J and X are sets and $A\colon J \to \mathbb{P}(X)$ is an indexed collection of nonempty sets then there is a function $f\colon J \to X$ such that $f(\beta) \in A_\beta \ \forall \beta \in J$. A function with this property is called a **choice function** for A.

Essentially, this axiom states that given any generic set $\mathbb{S}$ of nonempty sets, there is a way of selecting one element from each member of $\mathbb{S}$. The other axioms do not imply that such a selection can be made, unless every member of $\mathbb{S}$ has an element with some unique property, which would allow it to be singled out.

> **Zorn's Lemma**: If S is a set with a partial order and if every chain in S possesses an upper bound in S, then S has a maximal member.
>
> **Zermelo's Theorem**: Every nonempty set can be well ordered.
>
> **Kuratowski's Lemma**: Each chain in a partially ordered set S is contained in a **maximal chain** in S (that is, a chain in S not contained in any other chain in S.)

Kuratowski's Lemma is also often called **The Hausdorff Maximal Principle.**

That Zorn's Lemma implies Kuratowski's Lemma is immediate. Suppose S is a set with a partial order and and C is a chain in S. Let $\mathbb{W}$ denote the set of all chains in S which contain the chain C, ordered by containment. Any chain in $\mathbb{W}$ is bounded above by the union of the chain, so Zorn's Lemma implies that $\mathbb{W}$ contains a maximal member. That maximal member is a chain in S not properly contained in any other chain in S.

On the other hand, assuming Kuratowski's Lemma to be true, suppose S is a set with a partial order and that every chain in S possesses an upper bound in S. This time let $\mathbb{W}$ denote the set of *all* chains in S. Let X denote a maximal member of $\mathbb{W}$. So X is a chain in S not contained in any other chain. Let M be any upper bound for X. By maximality of X, M must actually be in X and cannot be less than any other member of S: that is, M is maximal in S. So Zorn's Lemma is true.

In the last two paragraphs we have shown that Zorn's Lemma and Kuratowski's Lemma are equivalent statements.

We will now show that Zorn's Lemma implies AC. Suppose $\mathbb{S}$ is any nonempty set of nonempty sets and X is the union of all the sets in $\mathbb{S}$. Let $B = \mathbb{S} \times X$. Now let Q denote the set of all subsets of $\mathbb{P}(B)$ which are choice functions on their domains: that is, $T \in Q$ exactly when T is nonempty and there is at most one ordered pair in T whose first component is any particular member of $\mathbb{S}$, and also $s \in A$ whenever $(A, s) \in T$. These are called "**partial choice functions**." Order Q by containment. The union of any chain in Q is a member of Q so Zorn's Lemma implies that Q has a maximal member. This maximal member is a choice function on its domain, which must by maximality be all of $\mathbb{S}$.

The fact that Zermelo's Theorem implies AC is also straightforward: given any nonempty set $\mathbb{S}$ of nonempty sets, well order the set $X = \bigcup_{S\in\mathbb{S}} S$. For each $S \in \mathbb{S}$ let $f(S)$ be the least element of S with respect to this ordering. f is the requisite choice function.

The opposite implication is a bit trickier. It involves using a choice function to create the well order.

Suppose set A has more than one element and $f \colon \mathbb{P}(A) - \{\varnothing\} \to A$ is a choice function: that is, $f(B) \in B$ whenever $\varnothing \neq B \subset A$.

Let $\mathbb{B}$ denote the set of all nonempty containment-chains in $\mathbb{P}(A) - \{\varnothing\}$ which are well ordered and satisfy the condition:

> Whenever I_K is an initial segment of one of these chains and if J is the union of all the sets in I_K then $J \neq A$ and $K = J \cup \{f(A - J)\}$.

$\mathbb{B}$ is nonempty: for example, $\{\,\{f(A)\}, \{\,f(A), f(A - \{f(A)\})\,\}\,\}$ is in $\mathbb{B}$.

The condition above implies that each of the chains in $\mathbb{B}$ must start with the set $\{f(A)\}$, and the successor to any set K in such a chain (if, of course, K is not the last set in the chain) has exactly one more member than does K. It also implies directly that if two different chains X and W of this kind have a common initial segment, so that $I_K \subset X$ and $I_G \subset W$ and $I_K = I_G$ then $K = G$. In other words, the least successor of an initial segment is determined by the sets in the initial segment, and *not* by the specific chain within which the initial segment sits.

Suppose that X is one of these chains. We will call S a "starting chunk" of X if $\varnothing \neq S \subset X$ and whenever $B, C \in X$ the condition $B \in S$ and $C \subset B$ implies $C \in S$. Now it might be that a starting chunk is as short as $\{f(A)\}$ or it could, possibly, be all of X. But if it is *not* all of X then because X is well ordered there is a least member K of X not in S and so S contains all members of X less than K. That is, $S = I_K$ for some $K \in X$. So starting chunks are either initial segments or the entire chain.

Now suppose X and W are unequal chains, members of $\mathbb{B}$. Then one, say X, would contain a least set K not in the other. The initial segment I_K of X is contained in W. If there were a set in W not in I_K but less than some member of I_K then there would be a least member of W of this kind. Call that least member G. But then the initial segment I_G of W would be a starting chunk of X and by the above remark we would have $G \in X$, contrary to its definition.

So there are no missing members of W between members of I_K, which is therefore a starting chunk of W. Since $K \notin W$ we must have $I_K = W$, and conclude that W is an initial segment of X.

To reiterate: for each pair of members of $\mathbb{B}$, one is an initial segment of the other.

Now let S be the union of all the members of $\mathbb{B}$. Each set in S comes from a member of $\mathbb{B}$ and since one of any pair of members of $\mathbb{B}$ is an initial segment of the other we conclude that S itself is a chain, and well ordered too.

Let J denote the union of all the sets in S. If $J \neq A$ then we could extend S to $S \cup \{J \cup \{f(A - J)\}\}$ which satisfies the conditions for membership in $\mathbb{B}$ but is strictly longer than its longest member, a contradiction. We conclude that $J = A$.

So we can use S to create an order on A. If a and b are members of A there is a least member S_a of S containing a and a least member S_b containing b. Declare $a \leq b$ precisely when $S_a \subset S_b$. If J is the union of the sets in the initial segment determined by S_a then $a \notin J$ so it must be that $a = f(A - J)$. So this relation makes A into a total order. Further, if $\varnothing \neq T \subset A$ then the collection of all of the S_t with $t \in T$ has a least member, which produces a least member of T. So the order on A is a well order.

We conclude that the existence of a choice function on $\mathbb{P}(A) - \{\varnothing\}$ implies that A can be well ordered. So AC implies Zermelo's Theorem.

Upon accepting the Axiom of Choice, as we will do throughout this book, well ordered sets are plentiful and can be used.

At this point we have shown the following implications among the conditions which we claim to be equivalent to the Axiom of Choice.

$$\begin{array}{ccc} \text{Zorn} & \Longleftrightarrow & \text{Kuratowski} \\ \Downarrow & & \\ \text{AC} & \Longleftrightarrow & \text{Zermelo} \end{array}$$

The **Principles of Induction** and **Recursive Definition** are incredibly powerful and useful techniques, extending the idea of Induction on the Integers to many more well ordered sets and situations more varied than merely checking if an indexed set of propositions are all true. The methods are detailed in Section **??**. It is important to note, and the reader should check, that the proof of the version of Recursive Definition we use here does *not* require AC.

We will now use Induction and Recursive Definition to show that Zermelo's Theorem implies Kurotowski's Lemma, thereby proving that any of the four conditions listed above implies the others. The discussion below is a typical usage of this type of argument. It uses first a recursive definition to deduce that a certain function exists, and then induction to confirm various properties of that function.

We suppose we have a chain in a partially ordered set. We will line up the members of the set not already in the chain and test them one at a time. When it is an element's turn, if it can be added to yield a bigger chain than we have up to that point we select it. Otherwise we discard it. Then we go on to the next element and repeat until we have exhausted the possibilities. The product is a maximal chain. A rigorous justification can be produced after digesting the result in **??**

Assume Zermelo's Theorem to be true, and suppose H is a nonempty chain in set K with partial order $\precsim$. Suppose $B = K - H$ is nonempty. There is a well order $\leq$ for B. Since we have two orders in hand, we will use prefixes to describe which order is in use. We will let I_β stand for a $\leq$-initial segment for any $\beta \in B$.

Suppose y is a fixed element of H. For the $\leq$-first element α of B, let $P(\alpha)$ equal α if $H \cup \{\alpha\}$ is a $\precsim$-chain, and let $P(\alpha)$ be y otherwise.

Having found $P(\beta)$ for all $\beta \in B$ with $\alpha \leq \beta < \gamma$ for some $\gamma \in B$ define $P(\gamma)$ to be γ if $H \cup \{\gamma\} \cup P(I_\gamma)$ is a $\precsim$-chain, and let $P(\gamma)$ be y otherwise.

This serves to define $P(\gamma)$ for each $\gamma \in B$.

$H \cup P(B)$ must be a $\precsim$-chain: if not it must contain two $\precsim$-incomparable members s and t, which cannot both be in H. If one of the two, say s, is in H then there is a $\beta \in B$ with $P(\beta) = t = \beta$. But then $H \cup \{\beta\} \cup P(I_\beta)$ is not a chain, violating the defining condition for $P(\beta)$. A similar contradiction occurs if neither s nor t are in H, by examining the point at which the *second* of the two points would have been added. So in fact $H \cup P(B)$ must be a $\precsim$-chain.

No additional members of K can be added to $H \cup P(B)$ without causing the resulting set to fail to be a $\precsim$-chain: once again, letting γ be the $\leq$-least member of B which *could* be added, if any, yields a contradiction. That member *would* have been added at stage γ.

So $A \cup P(B)$ is a maximal $\precsim$-chain in K, and Kuratowski's Theorem holds.

8.1. ***Exercise.*** *Fill in the details of a direct proof using Induction and Recursive Definition that Zermelo's Theorem implies Zorn's Lemma. We assume that K is a set with partial order $\precsim$ for which every chain has an upper bound. We assume also that K has a well order $\leq$ with $\leq$-first member α.*

We would like to conclude that K has a $\precsim$-maximal element.

Let α denote the $\leq$-first member of K and define $G(\alpha) = \alpha$. Having defined G on $\leq$-initial segment I_β for $\beta > \alpha$ let $G(\beta) = \beta$ if β is a $\precsim$-upper bound for $G(I_\beta)$, and otherwise let $G(\beta) = \alpha$.

Show that $G(K)$ is a chain and that $G(K)$ has a $\precsim$-last member, which is $\precsim$-maximal in K.

8.2. ***Exercise.*** *(i) An axiom equivalent to our Axiom of Choice is produced if we add to that axiom the condition that $A_\alpha \cap A_\beta = \varnothing$ whenever $\alpha \neq \beta$.*

(ii) Consider the statement: "Whenever $\mathbb{B}$ is a nonempty set of nonempty pairwise disjoint sets, there is a set S for which $S \cap x$ contains a single element for each $x \in \mathbb{B}$." Show that this statement is equivalent to the Axiom of Choice.

*(iii) Let $\mathbb{B}$ be a (nonempty) set of sets. $\mathbb{B}$ is said to have **finite character** provided that $A \in \mathbb{B}$ if and only if every finite subset of A is in $\mathbb{B}$. **Tukey's Lemma** states that every set of sets of finite character has a maximal member: a set not contained in any other member. Show that Tukey's Lemma is equivalent to the Axiom of Choice. (hint: To prove that Tukey's Lemma implies the Axiom of Choice examine the set of partial choice functions and note that it satisfies the conditions of Tukey's Lemma.)*

The use of AC in the formation of mathematical arguments has historically been the subject of controversy centered around the nebulous nature of the objects whose existence is being asserted in each case. In applications the axioms of set theory are usually used to affirm the existence of one precisely defined set whose elements share an explicit property. That is less obviously the case when AC is invoked.

Applications which require less than the full strength of AC are common. In an effort to control, or at least record, how the axiom is being used, weaker variants have been created. Some mathematicians award "style points" to proofs using one of these, or which avoid AC altogether. We list two of these weaker versions of AC below.

> **The Axiom of Dependent Choice**: If X is a nonempty set and $R \subset X \times X$ is a binary relation with domain all of X, then there is a sequence $r \colon \mathbb{N} \to X$ for which $(r_k, r_{k+1}) \in R \ \forall k \in \mathbb{N}$.
>
> **The Axiom of Countable Choice**: If X is a nonempty set and $r \colon \mathbb{N} \to \mathbb{P}(X)$ is a sequence of nonempty subsets of X then there is a sequence $f \colon \mathbb{N} \to X$ such that $f(n) \in A_n \ \forall n \in \mathbb{N}$.

These axioms are frequently abbreviated to **DC** and $\mathbf{AC}_\omega$, respectively.

8.3**. *Exercise.*** *(i) Prove the implications $AC \Rightarrow DC \Rightarrow AC_\omega$.*

*(ii) Suppose X is infinite. For each $k \in \mathbb{N}$ let S_k denote the set of all subsets of X which have 2^k elements. ZF alone implies that S_k is nonempty for each k, and you may assume this. Let S denote the set of all the S_k. Use AC_ω twice to prove that there is a one-to-one function $f \colon Y \to \mathbb{N}$ for an infinite subset Y of X. Any set Y (infinite or not) for which there is a function of this kind is called **countable**, and the result here may be paraphrased as "Any infinite set has an infinite countable subset in ZF+AC_ω."*

(iii) Sometimes the use of an axiom, particularly a variant of the Axiom of Choice, is hard to spot in an argument. It seems so reasonable, it is hard to see you are assuming anything. The theorem that "The union of a countable set of countable sets is countable." is an example.

Suppose A is a countable set of countable sets, and let B denote the union of all the members of A. Because A is countable, there exists one-to-one $T \colon A \to \mathbb{N}$. Because each member of A is countable, for each nonempty set $S \in A$ there is a nonempty set F_S consisting of all one-to-one functions from S to $\mathbb{N}$. Using T, this collection of sets of functions is seen to be countable, so AC_ω guarantees that we *can pick a function from each. It is easy to overlook this step, and merely assert "Because each member of A is countable there exists one-to-one $G_S\colon S \to \mathbb{N}$ for each $S \in A$." and get on with the discussion using these selected functions. But it is AC_ω which endorses this selection.*

To finish the argument, for each $x \in B$ we let $A_x = \{\, S \in A \mid x \in S \,\}$ and define i_x to be the least integer in $T(A_x)$. We define W_x to be that member of A_x with $T(W_x) = i_x$. The function $H\colon B \to \mathbb{N}$ given by

$$H(x) = 2^{i_x} \cdot 3^{G_{W_x}(x)}$$

is one-to-one, so B is countable.

8.4**.** ***Exercise.*** *(i) Any chain in a tree is contained in a branch.*

(ii) Prove ***König's Tree Lemma****: If S is an infinite rooted tree but each $t \in S$ is the immediate predecessor of only finitely many members of S then S has an infinite branch. (hint: Let K denote those members of S with an infinite number of successors and for each $t \in K$ let $M_t = T_t \cap K - \{t\}$. Let f denote a choice function for these sets: $f(t) \in M_t \ \forall t \in K$. Use induction on $\mathbb{N}$ to create an infinite chain.)*

The next section contains another important consequence of the Axiom of Choice. Many more can be found scattered in appendices and chapters throughout this book.

Those who want a slightly more detailed look at the ZF axioms can find them listed in Sections **??** and **??**. The discussions there are rudimentary but, I hope, a practical guide providing a taste of modern set theory.

9. Nets and Filters

Suppose $r\colon J \to X$ is a net. Recall that this means that J is preordered and there is an upper bound in J for each two-element subset of J.

If $A \subset X$, r is said to be **in** A if $r(J) \subset A$. r is said to be **eventually in** A if there is a terminal segment $T_\alpha \subset J$ such that $r(T_\alpha) \subset A$. r is said to be **frequently in** A if $r(T_\alpha) \cap A \neq \varnothing \ \forall$ terminal segments T_α in J. Obviously, if r is eventually in A then r is frequently in A.

A **subnet of** r is a net $s\colon K \to X$ such that $\exists f\colon K \to J$ for which $s = r \circ f$ and $\forall m \in J \ \exists n \in K$ such that $f(T_n) \subset T_m$. Note that f is not presumed to be nondecreasing. It is simply eventually in any terminal segment of J. A subnet of a subnet is also a subnet of the original net.

A net in a set X is called **universal** if the net is eventually in A or eventually in A^c for all $A \in \mathbb{P}(X)$.

9.1**.** ***Proposition.*** *Each net $r\colon D \to X$ has a universal subnet.*

Proof. Let $\mathbb{M} \subset \mathbb{P}^2(X)$ be the set of all those $\mathbb{G} \subset \mathbb{P}(X)$ such that r is frequently in each member of $\mathbb{G}$ and also if $A, B \in \mathbb{G}$ then $A \cap B \in \mathbb{G}$.

Obviously $\{X\} \in \mathbb{M}$ so $\mathbb{M} \neq \varnothing$, and chains in $\mathbb{M}$ ordered by inclusion have upper bounds in $\mathbb{M}$ (the union of the chain) so $\mathbb{M}$ contains a maximal member $\mathbb{K}$.

If r is eventually in A or eventually in A^c then the maximality of $\mathbb{K}$ guarantees that one or the other is in $\mathbb{K}$. It remains to consider the case where r is frequently in A but $A \notin \mathbb{K}$. By maximality of $\mathbb{K}$ there must be some $S \in \mathbb{K}$ so that r is not frequently in $A \cap S$: that is, r is eventually in $(A \cap S)^c$. Now, if T is any member of $\mathbb{K}$, r is frequently in $T \cap S = (T \cap S \cap A) \cup (T \cap S \cap A^c)$ so r must be frequently in $T \cap S \cap A^c \subset T \cap A^c$. This is true for any $T \in \mathbb{K}$ so by maximality of $\mathbb{K}$, $A^c \in \mathbb{K}$.

We have just shown that either A or $A^c \in \mathbb{K}\ \forall A \in \mathbb{P}(X)$.

Now let $E = \{\, (\alpha, B) \mid \alpha \in D,\ B \in \mathbb{K} \text{ and } r(\alpha) \in B \,\}$ directed by $(\alpha, B) \leq (\beta, C)$ precisely when $\alpha \leq \beta$ and $B \supset C$. The net $s\colon E \to X$ defined by $s((\alpha, B)) = r(\alpha)$ is a subnet of r and universal by construction. $\square$

We now move on to the next idea of this section.

A nonempty subset $\mathbb{F}$ of $\mathbb{P}(X)$ is called a **filterbase on** X if

$$(a) \qquad \varnothing \notin \mathbb{F} \qquad \text{and}$$
$$(b) \qquad A, B \in \mathbb{F} \Rightarrow A \cap B \in \mathbb{F}.$$

If the additional condition

$$(c) \qquad A \in \mathbb{P}(X),\ B \in \mathbb{F} \Rightarrow A \cup B \in \mathbb{F}$$

holds, $\mathbb{F}$ is called a **filter on** X .

Given any nonempty subset $\mathbb{F}$ of $\mathbb{P}(X)$ for which finite intersections of members of $\mathbb{F}$ are nonempty there is a unique smallest filterbase containing $\mathbb{F}$. Each filterbase is contained in a unique smallest filter. This filterbase and this filter are said to be **generated by** $\mathbb{F}$.

The most common example of a filterbase is the collection of all open sets containing a particular point of a topological space. A filter containing this filterbase would be the set of all neighborhoods of that point.

Another filterbase would be $\{\, (0, a) \subset (0, \infty) \mid a > 0 \,\}$.

Yet another example is given by the following: Let $r\colon D \to X$ be a net in X. Let $\mathbb{F} = \{\, r(T_d) \mid d \in D \,\}$, where each T_d is a terminal segment of D. $\mathbb{F}$ is a filterbase. The collection of all sets containing any terminal segment, $\mathbb{G} = \{\, A \in \mathbb{P}(X) \mid r(T_d) \subset A \text{ for some } d \in D \,\}$, is the smallest filter containing $\mathbb{F}$.

Let $\boldsymbol{\mathfrak{F}}$ denote the set of filters on X. $\boldsymbol{\mathfrak{F}}$ is partially ordered by containment. If $\{\, \mathbb{F}_\alpha \mid \alpha \in J \,\}$ is any chain of filters then $\bigcup_{\alpha \in J} \mathbb{F}_\alpha$ is also a filter and an upper bound for the chain. So $\boldsymbol{\mathfrak{F}}$ possesses maximal members called **ultrafilters.** When $\mathbb{G}$ is a filterbase, $\{\, \mathbb{F} \in \boldsymbol{\mathfrak{F}} \mid \mathbb{F} \supset \mathbb{G} \,\}$ is nonempty and possesses maximal members, which are maximal in $\boldsymbol{\mathfrak{F}}$ as well. So any filterbase is contained in an ultrafilter.

It is not hard to show that a filter $\mathbb{F}$ on X is an ultrafilter if and only if whenever $A \in \mathbb{P}(X)$ then $A \in \mathbb{F}$ or $A^c \in \mathbb{F}$.

This provides a link between universal nets and ultrafilters.

If $\mathbb{F}$ is the filterbase on X formed from the net r as above, let $s\colon E \to X$ be a universal subnet of r. Let $\mathbb{K} = \{\, A \in \mathbb{P}(X) \mid s(T_d) \subset A \text{ for some } d \in E \,\}$. Since s is universal, either A or A^c is in $\mathbb{K}$ $\forall A \in \mathbb{P}(X)$. $\mathbb{K}$ is an ultrafilter containing $\mathbb{F}$.

Alternatively, suppose $\mathbb{F}$ is any filter and $\mathbb{G}$ is the ultrafilter generated by $\mathbb{F}$. Let $J = \{\, (x, A) \mid x \in A \in \mathbb{F} \,\}$ and $K = \{\, (x, A) \mid x \in A \in \mathbb{G} \,\}$. Direct J and K by $(x, A) \le (y, B)$ if and only if $A \supset B$. We define $r\colon J \to X$ by $r((x, A)) = x$ and $s\colon K \to X$ by $s((x, A)) = x$. The filters $\mathbb{F}$ and $\mathbb{G}$ are precisely the sets formed from terminal segments of J and K by r and s, respectively. If we define $f\colon K \to J$ by $f((x, A) = (x, X)$, then $s = r \circ f$ and it follows that s is a subnet of r. Moreover, s is a universal net.

Suppose X is a nonempty set and $p \in X$. Let $\mathbb{F}_p$ denote the collection of all subsets of X containing p. $\mathbb{F}_p$ is an ultrafilter, and ultrafilters of this type are called **principal**. Other kinds of ultrafilters are called **free**.

If X is nonempty let $\mathbb{K}$ denote the set of **cofinite** subsets of X: that is, all subsets S of X for which $X - S$ is a finite set. If X is finite, $\mathbb{K} = \mathbb{P}(X)$. But if X is infinite, $\mathbb{K}$ is a filter on X, the **filter of cofinite subsets of X**.

9.2. ***Exercise.*** *(i) If $V_1, V_2, \ldots, V_n$ is a finite partition of X and $\mathbb{F}$ is an ultrafilter on X then $\mathbb{F}$ contains exactly one of the V_i.*

(ii) If an ultrafilter on X contains a finite set it contains a one point set, and is principal.

(iii) Suppose $\mathbb{U}$ is an ultrafilter on infinite X. $\mathbb{U}$ is free exactly when $\mathbb{U}$ contains all cofinite subsets of X.

(iv) There is a free ultrafilter $\mathbb{U}$ on $\mathbb{N}$ containing the set of even natural numbers. There is another containing the set of odd natural numbers. In fact if A is any infinite subset of $\mathbb{N}$ there is a free ultrafilter on $\mathbb{N}$ containing A.

10. Rings and Algebras of Sets

Consider $\mathbb{P}(X)$, the power set on the set X. When there is no danger of ambiguity and $A \subset X$, the notation A^c is often seen in place of $X - A$. $\mathbb{P}(X)$ is partially ordered by containment. $\mathbb{P}(X)$ is a lattice, with $A \wedge B = A \cap B$ and $A \vee B = A \cup B$. There is also additional structure on subsets of $\mathbb{P}(X)$.

$\mathbb{G} \subset \mathbb{P}(X)$ will be referred to as a **ring in X** if

(i) $\quad \varnothing \in \mathbb{G}$

(ii) $\quad A, B \in \mathbb{G} \Rightarrow A - B \in \mathbb{G}$ and

(iii) $\quad A, B \in \mathbb{G} \Rightarrow A \cup B \in \mathbb{G}$.

The last two items can be rephrased by saying that $\mathbb{G}$ is **closed** with respect to the operations $\cup$ and $-$.

$\mathbb{G} \subset \mathbb{P}(X)$ will be referred to as an **algebra on X** if, in addition

(iv) $\quad X \in \mathbb{G}$

Items (ii) and (iv) imply that $\mathbb{G}$ is **closed** with respect to the operations $\cup$ and ^{c}. In the presence of (iii), this last statement implies (ii).

It is apparent that item (i) is redundant in the presence of (ii) and (iv). Also, if $\mathbb{G}$ is a ring in X and A and B are in $\mathbb{G}$ then so is $A \cap B$. In fact, item (iii) could be replaced by "if A and B are in $\mathbb{G}$ then so is $A \cap B$" to yield equivalent definitions for a ring in a set.

10.1. ***Exercise.*** *(i) Show that the smallest algebra on X containing a topology for X consists of all sets $A \cap B$ or $A \cup B$ where A is an open set and B is a closed set.*

(ii) If $\mathbb{G}$ is a ring in X then both $\{ A \in \mathbb{P}(X) \mid A \in \mathbb{G}$ or $A^c \in \mathbb{G} \}$ and $\{ A \in \mathbb{P}(X) \mid A \cap B \in \mathbb{G}$ whenever $B \in \mathbb{G} \}$ are algebras on X.

*(iii) The set of "**clopen sets**" (that is, sets that are both open and closed) in a topology for X constitutes an algebra on X.*

10.2. ***Exercise.*** *(i) Suppose $\mathbb{G}$ is a ring in X. Define multiplication in $\mathbb{G}$ by $A \cdot B = A \cap B$ and addition by **symmetric difference**: $A \triangle B = (A - B) \cup (B - A)$. Show that $\mathbb{G}$ is a commutative (algebraic) ring with these operations. This ring has identity when the union of all sets in $\mathbb{G}$ is a member of $\mathbb{G}$. An additive subgroup $\mathbb{S}$ of this ring is an ideal exactly when $A \subset B$, $A \in \mathbb{G}$ and $B \in \mathbb{S}$ imply $A \in \mathbb{S}$.*

(ii) Suppose X is infinite and $\mathbb{G}$ is a ring in X. Let $\mathbb{S}$ denote the finite members of $\mathbb{G}$. Then $\mathbb{S}$ is an ideal.

When thinking of rings and algebras of sets, bear in mind two basic examples. One possible ring in $\mathbb{R}$ would be all finite unions of bounded subintervals of $\mathbb{R}$. An algebra on $\mathbb{R}$, obviously closely related to this ring, would be all finite unions of subintervals (bounded or not) of $\mathbb{R}$.

Apart from its raw defining qualities, an algebra on a set has useful properties which will be used throughout this work. Turn to Section **??** on Boolean Algebras and Rings to find out how some of these properties, when extracted and studied on their own, necessarily return to their roots as an algebra on a set.

Because of the way we handle the material of later chapters, rings in a set will be far less common than algebras on a set.

Topology is a huge subject and the outline found below only touches on some highlights of what is often called point-set topology. For more on point-set topology see Kelley, *General Topology* [**?**], Dugundji, *Topology* [**?**], Steen and Seebach, *Counterexamples in Topology* [**?**] and Engelking, *General Topology* [**?**].

11. Essential Definitions and Properties

A **topology** on a nonempty set Y is a family of sets $\mathbb{T} \subset \mathbb{P}(Y)$ that is closed under arbitrary unions and finite intersections and contains both $\varnothing$ and Y. A **topological space** is a pair $(\mathbf{Y}, \mathbb{T})$ where Y is a set and $\mathbb{T}$ is a topology on Y. Often the phrase "Y is a topological space with topology $\mathbb{T}$" is used to indicate this pair. Sometimes we will write "Y is a topological space" and a topology is simply presumed to exist. Members of $\mathbb{T}$ are called **open sets** and the complements of open sets are called **closed sets.** If B contains an open set containing $x \in Y$, B is called a **neighborhood of** x.

A subset A of a topological space is called an $\mathbf{F}_\sigma$ set if it is the union of a countable family of closed sets. It is called a $\mathbf{G}_\delta$ set if it is the intersection of a countable family of open sets. A set is a G_δ set if and only if its complement is an F_σ set.

If A is a nonempty subset of Y, the **subspace topology** on A consists of all sets of the form $A \cap B$ for $B \in \mathbb{T}$, and can be denoted $\mathbb{T}|_{\mathbf{A}}$. Some texts call this the **relative topology** on A. Members of $\mathbb{T}|_A$ are called **relatively open**, while the complements in A of relatively open sets are called **relatively closed**.

In $\mathbb{R}$ the **usual topology** consists of the empty set together with all sets that can be formed as unions of open intervals.

For a nonempty set X, $\mathbb{P}(X)$ is called the **discrete topology**, while $\{\varnothing, X\}$ is called the **indiscrete topology**. A subset $A \subset Y$ is called **discrete** if it is discrete with subspace topology.

The set of topologies on Y forms a lattice with containment order. In fact any set of topologies on Y has a least upper bound and a greatest lower bound in the set of topologies on Y. If $\mathbb{T}$ and $\mathbb{B}$ are two topologies on Y and $\mathbb{T} \subset \mathbb{B}$ we say that $\mathbb{T}$ is **coarser** than $\mathbb{B}$ and that $\mathbb{B}$ is **finer** than $\mathbb{T}$.

If $r\colon J \to Y$ is a net we call $x \in Y$ a **limit of the net** r and write $r(\alpha) \xrightarrow{\alpha} x$ if and only if r is eventually in any neighborhood of x. When limits of a net exists we say the **net converges** or, when we have a specific limit in mind, **converges to** x. These definitions agree with our earlier definition from page 14 of limits for nets in $\mathbb{R}$, when $\mathbb{R}$ is given the usual topology.

When the directed set J is finite every net defined on J is convergent. We are not particularly interested in nets of this kind, though they are not excluded.

x is called a **cluster point** of r if r is frequently in any neighborhood of x. x is called an **accumulation point** of r if r is frequently in $N - \{x\}$ for every neighborhood N of x. We note that if J is infinite and if a net r has no cluster points then $r(J)$ must be infinite too.

If $A \subset Y$ the point $x \in Y$ is called an **adherent point of A** if it is a cluster point of some net in A. An adherent point x for which every neighborhood of x contains a member of $A - \{x\}$ is called a **limit point of A**. This is the same as saying that x is an accumulation point of some net in A. The set of limit points is called the **derived set of A**, and can be denoted $\mathbf{A'}$. A member of A that is not a limit point of A is called an **isolated point of A**. An isolated point is a member of A that has a neighborhood containing no other point of A. The set of isolated points of A, if nonempty, is discrete. A closed set with no isolated points is called **perfect**.

11.1. ***Proposition.*** *Suppose $r\colon J \to Y$ is a net in the topological space $(Y, \mathbb{T})$.*

(*i*) *Every subnet of a convergent net converges, and to all the same limits.*

(*ii*) *x is a cluster point of r if and only if r has a subnet converging to x.*

(*iii*) *x is an accumulation point of r if and only if r has a subnet in $Y - \{x\}$ converging to x.*

Proof. We will prove the second of these statements, and leave the other two to the reader.

Let $K = \{\, (\alpha, A) \mid \alpha \in J,\ A \in \mathbb{T},\ x \in A \text{ and } r(\alpha) \in A \,\}$. Direct K by $(\alpha, A) \le (\beta, B)$ precisely when $\alpha \le \beta$ and $B \subset A$. Let $s\colon K \to Y$ be the net $s((\alpha, A)) = r(\alpha)$ The function $f\colon K \to J$ defined by $f((\alpha, A)) = \alpha$ establishes s as a subnet of r. s converges to x. The converse is obvious. □

If $A \subset Y$, let $\overline{\mathbf{A}}$ be the intersection of all closed sets containing A. $\overline{A}$ is called the **closure of** A. $\overline{A}$ is itself closed, so A is closed if and only if $A = \overline{A}$. A point p is in $\overline{A}$ if and only if every open set containing p contains a point of A.

The **interior of** A is the union of all the open sets contained in A. Alternatively, it is $\overline{A^c}^{\,c}$. The interior of A can be denoted $\mathbf{A^o}$. A^o is open so A is open exactly when $A = A^o$.

The **boundary of** A is $\overline{A} \cap \overline{A^c}$. The boundary of A is denoted $\partial\mathbf{A}$. Boundary points of A are characterized by the property that every neighborhood of the point contains a member of A and a member of A^c. Note that $\partial\partial A - \partial A = \varnothing$ and $\partial A = \overline{A} - A^o$.

11.2. ***Proposition.*** *Suppose A is a nonempty subset of topological space.*

(*i*) *$x \in \overline{A}$ if and only if there is a net in A converging to x.*

(*ii*) *$x \in A'$ if and only if there is a net in $A - \{x\}$ converging to x.*

(*iii*) *$x \in \partial A$ if and only if there are nets in both A and A^c converging to x.*

(*iv*) *A is open if and only if no net in A^c converges to a point of A.*

(*v*) *A is open exactly if any net converging to a point of A is eventually in A.*

Proof. We will prove only (i), leaving the rest as an exercise. Suppose $x \in \overline{A}$. Let J consist of the open sets containing x ordered by reverse containment. If $B \in J$, $A \cap B$ cannot be empty, for if it were $Y - B$ would be a closed set containing A, contrary to the assumption that x is in the intersection of all such sets. So let $r(B)$ be the choice of an element of $A \cap B$ for each $B \in J$. The net r converges to x.

On the other hand if $x \notin \overline{A}$ then $\overline{A}^{\,c}$ is a neighborhood of x which does not intersect A so no net in A could converge to x. □

11.3. ***Exercise.*** *(i) If A is any set, A' is closed.*
(ii) If A is closed, $A' \subset A$.
(iii) If Y is without isolated points and A is any set in Y, $A^o \subset A'$.
(iv) $A = A'$ if and only if A is perfect.
(v) $(C \cup D)' = C' \cup D'$.
(vi) $(A')' \subset A \cup A'$.
(vii) Suppose $\aleph$ is a cardinal number exceeding the cardinality of A'. Define $A^{(0)}$ to be A'. Having defined $A^{(\beta)}$ for all ordinals β with $\beta < \gamma$ let $A^{(\gamma)}$ be either $\left(A^{(\beta)}\right)'$ if $\gamma = \beta + 1$ or, if γ is a limit ordinal, $\bigcap_{\beta<\gamma} A^{(\beta)}$. This serves, through a recursive definition, to define $A^{(\gamma)}$ for all $\gamma \leq \aleph$. For each $\beta \leq \gamma$ we have $A^{(\beta)} \supset A^{(\gamma)}$, so these sets are nested and must, due to the size of $\aleph$, be constant on some terminal segment T_γ of $\aleph$. $\left(A^{(\gamma)}\right)' = A^{(\gamma)}$, so $A^{(\gamma)}$ is perfect. The ordinal γ is called the ***Cantor-Bendixson rank*** *of A.*

The ideas behind the Cantor-Bendixson rank of a set, defined above, stem from the very beginnings of modern set theory in the early 1870s. Cantor was considering convergence of Fourier series, and the set of exceptional points of those series. He showed that if two (potentially different) Fourier Series converge to the same value except possibly at a set of points A for which $A^{(n)} = \varnothing$ for some *finite* n then these series must be, actually, the same series. Contemplating the potential complexities of sets of this type (outside the context of Fourier Series now) led him to imagine iterating the derivation process more than finitely many times, and on to thoughts about well ordering and cardinality.

If X is also a topological space with topology $\mathbb{S}$, a function $g\colon Y \to X$ is called **continuous** when $g^{-1}(\mathbb{S}) \subset \mathbb{T}$. Note that g is continuous if and only if $g^{-1}(K)$ is closed in Y whenever K is closed in X.

The function g is called **open** if $g(\mathbb{T}) \subset \mathbb{S}$. The function g is called **closed** if $g(K)$ is closed for every closed subset K in Y. Whenever the function g has an inverse function, the properties of being an open function or a closed function are equivalent.

If g is continuous and open and if g has an inverse function it is called a **homeomorphism**. The inverse is also a homeomorphism. Two topological spaces are called **homeomorphic** by virtue of the existence of a homeomorphism between them.

A property which can be possessed by a topological space is called **topological** if pairs of homeomorphic spaces either both possess the property or both do not. In other words, homeomorphic spaces have identical topological properties.

A subset A of Y is called **dense in Y** if it has nonvoid intersection with every nonempty open set. This is the same as saying that $\overline{A} = Y$. It also is equivalent to saying that each point of Y is either a point or a limit point of A.

Y is called **separable** if there is a countable dense subset of Y.

A subset B of Y is called **nowhere dense** if $\overline{B}^o = \varnothing$. It is obvious that a nowhere dense set has empty interior. This happens exactly when the open set $\overline{B}^c$ is dense in Y.

A subset A of Y is said to be of **first category** or **meager** if it is the union of a countable collection of nowhere dense subsets. Y itself will be of first category if $Y = \bigcup_{i=1}^{\infty} B_i$ where each $\overline{B_i}^{o} = \varnothing$. Each $\overline{B_i}^{c}$ is open and dense so

$$Y^c = \varnothing = \bigcap_{i=1}^{\infty} B_i^c \supset \bigcap_{i=1}^{\infty} \overline{B_i}^{c}$$

which means that the intersection of the open dense sets $\overline{B_i}^{c}$ is empty. Conversely, *any* countable collection of open dense subsets with empty intersection can be used to represent Y as a countable union of nowhere dense (closed) sets. We conclude that Y is first category if and only if there is a countable collection of open dense subsets of Y with empty intersection.

A subset A of Y is said to be of **second category** if it is **not** first category. So for a second category set, every countable collection of open dense sets has nonempty intersection.

A **neighborhood base at** $x \in Y$ is a collection of neighborhoods of x such that every neighborhood of x contains one of these sets. A **neighborhood subbase at** $x \in Y$ is a collection of neighborhoods of x whose finite intersections comprise a neighborhood base at x.

A **neighborhood base for the topology** is a collection of neighborhoods which is a neighborhood base at each point in Y, and a **neighborhood subbase for the topology** is a collection of neighborhoods whose finite intersections form a neighborhood subbase at each point.

Sometimes adjectives are applied to these. For example a closed neighborhood base would be a neighborhood base consisting entirely of closed sets.

A **base for the topology on** Y is an open neighborhood base for that topology. A **subbase for the topology on** Y is a collection of open sets whose finite intersections constitute a base. Some sources use the variant vocabulary "basis" and "subbasis" for these concepts.

A function $g\colon Y \to X$ is called **continuous at** $\mathbf{a} \in \mathbf{Y}$ if whenever r is a net converging to a in Y the net $g \circ r$ converges to $g(a)$.

11.4. ***Exercise.*** *(i) g is continuous exactly when g is continuous at a $\forall a \in Y$.*

(ii) g is continuous if and only if $g^{-1}(\mathbb{G})$ is a collection of open sets for any subbase $\mathbb{G}$ of the range topology.

(iii) g is continuous exactly when $g\left(\overline{K}\right) \subset \overline{g(K)}$ for each subset K of the domain.

(iv) g is an open function if and only if $g(\mathbb{F})$ is a collection of open sets for any base $\mathbb{F}$ of the domain topology.

11.5. ***Exercise.*** *A continuous function $g\colon Y \to X$ is determined by its values on any dense subset of Y.*

*If $A \subset Y$ let $\mathbf{h}|_{\mathbf{A}}$ denote the function defined only on A but whose values there agree with h. $h|_A$ is called the **restriction of** $\mathbf{h}$ **to** $\mathbf{A}$. Whenever h is continuous, $h|_A$ is continuous when A has the subspace topology.*

Rephrasing the first sentence using this notation we have: If $\overline{A} = Y$ and both $g\colon Y \to X$ and $h\colon Y \to X$ are continuous and $g|_A = h|_A$ then $g = h$.

We further explore the concept of continuity with several useful results involving functions with range $\mathbb{R}$ or $[-\infty, \infty]$ with the usual topologies. Nontrivial continuous functions with $\mathbb{R}$ as either domain or range are particularly important, and through their very existence convey many convenient properties of $\mathbb{R}$ into the other space involved.

11.6. ***Exercise.*** *Suppose f and g are real valued continuous functions on Y and c is a real constant. The constant function defined by $h(x) = c$ for all $x \in Y$ is continuous. The sum and product functions $f + g$ and fg are continuous. If f is never 0 then $\frac{1}{f}$ is continuous.*

11.7. ***Exercise.*** *Suppose $f : Y \to [-\infty, \infty]$ and $g : Y \to [-\infty, \infty]$. Define $\mathbf{f} \vee \mathbf{g} : Y \to [-\infty, \infty]$ to be the larger of f or g at each $x \in Y$: specifically, $(f \vee g)(x) = f(x)$ if $f(x) > g(x)$ and $g(x)$ otherwise. Define $\mathbf{f} \wedge \mathbf{g}$ to be the lesser of f or g. Wherever both are finite, this is $f(x) + g(x) - (f \vee g)(x)$. If f and g are both continuous then $f \vee g$ and $f \wedge g$ are too.*

Suppose X is any topological space and D is a directed set and n is a net, with directed set D, of real valued functions with domain X. For each $x \in X$ the net $n(x)$ will be the net formed by evaluating, for each $\alpha \in D$, the function n_α at x. We say the net n **converges pointwise to a real valued function f** defined on X if $n(x)$ converges to $f(x)$ for each $x \in X$. This means that for each x and any $\varepsilon > 0$ the net $n(x)$ is eventually in $(f(x) - \varepsilon, f(x) + \varepsilon)$. More precisely, for each $\varepsilon > 0$ and each $x \in X$ there is a terminal segment $T_{\varepsilon,x}$ of D so that

$$n_{T_{\varepsilon,x}}(x) \subset (f(x) - \varepsilon, f(x) + \varepsilon).$$

We say the net **converges uniformly to f** if for each ε the terminal segment may be chosen independently of x: the same terminal segment "works" for each domain member. More precisely,

$$\forall \varepsilon > 0 \ \exists \text{ terminal segment } T_\varepsilon \text{ so that } n_{T_\varepsilon}(x) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \ \forall x \in X.$$

11.8. ***Lemma.*** *If n is a net of real valued functions converging uniformly to f as above and if each n_α is continuous then f is continuous.*

Proof. Suppose $\varepsilon > 0$. Select $\beta \in D$ so that $|f(x) - n_\beta(x)| < \varepsilon/3$ for all $x \in X$. Now pick $x \in X$ and suppose $r : E \to X$ is any net in X converging to x. Since n_β is continuous there is $e \in E$ so that $k \geq e$ implies $|n_\beta(r(k)) - n_\beta(x)| < \varepsilon/3$. So

$$\begin{aligned} |f(x) - f(r(k))| &\leq |f(x) - n_\beta(x)| + |n_\beta(x) - n_\beta(r(k))| + |n_\beta(r(k)) - f(r(k))| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since this can be done for each $\varepsilon > 0$ we conclude that the net $f \circ r$ converges to $f(x)$. Since this can be done for any net converging to x and any $x \in X$ we conclude that f is continuous. □

11.9. ***Corollary. The Weierstrass "M" Test*** *Suppose X is a topological space and $f_i : X \to \mathbb{R}$ for $i \in \mathbb{N}$ is a sequence of continuous functions.*

Suppose further that for each i the value of $|f_i(x)|$ can never exceed M_i and that $\sum_{i=0}^{\infty} M_i < \infty$.

Then $\sum_{i=0}^{\infty} f_i$ exists: that is, the partial sums form a sequence that converges pointwise. The convergence is uniform, and the limit function is a continuous function.

Proof. The proof is an exercise. □

11.10**.** ***Exercise.*** *Suppose* $\nu\colon D \to Y$ *is a net in topological space* Y*. For each terminal segment* $T_d \subset D$ *we see that* $\nu\,(T_b) \subset \nu\,(T_a)$ *when* $a < b$*.*

We now consider the case of $Y = [-\infty, \infty]$*. So* $l(d) = \inf \nu\,(T_d)$ *and* $u(d) = \sup \nu\,(T_d)$ *are extended real numbers for each* $d \in D$*. The net* l *is nondecreasing while the net* u *is nonincreasing. So both converge in* $[-\infty, \infty]$*. We define* $\liminf \nu$ *to be the limit of the net* l *and* $\limsup \nu$ *to be the limit of the net* u*.*

Show that ν *converges to* $c \in [-\infty, \infty]$ *exactly when* $\liminf \nu = \limsup \nu = c$*.*

A function $f\colon X \to [-\infty, \infty]$ is called **lower semicontinuous** provided $f^{-1}((b, \infty])$ is open in X for all real b. Equivalently, f is lower semicontinuous exactly when $f^{-1}([-\infty, b])$ is closed in X for all real b.

f is called **upper semicontinuous** exactly when $f^{-1}([-\infty, b))$ is open in X for all real b. Equivalently, f is upper semicontinuous exactly when $f^{-1}([b, \infty])$ is closed in X for all real b.

11.11**.** ***Exercise.*** *(i) An extended real valued function is continuous exactly when it is both upper and lower semicontinuous.*

(ii) The characteristic function of an open set is lower semicontinuous. The characteristic function of a closed set is upper semicontinuous.

(iii) Now suppose $f\colon X \to [-\infty, \infty]$*. Show that* f *is lower semicontinuous exactly when* $\liminf f \circ \mu \geq f(p)$ *whenever* μ *is a net in* X *converging to* p*. This can be (very) loosely paraphrased as "The least values of* f *near* p *are not much less than* $f(p)$*."*

(iv) Suppose f *and* g *are lower semicontinuous functions and* c *is a real number.* $f \wedge g$ *and* $f \vee g$ *are both lower semicontinuous.* cf *is lower semicontinuous if* $c > 0$ *and upper semicontinuous if* $c < 0$*.* fg *is lower semicontinuous when* f *and* g *are nonnegative.* $f + g$ *is lower semicontinuous if* f *and* g *are never oppositely infinite.*

(v) Suppose $\mathbf{S}$ *is a set of real valued functions with domain* X *and each member of* $\mathbf{S}$ *is lower semicontinuous. Suppose also that for each* x *the there is a number* $f(x) = \sup\{\, g(x) \mid g \in \mathbf{S}\,\}$*. A typical situation has* $\mathbf{S} = \{\, f_n \mid n \in \mathbb{N}\,\}$ *where for each* x *the numbers* $f_n(x)$ *form a nondecreasing sequence which converges to a number* $f(x)$*. However this problem pertains to any set of lower semicontinuous functions bounded above at each point in* X*. Show that the function* f *is also lower semicontinuous.*

(vi) Formulate and prove statements similar to (iii), (iv) and (v) for upper semicontinuous functions.

Y is called **first countable**, or simply $\mathbf{C_I}$, if for each point x in Y there is a countable neighborhood base at x. If there is a countable neighborhood base at a point, the neighborhoods comprising the base at this point can be chosen to be a countable chain of open sets. A chain of sets is sometimes described, colloquially, as **nested**.

A topological space is called **second countable**, or simply $\mathbf{C_{II}}$, if there is a countable base (or subbase) for the topology. It is obvious that every C_{II} space is C_I. If Y is C_{II} it is clearly separable, but the converse is false.

11.12. ***Exercise.*** *Give $\mathbb{R}$ the topology with base $\{\, \{0, r\} \mid r \in \mathbb{R} \,\}$. Every nonempty open set contains 0 so the topology is separable but obviously not C_{II}.*

If $r\colon \mathbb{N} \to Y$ is a sequence we define the sequence $s\colon \mathbb{N} \to Y$ to be a **subsequence of** r when there is an **order preserving** $f\colon \mathbb{N} \to \mathbb{N}$ for which $s = r \circ f$ and $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}$ such that $f(T_n) \subset T_m$. It is worth noting that, although every sequence is a net, a subnet of a sequence need not be a subsequence even if its domain is $\mathbb{N}$.

The characterization of the concepts of closed, open, continuity and boundary, given above in terms of nets, can be restated in terms of sequences and subsequences in C_I spaces. This is handy because sequences and subsequences are often easier to think about than generic nets.

11.13. ***Exercise.*** *Suppose Y is a C_I topological space. Prove the following:*

(i) x is a cluster point of a sequence r if and only if r has a subsequence converging to x.

(ii) x is an accumulation point of a sequence r if and only if r has a subsequence in $Y - \{x\}$ converging to x.

(iii) $x \in \overline{A}$ if and only if there is a sequence in A converging to x.

(iv) $x \in \partial A$ exactly when there are sequences in A and in A^c converging to x.

(v) A is open if and only if no sequence in A^c converges to a point of A.

(vi) A is open if and only if every sequence converging to any point of A is eventually in A.

(vii) If X is another topological space then $g\colon Y \to X$ is continuous if and only if for each a in Y, $g \circ r$ converges to $g(a)$ whenever r is a sequence in Y converging to a.

Two nonempty sets A and B are called **separated** if $\overline{A} \cap B = \varnothing$ and $\overline{B} \cap A = \varnothing$. This is the same as saying that no net in B converges to any point of A, and no net in A converges to any point in B. When there exist sets A and B with $C \subset A \cup B$ and if $C \cap A \neq \varnothing$ and $C \cap B \neq \varnothing$ and if A and B are separated we say that C itself is **separated**. C is said to be **separated by A and B** in this case. In a different but similar usage, if R and S are separated and $C \cap R \neq \varnothing \neq C \cap S$ the subsets $C \cap R$ and $C \cap S$ are said to be **separated by R and S**.

A **Urysohn function** is a continuous function $f\colon Y \to [0, 1]$. The set of Urysohn functions for a topological space will appear several times in different contexts in these notes.

A Urysohn function f is said to be a **Urysohn function for sets A and B** provided $\{f(A), f(B)\} = \{\{0\}, \{1\}\}$. The sets A and B are said to be **separated by a Urysohn function** in case such a function exists. We will say that the Urysohn function **precisely separates A and B** if $\{A, B\} = \{f^{-1}(0), f^{-1}(1)\}$.

11.14. ***Exercise.*** *If A and B are separated by a Urysohn function then A and B are separated. In fact, more is true: there are nonintersecting open sets V and W with $\overline{A} \subset V$ and $\overline{B} \subset W$ and $\overline{V} \cap \overline{W} = \varnothing$.*

12. Tools to Construct Examples

(i) Suppose X is a set and $\mathfrak{F} = \{f_\alpha \colon X \to Y_\alpha \mid \alpha \in A\}$ is an indexed family of functions where each Y_α possesses a topology $\mathbb{G}_\alpha$. Each $f_\alpha^{-1}(\mathbb{G}_\alpha)$ is a topology on X. The coarsest topology on X containing *all* these topologies is called the **initial topology induced by $\mathfrak{F}$**. It is the smallest topology with respect to which *all* of the f_α are continuous.

12.1. ***Exercise.*** *If Z is any topological space and X has this initial topology, a function $G \colon Z \to X$ is continuous when and only when $f_\alpha \circ G \colon Z \to Y_\alpha$ is continuous for every $\alpha \in A$.*

On the other hand, a function $H \colon X \to Z$ is continuous exactly when, for every open U in Z and $x \in H^{-1}(U)$ there are a finite number of members $f_{\alpha_i} \in \mathfrak{F}$ and $U_i \in \mathbb{G}_{\alpha_i}$ $(i = 1, \dots, n)$ so that $x \in \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_i) \subset H^{-1}(U)$.

The result remains true if U and the U_i are required to be drawn from subbases for the relevant topologies.

(ii) Suppose Y is a set and $\mathfrak{G} = \{g_\alpha \colon X_\alpha \to Y \mid \alpha \in A\}$ is an indexed family of functions where each X_α possesses a topology $\mathbb{G}_\alpha$. Each g_α induces a topology $\mathbb{F}_\alpha$ on Y by declaring $S \in \mathbb{F}_\alpha$ whenever $g_\alpha^{-1}(S) \in \mathbb{G}_\alpha$. This is the largest topology on Y with respect to which g_α is continuous. The intersection of *all* these topologies is called the **final topology induced by $\mathfrak{G}$**. It is the finest topology on Y with respect to which *all* of the g_α are continuous.

12.2. ***Exercise.*** *If Z is any topological space and Y has this final topology, a function $G \colon Y \to Z$ is continuous when and only when $G \circ g_\alpha \colon X_\alpha \to Z$ is continuous for every $\alpha \in A$.*

On the other hand, a function $H \colon Z \to Y$ is continuous exactly when $H^{-1}(U)$ is open whenever $g_\alpha^{-1}(U)$ is open for every α.

(iii) Let A be any nonempty indexing set and for each a in A let X_a be a set with topology $\mathbb{T}_a$. Define $Y = \prod_{\mathbf{a} \in \mathbf{A}} \mathbf{X_a}$ to be $\{\, f \colon A \to \bigcup_{a \in A} X_a \mid f(a) \in X_a \ \forall a \in A \,\}$.

For each $a \in A$ we define $\pi_a \colon Y \to X_a$ by $\pi_a(f) = f(a)$. This is called the **projection map** onto the **factor space** X_a of the **product space** Y. It is also called **the evaluation map at the index a**.

Suppose $f \in Y$ is selected. For each $a \in A$ we define $\iota_a^f \colon X_a \to Y$ by $\iota_a^f(x)(b) = f(b)$ whenever $b \neq a$ and $\iota_a^f(x)(a) = x$. It leaves f alone except at index a, and inserts the value x there. This is called the **injection map with base point f of the factor space $\mathbf{X_a}$ into the product space Y**.

An **open cylinder** in Y is a subset C of Y of the form $C = \bigcap_{i=1}^n \pi_{a_i}^{-1}(B_i)$ for a *finite* selection of $a_i \in A$ and $B_i \in \mathbb{T}_{a_i}$ for $i = 1, \dots, n$. This implies $\pi_a(C) = X_a$ for all but a finite number of indices.

Although obvious, it bears mentioning that the condition $\pi_a(K)$ be open for all a and $\pi_a(K) = X_a$ for all but a finite number of a does not imply K is open. For instance, consider the product space $\mathbb{R}^2$ and let K be the graph of $y = x$ on the interval $(0, 1)$. So $\pi_x(K) = \pi_y(K) = (0, 1)$ but K is not open.

The collection of open cylinders in Y forms a base for a topology on Y, called the **product topology**. A subbase for this topology is the collection of all $\pi_a^{-1}(S)$ for $a \in A$ and $S \in \mathbb{T}_a$. The product topology is the initial topology induced by the set of projections $\{\pi_a \colon Y \to X_a \mid a \in A\}$. Y with this topology is called a **product space**. It is the coarsest topology on Y with respect to which *all* of the projections π_a are continuous.

12.3**. *Exercise.*** *Suppose $f \colon Z \to Y$ where Z is a topological space and $Y = \prod_{a\in A} X_a$ is a product space. Show that f is continuous exactly when $\pi_a \circ f$ is continuous for every $a \in A$. Show that a net ν in Y converges exactly when every coordinate net $\pi_a \circ \nu$ converges in X_a.*

When all X_a are equal to some set X with common topology $\mathbb{T}$ we use X^A to denote $\prod_{a\in A} X_a$.

12.4**. *Exercise.*** *If X^A and X^B are two product spaces and $\phi \colon A \to B$ is any function, the function $\Phi \colon X^B \to X^A$ defined by $\Phi(g) = g \circ \phi$ is continuous. (hint: Suppose n is any net in X^B. n is convergent exactly when every $\pi_b \circ n = n(b)$ is convergent in X. The function Φ is continuous exactly when $\Phi \circ n$ converges for every convergent net n. This happens exactly when $\pi_a \circ \Phi \circ n = n(\phi(a))$ converges in X for every a.)*

The **box topology** on the product set Y as above is that formed using as base those subsets K of Y of the form $K = \bigcap_{a\in A} \pi_a^{-1}(B_a)$ with $B_a \in \mathbb{T}_a \forall a \in A$. A set of this kind is called an **open box.**

The box and product topologies differ unless all but finitely many of the factor spaces are indiscrete and, generally, the product topology is more useful. There are so many open sets in the box topology that convergence is too strong a condition for most purposes. For instance, with the box topology on $\mathbb{R}^{\mathbb{N}}$ the function $G \colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ given by

$$G(t) = (t,\, t,\, t, \dots)$$

is not continuous. However $\pi_n \circ G$ is continuous for each n so G is continuous when its range has the product topology.

The projections π_a and injections ι_a^f are all continuous when Y has either product or box topologies.

Suppose Z is a topological space and $H \colon Y \to Z$. If H is continuous with the product topology on Y it is called **jointly continuous,** and this vocabulary is usually used when the function H is being built in some way from functions defined on the factor spaces.

If $H \circ \iota_a^f$ is continuous for every $f \in Y$ and $a \in A$ then H is called **continuous in each factor** or **continuous along each slice.**

12.5**. *Exercise.*** *We suppose Y is a product space as above with injections $\iota_a^f \colon X_a \to Y$ for indices $a \in A$ and $f \in Y$ and that Z is a topological space and $H \colon Y \to Z$.*

Even in the case of two factors it may be that $H \circ \iota_a^f \colon X_a \to Z$ is continuous for every f and a, yet H itself fails to possess the virtue of joint continuity. Find an example where $Z = \mathbb{R}$ and $Y = \mathbb{R}^2$. (hint: Examine the function given by $f(0,0) = 0$ and otherwise $f(x,y) = xy/(x^2+y^2)$.)

(iv) If X has topology $\mathbb{F}$ and Y has topology $\mathbb{G}$ and if $X \cap Y = \varnothing$ we define the **free union topology** on $X \cup Y$ to be the topology, denoted $\mathbb{F} + \mathbb{G}$, consisting of the sets $\{S \subset X \cup Y \mid S \cap X \in \mathbb{F} \text{ and } S \cap Y \in \mathbb{G}\}$.

More generally, if Y_α is a nonempty set with topology $\mathbb{G}_\alpha$ for each α in a nonempty index set A and if $Y_\alpha \cap Y_\beta = \varnothing$ when $\alpha \neq \beta$ we define the **free union topology** on $\bigcup_{\alpha \in A} Y_\alpha$ to be the topology, denoted $\oplus_{\alpha \in \mathbf{A}} \mathbb{G}_\alpha$, consisting of sets $\{S \subset \bigcup_{\alpha \in A} Y_\alpha \mid S \cap Y_\alpha \in \mathbb{G}_\alpha \text{ for all } \alpha \in A\}$.

(v) Suppose $f \colon X \to Y$ is onto Y and X has topology $\mathbb{F}$. The final topology induced by f on Y is called the **identification topology for f**. Note: this vocabulary is only used when f is onto Y.

If $g \colon X \to Y$ and g is onto Y and X has topology $\mathbb{F}$ and Y has topology $\mathbb{G}$ the function g is called an **identification map** provided the identification topology for g is $\mathbb{G}$.

(vi) Suppose P is a partition of a set X with topology $\mathbb{F}$. For each $x \in X$ let $[x]$ denote the member of P containing x.

The function $f \colon X \to P$ defined by $f(x) = [x]$ is called the **quotient function for P**. The identification topology on P, denoted $\mathbb{F}/\mathbf{P}$, is called the **quotient topology** and P with this topology is called a **quotient space of the topological space X with topology** $\mathbb{F}$.

The quotient function f is open exactly when $\bigcup_{x \in A} [x]$ is open in X for every open A in X.

The quotient function f is closed exactly when $\bigcup_{x \in A} [x]$ is closed in X for every closed A in X.

We will be interested in partitions P where the classes $[x]$ have certain useful properties. For example, we might require that each $[x]$ be closed in X.

It is an obvious but useful fact that if $f \colon X \to P$ is the quotient function for partition P with quotient topology, and $g \colon P \to Z$ then g is continuous exactly if $g \circ f \colon X \to Z$ is continuous.

(vii) Suppose $X \cap Y = \varnothing$ and X has topology $\mathbb{F}$ and Y has topology $\mathbb{G}$ and $X \cup Y$ has the free union topology $\mathbb{F} + \mathbb{G}$. Suppose further that $A \subset X$ and $B \subset Y$ and A is nonempty and closed and $g \colon A \to B$ is continuous with respect to subspace topologies and onto B.

Create a partition P on $X \cup Y$ by

$$P = \{\{x\} \mid x \in (X - A) \cup (Y - B)\} \bigcup \{\{b\} \cup g^{-1}(b) \mid b \in B\}.$$

Points not in A or B are related only to themselves, while points in A or B are also related to each other through the services of g.

The set $X \cup Y$ with topology $(\mathbb{F} + \mathbb{G})/P$ is called, awkwardly, **X attached to Y by g** and the process is called **attachment**.

If g is one-to-one, the process stitches X and Y together along the "seam" consisting of points paired by g.

If Y (and hence B) is a one point set, the process essentially collapses A to a point. More generally, if $Y = B$ is discrete then the process collapses each

$g^{-1}(b) \subset A \subset X$ to a point for each $b \in B$. Any subset J in discrete B is both closed and open. We have $g^{-1}(J)$ relatively closed in closed A, and hence closed in X. Since $g^{-1}(J)$ is also open in A there is an open set $S_J \subset X$ with $g^{-1}(J) = S_J \cap A$.

13. Separation

A topological space is called $\mathbf{T_0}$ if and only if given any two distinct points at least one of the two has a neighborhood not containing the other. So for distinct points x and y, $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

A topological space is called $\mathbf{T_1}$ if and only if given any two distinct points each one has a neighborhood not containing the other. So for distinct points x and y we have $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$: that is, single point sets are all closed. This property can be rephrased by saying that *distinct points are separated.*

The topology $\{\varnothing,\ \{a\},\ \{a,b\}\}$ is T_0 but not T_1.

A topological space is called **Hausdorff** or, synonymously, $\mathbf{T_2}$ if and only if any two distinct points have nonintersecting neighborhoods. So Y is T_2 if and only if every convergent net in Y has one limit. This property can be rephrased by saying that *distinct points are separated by open sets.*

Let X be the set $\mathbb{R} \cup \{*\}$ where the $*$ is not a real number. The open sets in X are the usual open sets in $\mathbb{R}$ together with any set that can be obtained by taking one of these open sets which contains 0 and either adding $*$ to it or replacing 0 by $*$. So every point in this set has a neighborhood which is homeomorphic to $\mathbb{R}$. Still, this topology is T_1 but not T_2.

13.1**. *Exercise.*** *Suppose Y is an infinite set. Give Y the topology consisting of the empty set and the complements of finite subsets in Y. This is called the **cofinite topology** on Y. This topology is T_1 but not T_2.*

13.2**. *Exercise.*** *(i) A finite T_2 topology is discrete. In fact, in a T_2 space any set that is not closed is infinite.*

(ii) Y is T_2 if and only if for each x and each y with $x \neq y$ there is an open set V_y containing y with $x \notin \overline{V_y}$.

(iii) Suppose $f \colon X \to Y$ is continuous and Y is T_2. If $a, b \in X$ and if a and b cannot be separated by open sets then $f(a) = f(b)$.

(iv) Suppose $f \colon X \to Y$ and $g \colon X \to Y$ are continuous and Y is T_2. Then the set of all $x \in X$ for which $f(x) = g(x)$ is closed in X.

Y is called $\mathbf{T_3}$ if $\{x\}$ and A can be separated by open sets for each point $x \in Y$ and each closed set A not containing x. Y is called **regular** if it is T_0 and T_3.

13.3**. *Exercise.*** *(i) Y is T_3 exactly when for each $p \in Y$ and open set W containing p there is an open set V with $p \in V \subset \overline{V} \subset W$. So Y is T_3 exactly when there is a closed neighborhood base for every point in Y.*

(ii) Let $\mathbb{B}$ be a subbase for the topology on Y. Y is T_3 if and only if whenever $p \in W \in \mathbb{B}$ there is open V with $p \in V \subset \overline{V} \subset W$.

(iii) If Y is regular and p, q are distinct points in Y then there are open sets V_p and V_q containing p and q respectively with $\overline{V_p} \cap \overline{V_q} = \varnothing$.

(iv) Any subset of a T_i space with subspace topology is T_i for $i = 0, 1, 2$ or 3.

13.4. ***Exercise.*** *Consider* $D = (\mathbb{R}\times\mathbb{R})\cup\{*\}$ *Where* $*$ *is a point not in* $\mathbb{R}\times\mathbb{R}$.

For each (a,b) *in* $\mathbb{R}\times\mathbb{R}$ *with* $(a,b)\neq(0,0)$ *define* $\mathbb{A}_{(a,b)}$ *to be the set of all* $(r,s)\times(t,u)\subset\mathbb{R}\times\mathbb{R}$ *which contain* (a,b) *but do not contain* $(0,0)$. *Let* $\mathbb{A}$ *be the union of all the* $\mathbb{A}_{(a,b)}$ *for* $(a,b)\neq(0,0)$.

Form $\mathbb{A}_{(0,0)}$ *as the set of all* $\{(0,0)\}\cup((-r,r)\times(0,r))\subset\mathbb{R}\times\mathbb{R}$ *for* $r>0$.

Define $\mathbb{A}_*$ *to be the set of all* $\{*\}\cup((-r,r)\times(-r,0))\subset D$ *for all* $r>0$.

$\mathbb{A}\ \cup\ \mathbb{A}_{(0,0)}\cup\ \mathbb{A}_*$ *constitutes a base for a topology on* D *called by Steen and Seebach* [**?**] *the* ***double origin topology.***

Show that the double origin topology is T_2 *but not* T_3. *(hint: The closure of disjoint neighborhoods of* $(0,0)$ *and* $*$ *must overlap.)*

13.5. ***Exercise.*** *If* X *is* T_3 *and* C_{II} *then every closed set is a* G_δ. *(hint: Let* $\mathbb{B}$ *be a countable base for the topology and suppose* K *is closed. For each* $p\in K^c$ *there is an open set* V_p *with* $\overline{V_p}\cap K=\varnothing$. *Let* $\mathbb{A}$ *denote those members of* $\mathbb{B}$ *which are in at least one* V_p. *Let* $A_0, A_1, \ldots$ *denote an enumeration of the members of* $\mathbb{A}$. *Any open subset of* K^c *is a union of these* A_i. *Well order* K^c. *For each* i *there is at least one but possibly many members* p *of* K^c *for which* V_p *contains* A_i. *Let* p_i *be the least of these. Let* $M_i=\bigcup_{0\le j\le i}\overline{V_{p_i}}$ *and* $O_i=M_i^c$. *The* O_i *are nested and decrease in size as* i *grows. The closed set* K *is in every* O_i *and so is in the intersection of all of them. On the other hand, each member* p *of* K^c *is in at least one of the* A_i *so* p *is not in* O_i. *So* $K=\bigcap_{i\in\mathbb{N}}O_i$.*)*

Y is called $\mathbf{T_4}$ if each pair of nonintersecting closed sets can be separated by open sets. Y is called **normal** if it is T_1 and T_4.

13.6. ***Exercise.*** *We will define a topology called by Steen and Seebach* [**?**] *the* ***either-or topology.*** *Declare a subset of the interval* $[-1,1]$ *to be open if it contains either* 0 *or the entire interval* $(-1,1)$. *This is a topology on* $[-1,1]$ *and it is* T_0 *and* T_4. *It is not* T_1.

13.7. ***Exercise.*** *(i)* X *is* T_4 *exactly when for each closed* $B\subset X$ *and open set* W *containing* B *there is an open set* V *with* $B\subset V\subset\overline{V}\subset W$.

(ii) Y *is* T_4 *exactly when for each pair of closed nonintersecting subsets* A *and* B *there are open sets* V_A *and* V_B *containing* A *and* B *respectively and with* $\overline{V_A}\cap\overline{V_B}=\varnothing$.

(iii) A closed subset of a T_4 *space with the subspace topology is* T_4. *See Exercise 17.14 for an example of an open subset of a normal space which is not* T_4.

13.8. ***Exercise.*** *(i) Suppose that whenever* $x,y\in Y$ *and* $x\neq y$ *then* $\{x\}$ *can be separated from* $\{y\}$ *by a Urysohn function. Then* Y *is* T_2.

(ii) Suppose that whenever $x\in Y$ *and* A *is closed and nonempty and* $x\notin A$ *then* $\{x\}$ *can be separated from* A *by a Urysohn function. This condition implies that* Y *is* T_3. *When this condition holds the space is called* ***completely regular***. *We will usually refer to completely regular spaces as* $\boldsymbol{\mathcal{CR}}$. *If the space is* T_0 *and* $\mathcal{CR}$ *it is also* T_2, *and called a* ***Tychonoff space***.

(iii) A subset of a $\mathcal{CR}$ *space with subspace topology is* $\mathcal{CR}$.

(iv) If all pairs of disjoint nonempty closed sets can be separated by a Urysohn function then Y *is* T_4. *If, in addition,* Y *is* T_1 *then it is normal.*

(v) If X is a $\mathcal{CR}$ space then there is a set of real valued functions on X so that the topology on X is the initial topology for this set of functions.

We note that some books switch the meaning of normal and T_4. Those books also switch the meaning of regular and T_3. Further, some books do not identify the T_3 or T_4 properties except in the context of Hausdorff spaces: they either do not identify spaces as T_3 or T_4 at all, using the vocabulary "regular" or "normal" alone, or insist that T_3 or T_4 spaces must be T_2 and do not distinguish them from regular or normal spaces. In *General Topology* [**?**], Bourbaki says that our T_3 spaces "satisfy Axiom O_{III}," while our T_4 spaces "satisfy Axiom O_V." Some sources call completely regular spaces $T_{3\frac{1}{2}}$ while Bourbaki says they "satisfy Axiom O_{IV}." The reader is advised to watch for (that is, expect) variant vocabulary.

13.9**.** ***Exercise.*** *We collect here some of the relationships among the separation properties. $\Rightarrow$ means "implies" while $\not\Rightarrow$ means "does not imply." Justify each line.*

(i) $T_2 \Rightarrow T_1 \Rightarrow T_0$.

(ii) $T_0 + T_3 \Rightarrow T_1 + T_2$.

(iii) $\mathcal{CR} \Rightarrow T_3$.

(iv) $T_1 + T_4 \Rightarrow T_0 + T_2 + T_3 + \mathcal{CR}$. *See Exercise 16.6.*

(v) $T_0 + T_4 \not\Rightarrow T_1$ *or* T_2 *or* T_3. *See Exercise 13.6 and (ii) above.*

(vi) $T_1 \not\Rightarrow T_2$. *See Exercise 13.1.*

(vii) $T_2 \not\Rightarrow T_3$ *or* T_4. *See Exercise 13.4.*

(viii) $T_2 + \mathcal{CR} \not\Rightarrow T_4$. *See Exercise 17.14.*

(ix) $T_3 + T_4 \not\Rightarrow T_0$ *or* T_1 *or* T_2. *Example: the indiscrete topology* $\{\varnothing,\ \{a, b\}\}$.

13.10**.** ***Exercise.*** *(i) Suppose $Y = \prod_{a\in A} X_a$ has product topology. Y is T_2, T_3 or $\mathcal{CR}$ if and only if every factor space has the corresponding property. (hint: To show the T_3 property or complete regularity holds in the product space, consider the case of a point $p \in V$ where V is a cylinder in the product space.)*

(ii) See Exercise 16.5 for a counterexample to show a product of two normal spaces need not be T_4.

(iii) There exist T_3 spaces which are not $\mathcal{CR}$. Go to the general topology section of your library and find two examples of this phenomenon. You might wish to delay this "literature familiarization" exercise until after Section 18. By then you will have constructed a small library of spaces made to expose other qualities but which are likely to be used to form the examples in your library sources.

14. A Cornucopia of Covers

A collection $\mathbb{G}$ of subsets of Y is called a **cover of Y** if every point of Y is in at least one member of $\mathbb{G}$.

The collection is called an **open cover of Y** if it is comprised of open sets. The cover is called **closed** if it is comprised of closed sets.

$\mathbb{G} \subset \mathbb{P}(Y)$ is called **point finite** if each point of Y is in only finitely many members of $\mathbb{G}$. We do not presume that $\mathbb{G}$ covers Y.

$\mathbb{G} \subset \mathbb{P}(Y)$ is called **locally finite** if each point of Y has a neighborhood intersecting only finitely many—possibly none—of the members of $\mathbb{G}$.

$\mathbb{G} \subset \mathbb{P}(Y)$ is called **σ-locally finite** if it is the union of countably many locally finite subsets of $\mathbb{P}(Y)$. We do not presume that any of these, or $\mathbb{G}$ itself, are covers.

$\mathbb{G} \subset \mathbb{P}(Y)$ is called **discrete** if each point of Y has a neighborhood intersecting no more than one of the members of $\mathbb{G}$.

$\mathbb{G} \subset \mathbb{P}(Y)$ is called **σ-discrete** if it is the union of countably many discrete subsets of $\mathbb{P}(Y)$. We do not presume that any of these, or $\mathbb{G}$ itself, are covers.

A cover $\mathbb{B}$ of Y is called a **refinement of the cover** $\mathbb{G}$ provided each member of $\mathbb{B}$ is a subset of a member of $\mathbb{G}$. Note: we only define refinements for covers, and refinements are themselves covers. If $\mathbb{B}$ is a refinement of $\mathbb{G}$ we say that $\mathbb{B}$ refines $\mathbb{G}$ and can write either $\mathbb{B} \ll \mathbb{G}$ or $\mathbb{G} \gg \mathbb{B}$ to denote the situation.

The cover $\mathbb{B}$ of Y is called a **subcover** of the cover $\mathbb{G}$ of Y if $\mathbb{B} \subset \mathbb{G}$. A subcover of $\mathbb{G}$ is, of course, a refinement of $\mathbb{G}$.

Covers of various types, particularly open covers, provide a means of localizing the topological properties of sets to smaller and (presumably) more manageable bits.

14.1. ***Exercise.*** *Suppose $\mathbb{G} \subset \mathbb{P}(Y)$ is locally finite. Then $\left\{\overline{G} \mid G \in \mathbb{G}\right\}$ is also locally finite. (hint: Let A be an open set which intersects only the members $\mathbb{H} = \{G_1, \ldots, G_n\} \subset \mathbb{G}$. If A contained any point of any $\overline{G}$ for $G \in \mathbb{G} - \mathbb{H}$ there would be a net in G converging to a point in A and since A is open this net must eventually be in A, contrary to our assumption that $A \cap G = \varnothing$.)*

14.2. ***Exercise.*** *(i) Suppose $\mathbb{G} \subset \mathbb{P}(Y)$ is locally finite and $\mathbb{H} \subset \mathbb{G}$. Then $K = \bigcup_{G \in \mathbb{H}} \overline{G}$ is closed. (hint: Let $p \in \overline{K}$ and let n be a net in K converging to p. There is a neighborhood V of p which intersects only finitely many $\overline{G}$ for G in $\mathbb{H}$. Since the net is eventually in V it must frequently be in $\overline{G}$ for one member G of $\mathbb{H}$. So there is a subnet in closed $\overline{G}$ converging to p: that is, $p \in \overline{G}$.)*

(ii) Suppose $\mathbb{G}$ is any open cover of Y. V is open in Y exactly when $V \cap G$ is open for all $G \in \mathbb{G}$.

(iii) Suppose $\mathbb{G}$ is any open cover of Y. K is closed in Y exactly when $K \cap \overline{G}$ is closed for all $G \in \mathbb{G}$.

(iv) In general if $\mathbb{G}$ is any open cover and $K \subset Y$

$$\overline{K} = \bigcup_{G \in \mathbb{G}} \overline{K \cap G}.$$

(hint: Consider a net n in K converging to a point p. This net is eventually in some open $G \in \mathbb{G}$, so it will be in $K \cap G$.)

(v) Suppose $\mathbb{G}$ is any locally finite closed cover of Y. K is closed in Y exactly when $K \cap G$ is closed for all $G \in \mathbb{G}$.

(vi) Suppose $\mathbb{G}$ is any locally finite closed cover of Y and $K \subset Y$.

$$K^o = \bigcup_{G \in \mathbb{G}} (K \cap G)^o.$$

14.3. ***Exercise.*** *Suppose $\mathbb{G}$ is a cover of Y. $\mathbb{G}$ is called a **minimal cover** if the removal of any single set from $\mathbb{G}$ is no longer a cover of Y.*

*(i) Let $\mathbf{S}$ denote the set of all subcovers of a **point finite cover** $\mathbb{G}$. Chains in $\mathbf{S}$ have lower bounds in $\mathbf{S}$ (the intersection of the chain) so $\mathbf{S}$ has minimal elements called **minimal subcovers**. (hint: The intersection of any chain of subcovers of $\mathbb{G}$ is a collection of sets from $\mathbb{G}$. If this collection is not a cover there will be some point p not in any of these sets. Only finitely many members of $\mathbb{G}$ contain p and there must be a member of the chain that does not contain any of them.)*

(ii) It is not true that the intersection of any chain of subcovers of a generic open cover is a cover. For example let $\mathbb{G}$ be the usual topology on the real line and define for integer $n > 0$ the open cover

$$\mathbb{G}_n = \left\{ \left(q - \frac{1}{j}, q + \frac{1}{j} \right) \,\middle|\, j \text{ is an integer greater than } n \text{ and } q \in \mathbb{Q} \right\}.$$

So the intersection of the $\mathbb{G}_n$ is empty.

The following Lemma, though not hard, is very important for us. A variety of later proofs have been organized to take advantage of the construction found here.

14.4. ***Lemma. The Refinement Lemma*** *Suppose $\mathbb{G}$ is a cover of Y and $\mathbb{H}$ is a refinement of $\mathbb{G}$. Then there is a subcover $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ and a refinement $\widetilde{\mathbb{H}}$ of $\widetilde{\mathbb{G}}$, where the members of $\widetilde{\mathbb{H}}$ are formed as unions of the sets in members of a partition of $\mathbb{H}$, and a one-to-one and onto function $f : \widetilde{\mathbb{G}} \to \widetilde{\mathbb{H}}$ for which $f(K) \subset K$ for all $K \in \widetilde{\mathbb{G}}$.*

Proof. Assign to each $K \in \mathbb{H}$ a set $\Phi(K) \in \mathbb{G}$ for which $K \subset \Phi(K)$. So $\Phi : \mathbb{H} \to \mathbb{G}$, but Φ need not be either one-to-one or onto.

Let $\widetilde{\mathbb{G}} = \Phi(\mathbb{H})$. Since $\mathbb{H}$ is a cover so too is $\widetilde{\mathbb{G}}$, and a subcover of $\mathbb{G}$ as well.

For each $L \in \widetilde{\mathbb{G}}$ define $f(L)$ to be the union of the members of $\Phi^{-1}(L)$ and define $\widetilde{\mathbb{H}} = f(\widetilde{\mathbb{G}})$. Every member of $\mathbb{H}$ is involved in exactly one of these unions. So $\widetilde{\mathbb{H}}$ is a cover of Y, and is obviously a refinement of $\widetilde{\mathbb{G}}$. □

14.5. ***Exercise.*** *(i) In Lemma 14.4 if $\mathbb{H}$ is either point finite or locally finite then, since members of $\widetilde{\mathbb{H}}$ are formed by agglomerating members of a partition of $\mathbb{H}$, $\widetilde{\mathbb{H}}$ has the corresponding property too.*

(ii) If $\mathbb{H}$ is an open cover or a closed locally finite cover then so is $\widetilde{\mathbb{H}}$.

(iii) If $\mathbb{G}$ is a minimal cover of Y then $\mathbb{G} = \widetilde{\mathbb{G}}$ and each $f(L) \in \widetilde{\mathbb{H}}$ contains a point which is in L but in no other member of $\mathbb{G}$. The refinement $\widetilde{\mathbb{H}}$ is itself, therefore, a minimal cover.

(iv) If $\mathbb{G}$ is σ-discrete or σ-locally finite then so is the subcover $\widetilde{\mathbb{G}}$. Since $f(L) \subset L$ for each $L \in \widetilde{\mathbb{G}}$ it follows that $\widetilde{\mathbb{H}}$ also shares these properties with $\mathbb{G}$.

*(v) Suppose $\widetilde{\mathbb{H}}$ is point finite. This will happen, for example, if $\mathbb{H}$ is point finite. So $\widetilde{\mathbb{H}}$ has a minimal subcover $\widetilde{\mathbb{K}}$. It is **not** true that $f^{-1}(\widetilde{\mathbb{K}})$ must be a minimal subcover of $\mathbb{G}$.*

14.6. ***Exercise.*** *Suppose $\mathbb{U}$ is either an open cover or a closed locally finite cover of Y and X is a topological space. Suppose that for each $U \in \mathbb{U}$ there is a function $f_U : U \to X$ which is continuous with respect to subspace topology on U. Suppose* *also that for each pair $U, V \in \mathbb{U}$ and each $x \in U \cap V$ we have $f_U(x) = f_V(x)$. Then the function $f\colon Y \to X$ defined by $f(x) = f_U(x)$ whenever $x \in U$ is continuous.*

15. Barycentric and Star Refinements

We assemble here a number of related results and definitions which otherwise would find themselves embedded, rather awkwardly, in the proofs of major theorems scattered throughout the remainder of this appendix. Our treatment of covers is (largely) adapted from the approach in Dugundji [**?**]. A few of these results have independent interest. It would make sense to skim this section and come back for the details as needed.

Suppose $\mathbb{A}$ is a cover of X. For $B \subset X$ define

$$\mathbb{A}(B) = \{V \in \mathbb{A} \mid B \cap V \neq \varnothing\} \quad \text{and} \quad Star_{\mathbb{A}}(B) = \bigcup_{V \in \mathbb{A}(B)} V$$

$$Bary(\mathbb{A}) = \{Star_{\mathbb{A}}(\{x\}) \mid x \in X\} \quad \text{and} \quad Star(\mathbb{A}) = \{Star_{\mathbb{A}}(B) \mid B \in \mathbb{A}\}.$$

$\mathbb{A}(B)$ assembles the members of $\mathbb{A}$ which touch B. $Star_{\mathbb{A}}(B)$ agglomerates all these into a single set. $Bary(\mathbb{A})$ is a new cover formed from all "one point centers," while $Star(\mathbb{A})$ is a new cover formed using as "centers" the various members of $\mathbb{A}$ itself.

If $\mathbb{A}$ is an open cover, both $Bary(\mathbb{A})$ and $Star(\mathbb{A})$ are open covers of X.

Suppose $\mathbb{B} \ll \mathbb{A}$. Then for each $B \subset X$, each member of $\mathbb{B}(B)$ is contained in a member of $\mathbb{A}(B)$, so $Star_{\mathbb{B}}(B) \subset Star_{\mathbb{A}}(B)$. It now follows immediately that $Bary(\mathbb{B}) \ll Bary(\mathbb{A})$. Similarly, $Star(\mathbb{B}) \ll Star(\mathbb{A})$.

It is obvious but worth noting that $\mathbb{A} \ll Bary(\mathbb{A}) \ll Star(\mathbb{A})$.

The cover $\mathbb{A}$ is called a **barycentric refinement of the cover** $\mathbb{B}$ if $Bary(\mathbb{A}) \ll \mathbb{B}$. Any refinement of a barycentric refinement of $\mathbb{B}$ is a barycentric refinement of $\mathbb{B}$. If $\mathbb{A}$ is a barycentric refinement of $\mathbb{B}$ and $\mathbb{B} \ll \mathbb{C}$ then $\mathbb{A}$ is a barycentric refinement of $\mathbb{C}$.

The cover $\mathbb{A}$ is called a **star refinement of the open cover** $\mathbb{B}$ if $Star(\mathbb{A}) \ll \mathbb{B}$. Any refinement of a star refinement of $\mathbb{B}$ is a star refinement of $\mathbb{B}$. If $\mathbb{A}$ is a star refinement of $\mathbb{B}$ and $\mathbb{B} \ll \mathbb{C}$ then $\mathbb{A}$ is a star refinement of $\mathbb{C}$.

In this appendix, we only use the vocabulary of barycentric or star refinements when all covers involved are open. However we find occasion in Appendix **??** to consider other covers.

15.1**. *Exercise.*** *If $\mathbb{A}$ is a barycentric refinement of $\mathbb{B}$ and $\mathbb{B}$ is a barycentric refinement of $\mathbb{C}$ then $\mathbb{A}$ is a star refinement of $\mathbb{C}$. (hint: Consider first a slightly different problem. The sets in $Star(\mathbb{A})$ are unions of all sets in $\mathbb{A}$ that touch individual members of $\mathbb{A}$. These sets are actually (potentially) smaller than the sets in $Bary(Bary(\mathbb{A}))$. That is because members of $Bary(\mathbb{A})$ are unions of all sets in $\mathbb{A}$ touching some x so members of $Bary(Bary(\mathbb{A}))$ are unions of all members of $\mathbb{A}$ which touch **any** member of $\mathbb{A}$ containing some x, not just one of them. We conclude that $Star(\mathbb{A}) \ll Bary(Bary(\mathbb{A}))$.)*

We refer to a topological space Y as a $\mathbf{T}_*$ space if every open cover has an open star refinement. From the last exercise we see that this is equivalent to saying every open cover has an open barycentric refinement.

Some sources refer to spaces which are both T_* and T_1 as **fully normal.**

15.2. ***Lemma.*** *If X is T_1 and T_* then X is regular.*

Proof. Suppose $p \in X$ and B is a closed set not containing p. Then $\mathbb{A} = \{X - \{p\},\ B^c\}$ is an open cover of X. There is an open star refinement $\mathbb{B}$ of $\mathbb{A}$. $Star_{\mathbb{B}}(\{p\})$ and $Star_{\mathbb{B}}(B)$ are both open and contain p and B respectively. If they had nonempty intersection then there would be a member of $\mathbb{B}(B)$ which touches a member $C \in \mathbb{B}(\{p\})$. But then $Star_{\mathbb{B}}(C)$ contains both p and points of B, contrary to our choice of $\mathbb{B}$ as a star refinement $\mathbb{A}$. $\square$

Later we will show that spaces which are both T_1 and T_* are actually normal. This lemma is used in the proof of that fact.

Finally, the sequence $\mathbb{A}_n$, for $n \in \mathbb{N}$, of open covers of X is called **locally starring for the open cover** $\mathbb{B}$ provided for each $x \in X$ there is an integer n and an open neighborhood V_x of x and a member B_x of $\mathbb{B}$ for which $Star_{\mathbb{A}_n}(V_x) \subset B_x$.

In words: each point has a neighborhood so that all the sets touching that neighborhood from at least one of the $\mathbb{A}_n$ are in one member of $\mathbb{B}$.

15.3. ***Exercise.*** *Suppose $\mathbb{A}_n$, for $n \in \mathbb{N}$, is locally starring for the open cover $\mathbb{O}$. Then there is a sequence $\mathbb{B}_n$, for $n \in \mathbb{N}$, which is also locally starring for the open cover $\mathbb{O}$ and with $\mathbb{B}_{n+1} \ll \mathbb{B}_n$ for each n. (hint: Let $\mathbb{B}_0 = \mathbb{A}_0$. Having found $\mathbb{B}_n$ let $\mathbb{B}_{n+1} = \{A \cap B \mid A \in \mathbb{A}_{n+1}$ and $B \in \mathbb{B}_n\}$. So each $\mathbb{B}_{n+1}$ is a refinement of both $\mathbb{B}_n$ and $\mathbb{A}_{n+1}$.)*

15.4. ***Lemma.*** *X is T_* if and only if there is a locally starring sequence for every open cover.*

Proof. If $\mathbb{B}$ is an open star refinement of the open cover $\mathbb{O}$ the sequence of covers $\mathbb{B}_n = \mathbb{B}$ for each n is locally starring for $\mathbb{O}$. So if X is T_* every open cover has a locally starring sequence.

We now prove the converse. Suppose $\mathbb{B}_n$, for $n \in \mathbb{N}$, is a locally starring sequence for the open cover $\mathbb{O}$. We may presume that $\mathbb{B}_{n+1} \ll \mathbb{B}_n$ for each n. (See Exercise 15.3.)

Let $\mathbb{S}$ denote the collection of all open sets V for which there exists an integer n and sets H and G with $V \subset H \in \mathbb{B}_n$ and also $Star_{\mathbb{B}_n}(V) \subset G \in \mathbb{O}$.

$\mathbb{S}$ is an open cover because the $\mathbb{B}_n$ form a locally starring sequence for $\mathbb{O}$.

For each $V \in \mathbb{S}$ define $n(V)$ to be least among all integers k for which there are sets H in $\mathbb{B}_k$ and G in $\mathbb{O}$ satisfying the membership condition for V. For each point p define $n(p)$ to be the least integer among the $n(V)$ where V is a neighborhood of p and in $\mathbb{S}$. For each p select a neighborhood V_p of p in $\mathbb{S}$ with $n(p) = n(V_p)$ and select $G_p \in \mathbb{O}$ with $Star_{\mathbb{B}_{n(p)}}(V_p) \subset G_p$.

We know that $Star_{\mathbb{B}_k}(C) \supset Star_{\mathbb{B}_n}(C)$ for any set C when $n > k$. In particular, if $p \in V \in \mathbb{S}$ then V is a subset of some $H \in \mathbb{B}_n$ for some $n \geq n(p)$. This H is a subset of some $W \in \mathbb{B}_{n(p)}$.

We conclude that

$$Star_{\mathbb{S}}(\{p\}) \subset Star_{\mathbb{B}_{n(p)}}(\{p\}) \subset Star_{\mathbb{B}_{n(p)}}(V_p) \subset G_p.$$

So $\mathbb{S}$ is a barycentric refinement of $\mathbb{O}$. If every open cover has a locally starring sequence then we can create such a sequence for $\mathbb{S}$ which would generate, by a duplication of the argument found above, an open star refinement of $\mathbb{O}$. We conclude that if every open cover has a locally starring sequence then X is T_*. □

Combining several of these results into a single package, we have the following list of equivalent conditions, any one of which can be used to define T_* spaces.

15.5**.** ***Corollary.*** *The following conditions are equivalent:*

Every open cover of X is has an open barycentric refinement.

Every open cover of X is has an open star refinement.

Every open cover of X is has a locally starring sequence.

Proof. See above. □

16. Properties and Characterizations of T_4 Spaces

16.1**.** ***Proposition. Urysohn's Lemma*** *If X is T_4 then each pair of disjoint nonempty closed sets can be separated by a Urysohn function.*

Proof. Suppose A and B are an arbitrary pair of nonempty disjoint closed sets in the T_4 space X.

Let D denote the dyadic rationals $\{p\, 2^{-q} \mid p, q \text{ are nonnegative integers}\}$.

Define $F\colon D \to \mathbb{P}(X)$ as follows. First define $F(0) = N_A$ where N_A is an open set containing A with $\overline{N_A} \cap B = \varnothing$. Then for all dyadic $t > 1$ define $F(t) = X$ and $F(1) = X - B$.

Having chosen $F(t)$ for $t = k/2^n$ and $k = 0, \ldots, 2^n$ and for which $\overline{F(t)} \subset F(s)$ whenever $t < s$ (as we just did for $n = 0$) select for each $M = 0, \ldots, 2^n - 1$ a set $F\left(\frac{2M+1}{2^{n+1}}\right)$ to be an open and satisfy

$$\overline{F\left(\frac{M}{2^n}\right)} \subset F\left(\frac{2M+1}{2^{n+1}}\right) \subset \overline{F\left(\frac{2M+1}{2^{n+1}}\right)} \subset F\left(\frac{M+1}{2^n}\right).$$

This can be accomplished for each M because $\overline{F\left(\frac{M}{2^n}\right)}$ is closed and contained in open $F\left(\dfrac{M+1}{2^n}\right)$.

We conclude that the process defines the function F on all of D by induction. We now define ϕ on X by

$$\phi(x) = \inf\{t \in D \mid x \in F(t)\}.$$

$\phi(A) = \{0\}$ because $A \subset F(0)$. Also $\phi(B) = \{1\}$. If ϕ is continuous it is a Urysohn function of the type we were looking for.

Pick $s \in [0, 1]$. So

$$\begin{aligned}\phi^{-1}([0, s]) &= \{x \mid x \in F(t) \text{ for all dyadic } t \text{ with } t > s\} \\ &= \{x \mid x \in \overline{F(t)} \text{ for all dyadic } t \text{ with } t > s\} = \bigcap_{t \in D \text{ and } t > s} \overline{F(t)}.\end{aligned}$$

So $\phi^{-1}((s,1])$ is open for each s. Similarly,

$$\phi^{-1}([0,s)) = \{x \mid x \in F(t) \text{ for some dyadic } t \text{ with } 0 \le t < s\} = \bigcup_{t \in D \text{ and } 0 \le t < s} F(t).$$

As the union of open sets, it too is open. So ϕ^{-1} takes all open sets in a subbase of the topology on $[0,1]$ to open sets in X and therefore ϕ is continuous. □

Obviously, the endpoints 0 and 1 in the range interval can be modified to match any preferred constants.

16.2**. *Exercise*.** *From this last lemma we conclude that in any normal space disjoint closed sets can be separated by a Urysohn function, and since points are closed in a normal space we have the Tychonoff property and then regularity. Since any subspace of a $\mathcal{CR}$ space is $\mathcal{CR}$ we conclude that any subspace of a normal space is Tychonoff.*

Urysohn's Lemma implies another result called the Tietze Extension Theorem.

16.3**. *Proposition. Tietze's Extension Theorem*** *If X is T_4 and $A \subset X$ is closed and $\tau\colon A \to [a,b]$ is continuous, where A is endowed with the subspace topology, then there is a continuous function $f\colon X \to [a,b]$ with $f(x) = \tau(x)$ for all $x \in A$.*

Proof. The result is trivial if τ is constant or $X = A$, so suppose otherwise. We will also suppose for convenience that $a = -1 = \inf\{\tau(x) \mid x \in A\}$ and that $b = 1 = \sup\{\tau(x) \mid x \in A\}$. Once the result is proved in this case, the more general situation follows instantly.

Define $A_0 = \tau^{-1}\left(\left[-1, \frac{-1}{3}\right]\right)$ and $B_0 = \tau^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$.

Since X is T_4 there is a continuous function $\phi_0\colon X \to \left[\frac{-1}{3}, \frac{1}{3}\right]$ with $\phi_0^{-1}\left(\frac{-1}{3}\right) \supset A_0$ and $\phi_0^{-1}\left(\frac{1}{3}\right) \supset B_0$. Note that for all $x \in A$ we have $|\tau(x) - \phi_0(x)| \le \frac{2}{3}$.

Suppose we have in hand for $0 \le i < k$ continuous functions $\phi_i\colon X \to \left[\frac{-1}{3^{i+1}}, \frac{1}{3^{i+1}}\right]$ for which for each $x \in A$ the value of $|\tau(x) - \mu_i(x)|$ does not exceed $\left(\frac{2}{3}\right)^{i+1}$ where we define $\mu_i = \sum_{j=0}^{i} \phi_j\colon X \to \left[-1 + \left(\frac{2}{3}\right)^{i+1},\ 1 - \left(\frac{2}{3}\right)^{i+1}\right]$. We will also be interested in the restriction of μ_i to A and we will denote this function $\mu_i|_A$. We created just this scenario with $k = 1$ above.

Define

$$A_k = (\tau - \mu_{k-1}|_A)^{-1}\left(\left[\frac{-2^k}{3^k}, \frac{-2^k}{3^{k+1}}\right]\right) \text{ and } B_K = (\tau - \mu_{k-1}|_A)^{-1}\left(\left[\frac{2^k}{3^{k+1}}, \frac{2^k}{3^k}\right]\right).$$

There is a continuous function $\phi_k\colon X \to \left[\frac{-1}{3^{k+1}}, \frac{1}{3^{k+1}}\right]$ defined on all of X with $\phi_k(A_k) = \left\{\frac{-2^k}{3^{k+1}}\right\}$ and $\phi_k(B_k) = \left\{\frac{2^k}{3^{k+1}}\right\}$.

Verify that for each $x \in A$ the value of $|\tau(x) - \mu_k(x)|$ cannot exceed $\left(\frac{2}{3}\right)^{k+1}$ where we define $\mu_k = \sum_{j=0}^{k} \phi_j\colon X \to \left[-1 + \left(\frac{2}{3}\right)^{k+1}, 1 - \left(\frac{2}{3}\right)^{k+1}\right]$.

The sequence μ converges uniformly by the Weierstrass "M" test to a continuous function $f\colon \to [-1,1]$. f evidently equals τ on A. □

The situation of the proposition is described by saying that any function $\tau\colon A \to [a,b]$ defined and continuous on a closed subset A of X can be **extended** to a continuous function defined on all of X or, phrased another way, that τ is the **restriction** of a continuous function defined on all of X. Further, such an extension exists whose function values lie in the interval $[a,b]$.

16.4. ***Exercise.*** *Define*

$$g\colon \mathbb{R} \to (-1,1) \text{ by } g(x) = \begin{cases} \frac{x}{1+x}, & \text{if } x \geq 0; \\ \frac{x}{1-x}, & \text{if } x < 0. \end{cases}$$

Show that g is a homeomorphism. Use g to create a continuous extension for unbounded continuous τ defined on a closed subset of normal X. In particular, create f so that if $\tau(A) \subset (a,b)$ then $f(X) \subset (a,b)$, where a can be $-\infty$ and b can be ∞.

If any bounded function defined on a closed subset can be extended to all of X and if A and B are disjoint nonempty closed sets then the function τ defined by $\tau(x) = 0$ if $x \in A$ and $\tau(x) = 1$ if $x \in B$ is continuous on $A \cup B$ with subspace topology, and therefore extends to a Urysohn function separating A and B on X, which implies that X is T_4. If X is T_1 we have normality. So the "Tietze function extension property" characterizes T_4 spaces.

The Tietze extension theorem has many uses. Here is an application to a question of normality.

16.5. ***Exercise.*** *(i) Let X denote the topological space with $\mathbb{R}$ as the underlying set but with topology created using base consisting of all $[a,b)$ where $-\infty < a < b < \infty$. It is called the* ***right half open interval topology****. This topology is finer than the usual topology, so it is T_2. It is also separable, since there is a rational number in every basic open set.*

X is C_I: the set of $[x, x+q)$ where q is a positive rational is a neighborhood base at x. However X is not C_{II}, for if $\mathbb{B} = \{[x, y_x) \mid x \in S\}$ is any collection of base members and S is countable there is a member of X not in S, because X is uncountable. But then $[x, x+\varepsilon)$ is not the union of members of $\mathbb{B}$, since any member of $\mathbb{B}$ containing x must contain points to the left of x too.

It is also interesting to note that all sets of the form $(-\infty, x)$ or $[x,y)$ for $-\infty < x < y \leq \infty$ are both open and closed in X.

Suppose A and B are closed nonempty and disjoint sets in X. Since both A^c and B^c are open, for each $a \in A$ there is a basic open set $[a, x_a) \subset B^c$ and for each $b \in B$ there is a basic open set $[b, y_b) \subset A^c$.

Let $\widetilde{A} = \bigcup_{a\in A}[a, x_a)$ and $\widetilde{B} = \bigcup_{b \in B}[b, y_b)$. These sets are open and contain A and B respectively. If there were any point $p \in \widetilde{A} \cap \widetilde{B}$ then there would be open intervals $[a, x_a)$ and $[b, y_b)$ with $p \in [a, x_a) \cap [b, y_b)$. So either $a \in [b, y_b)$ or $b \in [a, x_a)$, which contradicts their definition. We conclude that X is normal.

(ii) Let Y be the topological product space $X \times X$ called by Steen and Seebach [**?**] ***Sorgenfrey's half open square topology****. Y is separable, T_2 and $\mathfrak{CR}$.*

Recall that any continuous function on X is determined by its values on any dense subset. The separability of Y puts a limit on the cardinality of the set of *continuous functions from Y into any fixed range space. In particular, the set of all continuous real valued functions on Y cannot have cardinality exceeding $\mathbb{R}^{\mathbb{N}}$ which has the same cardinality as $\mathbb{R}$.*

Let $D = \{(x,-x) \in Y \mid x \in X\}$. Since $D \cap ([x, x+\varepsilon) \times [-x, -x+\varepsilon)) = \{(x,-x)\}$ for any positive ε, the set D is discrete and therefore closed. This means that any real valued function on D is continuous with subspace topology, and the cardinality of this set of functions is the same as that of $\mathbb{R}^{\mathbb{R}}$ which exceeds the cardinality of $\mathbb{R}$. So there are continuous functions on D which cannot be extended to all of Y: there simply are not enough continuous functions on Y to accommodate all the continuous functions on D. So Y cannot be normal. This is an example that the product of two normal spaces need not be T_4.

16.6. ***Exercise.*** *If Y is T_3 and T_4 then Y is $\mathcal{CR}$.*

16.7. ***Exercise.*** *Suppose X is T_4. Show that closed nonempty disjoint A and B can be precisely separated by a Urysohn function ϕ if and only if A and B are G_δ set. (hint: First, if a Urysohn function of the specified kind exists, then $A = \bigcap_{i=1}^{\infty} \phi^{-1}([0, \frac{1}{n}))$ demonstrates that A is a G_δ set. B is handled similarly. Conversely, suppose S_i for $i \in \mathbb{N}$ is a chain of open sets and $A = \bigcap_{i \in \mathbb{N}} S_i$. We may suppose that $B \cap S_i = \varnothing$ for all i. Let $T_i = X - S_i$ for each i. The T_i are closed and contain B and are nested, eventually containing any particular point not in A. Create Urysohn functions ϕ_i with $A \subset \phi_i^{-1}(0)$ and $T_i \subset \phi_i^{-1}(1)$. Examine $\phi = \frac{1}{3}\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \phi_i$. Then extend the argument to deal with B.)*

16.8. ***Lemma. Shrinking Lemma*** *Suppose X is T_4 and $\mathbb{O}$ is any locally finite open cover. Then there is a locally finite open refinement $\mathbb{H} = \{H_G \mid G \in \mathbb{O}\}$ of $\mathbb{O}$ with $\overline{H_G} \subset G$ $\forall G \in \mathbb{O}$.*

Proof. Well order $\mathbb{O}$. If G_1 is the first member of $\mathbb{O}$ then $X_{G_1} = X - \bigcup_{F \in \mathbb{O} \text{ and } F > G_1} F$ is closed and inside open G_1 in normal X. So there is open H_{G_1} containing X_{G_1} with $\overline{H_{G_1}} \subset G_1$.

Note that $\mathbb{O}_{G_1}$, defined by replacing G_1 by H_{G_1} in $\mathbb{O}$, is also a locally finite open cover. If $\mathbb{O}$ has a single member we are done. Otherwise we proceed as below.

Suppose we have found for all $G < K$ open H_G with $\overline{H_G} \subset G$ and so that for each such G the collection of sets $\mathbb{O}_G$, defined by replacing B by H_B in $\mathbb{O}$ for all $B \leq G$, is also a locally finite open cover. This is exactly the situation we created in the last paragraph, where K is the second member of $\mathbb{O}$.

Define

$$X_K = X - \left(\left(\bigcup_{F \in \mathbb{O} \text{ and } F < K} H_F \right) \cup \left(\bigcup_{F \in \mathbb{O} \text{ and } F > K} F \right) \right).$$

X_K is closed and inside open K in normal X. So there is open H_K containing X_K with $\overline{H_K} \subset K$.

Let $\mathbb{O}_K$ be defined by replacing B by H_B in $\mathbb{O}$ for all $B \leq K$. $\mathbb{O}_K$ is also a locally finite open cover.

This process allows us to infer the existence of the desired open cover $\mathbb{H}$ through a transfinite construction, as in Section **??**. $\square$

16.9. ***Exercise.*** *We note that in Lemma 16.8 if* $\mathbb{O}$ *is a minimal open cover then so is* $\mathbb{H}$. *In that case each* H_G *must be a nonempty subset of* G *and* H_G *is contained in no member of the partition* $\mathbb{O}$ *other than* G.

16.10. ***Exercise.*** *If the conclusion of the Shrinking Lemma is true then* X *is* T_4. *(hint: If* A *and* B *are closed, nonempty and disjoint then* $\{X - A,\ X - B\}$ *is an open cover.) So the "Shrinking Lemma property" characterizes* T_4 *spaces.*

A **partition of unity** for a topological space X is a nonempty set $\mathcal{F}$ of Urysohn functions on X with only finitely many of the members of $\mathcal{F}$ nonzero at each $x \in X$ and $1 = \sum_{f\in\mathcal{F}} f(x)$.

The **support** of a real valued function f defined on X is

$$Support(f) = \overline{X - f^{-1}(0)}.$$

It is, obviously, a closed set and $f^{-1}(0) \cap Support(f) = \partial(Support(f))$.

If $\mathbb{O}$ is an open cover of X and for each $f \in \mathcal{F}$ there is a member of the open cover containing $Support(f)$ we say that the partition of unity is **subordinate to** $\mathbb{O}$. Partitions of unity are very important. They serve as a tool to localize our thinking and calculations which can then be reassembled for global consequences.

16.11. ***Proposition.*** *Suppose* X *is* T_4 *and* $\mathbb{O}$ *is a locally finite open cover of* X. *Then there is a partition of unity subordinate to* $\mathbb{O}$.

Proof. By applying the Shrinking Lemma we can create a locally finite open refinement $\mathbb{V} = \{V_G \mid G \in \mathbb{O}\}$ with $V_G \subset \overline{V_G} \subset G$ for every $G \in \mathbb{O}$.

Since X is T_4, for each G there is a Urysohn function f_G with $f_G(\overline{V_G}) = \{1\}$ and $f_G(X - G) = \{0\}$.

Since $\mathbb{G}$ is locally finite only finitely many $f_G(x)$ are nonzero for each $x \in X$. So $F = \sum_{G\in\mathbb{O}} f_G$ is defined on X.

Since $\mathbb{V}$ is a cover, $F(x) \geq 1$ for every $x \in X$.

The set of all $g_G = \dfrac{f_G}{F}$ is the partition of unity we want. □

16.12. ***Exercise.*** *Say that any space in which the conclusion of Proposition 16.11 holds has "the partition of unity property for locally finite open covers." Show that this property characterizes* T_4 *spaces.*

The following proof of a characterization of T_4 spaces in terms of semicontinuous functions is outlined in a problem in Engelking [**?**]. The result is due to Tong and Katětov, who adapt earlier results of Dieudonné and Hahn. Similar characterizations of the perfectly T_4 (see definition below) and $\mathcal{CR}$ properties can be found in Engelking as well.

16.13. ***Proposition.*** X *is* T_4 *exactly when for each pair of real valued functions* f *and* g *on* X *with* $f(x) \leq g(x)$ *for all* x *and where* f *is upper semicontinuous and* g *is lower semicontinuous there is a continuous real function* h *on* X *with* $f(x) \leq h(x) \leq g(x)$ *for all* x.

Proof. Suppose continuous h exists for each f and g as described. Suppose A and B are two nonempty disjoint closed sets in X. Define f and g by

$$f(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & \text{if } x \in B; \\ 1, & \text{otherwise.} \end{cases}$$

So f is upper semicontinuous, g is lower semicontinuous and $f(x) \leq g(x)$ for all x. Any continuous h between f and g will be a Urysohn function separating A and B.

We now go to work on the converse, supposing X to be T_4. Note first that $\mathbb{R}$ is homeomorphic via an increasing homeomorphism to $(0, 1)$ so if we can find continuous h between upper semicontinuous f and lower semicontinuous g when $0 < f(x) \leq g(x) < 1$ we will have proved the proposition.

For each $i, j \in \mathbb{N}$ with $0 < i < j$ define

$$A_{i,j} = g^{-1}\left(\left[0, \ \frac{i}{j} \right] \right) \quad \text{and} \quad B_{i,j} = f^{-1}\left(\left[\frac{i}{j} + \frac{1}{2j}, \ 1 \right] \right).$$

For each i, j these are disjoint closed sets in a T_4 space. For fixed j the $A_{i,j}$ are nested increasing while the $B_{i,j}$ are nested decreasing.

When $A_{i,j} = \varnothing$ define the function $F_{i,j}$ to be identically 1 on X.

When $A_{i,j} \neq \varnothing$ but $B_{i,j} = \varnothing$ let the function $F_{i,j}$ be identically $\frac{i}{j} + \frac{1}{2j}$ on X.

When both sets are nonvoid there is a continuous function $F_{i,j} \colon X \to \left[\frac{i}{j} + \frac{1}{2j}, 1 \right]$ with $F_{i,j}(A_{i,j}) = \left\{ \frac{i}{j} + \frac{1}{2j} \right\}$ and $F_{i,j}(B_{i,j}) = \{1\}$.

For $j \geq 2$ let F_j denote the continuous function $F_{1,j} \wedge \cdots \wedge F_{j-1,j}$. For each i, j and x, $f(x) \leq F_{i,j}(x)$ so $f(x) \leq F_j(x)$ for all x and j.

It is also true that $F_j(x) < g(x) + \frac{3}{2j}$. To see this examine the two cases: first, $x \in A_{i,j}$ for some **least** i and second, x is in **none** of the $A_{i,j}$.

So if we define for $n > 1$ and x the number $S_n(x) = F_2(x) \wedge \cdots \wedge F_n(x)$ the sequence of numbers $S_n(x)$ is nonincreasing and $f(x) \leq S_n(x) < g(x) + \frac{3}{2n}$ for each n. So the sequence S_n of continuous functions converges to some limit F, which is therefore upper semicontinuous, and $f(x) \leq F(x) \leq g(x)$ for each x.

Note that the functions $1 - g$ and $1 - F$ have range in $(0, 1)$, $1 - g$ is upper semicontinuous and $1 - F$ is lower semicontinuous and $1 - g \leq 1 - F$. We can apply the procedure above to these functions to produce a **nondecreasing** sequence of continuous functions T_n for $n > 1$ with $F(x) - \frac{3}{2n} < T_n(x) \leq g(x)$ for each n and x. The limit function G is lower semicontinuous and $f(x) \leq F(x) \leq G(x) \leq g(x)$ for each x.

We now take advantage of the fact that F and G have been formed as specific limits of continuous functions to create a continuous function between F and G.

For each $i > 1$ let $K_i = (S_2 \wedge T_2) \vee \cdots \vee (S_i \wedge T_i)$.

So $K_i \leq K_{i+1} \leq G$ for each i and so the sequence of numbers $K_i(x)$ converges to some number $H(x) \leq G(x)$ for each x. The function H is lower semicontinuous.

Define, for each n, $L_n = K_n \vee S_n$. Because the S_i are decreasing, when $n \geq m$

$$\begin{aligned} K_n &\leq (S_2 \wedge T_2) \vee \cdots \vee (S_m \wedge T_m) \vee (S_{m+1} \wedge T_{m+1}) \vee \cdots \vee (S_n \wedge T_n) \vee S_m \\ &= (S_2 \wedge T_2) \vee \cdots \vee (S_m \wedge T_m) \vee S_m = K_m \vee S_m = L_m. \end{aligned}$$

So $H \leq K_m \vee S_m = L_m$ for each m.

Suppose for some x, $S_i(x) \geq T_i(x)$ for **all** i. It is easy to see that $F(x) = H(x) = G(x)$ at these points, and $K_n(x) = T_n(x)$ and $L_n(x) = S_n(x)$ for each n and the sequence $L_n(x)$ converges to $H(x)$.

On the other hand, suppose for some x we find $S_i(x) < T_i(x)$ for some i. Choose j to be the first subscript for which the inequality holds. For $n \geq j$ we find

$$K_n(x) = T_2(x) \vee \cdots \vee T_{j-1}(x) \vee S_j(x) \vee \cdots \vee S_n(x) = T_{j-1}(x) \vee S_j(x).$$

So $L_n(x) = K_n(x) \vee S_n(x) = K_j(x)$: the sequence $K_n(x) = L_n(x)$ is constant after j for this x. Once again we have $L_n(x)$ converging to $H(x)$.

So $H(x) = \inf\{L_n(x) \mid n \geq 2\}$ for each x and since each L_n is continuous we have H upper semicontinuous and the proof is complete. $\square$

16.14**. *Proposition.*** *In a T_4 space, any locally finite open cover has an open barycentric refinement.*

Proof. Suppose $\mathbb{O}$ is a locally finite open cover of the T_4 space X.

For each $x \in X$ let V_x be the intersection of all members of $\mathbb{O}$ containing x. Each V_x is open because $\mathbb{O}$ is point finite.

Apply the Shrinking Lemma 16.8 to create a locally finite refinement $\mathbb{H}$ of $\mathbb{O}$ consisting, as in that lemma, of open sets H_G with $\overline{H_G} \subset G$ for each $G \in \mathbb{O}$.

Let $\mathbb{H}_x$ denote the set of all members H of $\mathbb{H}$ for which x is **not** in $\overline{H}$. Let H_x denote the union of all $\overline{H}$ where $H \in \mathbb{H}_x$. By Exercise 14.2 we find that H_x is closed, so H_x^c is an open set containing x.

Now let $\mathbb{B} = \{V_x \cap H_x^c \mid x \in X\}$. $\mathbb{B}$ is an open cover and a refinement of $\mathbb{O}$.

Pick $x \in X$. So $x \in V_x \cap H_x^c$ and so $x \in G$ where G is one of the members of $\mathbb{O}$ whose intersection forms V_x. Now suppose $x \in V_p \cap H_p^c$. This implies two things.

First, x is in every member of $\mathbb{O}$ containing p so $V_x \subset V_p$.

Second, x **cannot** be in any $\overline{H_W}$ unless $p \in \overline{H_W}$. But x is in H_G so $p \in \overline{H_G} \subset G$. Since this can be done for each G involved in forming V_x we find that p is in every member of G containing x: that is, $V_p \subset V_x$. Coupled with the earlier remark we find that $V_x = V_p$ whenever $x \in V_p \cap H_p^c$.

In particular, $V_p \cap H_p^c \subset G$ where G is any particular member of $\mathbb{O}$ containing x. So $\mathbb{H}$ is an open barycentric refinement of $\mathbb{O}$. $\square$

A space is called **perfectly $\mathbf{T_4}$** if it is T_4 and every closed set is a G_δ. A space is called **completely $\mathbf{T_4}$** if any pair of separated subsets can be separated by open sets. It is obvious that a completely T_4 space is T_4.

16.15. ***Exercise.*** *A perfectly T_4 space is completely T_4. (hint: Suppose S and T are separated in the perfectly T_4 space X. Since $\overline{S}$ and $\overline{T}$ are G_δ and neither are all of X there must be a nonempty closed set A in $X - \overline{S}$ and a nonempty closed set B in $X - \overline{T}$. Create two Urysohn functions: α precisely separating the pair $\overline{S}$ and A and β precisely separating the pair $\overline{T}$ and B, with $\alpha(\overline{S}) = \{0\}$ and $\beta(\overline{T}) = \{0\}$. So α is less than β on S and β is less than α on T. Let $U = (\alpha - \beta)^{-1}([-1, 0))$ and $V = (\alpha - \beta)^{-1}((0, 1])$.)*

17. Compactness

If $(Y, \mathbb{T})$ is a topological space, a subset A of Y (including, potentially, Y itself) is called **compact** if and only if every open cover of A has a finite subcover. Specifically, if $\mathbb{F} \subset \mathbb{T}$ and $A \subset \bigcup_{B \in \mathbb{F}} B$ then there is an integer n and $\{\, B_i \in \mathbb{F} \mid i = 1, \ldots, n \,\}$ so that $A \subset \bigcup_{i=1}^{n} B_i$. Some sources require compact spaces to be T_2 but we do not.

Taking complements of these open sets, we have the following reformulation, called the **finite intersection property for closed sets**: A is compact if and only if, whenever $A \cap \left(\bigcap_{K \in \mathbb{G}} K\right) = \varnothing$, where all members of $\mathbb{G}$ are closed sets, there is a finite collection $K_1, \ldots, K_n$ of members of $\mathbb{G}$ with $A \cap \left(\bigcap_{i=1}^{n} K_i\right) = \varnothing$.

17.1. ***Exercise.*** *A third condition **equivalent to compactness** is the following. **X is compact if and only if every open cover has a minimal subcover.** (hint: Suppose every open cover of X has a minimal subcover. Let $\mathbb{O}$ be a minimal subcover that is not finite. Each member of $\mathbb{O}$ contains a point in no other member of $\mathbb{O}$. Let $A \colon \mathbb{N} \to \mathbb{O}$ be a one-to-one selection and let C be the union of all the members of $\mathbb{O}$ not in $A(\mathbb{N})$. Define for each $j \in \mathbb{N}$, $B(j) = \bigcup_{i \le j} A(j)$. So $\{C\} \cup B(\mathbb{N})$ is a cover of X without minimal subcover. The conclusion is that if every cover has a minimal subcover then all those subcovers are finite and X is compact.)*

17.2. ***Exercise.*** *(i) If $f \colon Y \to X$ is continuous and C is compact in Y then $f(C)$ is compact in X.*

(ii) Every closed subset of a compact set is compact.

(iii) In a T_2 space, compact sets are closed. (hint: If C is not closed in Y there is a member p of $\partial C - C$. Presuming Y to be T_2, for each $x \in C$ there is an open set V_x containing x and open set W_x containing p with $\overline{V_x} \cap W_x = \varnothing = \overline{W_x} \cap V_x$. Show that the cover consisting of all $Y - \overline{W_x}$ for $x \in C$ has no finite subcover.)

(iv) Give $\mathbb{N}$ the topology with base $\{\{0, n\} \mid n \in \mathbb{N}\}$. So $\{0\}$ is compact and $\overline{\{0\}} = \mathbb{N}$ which is not compact. Note that $\mathbb{N}$ with this topology is neither T_2 nor T_3.

(v) If $f \colon Y \to X$ is continuous and C is compact in Y and X is T_2 then $f(C)$ is closed.

(vi) A continuous function with a T_2 range and a compact domain is a closed function.

(vii) A continuous one-to-one function from a compact domain onto a T_2 range is a homeomorphism.

17.3. ***Exercise.*** *If A is compact in the T_3 space X then $\overline{A}$ is compact. (hint: Let $\mathcal{O}$ be an open cover of $\overline{A}$. Then $\mathcal{O}$ also covers A. Extract finite subcover $A_1, \ldots, A_n$. So $B = \overline{A} - \bigcup_{i=1}^{n} A_i$ is closed and consists entirely, if nonempty, of limits of nets in A. Suppose that B is nonempty. Then for each $a \in A$ there is an open set N_a containing a and an open set V_a containing B with $V_a \cap N_a = \varnothing$. The sets N_a cover A so there is a finite subcover $N_{a_1}, \ldots, N_{a_k}$. The set $\bigcap_{i=1}^{k} V_{a_i}$ is open and contains B but does not contain any elements of A. We conclude that B is empty so the sets $A_1, \ldots, A_n$ cover $\overline{A}$.)*

17.4. ***Exercise.*** *Suppose $Y = \prod_{a \in A} X_a$ is the product of factors infinitely many of which fail to be compact. Suppose also that C is a compact subset of Y. Then C has empty interior. In case Y is T_2 or T_3 this can be rephrased by saying that compact sets in Y are nowhere dense. (hint: The projections of Y onto the factor spaces are all continuous so the image of C under each projection must be compact. If C contains an open cylinder this requires all but a finite number of the factor spaces to be compact.)*

17.5. ***Exercise.*** *Compact subsets take the place of points with respect to the separation conditions T_2 and T_3 and the existence of Urysohn functions of certain types. Specifically, let H and J be nonintersecting compact subsets of Y and let A be a closed set not intersecting H.*

(i) If Y is T_2 $\exists$ open U and V with $U \cap V = \varnothing$ and $H \subset U$ and $J \subset V$.

(ii) If Y is T_3 $\exists$ open U and V with $U \cap V = \varnothing$ and $H \subset U$ and $A \subset V$.

(iii) If any two points can be separated by a Urysohn function in Y then there is a Urysohn function separating H from J.

(iv) If Y is $\mathcal{CR}$ then there is a Urysohn function separating H from A.

17.6. ***Proposition.*** *A is compact if and only if and every net in A has a cluster point in A.*

Proof. We first suppose r is a net in A with no cluster point in A. Then for every point x in A there is an open neighborhood N_x of x such that r is eventually in $A - N_x$. Then $\{ N_x \mid x \in A \}$ is an open cover with no finite subcover. So nets in compact sets all have cluster points in A.

On the other hand, suppose $\mathbb{F}$ is an open cover of A with no finite subcover. Let $\mathbb{G} = \{ A \cap B \mid B$ is a finite union of members of $\mathbb{F} \}$. $\mathbb{G}$ is directed by containment. $A \notin \mathbb{G}$ so for each $C \in \mathbb{G}$ we can choose $r(C) \in A - C$. r is a net in A. Now suppose $x \in A$. Then there is some $F \in \mathbb{F}$ with $x \in F$. But then $F \cap r(T_{A \cap F}) = \varnothing$. So x cannot be a cluster point for the net. So if A is not compact there is a net in A with no cluster point in A. □

If $f \colon X \to Y$ is a function between two topological spaces the **graph of f**, sometimes denoted $\gamma(\mathbf{f})$, is the topological space with set $\{(x, f(x)) \in X \times Y \mid x \in X\}$ (that is, the function itself) with subspace topology from the product space $X \times Y$.

When thought of as a set, there is no difference between f and $\gamma(f)$. Depending on the phrasing of results involving the graph, there may be no need to distinguish between them.

17.7. ***Exercise.*** *(i) If $f\colon X \to Y$ is continuous and Y is T_2 the graph is always closed as a subset of $X \times Y$. (hint: Suppose (x_n, y_n) is any net in the graph converging to a point (a, b). Then x_n converges to a and $y_n = f(x_n)$ converges to b. Continuity implies that $f(x_n)$ converges to $f(a)$ and since Y is T_2 we have $b = f(a)$ so (a, b) is in the graph.)*

(ii) Suppose $f\colon X \to Y$ where X is T_2 and Y is compact and T_2. If the graph of f is closed in $X \times Y$ then f is continuous. (hint: Suppose the graph is closed and x_n is a net converging to a in X. Since X is T_2 this is the only point to which the net converges. Since Y is compact the net $f(x_n)$ has a convergent subnet $f(y_n)$ and since Y is T_2 this net converges to a unique limit b. The net $(y_n, f(y_n))$ converges to (a, b) and since the graph is closed in the T_2 space $X \times Y$ we have $b = f(a)$. Conclude that f is continuous.)

17.8. ***Exercise.*** *A net with a cluster point has a subnet converging to that point. Every net has a universal subnet. A universal net converges to each of its cluster points. Therefore, a set A is compact if and only if each universal net in A converges to a point in A. It might, of course, also converge to points not in A.*

If the set A is known to be closed, specification of a limit point as a member of A is redundant. A is closed and compact if and only if each universal net in A converges.

*In a T_2 space, a net can converge to at most one point. So in a T_2 space, A is compact if and only if each universal net in A converges to a limit in A. This statement is a form of the **Bolzano-Weierstrass Theorem**.*

17.9. ***Proposition. Alexander's Subbase Lemma*** *Suppose $\mathbb{F}$ is a subbase for the topology on Y. $S \subset Y$ is compact if and only if every open cover of S by members of $\mathbb{F}$ has a finite subcover.*

Proof. We will prove that if every open cover $\mathbb{O} \subset \mathbb{F}$ of S has a finite subcover then S is compact. Suppose r is a universal net in Y. Let $\mathbb{D} = \{\, A \in \mathbb{O} \mid r \text{ is eventually in } A^c \,\}$. If $\mathbb{D}$ covers S then we can extract a finite subcover $D_1, \ldots, D_k$ and then r is eventually in $\bigcap_{i=1}^k D_i^c = \varnothing$, a contradiction. Therefore there is some $x \in S - \bigcup_{D \in \mathbb{D}} D$. By definition of $\mathbb{D}$, if $x \in A \in \mathbb{O}$ then r is eventually in A. So r converges to x. $\square$

17.10. ***Exercise.*** *Suppose Γ is any ordinal number. Recall (see Section **??**) that Γ **is** the set of all ordinal numbers less than Γ, and $\Gamma + 1 = \Gamma \cup \{\Gamma\}$.*

Define intervals

$$(\alpha, \Gamma] = \{\gamma \in \Gamma + 1 \mid \alpha < \gamma \le \Gamma\} \text{ and } [0, \alpha) = \{\gamma \in \Gamma + 1 \mid \alpha > \gamma\}.$$

These are the initial segments I_α and those terminal segments of the form $T_{\alpha+1}$ for $\alpha \in \Gamma$. Let $\mathbb{S}$ be the set of all these intervals for $\alpha \in \Gamma$. Open intervals (α, β) and closed or half-open intervals are defined in the obvious ways.

*$\mathbb{S}$ is a subbase for a topology on $\Gamma + 1 = \Gamma \cup \{\Gamma\} = [0, \Gamma]$ called the **order topology**. With this topology $\Gamma + 1$ is compact and T_2. (hint: Any open cover of $\Gamma + 1$ by members of $\mathbb{S}$ has a subcover consisting of two or one set.)*

More generally, a subinterval $[0, \beta)$ of $\Gamma + 1$ with subspace topology is compact if and only if β is not a limit ordinal.

If $\alpha < \beta < \Gamma$ the sets $(\alpha, \beta + 1)$ and $(\alpha, \beta]$ are equal. So any set of the form $(\alpha, \beta]$ is open, and sets of this kind (with $\beta = \Gamma$ allowed) together with $\{0\}$ comprise a base for the order topology on $[0, \Gamma]$.

Even though $[0, \beta)$ with subspace topology need not be compact, it will always be normal. To see this, suppose A and B are two closed, nonempty and disjoint subsets of $[0, \beta)$. For each $x \in A$ let $S_x = B \cap [0, x)$ and define $s_x = \sup S_x$. Since B is closed, $s_x \in B$. The set $\bigcup_{x \in A} (s_x, x]$ is open and contains A. Define for each $y \in B$ the set $T_y = A \cap [0, y)$ and define $t_y = \sup T_y$. Once again, A is closed so $t_y \in A$. The set $\bigcup_{y \in B} (t_y, y]$ is open and contains B. If these two open sets overlapped there would be $x \in A$ and $y \in B$ for which $(t_y, y] \cap (s_x, x] \neq \varnothing$ which would imply either $y \in (s_x, x]$ or $x \in (t_y, y]$ and both of these are impossible. We conclude that A and B can be separated by open sets. Note the similarity with the argument from Exercise 16.5.

17.11**. *Exercise*.** *Let Ω denote the first uncountable ordinal and give $[0, \Omega)$ the subspace topology from $[0, \Omega]$ with order topology. Let f be a continuous function on $[0, \Omega)$. We will show that there is a nonempty interval of the form $[\zeta, \Omega)$ upon which f is constant: that is, any real valued continuous function is eventually constant.*

We will assume that f is not eventually constant. Then for each ordinal β there is a least positive integer n_β for which there exists an ordinal z with $z > \beta$ and

$$| f(\beta) - f(z) | > \frac{1}{n_\beta}.$$

Let $\beta_1 = 0$ and define $n_1 = n_{\beta_1}$ and select β_2 to be an ordinal exceeding β_1 with $| f(\beta_1) - f(\beta_2) | > \frac{1}{n_1}$. Following this pattern, we use induction to define the sequence of ordinals β_i and integers n_i with $n_i = n_{\beta_i}$ and $\beta_{i+1} > \beta_i$ and $| f(\beta_i) - f(\beta_{i+1}) | > \frac{1}{n_i}$ for each i.

Let μ be the least ordinal not smaller than any of the β_i. μ is less than Ω because $[0, \mu) = \bigcup_{i \geq 1} [0, \beta_i]$, a countable union of countable sets and hence countable itself.

The sequence n_i assumes various positive integer values and it must assume at least one value k infinitely often. Otherwise, for all j there is an integer N_j for which $|f(\beta_n) - f(z)| < \frac{1}{j}$ whenever $n > N_j$ and $z > \beta_n$. Using the triangle inequality, we would have $|f(y) - f(z)| < \frac{2}{j}$ for all j and any pair y, z not less than μ, which would imply that f is constant on $[\mu, \Omega)$, contrary to assumption.

But if $n_i = k$ infinitely often, then infinitely many of the associated β_i are in any basic open $(\theta, \mu]$ which implies (the triangle inequality again) that f fails to be continuous at μ.

We conclude both that any continuous function on $[0, \Omega)$ is eventually constant and that the set of real valued continuous functions on $[0, \Omega)$ can be identified with the set of real valued continuous functions on $[0, \Omega]$ by restriction.

17.12**. *Proposition*. *Tychonoff's Theorem*** *Suppose Y is a product space with the product topology. Y is compact if and only if X_a is compact for every $a \in A$.*

Proof. Suppose each X_a is compact, and r is a universal net in Y. Then each $\pi_a \circ r$ is universal in X_a and so converges, for each a, to some $x_a \in X_a$. But then r converges to $y \in Y$ where $y(a) = x_a$ $\forall a \in A$. The converse is trivial. □

The proof of Tychonoff's Theorem given above relies on the Axiom of Choice, through the use of universal nets. It has been shown that the Axioms of Set Theory plus Tychonoff's Theorem implies the Axiom of Choice.

17.13. ***Exercise.*** *Show that the product of compact topological spaces need not be compact with respect to the box topology.*

17.14. ***Exercise.*** *If X and Y are normal then $X \times Y$ is Tychonoff, and so is any subset of $X \times Y$. So if we can produce a subset of $X \times Y$ which is not T_4 we would have an example to show that $T_3 + T_2 \not\Rightarrow T_4$. Actually, since subspaces of normal spaces are $\mathcal{CR}$ we would have more than that: an example to show $\mathcal{CR} + T_2 \not\Rightarrow T_4$.*

*Let ω be the first ordinal with an infinite set of predecessors (the first infinite ordinal) and Ω the first ordinal with an uncountable number of predecessors (the first uncountable ordinal.) With order topology, $[0, \omega]$ and $[0, \Omega]$ are both compact and T_2 so the product $[0, \omega] \times [0, \Omega]$ is compact and T_2 too. That means the product is normal and hence $\mathcal{CR}$, so all subspaces are T_2 and $\mathcal{CR}$. This construction is called the **Tychonoff plank** which we denote T.*

In the following few paragraphs we are going to use notation (α, β) to denote both open intervals of ordinal numbers and ordered pairs of ordinals. This is one of the few times when this traditional dual meaning can be confusing, so pay attention to context.

*Consider $\widetilde{T} = T - \{(\omega, \Omega)\}$ with subspace topology. This space is called the **deleted Tychonoff plank**.*

The sets $A = \{(\omega, \gamma) \in \widetilde{T} \mid \gamma < \Omega\}$ and $B = \{(\gamma, \Omega) \in \widetilde{T} \mid \gamma < \omega\}$ are relatively closed in $\widetilde{T}$ and disjoint. Suppose D is a relatively open set containing B.

For each γ with $(\gamma, \Omega) \in B$ there is a least $a_\gamma \in [0, \omega)$ with $\{\gamma\} \times (a_\gamma, \Omega) \subset D$.

Let a be the least ordinal with $a \geq a_\gamma$ for all γ. There are a countable number of these a_γ and each one has only a countable number of predecessors so $a < \Omega$. This means that $[0, \omega] \times (a, \Omega] - \{(\omega, \Omega)\}$ is a nonempty subset of D.

But $(\omega, a + 1)$ is in A. So no open set containing A can be disjoint from D and we have shown that the deleted Tychonoff plank is not T_4.

Compactness and the various separation properties are, at least superficially, in conflict in the sense that compactness puts limits on the number of open sets in a topology, while separation requires their existence. That conflict has an interesting expression in the following exercise, and serves to distinguish compact T_2 spaces as rather special.

17.15. ***Exercise.*** *Suppose $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \mathbb{T}_3$ are three **different** topologies on a set Y and that $(Y, \mathbb{T}_2)$ is compact and T_2. Show that $(Y, \mathbb{T}_1)$ is not T_2 and $(Y, \mathbb{T}_3)$ is not compact.*

(hint: The identity map $\alpha \colon (Y, \mathbb{T}_2) \to (Y, \mathbb{T}_1)$ is continuous. Let A be a member of $\mathbb{T}_2$ not in $\mathbb{T}_1$. So $Y - A$ is closed in compact $(Y, \mathbb{T}_2)$ and hence compact itself. So $\alpha(Y - A) = Y - A$ is compact in $(Y, \mathbb{T}_1)$. If $(Y, \mathbb{T}_1)$ were T_2 $Y - A$ would be closed in $(Y, \mathbb{T}_1)$, contrary to choice of A. To show that $(Y, \mathbb{T}_3)$ is not compact use a very similar argument applied to the continuous identity map $\beta \colon (Y, \mathbb{T}_3) \to (Y, \mathbb{T}_2)$.)

18. Other Conditions Related to Compactness

There are various weaker conditions similar to compactness that are frequently of use. A topological space is called **countably compact** if each countable open cover has a finite subcover. It is called **Lindelöf** if each open cover has a countable subcover, and **σ-compact** if it is the union of countably many compact sets. It is called **locally compact** if each point has a compact neighborhood. (This does not imply that there is a base for the topology consisting of open sets with compact closure, because the closure of a compact set need not be compact, in general.)

18.1**. *Exercise*.** *Compactness implies these other four conditions. The σ-compactness condition implies the Lindelöf condition. Also, a C_{II} space is Lindelöf. A Lindelöf and countably compact space is compact. A Lindelöf and locally compact space is σ-compact.*

18.2**. *Exercise*.** *In this exercise we explore some of the relationships among the four properties: Lindelöf, C_I, C_{II} and separable. Except for the fact that second countability implies the other three, there is no combination of properties that implies another.*

Let Ω denote the first uncountable ordinal and let $X = \Omega + 1 = \Omega \cup \{\Omega\} = [0, \Omega]$.

Let $*$ be any point not in X and define $X^* = \{*\} \cup X$.

Form topology $\mathbb{S}$ from subbasic sets $\{x \in X \mid y < x\}$ for $y \in X$ and let $\mathbb{D}$ be the discrete topology on X. Define $\mathbb{S}^* = \{\varnothing\} \cup \{A \cup \{*\} \mid A \in \mathbb{S}\}$ and $\mathbb{D}^* = \{\varnothing\} \cup \{A \cup \{*\} \mid A \in \mathbb{D}\}$.

(i) The subspace $X^* - \{\Omega\}$ of $\mathbb{S}^*$ is Lindelöf, C_I and separable but not C_{II}.

(ii) X^* with topology $\mathbb{S}^*$ is Lindelöf and separable but not C_I.

(iii) Ω with subspace topology from $\mathbb{S}$ is Lindelöf and C_I but not separable.

(iv) X^* with topology $\mathbb{D}^*$ is separable and C_I but not Lindelöf.

18.3**. *Exercise*.** *$\mathbb{R}$ is C_{II} and T_2. If a subset of $\mathbb{R}$ is unbounded it cannot be compact. A closed interval $[a, b]$ is compact. So a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. This last is the most common form of the **Heine-Borel Theorem**. $\mathbb{R}$ is locally compact and σ-compact.*

(hint: We will indicate how to prove that $[a, b]$ is compact. Suppose n is a universal net in $[a, b]$. Then n is eventually in $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$. Let $[a_1, b_1]$ be the leftmost subinterval $\left[a, \frac{a+b}{2}\right]$ if the net is eventually in that set and $\left[\frac{a+b}{2}, b\right]$ otherwise. Define Q_1 to be the set of rationals less than a_1. Iterate this process, in each case producing an interval $[a_i, b_i]$ half as big as before, contained in $[a_{i-1}, b_{i-1}]$, and with n eventually in the interval and a set of rationals Q_i defined to be those rationals less than a_i. Note $Q_i \subset Q_{i+1}$ for each i. The set $\bigcup_{i \geq 1} Q_i$ is a real number L and n is eventually in any open interval containing L, so n converges to L.)

18.4**. *Exercise*.** *Suppose $f \colon X \to \mathbb{R}$ and C is countably compact in X. Let $a = inf\ f(C)$ and $b = sup\ f(C)$. Then a and b are real (that is, f is bounded on C) and there are points x and y in C for which $f(x) = a$ and $f(y) = b$. The function* **f *actually attains both a maximum and a minimum value on* C**. *This is called the **Extreme Value Theorem** for countably compact sets.*

18.5. ***Exercise.*** *Examine the product topology on $\mathbb{R} \times \{a, b\}$ where $\{a, b\}$ is a two point set with indiscrete topology. Let $A = \{(t, a) \mid t \in [0, 1]\}$ and $B = \{(t, a) \mid t \in (0, 1)\} \cup \{(1, b), (0, b)\}$. This is an example to show that the intersection of two compact sets can fail to be compact. Note that these sets are not closed.*

Y is called **paracompact** if every open cover has a locally finite open refinement. Some sources require paracompact spaces to be T_2, but we do not. Trivially, every compact space is paracompact.

18.6. ***Exercise.*** *The existence of a locally finite open refinement for each open cover, guaranteed in paracompact spaces, implies that every open cover has a locally finite open refinement which is a minimal cover. See Exercise 14.5.*

18.7. ***Exercise.*** *A closed subset of a paracompact space is itself paracompact with subspace topology. More generally, an F_σ subset is paracompact.*

18.8. ***Exercise.*** *A separable paracompact space is Lindelöf.*

18.9. ***Exercise.*** *We will provide here an example to show that a normal space need not be paracompact.*

(i) Specifically, we saw in Exercise 17.10 that Ω with order topology is normal but not compact, where Ω is the first uncountable ordinal. We will show below that it actually fails to be paracompact. Consider the open cover $\mathbb{A} = \{[0, \beta) \mid 0 < \beta < \Omega\}$ and suppose $\mathbb{B}$ is any open refinement of $\mathbb{A}$. We will show that $\mathbb{B}$ is not point finite, and hence cannot be locally finite, and conclude that Ω is not paracompact. For each $x \in \Omega$ define a_x to be a member of Ω for which $(a_x, x]$ is a subset of some member of $\mathbb{B}$.

(ii) We claim that there is a $y \in \Omega$ so that for each $x \in \Omega$ there is an $h(x) \in \Omega$ for which $a_{h(x)} \leq y$.

If this were false then for every $y \in \Omega$ there would be a least $x \in \Omega$ so that $z \geq x$ implies $a_z > y$. We will indicate this least x by $f(y)$.

Let ω be the first infinite ordinal and define $g \colon \omega \to \Omega$ inductively by $g(1) = f(1)$ and, having defined $g(n)$, let $g(n+1) = f(g(n))$.

The set $S = g(\omega)$ is a countable subset of Ω and so has a least upper bound $\mu < \Omega$ in Ω. Note that if $\tau \geq g(n)$ then $a_\tau > g(n)$ and since $\mu \geq g(n+1)$ for every n we have $a_\mu \geq \mu$. But by definition we must have $a_\mu < \mu$. So the assumption that led to this contradiction is false and the claim is proved.

(iii) So let y be a member of Ω so that for each $x \in \Omega$ there is an $h(x) \in \Omega$ with $h(x) \geq x$ and $a_{h(x)} < y$. Using induction, infer the existence of an infinite sequence $x_1, x_2, \ldots$ of members of Ω for which $a_{h(x_i)} \leq y < h(x_i)$. Choose x_{i+1} to be large enough in each case so that there is a member of $\mathbb{B}$ containing $(a_{h(x_i)}, h(x_i)]$ but not $(a_{h(x_{i+1})}, h(x_{i+1})]$.

$y + 1$ is in infinitely many of intervals $(a_{h(x_i)}, h(x_i)]$ and therefore in infinitely many of the members of $\mathbb{B}$.

A space X is called **sequentially compact** if every sequence has a convergent subsequence. Although this property is similar to compactness, it is only related in the presence of additional conditions.

18.10**. *Exercise*.** *(i) The set Ω with order topology is sequentially compact but not compact.*

(ii) Give $[0,1]$ the usual topology and let $X = [0,1]^{[0,1]}$ with product topology. X is compact. Consider the sequence x in X defined for each $n \in \mathbb{N}$ by letting $x_n(t)$ be the n-th digit in the binary expansion of the number $t \in [0,1]$. Suppose $f\colon \mathbb{N} \to \mathbb{N}$ is any increasing function. So $y = x \circ f$ is a generic subsequence of x. Now define

$$t = \sum_{j=0}^{\infty} \frac{c(j)}{2^{f(j)+1}}$$

where $c(j) = 0$ if j is even, and $c(j) = 1$ if j is odd. So the real valued sequence $y(t)$ does not converge. This implies that X is not sequentially compact.

18.11**. *Exercise*.** *(i) A sequentially compact space is countably compact (hint: Suppose given any countable open cover V_i for $i \in \mathbb{N}$ with no finite subcover. Use the V_i to create a chain W_i for $i \in \mathbb{N}$ with $W_i \neq W_{i+1}$ for each i and $X = \bigcup_{i=0}^{\infty} W_i$. Select for each i a member r_i of $W_{i+1} - W_i$. Show that r has no convergent subsequence.)*

(ii) A paracompact and countably compact space is compact. (hint: Suppose $\mathbb{U}$ is an open cover in paracompact sequentially compact X. Let $\mathbb{O}$ be a locally finite open refinement of $\mathbb{U}$ which is a minimal open cover: each member of $\mathbb{O}$ has a point that is not in any other member of $\mathbb{O}$. If $\mathbb{O}$ is infinite there would be a sequence p_i formed by selecting one of these member-specific points from a different member O_i of $\mathbb{O}$ for each integer. Let A be the union of all the members of $\mathbb{O}$, if any, not used to create the O_i. So A together with the O_i form a countable cover with no finite subcover.)

(iii) If r is a sequence in countably compact X then r has a cluster point. (hint: Let $S_n = \{\, r_i \mid i \geq n \,\}$. If S_0 is finite then r equals some point p in X infinitely often, and p will be a cluster point. If S_0 is infinite, we may assume without loss (by going to a subsequence, if necessary) that r is one-to-one. Suppose r has no cluster point. Each $p \in X$ has an open neighborhood O_p that does not intersect S_{k_p} for some k_p. For each p let A_p denote the union of ***all*** *open neighborhoods of p that do not intersect S_{k_p}. Though X may be uncountable, the set of* ***distinct*** *A_p is countable. These sets constitutes a countable cover with no finite subcover.)*

18.12**. *Exercise*.** *(i) A countably compact and first countable space is sequentially compact. (hint: Suppose $y\colon \mathbb{N} \to X$ in countably compact and first countable X. If $y(\mathbb{N})$ is finite it is easy to produce a convergent subsequence. We may presume, by going to a subsequence if necessary, that y is one-to-one. Let $S_n = \{y_k \mid k \geq n\}$. If $\bigcap_{n\in\mathbb{N}} \overline{S_n} = \varnothing$ we could use the $\overline{S_n}$ to produce an countable open cover of X without a finite subcover. Conclude that there is some $p \in \bigcap_{n\in\mathbb{N}} \overline{S_n}$. Let $\{\mathbb{O}_n \mid n \in \mathbb{N}\}$ be a nested open countable neighborhood base at p. Deduce that $S_k \cap \mathbb{O}_n \neq \varnothing$ for all k, n and use that fact to create a subsequence of y converging to p.)*

(ii) A countably compact and first countable and T_1 space is regular. (hint: Suppose $p \in X$ and A is nonempty and closed in the countably compact and first countable and T_1 space X and $p \notin A$. Let $\{\mathbb{O}_n \mid n \in \mathbb{N}\}$ be a nested open countable neighborhood base at p. We may presume that $\mathbb{O}_1 \cap A = \varnothing$. Since X is T_1, $\{p\} = \bigcap_{n\in\mathbb{N}} \overline{\mathbb{O}_n}$ So $\{A^c\} \cup \{\overline{\mathbb{O}_n}^{\,c} \mid n \in \mathbb{N}\}$ is a countable open cover of X. So for some k, $\{A^c,\ \overline{\mathbb{O}_k}^{\,c}\}$ is a subcover. The result follows easily.)

18.13. ***Exercise.*** *The usual statement of the* ***Bolzano-Weierstrass Theorem*** *is: Any bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence. Prove it.*

19. Some Implications of the Weaker Compactness Conditions

19.1. ***Proposition.*** *A Lindelöf and* T_3 *space is* T_4*.*

Proof. Suppose A and B are closed, disjoint and nonempty in the Lindelöf and T_3 space X. Each $a \in A$ has an open neighborhood N_a with $\overline{N_a} \cap B = \varnothing$. Each $b \in B$ has an open neighborhood N_b with $\overline{N_b} \cap A = \varnothing$. The Lindelöf property implies there are sequences a_i in A and b_i in B with $A \subset \bigcup_{i\in\mathbb{N}} N_{a_i}$ and $B \subset \bigcup_{i\in\mathbb{N}} N_{b_i}$.

$$\text{Define } \mathcal{O}_1 = \bigcup_{i\in\mathbb{N}} \left(N_{a_i} - \bigcup_{j=0}^{i} \overline{N_{b_j}} \right) \text{ and } \mathcal{O}_2 = \bigcup_{i\in\mathbb{N}} \left(N_{b_i} - \bigcup_{j=0}^{i} \overline{N_{a_j}} \right).$$

These are open sets and $A \subset \mathcal{O}_1$ and $B \subset \mathcal{O}_2$. Moreover

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \bigcup_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \left[\left(N_{a_i} - \bigcup_{k=0}^{i} \overline{N_{b_k}} \right) \bigcap \left(N_{b_j} - \bigcup_{l=0}^{j} \overline{N_{a_l}} \right) \right] = \varnothing.$$

□

19.2. ***Exercise.*** *(i) Conclude from Proposition 19.1 that a Lindelöf regular space is normal.*

(ii) A C_{II} *and* T_3 *space is perfectly* T_4*. (hint: Use Exercises 13.5 and 16.7.)*

19.3. ***Exercise.*** *(i) Suppose* X *is either* T_2 *or* T_3 *and is locally compact. The set of closed compact neighborhoods of* x *is a neighborhood base for each* $x \in X$*. (hint: Suppose* A *is any open neighborhood of* x*. We need to find a closed compact neighborhood of* x *inside* A*. Let* $B = A \cap N_x^o$ *where* N_x *is a compact neighborhood of* x*. So* B *is open and contained in both* A *and* $\overline{N_x}$*. There are two cases. First, suppose* X *is regular. By Exercise 17.3* $\overline{N_x}$ *is compact. Since* $X - B$ *is closed there is an open set* $G \subset A$ *with* $x \in G$ *and* $\overline{G} \subset B \subset A$*.* $\overline{G}$ *is a closed subset of a compact set and so itself compact. Second, suppose* X *is* T_2*. This means* N_x *is closed and so* $\overline{B}$ *is a compact subset of* N_x *with* x *in its interior.* $\overline{B}$ *with subspace topology is compact* T_2 *hence normal. So there is relatively open* $G \subset B$ *with* $x \in G$ *and for which the relative closure of* G *is a subset of* B*. Since* G *is a relatively open subset of open* B *it is open in* X*. Since the relative closure of* G *is relatively closed in closed* $\overline{B}$ *it is closed in* X*: that is, the relative closure is actually* $\overline{G}$*. Since* $\overline{B}$ *is compact so is* $\overline{G}$*.)*

(ii) Let $\mathbb{B}$ *denote the collection of interiors of compact sets in* X *and* $\widetilde{\mathbb{B}} = \{\overline{A} \mid A \in \mathbb{B}\}$*. If* X *is locally compact* $\mathbb{B}$ *is a base for the topology. If* X *is also* T_2 *or* T_3 *then* $\widetilde{\mathbb{B}}$ *is a compact neighborhood base for the topology.*

19.4. ***Exercise.*** *(i) A* T_2 *locally compact space is Tychonoff. (hint: Suppose* $x \in X$ *and* K *is a closed set not containing* x*. From Exercise 19.3 we have an open neighborhood* G *of* x *with* $\overline{G}$ *compact and* $\overline{G} \cap K = \varnothing$*. Since* $\overline{G}$ *is compact it is normal with subspace topology so there is a relatively open neighborhood* H *of* x *and for which the relative closure of* H *is a subset of* G*. As in Exercise 19.3, since* H *is relatively open in open* G *it is actually open in* X*, and since the relative closure of* H *is relatively closed in closed* $\overline{G}$ *it is actually closed in* X*. Because* $\overline{G}$ *is normal there* *is a Urysohn function $g\colon \overline{G} \to [0,1]$ separating $\overline{H}$ from $\overline{G} - G$. Suppose $g(\overline{H}) = \{1\}$ and $g(\overline{G} - G) = \{0\}$. Define $f\colon X \to [0,1]$ by $f(p) = 0$ when $p \notin G$ and $f(p) = g(p)$ if $p \in G$. Show that f is continuous.)*

(ii) A T_3 locally compact space is $\mathcal{CR}$.

19.5. ***Proposition.*** *A T_2 paracompact space is normal.*

Proof. Suppose Y is T_2 and paracompact and $x \in Y - A$ and A is closed and nonempty in Y.

Because Y is T_2, for each $a \in A$ there is an open set V_a containing a but with $x \notin \overline{V_a}$. The collection of open sets consisting of $Y - A$ and all these V_a is an open cover $\mathbb{G}$ of Y. Because Y is paracompact there is an open locally finite refinement $\mathbb{H}$ of $\mathbb{G}$.

By Lemma 14.4 there is a subcover $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ with open locally finite refinement $\widetilde{\mathbb{H}}$ and a function $f\colon \widetilde{\mathbb{G}} \to \widetilde{\mathbb{H}}$ so that $f(K) \subset K\ \forall K \in \widetilde{\mathbb{G}}$.

Note that $Y - A \in \widetilde{\mathbb{G}}$ and $x \in f(Y - A) \subset Y - A$ but every other member of $\widetilde{\mathbb{H}}$ is of the form $f(V_a) \subset V_a$ for some $a \in A$.

Define the sets

$$S_A = \bigcup_{V_a \in \widetilde{\mathbb{G}}} f(V_a) \quad \text{and} \quad T_A = \bigcup_{V_a \in \widetilde{\mathbb{G}}} \overline{f(V_a)}$$
$$W_x = Y - S_A \quad \text{and} \quad Z_x = Y - T_A.$$

S_A consists of the union of all $f(V_a) \in \widetilde{\mathbb{H}}$, which are the only members of the open cover which could intersect A so S_A is an open set containing A.

By Exercise 14.1 we find $T_A = \overline{S_A}$. It is the locally finite nature of $\widetilde{\mathbb{H}}$ that permits this conclusion.

x is not in any $\overline{V_a} \supset \overline{f(V_a)}$ so Z_x is an open neighborhood of x. The closure of Z_x is W_x which is disjoint from S_A.

So the open sets S_A and Z_x separate the point x from the closed set A: we have just shown that Y is regular.

If A and B are two disjoint nonempty closed sets we can now duplicate this argument with B in place of x, finding for each $a \in A$ an open set V_a containing A for which $\overline{V_a} \cap B = \varnothing$. Using the paracompactness of Y just as before we produce open sets S_A and Z_B separating the closed sets A and B. □

19.6. ***Proposition.*** *A regular and Lindelöf space is paracompact.*

Proof. Suppose $\mathbb{G}$ is an open cover of regular and Lindelöf X.

For each p in each G in $\mathbb{G}$, regularity implies the existence of open $A_{G,p}$ with $p \in A_{G,p} \subset \overline{A_{G,p}} \subset G$. We can then find open $B_{G,p}$ with

$$p \in B_{G,p} \subset \overline{B_{G,p}} \subset A_{G,p} \subset \overline{A_{G,p}} \subset G.$$

The collection of all these $B_{G,p}$ form an open refinement of $\mathbb{G}$.

Extract the countable open subcover $\mathbb{B} = \{B_i \mid for i \in \mathbb{N}\}$ of this refinement, invoking the Lindelöf property. Let $\mathbb{A} = \{A_i \mid i \in \mathbb{N}\}$ be the open refinement of $\mathbb{G}$ consisting of the sets $A_{G,p}$ associated with each $B_{G,p}$ picked for membership in $\mathbb{B}$.

Create the countable collection of sets with $H_i = A_i - \bigcup_{j=0}^{i-1} \overline{B_j}$ for $i > 1$ and $H_0 = A_0$. $\mathbb{H}$, the set of these H_i, also forms a cover of X, because each point p in X occurs in A_i for some smallest i and then $p \in H_i$. Note also that p occurs in $\overline{B_k}$ for some smallest $k \geq i$ and then $p \notin H_j$ for $j \geq k+1$. Even more, $B_k \cap H_j = \varnothing$ for $j > k$. So $\mathbb{H}$ is a locally finite open refinement of $\mathbb{G}$. □

19.7. ***Proposition.*** *If X is either T_2 or T_3 and is both locally compact and paracompact then X is the free union of σ-compact spaces.*

Proof. In light of Exercise 19.3 (i) there is an open cover $\mathbb{S}$ of locally compact and paracompact X consisting of sets with compact closure. Let $\mathbb{G}$ be a locally finite open refinement of $\mathbb{S}$.

Each member A of $\mathbb{G}$ has compact closure. For each point of $\overline{A}$ there is a neighborhood touching only finitely many members of $\mathbb{G}$. Because $\overline{A}$ is compact a finite number of these neighborhoods cover $\overline{A}$ so A itself touches only a finite number of members of $\mathbb{G}$.

Create an equivalence relation on $\mathbb{G}$ as follows: A is related to B if and only if there is an integer n and a list A_i for $i = 1, \ldots, n$ of members of $\mathbb{G}$ with $A = A_1$ and $B = A_n$ and $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, \ldots, n-1$.

It is an exercise to show that the union of the members of each class is σ-finite. The result follows. □

19.8. ***Exercise.*** *A T_2 and σ-compact and locally compact space is both normal and paracompact.*

19.9. ***Exercise.*** *One reason paracompact spaces are important is because in T_2 paracompact spaces there is a partition of unity subordinate to every open cover. If, further, the space is locally compact the functions involved can be chosen to have compact support. Examine Proposition 16.11 and prove these statements.*

19.10. ***Exercise.*** *(i) $\mathbb{R}$ is paracompact. (hint: Given any open cover $\mathbb{G}$ create the refinement consisting of all $(n, n+3) \cap A$ for $A \in \mathbb{G}$ and all integers n. Then use compactness of each $[n+1, n+2]$ to create a locally finite subcover of this refinement.)*

(ii) Each open set in $\mathbb{R}$ is an F_σ set so each closed set is a G_δ. (hint: Any open set is the union of closed sets $[r_i, s_i]$ for rational r_i and s_i.)

(iii) These facts, coupled with Proposition 19.5 and Exercise 16.7, imply that any two disjoint nonempty subsets A and B in $\mathbb{R}$ can be precisely separated by a Urysohn function.

19.11. ***Proposition.*** *Suppose Y is locally compact and either T_2 or T_3, and A_i, $i \in \mathbb{N}$ is a countable set of open dense subsets. Then $\bigcap_{i=0}^{\infty} A_i$ is dense.*

Proof. Given the conditions, we know that Y is T_3 (since T_2 and local compactness imply regularity) and by Exercise 19.3 that Y has a neighborhood base of closed and compact sets for each point. We will have the result if we can conclude that $K \cap (\bigcap_{i=0}^{\infty} A_i)$ is nonempty for every closed compact K with nonempty interior.

We know that $P_0 = K^0 \cap A_0$ is open and nonempty in closed compact K. So there is a nonempty open set B_0 with $B_0 \subset \overline{B_0} \subset P_0 \subset K$. Having found nonempty open B_i with $\overline{B_i} \subset K$ note that $P_{i+1} = B_i \cap (\bigcap_{j=0}^{i+1} A_j)$ is open and nonempty. So there is a nonempty open set B_{i+1} with $B_{i+1} \subset \overline{B_{i+1}} \subset P_{i+1} \subset K$.

This process forms a nested sequence $\overline{B_i},\ i \in \mathbb{N}$ of nonempty subsets of compact K, so the intersection is nonempty and contains at least one member of $\bigcap_{i=0}^{\infty} A_i$. $\square$

19.12**. *Exercise. Baire Category Theorem (Part One)*** *A locally compact space Y which is either T_2 or T_3 is of second category. More generally, subsets of Y of first category have empty interior.*

19.13**. *Exercise.*** *Suppose X is T_2 and Y is a dense subset of X. If Y with subspace topology is locally compact then Y is an open subset of X.*

20. Paracompactness: Equivalent Conditions

20.1**. *Proposition.*** *A space X is T_3 and paracompact if and only if any of the following three equivalent conditions holds.*

(i) X is T_3 and each open cover has a σ-locally finite open refinement.

(ii) X is T_3 and each open cover has a locally finite refinement.

(iii) X is T_3 and each open cover has a locally finite closed refinement.

Proof. It is obvious that (i) is true if X is T_3 and paracompact.

So assume condition (i) to be true and suppose $\mathbb{O}$ is an open cover of X. Then by condition (i), $\mathbb{O}$ has an refinement $\mathbb{A} = \bigcup_{n \in \mathbb{N}} \mathbb{A}_n$ where each $\mathbb{A}_n$ is locally finite. Let X_n be the union of the members of $\mathbb{A}_n$. Each X_n is open and $X = \bigcup_{n=0}^{\infty} X_n$. Define for each n the set $Y_n = X_n - \bigcup_{i=0}^{n-1} X_i$. Finally, define $\mathbb{B}$ to be the set of all $Y_n \cap A$ for $A \in \mathbb{A}_n$ and $n \in \mathbb{N}$.

Suppose $p \in X$. Let i be the first integer with $p \in X_i$. So $p \in Y_i$ but $p \notin Y_n$ for $i \neq n$. By assumption, there is a neighborhood V which intersects only sets $A_1, \ldots, A_k$ in $\bigcup_{j=0}^{i} \mathbb{A}_j$. So $V \cap \left(\bigcup_{j=0}^{i} X_j \right)$ is an open neighborhood of p intersecting only the members of $\mathbb{B}$ created from the $A_1, \ldots, A_k$. We conclude that $\mathbb{B}$ is a locally finite refinement of $\mathbb{O}$ and condition (ii) follows.

Now assume condition (ii) to be true, and let $\mathbb{O}$ be an open cover of X. For each $p \in X$ select neighborhood V_p of p in $\mathbb{O}$. Since X is T_3 there is an open neighborhood K_p of p with $\overline{K_p} \subset V_p$ for each p. The open cover formed from all these K_p therefore has a locally finite refinement by property (ii). This implies by Exercise 14.5 that the cover formed all these K_p has a minimal subcover $\mathbb{B} = \{K_p \mid p \in S\}$ for a subset S of X. By the Refinement Lemma and (ii), $\mathbb{B}$ has a locally finite refinement $\mathbb{C}$ where each member of $\mathbb{C}$ is in exactly one member of $\mathbb{B}$. By Exercise 14.1 the cover $\mathbb{D}$ formed from the closure of the members of $\mathbb{C}$ is also locally finite, and a refinement of $\mathbb{O}$. This implies that (iii) is true.

It remains only to show that (iii) implies paracompactness.

Suppose (iii) to be true and $\mathbb{O}$ is an open cover of X. Let $\mathbb{A}$ be a closed neighborhood finite refinement of $\mathbb{O}$. By substituting a subcover of $\mathbb{O}$ for $\mathbb{O}$, if necessary, by Exercise 14.1 and the Refinement Lemma, we may without loss assume that there is a one-to-one and onto association between members A of $\mathbb{A}$ and O_A of $\mathbb{O}$ with $A \subset O_A$.

For each $p \in X$ select open neighborhood W_p of p that intersects only finitely many members of $\mathbb{A}$.

$\mathbb{V} = \{W_p \mid p \in X\}$ is an open cover which has, itself, a closed neighborhood finite refinement $\mathbb{B}$. We may presume $\mathbb{B}$ is in one-to-one correspondence with a subcover $\mathbb{W} = \{W_p \mid p \in S\}$ of $\mathbb{V}$. In particular, the Refinement Lemma lets us assume that for each $p \in S$ there is exactly one $B_p \in \mathbb{B}$ with $B_p \subset W_p$.

For each $p \in S$ the member B_p of $\mathbb{B}$ is in W_p, so can intersect only finitely many of the members of $\mathbb{A}$.

For each $A \in \mathbb{A}$ let $S_A = \{p \in S \mid B_p \cap A = \varnothing\}$ and $T_A = S - S_A$. Each T_A is finite.

Define $Z_A = \bigcup_{p \in S_A} B_p$. Z_A is closed, so $Y_A = X - Z_A$ is open for each A. Note $A \subset Y_A \subset \bigcup_{p \in T_A} B_p$ for every $A \in \mathbb{A}$. A point is in Y_A exactly when it is not in any of the members of $\mathbb{B}$ that fail to touch A.

Let $\mathbb{Y} = \{Y_A \cap O_A \mid A \in \mathbb{A}\}$. This is an open cover of X and obviously a refinement of $\mathbb{O}$.

Now suppose $x \in X$. Then there is an open neighborhood G of x intersecting only finitely many $B_{p_1}, \ldots, B_{p_L}$ of the members of $\mathbb{B}$. Each of these can touch only finitely many of the members of $\mathbb{A}$ so the entire list of members of $\mathbb{A}$ touched by any of them is itself finite. If a member Y_A of $\mathbb{Y}$ touches G then it intersects one of the B_{p_j}. So there is a point in B_{p_j} that is not in any of the members of $\mathbb{B}$ that fail to touch A. So B_{p_j} touches A, and A is "on the list." So the cover $\mathbb{Y}$ is locally finite and we conclude that X is paracompact. □

20.2. ***Theorem.*** *A space is T_2 and paracompact if and only if it is T_1 and T_*.*

Proof. Since T_2 paracompact spaces are normal, and in T_4 spaces all open covers have open barycentric refinements, we know such refinements always exist for a T_2 paracompact space so these space are T_* and, obviously, T_1.

Suppose, conversely, that X is T_1 and every open cover $\mathbb{O}$ has an open star refinement. Recall from Lemma 15.2 that X is regular, so we will create a (non-open) locally finite refinement for $\mathbb{O}$ and invoke Proposition 20.1.

Start by using a standard induction argument to assert the existence of a sequence $\mathbb{A}_n$, for $n \in \mathbb{N}$, of open covers for which $\mathbb{A}_{n+1}$ is an open star refinement of $\mathbb{A}_n$ for each $n \in \mathbb{N}$ and $\mathbb{A}_0$ is an open star refinement of $\mathbb{O}$.

Now create the sequence $\mathbb{B}_n$, for $n \in \mathbb{N}$, by

$$\mathbb{B}_0 = \mathbb{A}_0, \quad \mathbb{B}_1 = \mathbb{A}_1 \quad \text{and for } n > 1 \quad \mathbb{B}_n = \{Star_{\mathbb{A}_n}(V) \mid V \in \mathbb{B}_{n-1}\}.$$

It is obvious that $\mathbb{B}_n \ll \mathbb{A}_0$ for $n = 0,\ 1$ or 2. We will show that each $\mathbb{B}_n \ll \mathbb{A}_0$ for all n. In fact, we will show that $\{Star_{\mathbb{A}_{n-1}}(V) \mid V \in \mathbb{B}_{n-1}\} \ll \mathbb{A}_0$ for each $n \geq 2$, which will imply that $\{Star_{\mathbb{A}_n}(V) \mid V \in \mathbb{B}_{n-1}\} \ll \mathbb{A}_0$ because $\mathbb{A}_n \ll \mathbb{A}_{n-1}$.

Suppose we have $\{Star_{\mathbb{A}_{k-1}}(V) \mid V \in \mathbb{B}_{k-1}\} \ll \mathbb{A}_0$ for some particular $k \geq 2$. We know, for example, that this is true for $k = 2$. Any member $Star_{\mathbb{A}_k}(W)$ of $\{Star_{\mathbb{A}_k}(V) \mid V \in \mathbb{B}_k\}$ is of the form $Star_{\mathbb{A}_k}(Star_{\mathbb{A}_k}(L))$ for some $L \in \mathbb{B}_{k-1}$. But

$$Star_{\mathbb{A}_k}(Star_{\mathbb{A}_k}(L)) = \bigcup_{H \in \mathbb{A}_k(L)} Star_{\mathbb{A}_k}(H).$$

Each $Star_{\mathbb{A}_k}(H)$ in the union on the right is a subset of a member of $\mathbb{A}_{k-1}$, and each $H \in \mathbb{A}_k(L)$ is a subset of a member of $\mathbb{A}_{k-1}(L)$. We conclude that the union is a subset of $Star_{\mathbb{A}_{k-1}}(L)$ which, by hypothesis, is a subset of a member of $\mathbb{A}_0$. The conclusion that each $\mathbb{B}_n$ refines $\mathbb{A}_0$ follows.

Consider X as a well ordered set and for each $n \geq 1$ and $p \in X$ define

$$S_n(p) = Star_{\mathbb{B}_n}(\{p\}) - \bigcup_{x<p} Star_{\mathbb{B}_{n+1}}(\{x\}) \quad \text{and} \quad \mathbb{M} = \{S_n(p) \mid p \in X,\ n \geq 1\}.$$

For each $p \in X$, define

$$Q_p = \{x \mid p \in Star_{\mathbb{B}_n}(\{x\}) \text{ for some } n \geq 1\}.$$

The set Q_p is not empty, because (at least) $p \in Q_p$. Let q_p denote the first member of Q_p, and let n_p be the least integer with $n_p \geq 1$ and $p \in Star_{\mathbb{B}_{n_p}}(\{q_p\})$. So if $x \in X$ and $x < p$ then p cannot be in any $Star_{\mathbb{B}_{n+1}}(\{x\})$ and, in particular, $p \notin Star_{\mathbb{B}_{n_p+1}}(\{x\})$. We conclude that $p \in S_{n_p}(q_p)$ and, finally, that $\mathbb{M}$ is a cover of X. By construction $\mathbb{M} \ll \mathbb{A}_0$.

Suppose $B \in \mathbb{A}_{n+1}$ for some $n \geq 1$ and that $B \cap S_n(p) \neq \varnothing$ for $p \in X$. By definition, there must be a set $C \in \mathbb{B}_n$ with $p \in C$ and $C \cap B \neq \varnothing$. So there is a member $D \in \mathbb{B}_{n+1}$ containing $C \cup B$ and hence p. This means $B \subset Star_{\mathbb{B}_{n+1}}(\{p\})$. Examining the definition of $S_n(p)$, we conclude that $B \cap S_n(x) = \varnothing$ whenever $x > p$: in other words, B intersects only one set $S_n(p)$ for each p.

We now define

$$H_n(p) = Star_{\mathbb{A}_{n+2}}(S_n(p)) \quad \text{and} \quad \mathbb{H} = \{H_n(p) \mid n \geq 1,\ p \in X\}.$$

$\mathbb{H}$ is clearly an open cover of X. Also, because $\mathbb{M}$ and $\mathbb{A}_{n+2}$ both refine $\mathbb{A}_0$ which star refines $\mathbb{O}$ we know that $\mathbb{H} \ll \mathbb{O}$.

Finally, for each n the subcollection $\{H_n(p) \mid\ p \in X\}$ is locally finite: $B \in \mathbb{A}_{n+2}$ and $B \cap H_n(p) \neq \varnothing$ if and only if $S_n(p) \cap Star_{\mathbb{A}_{n+2}}(B) \neq \varnothing$. But $Star_{\mathbb{A}_{n+2}}(B)$ is contained in a set $C \in \mathbb{A}_{n+1}$, and we know members of $\mathbb{A}_{n+1}$ touch at most one set of the form $S_n(p)$ for each n. So $\mathbb{H}$ is a σ-locally finite open refinement of $\mathbb{O}$. $\square$

21. A Synopsis of Dependencies

$$T_2 \Rightarrow T_1 \Rightarrow T_0 \qquad\qquad T_0 + T_3 \Rightarrow T_2 \qquad\qquad \mathcal{CR} \Rightarrow T_3$$

$$T_0 + \mathcal{CR} \Rightarrow T_2 \qquad\qquad T_3 + C_{II} \overset{13.5}{\Rightarrow} \text{All Closed Sets are } G_\delta$$

$$T_* \ = \begin{array}{l}\text{Every Open Cover}\\ \text{Has An Open}\\ \text{Star Refinement}\end{array} \overset{15.1}{\Leftrightarrow} \begin{array}{l}\text{Every Open Cover}\\ \text{Has An Open Bary-}\\ \text{centric Refinement}\end{array} \overset{15.4}{\Leftrightarrow} \begin{array}{l}\text{Every Open Cover}\\ \text{Has A Locally}\\ \text{Starring Sequence}\end{array}$$

$T_4 \overset{16.3}{\Leftrightarrow}$ Tietze Function Extension Property $\overset{16.1}{\Leftrightarrow}$ Urysohn Functions Separate Disjoint Closed Sets $\overset{16.8}{\Leftrightarrow}$ Shrinking Lemma Property

$\overset{16.11}{\Leftrightarrow}$ A Partition of Unity Exists Subordinate To Each Locally Finite Cover $\overset{16.13}{\Leftrightarrow}$ If $f \leq g$ for U.S.C f and L.S.C. g there is a continuous function between f and g

Normal $\overset{16.2}{\Rightarrow}$ Tychonoff

$T_3 + T_4 \overset{16.6}{\Rightarrow} \mathcal{CR}$

$T_4 \overset{16.14}{\Rightarrow}$ Locally Finite Open Covers Have Open Barycentric Refinements

Perfectly $T_4 \overset{16.15}{\Rightarrow}$ Completely $T_4 \Rightarrow T_4$

T_2 or $T_3 \overset{17.2,17.3}{\Rightarrow}$ Compact Sets Have Compact Closure

Compact $\Rightarrow$ σ-Compact + Lindelöf + Countably Compact + Locally Compact + Paracompact

$C_{II} \overset{18.2}{\Rightarrow}$ Lindelöf + C_I + Separable

Separable + Paracompact $\overset{18.8}{\Rightarrow}$ Lindelöf

Sequentially Compact $\overset{18.11}{\Rightarrow}$ Countably Compact

Paracompact + Countably Compact $\overset{18.11}{\Rightarrow}$ Compact

Countably Compact + $C_I \overset{18.12}{\Rightarrow}$ Sequentially Compact

Countably Compact + C_I + $T_1 \overset{18.12}{\Rightarrow}$ Regular

σ-Compact $\Rightarrow$ Lindelöf

Lindelöf + Locally Compact $\Rightarrow$ σ-Compact

T_3 + Lindelöf $\overset{16.6,19.1}{\Rightarrow}$ T_4 + $\mathcal{CR}$

T_3 + $C_{II} \overset{13.5,19.2}{\Rightarrow}$ perfectly T_4

(T_2 or T_3) + Locally Compact $\overset{19.4,19.3,19.12}{\Rightarrow}$ $\mathcal{CR}$ + There Is A Closed and Compact Neighborhood Base + First Category Subsets Are Nowhere Dense

T_2 + Paracompact $\overset{19.5}{\Rightarrow}$ Normal + There Is A Partition Of Unity Subordinate To Each Cover

Regular + Lindelöf $\overset{19.6}{\Rightarrow}$ Paracompact

$$(\,T_2 \text{ or } T_3\,) + \begin{array}{l}\text{Paracompact +}\\ \text{Locally Compact}\end{array} \overset{19.7}{\Rightarrow} \begin{array}{l}\text{Free Union of}\\ \sigma\text{-Compact}\end{array}$$

$$T_2 \;+\; \sigma\text{-Compact} \;+\; \text{Locally Compact} \overset{19.8}{\Rightarrow} \text{Normal} \;+\; \text{Paracompact}$$

$$\begin{array}{l}T_3 \,+\\ \text{Para-}\\ \text{compact}\end{array} \overset{20.1}{\Leftrightarrow} \begin{array}{l}T_3 + \text{Every Open}\\ \text{Cover Has A}\\ \sigma\text{-Locally Finite}\\ \text{Open Refinement}\end{array} \Leftrightarrow \begin{array}{l}T_3 + \text{Every Open}\\ \text{Cover Has A}\\ \text{Locally Finite}\\ \text{Refinement}\end{array} \Leftrightarrow \begin{array}{l}T_3 + \text{Every Open}\\ \text{Cover Has A}\\ \text{Locally Finite}\\ \text{Closed Refinement}\end{array}$$

$$T_1 \;+\; T_* \overset{20.2}{\Leftrightarrow} \begin{array}{c}T_2 \,+\\ \text{Paracompact}\end{array}$$

22. Compactification

Compactness is such a useful condition that, in its absence, mathematicians sometimes try to embed a space as a subspace of a compact space and try to make inferences about the embedded image from properties of the larger space. Here is an example, called the **one point compactification**.

22.1. ***Exercise.*** *Suppose $(Y, \mathbb{T})$ is a topological space which is not compact. Let Y^* be the set Y together with a distinguished point $* \notin Y$. Declare $A \in \mathbb{T}^* \subset \mathbb{P}(Y^*)$ when $A \in \mathbb{T}$ or $Y - A$ is a closed compact set in $(Y, \mathbb{T})$.*

(i) $(Y^*, \mathbb{T}^*)$ is a compact space, the one point compactification of $(Y, \mathbb{T})$.

(ii) $(Y^*, \mathbb{T}^*)$ is T_2 if and only if $(Y, \mathbb{T})$ is T_2 and locally compact.

(iii) Suppose $f^* \colon (Y^*, \mathbb{T}^*) \to (X, \mathbb{W})$ is continuous for some topological space $(X, \mathbb{W})$. The restriction of f^* to $(Y, \mathbb{T})$ is continuous.

(iv) There is no reason to think that in general a continuous function $f \colon (Y, \mathbb{T}) \to (X, \mathbb{W})$ is the restriction of any continuous f^* as in (iii). For example, a continuous real valued function f on the real line will have a continuous extension to the one point compactification precisely when both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are equal.

We define a **compactification of** Y to be a homeomorphism $\beta \colon Y \to X$ where X has the subspace topology from a compact space Z and $\overline{X} = Z$. We will primarily be interested in the case where Z, and consequently Y, is T_2. We will construct a compactification that allows the richest possible class of continuous functions to be extended from Y, or more exactly its homeomorphic image X embedded in Z, to continuous functions on all of Z. But first we must do some spadework.

Suppose for the following discussion that F is a set of **continuous** functions from topological space Y to topological space W. F is said to **distinguish points** if for distinct points $x, y \in Y$ there is an $f \in F$ with $f(x) \neq f(y)$. F is said to **distinguish points from closed sets** if for points $x \in Y$ and closed nonempty A with $x \notin A$ there is an $f \in F$ with $f(x) \notin \overline{f(A)}$. We call F a **separating set of functions** if F distinguishes points and distinguishes points from closed sets.

Consider W^F with product topology. Define $e \colon Y \to W^F$ to be the evaluation map $e(y)(f) = f(y)$ for all $f \in F$. If n is any net in Y converging to a point p

then it is easy to show that $e \circ n$ converges to $e(p)$ in the product topology so e is continuous.

Recall that for each $f \in F$ the projection onto the factor space of W^F corresponding to f is $\pi_f : W^F \to W$ defined by $\pi_f(\Psi) = \Psi(f)$. The members of W^F of the form $e(y)$, defined as they are as the evaluation functions $e(y)(f) = f(y)$, constitute a minute fragment of all potential W valued functions on F. One interpretation of the overarching purpose of these notes (integration theory) is to carefully examine a broader class of these functions, when $W = \mathbb{R}$ and F is a vector lattice.

22.2**. *Proposition.*** *Suppose given Y, F and evaluation map e as above. If F is a separating set of functions then e provides a homeomorphism of Y onto $e(Y)$.*

Proof. From the remark above, e is continuous. Since F distinguishes points, e is one-to-one. We will show that $e(A)$ is closed in $e(Y)$ (with subspace topology from W) when A is closed in Y, and this will imply e is a homeomorphism onto $e(Y)$.

If ν is any net in $e(Y)$, the fact that e is one-to-one implies that there is a unique net n in Y with $\nu = e \circ n$. In fact, for any $S \subset Y$ if ν is a net in $e(S)$ then n will be a net in S.

Suppose ν is any net in $e(A)$ for closed A converging to a limit $e(p)$. This happens exactly when $\pi_f(\nu) = \pi_f(e \circ n) = f \circ n$ converges to $\pi_f(e(p)) = e(p)(f) = f(p)$ for all $f \in F$. In other words, for every $f \in F$ we have a net n in A for which $f \circ n$ converges to $f(p)$ in W.

Since F distinguishes points and closed sets, if $p \notin A$ there would be an $f \in F$ with $f(p) \notin \overline{f(A)}$. This contradicts the convergence of $f \circ n$ to $f(p)$ for every $f \in F$, and we conclude that no net in $e(A)$ can converge to a point in $e(Y - A) = e(Y) - e(A)$. So $e(A)$ is relatively closed in $e(Y)$. $\square$

If W is compact and F is a separating family of continuous functions, the evaluation $e : Y \to e(Y) \subset W^F$ is a compactification of Y. $\overline{e(Y)}$ is a closed subset of a compact space W^F and so is itself compact. Note that the topology on W^F has nothing to do with the topology on Y, since F is used only as an indexing set in the creation of the product topology. The existence of this compactification depends entirely on the existence of the separating set of continuous functions. We have seen that Tychonoff spaces possess such a family, where W is taken to be the closed unit interval $[0, 1]$.

The **Stone-Čech compactification** is obtained when the family F consists of *all* continuous functions from Y into $[0, 1]$ for Tychonoff Y.

22.3**. *Exercise.*** *Suppose C is compact T_2. So it is Tychonoff by Proposition 19.5. So the family F_C of all continuous functions $f : C \to [0, 1]$ is separating. Let $e_C : C \to [0, 1]^{F_C}$ be the Stone-Čech compactification of C.*

C is compact so $e_C(C)$ is compact in T_2 $[0, 1]^{F_C}$ and so closed. We come to the unsurprising conclusion that a compact T_2 space is isomorphic to the image of its Stone-Čech compactification. This observation is (closely) related to the Gelfand-Kolmogoroff Theorem, an algebraic reformulation found in Proposition **??**.

22.4. *Proposition.* *Suppose $e_Y \colon Y \to [0,1]^{F_Y}$ is the Stone-Čech compactification of Y where Y is Tychonoff and the family F_Y consists of the continuous functions from Y into $[0,1]$.*

Suppose $g \colon Y \to C$ is continuous where C is a compact T_2 space. g induces a continuous function $G \colon e_Y(Y) \to C$ where $G = g \circ e_Y^{-1}$, and since Y and $e_Y(Y)$ are homeomorphic via the homeomorphism e_Y we may regard g and G as, essentially, identical.

G has a unique continuous extension $\overline{G} \colon \overline{e(Y)} \to C$.

Proof. First, since $e_Y(Y)$ is dense in $\overline{e_Y(Y)}$ if there is a continuous extension at all it must be unique. So we will concentrate on existence.

Let $e_C \colon C \to [0,1]^{F_C}$ denote the Stone-Čech compactification of C.

Define $\widetilde{g} \colon F_C \to F_Y$ given by $\widetilde{g}(f) = f \circ g$.

Now define $g^* \colon [0,1]^{F_Y} \to [0,1]^{F_C}$ by $g^*(\Psi) = \Psi \circ \widetilde{g}$. We saw in Exercise 12.4 that g^* is continuous.

So g^* restricted to $\overline{e_Y(Y)}$ is continuous, and in fact the values of g^* on $e_Y(Y)$ determine its values on $\overline{e_Y(Y)}$.

If we could conclude that $g^*(e_Y(p)) = e_C(g(p))$ for each $p \in Y$ then we would know that $g^*(e_Y(Y)) \subset e_C(C)$, which is closed. So we would know that $g^*\left(\overline{e_Y(Y)}\right) \subset e_C(C)$ and so $e_C^{-1} \circ g^* \colon \overline{e_Y(Y)} \to C$ would be defined, and would obviously agree with G on $e_Y(Y)$. We would have $\overline{G} = e_C^{-1} \circ g^*$.

We proceed with the calculation:

$$\begin{aligned} g^*(e_Y(p)) =& g^*(\text{“evaluate members of } F_Y \text{ at } p\text{”}) \\ =& \text{“evaluate members of } F_C \text{ at } g(p)\text{”} = e_C(g(p)). \end{aligned}$$

□

This proposition implies that the Stone-Čech compactification is the “richest possible” compactification in the sense that it is least restrictive about how a function defined only on Y approaches the “edge” of Y if it is to be extended to a compact superset. Functions on Y embedded in *any* other dense subset of a compact set which can be extended to the compact set can always be extended using the Stone-Čech compactification.

22.5. *Exercise.* *The one point compactification of $\mathbb{R}$ (or rather the closure of the homomorphic image of $\mathbb{R}$ under the compactification) is homeomorphic to the unit circle, or if you prefer to $[0,1]$ with the identification topology collapsing $\{0,1\}$ to a point. So a function on $\mathbb{R}$ can be extended to the closure of this compactification if its values approach a limit as $|x|$ gets large. One can create a two point compactification of $\mathbb{R}$ where functions can be extended if they approach different limits as x gets large positive versus negative. This compactification essentially identifies $\mathbb{R}$ with $(0,1)$ inside compact $[0,1]$. Neither compactification has a rich enough “edge” to allow the extension of the function $f(x) = sin(x)$ to the closure. The Stone-Čech compactification does, as well as every other continuous function g for which $g(\mathbb{R})$ is a subset of a compact set in the range space. It is difficult to imagine what the boundary of $\mathbb{R}$ looks like in the Stone-Čech compactification. Try.*

23. Connectedness

Any subset C of a topological space Y is called **connected** if it cannot be separated by a pair of subsets of C. This means it cannot be written as $C = A \cup B$ where $\overline{A} \cap B = \varnothing = A \cap \overline{B}$ and $A \neq \varnothing \neq B$.

So the space Y itself is connected exactly when the only sets both closed and open are $\varnothing$ and Y. A subset C is connected if it cannot be written as a union of nonintersecting nonempty relatively open sets. Equivalently, C is connected if it cannot be written as a union of nonintersecting nonempty relatively closed sets.

23.1. ***Exercise.*** *Suppose Y is an infinite set with cofinite topology from Exercise 13.1. Y is connected with this topology.*

A **component** of Y is a maximal connected subset of Y.

23.2. ***Exercise.*** *(i) Suppose $\mathbb{G}$ is any collection of connected subsets of Y and suppose there is at least one point common to all members of $\mathbb{G}$. Then the union of all members of $\mathbb{G}$ is connected.*

(ii) Any nonempty connected set is contained in a component. In particular, each single point set is contained in a component. So the components form a partition of the space. (hint: Let M be the union of all connected sets containing connected and nonempty B. If $M = E \cup F$ where $\overline{E} \cap F = \varnothing = E \cap \overline{F}$ then because B is connected B must be a subset of either E or F. Say $B \subset E$. If F is nonempty there must be a connected set A containing B with $A \cap F \neq \varnothing$.)

(iii) The closure of a connected set is connected so components are closed. (hint: Suppose B is connected and nonempty and $\overline{B} = E \cup F$ where $\overline{E} \cap F = \varnothing = E \cap \overline{F}$. Because B is connected B must be a subset of either E or F. Say $B \subset E$. So $\overline{B} \subset \overline{E}$. Conclude that $F = \varnothing$.)

(iii) So if C_1 and C_2 are distinct components they are separated.

*(iv) Suppose S and T are subsets of a topological space X. If T intersects both S and $X - S$ and if $T \cap \partial S = \varnothing$ then T is **not** connected.*

(v) If $S \subset T \subset \overline{S}$ and S is connected then T is connected.

(vi) Suppose X is connected and $\varnothing \neq A \subset X$ and $A \neq X$. Then $\partial A \neq \varnothing$.

(vii) Suppose X is compact and T_2 and C is a component in X. Then $C = \bigcap_{S \in \mathbb{K}} S$ where $\mathbb{K}$ is the collection of sets in X which are both closed and open and contain C. (hint: Let B be the indicated intersection. So $A \subset B$. If B is connected then $A = B$, so we will show that B must be connected. Since B is the intersection of closed sets it is itself closed. Suppose B is not connected. Then there are closed nonempty sets D and E inside B with $B = D \cup E$ and $D \cap E = \varnothing$. Since $A \subset B \subset D \cup E$ and A is connected we have $A \subset D$ or $A \subset E$. Suppose $A \subset D$. Since $\varnothing = \bigcap_{S \in \mathbb{K}}(S - (D \cup E))$ is the intersection of closed subsets in compact X there is a finite intersection among them which is void. Since a finite intersection of closed and open sets is closed and open, there is a single closed and open set S containing C with $S \subset D \cup E$. Since X is compact and T_2 it is normal so there are disjoint open sets D^* and E^* with $D \subset D^*$ and $E \subset E^*$ and $\overline{D^*} \cap \overline{E^*} = \varnothing$. Let $D^{**} = S \cap D^*$ and $E^{**} = S \cap E^*$. Both D^{**} and E^{**} are open and contain points of B, and $A \subset D^{**}$. If we can show that D^{**} is also closed, then $D^{**} \in \mathbb{K}$, which would imply $B \subset D^{**}$, contradicting the statement that part of B is in E.

So suppose $p \in \overline{D^{**}} = \overline{S \cap D^*}$. *Since* S *is closed,* $p \in S$. *Since* $S \subset D \cup E$, p *is in* D *or* E. *If* $p \in E$ *then* p *would be a point in* $\overline{D^*} \cap E$, *which is supposed to be empty. So* $p \in D$ *and hence in* D^{**}. *So* D^{**} *is closed. We conclude that* B *must be connected.)*

If the component containing $\{x\}$ is $\{x\}$ for each $x \in Y$ we call Y **totally disconnected.** Obviously, the discrete topology on any set gives a totally disconnected topological space.

Consider the set given by

$$\{1\} \bigcup \left\{ \sum_{i=0}^{\infty} \frac{x_i}{3^i} \;\middle|\; x_i = 0 \text{ or } 2, \text{ and 0 occurs infinitely often} \right\}$$

with subspace topology from $[0,1]$. This subset of $[0,1]$ is called the **Cantor set**.

23.3**. *Exercise*.** *The Cantor set is a closed subset of* $[0,1]$. *It is totally disconnected but not discrete. In fact, it has no isolated points. The same is true of* $\mathbb{Q}$ *as a subspace of* $\mathbb{R}$ *and the set* $\mathbb{R}$ *itself with the right half open interval topology from Exercise 16.5.*

A common source of confusion concerns the distinction between "open and disjoint" in a subspace topology and in the original space. A subset is connected if it is connected in the *subspace* topology. Two sets $C \cap A$ and $C \cap B$ relatively open in C with subspace topology might have empty intersection but "come from" two open sets A and B in Y which do intersect outside of C. The point is that it is easier for C to be separated in C by a pair of relatively open subsets than for these two parts of C to lie in different "halves" of a separation of Y. The example below illustrates the difference between connected component and separation in the larger space.

Let X be the subset of $\mathbb{R}^2$ formed from

$$\{\ (0,1),\ (0,0)\ \} \cup \left\{ \left(\frac{1}{n+1}, t \right) \;\middle|\; n \in \mathbb{N} \text{ and } t \in [0,1] \right\}.$$

So $(0,1)$ and $(0,0)$ are each components of X, but both points lie in the same "half" of any separation of X.

23.4**. *Exercise*.** *Intervals (bounded or otherwise) in* $\mathbb{R}$ *are connected. They are the only connected subsets of* $\mathbb{R}$. *(first hint: If a set* $C \subset \mathbb{R}$ *containing at least two points* a *and* b *with* $a < b$ *is missing even a single point* c *with* $a < c < b$ *then* $S \cap (-\infty, c)$ *and* $S \cap (c, \infty)$ *separate* S. *second hint: Suppose* S *is an interval with endpoints* a *and* b *with* $a < b$ *and* $A \cap S$ *and* $B \cap S$ *are nonempty and relatively open in* S *and have empty intersection. Suppose* $t \in B \cap S$ *and* $s \in A \cap S$ *and* $s < t$. *Look at* $w = \sup\{p \in A \cap S \mid p < t\}$. *)*

23.5**. *Exercise*.** *If* $A \subset Y \cup Z = X$ *in topological space* X *and the two sets* $Y - Z$ *and* $Z - Y$ *are separated then*

$$\overline{A} = \left(Y \cap \overline{A \cap Y}\right) \cup \left(Z \cap \overline{A \cap Z}\right).$$

Conclusion: A *is closed exactly when both* $A \cap Y$ *is closed in* Y *and* $A \cap Z$ *is closed in* Z. *(hint: The limit of a net in* A *must be in* $Y - Z$, $Z - Y$ *or* $Z \cap Y$.*)*

23.6. ***Exercise.*** *(i) The definition of "connected" is given by denying the existence of certain subsets. A positive reformulation in terms of covers is the following. Y is connected if and only if for every open cover $\mathbb{G}$ of Y and each pair A and B of nonempty members of $\mathbb{G}$ there is a finite list $G_i \in \mathbb{G}$, $i = 1, \ldots, n$ for some integer n with $A \cap G_1 \neq \varnothing \neq B \cap G_n$ and $G_i \cap G_{i+1} \neq \varnothing$ for $i = 1, \ldots, n-1$. (hint: if there is a cover and a pair A, B violating this condition look at the set of all finite lists of overlapping members of $\mathbb{G}$ that touch A. Form the union of those members. That union is both open and closed.)*

(ii) Suppose $G_i \cap G_{i+1} \neq \varnothing$ for $i = 1, \ldots, n-1$ and each G_i is connected. Then the union of the G_i is connected.

(iii) If X is connected and $f\colon X \to Y$ is continuous then $f(X)$ is connected.

*(iv) If $f\colon X \to \mathbb{R}$ is continuous and C is connected in X then $f(C)$ is an interval. This is called the **Intermediate Value Theorem.***

(v) If $f\colon X \to Y$ is a homeomorphism then $f(C)$ is a component of Y for every component C of X, and f restricted to each C establishes a homeomorphism onto $f(C)$: that is, f associates each component of X to a unique component of Y and f generates a homeomorphism between each associated pair of components.

23.7. ***Exercise.*** *Let C denote the Cantor set described in Exercise 23.3. Consider the function $h\colon C \to [0,1]$ defined by $h(1) = 1$ and*

$$h\left(\sum_{i=0}^{\infty} \frac{x_i}{3^i}\right) \;=\; \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}$$

*where the binary representative of a point in the domain C is chosen so that $x_i = 0$ or 2, and 0 occurs infinitely often. So h is one-to-one and monotone and onto $[0,1]$. h is continuous. The inverse of h is **not** continuous.*

Dugundji [**?**] provides a counterexample of the "continuous analog" of the Schröder-Bernstein Theorem. It is not true that the existence of continuous and one-to-one functions $f\colon X \to Y$ and $g\colon Y \to X$ implies that X and Y are homeomorphic. Even more, in the example below both f and g are onto!

Define sets X and Y by

$$\begin{aligned} X =&\{\, 3n+2 \mid n \in \mathbb{N}\,\} \cup \left(\bigcup_{n\in\mathbb{N}} (3n+3, 3n+4)\right) \cup [0,1) \text{ and} \\ Y =&\{\, 3n+5 \mid n \in \mathbb{N}\,\} \cup \left(\bigcup_{n\in\mathbb{N}} (3n+3, 3n+4)\right) \cup [0,1] \end{aligned}$$

with subspace topologies from $\mathbb{R}$. These sets cannot be homeomorphic because the compact component $[0,1]$ in Y is not homeomorphic to any of the components of X. Now define

$$f\colon X \to Y \text{ by } f(x) = \begin{cases} x, & \text{if } x \neq 2; \\ 1, & \text{if } x = 2 \end{cases}$$

$$\text{and } g\colon Y \to X \text{ by } g(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq 1; \\ \frac{x-2}{2}, & \text{if } 3 < x < 4; \\ x-3, & \text{otherwise .} \end{cases}$$

Both f and g are continuous, one-to-one and onto.

23.8. ***Exercise.*** *(i) Give $\mathbb{N}$ the topology generated by basic open sets $\{an+b \mid n \in \mathbb{N}\}$ for $a, b \in \mathbb{N}$. With this topology $\mathbb{N}$ is T_2 and connected and C_{II}. Since $\mathbb{N}$ is countable and connected it cannot be $\mathcal{CR}$. Even more, it is not T_3 because a T_3 Lindelöf space is T_4, and a normal space is $\mathcal{CR}$.*

(ii) If X is $\mathcal{CR}$ and connected then X is indiscrete or X is uncountable.

(iii) If X is T_4 and connected and if there exist two nonempty disjoint closed sets in X then X is uncountable.

X is called **locally connected** when the set of connected open sets form a base for the topology. So if X is locally connected the components of X, which must be closed in any case, are also open. This implies, for example, that if x and y are in different components then x and y lie in different "halves" of at least one separation of X. A subset of X is called locally connected if it is locally connected with subspace topology.

23.9. ***Exercise.*** *(i) Suppose X is locally connected and $A \subset X$. If ∂A is locally connected then so is $\overline{A}$.*

(ii) Suppose X is locally connected and A and B are closed subsets with $X = A \cup B$ and suppose $A \cap B$ is locally connected. Then both A and B are locally connected.

A subset A of X is called **path connected** if for each pair of distinct points x and y there is a continuous function $f\colon [0,1] \to A$ with $f(0) = x$ and $f(1) = y$, where A has subspace topology. In this context we refer to these functions as **paths in A** or specifically a **path connecting x to y in A**. A **path component of X** is a maximal path connected subset of X.

Each path component of X is contained in a component, and the path components form a partition of X.

Let X be the subset of $\mathbb{R}^2$ formed from

$$\{\ (0,t) \,\Big|\, t \in [0,1]\} \cup \left\{ \left(t, sin\left(\frac{1}{t}\right)\ \right) \mid t \in (0,1] \right\}.$$

This example has two path components but is connected. It shows that path components need not be closed.

23.10. ***Exercise.*** *(i) A product space is connected if and only if each factor space is connected.*

(ii) A product space is path connected if and only if each factor space is path connected.

(iii) A product space is locally connected if and only if each factor space is locally connected and all but a finite number are connected.

23.11. ***Exercise.*** *(i) X is path connected exactly when it is connected and each point has a path connected neighborhood.*

(ii) Each point in X has a path connected neighborhood exactly when the path components and the components coincide.

(iii) Each point in X has a path connected neighborhood exactly when the path components are all open.

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