WHAT ARE WE (MATHEMATICIANS) DOING?

LARRY SUSANKA

At a certain point every working mathematician asks him-or-herself the question "What the heck am I doing?" or some similar puzzler. I am not referring here to the details of employment or collecting the properties of finite groups. I refer to the long chains of incredibly abstract reasoning to which one is bound to ascend, from time to time, in most areas of modern mathematics. I refer, for instance, to thinking about and using infinite sets whose members can—provably, if mathematics itself is consistent—never be identified. Or using, as a logical tool, the idea that all "properly formed" statements are either true or false, ignoring the possibility that in many cases (such as the continuum hypothesis, famously) there is good reason to adopt the point of view that there simply is no "fact of the matter" at all.

As viewed by most practitioners of mathematics itself (as distinct from logicians or philosophers) the currently fashionable version of mathematics, founded entirely on set theory, has an intrinsically Platonist ontology. Infinite objects of many (often indescribable) varieties are simply *presumed* to exist in some ideal netherworld, and mathematics is all about identifying the properties of certain of these objects with particularly interesting features.

The epistemology of mathematics has two parts. First we have axioms, facts known to be true by intuition, usually taken to be the Zermelo-Fraenkel Axioms plus the Axiom of Choice: ZFC. Then we fold in first-order logic (including the law of the excluded middle) and derivations of true statements from assumptions by methods implied by the chosen logic. Neither the axioms nor the logic are contingent on today's weather or the White Sox pennant chances. They refer to abstract things that seem to be outside of humans or the apparent causal structure of the physical world we see around us. If they exist at all, it would seem we need an account of how humans come to learn anatomic details of such slippery fish.

These axioms, unproven presumed truths about objects we call sets, were chosen because, firstly, our (or someone's) vision of the universe of sets forcefully suggests or even requires that they be true and, secondly, they combine to form a framework strong enough to discuss and prove many properties of sets which correspond to facts mathematicians would—very much—like to know.

From these axioms existence and properties of sets of interest are inferred, *often merely because the contrary conclusion would contradict an "assumed truth."*

Axioms are supposed to be so pellucidly, limpidly clear that any fool, after understanding our interpretation of their meaning in ordinary mathematical usage, would acclaim their necessity and validity.

Date: April 30, 2010.

But what if some of the fools find some of these axioms less than compelling in this way? Compelling for their consequences, yes indeed. Consistent: probably you cannot produce a directly contradictory statement using them.

But "obviously true?" The existence of completed infinite objects? Show me one, and I will then show you a couple of angels engaged in the "macarena" on the head of a pin.

Perhaps—just perhaps—we want the consequences so badly we are willing to accept these ... foundations.

Some philosophers argue that the utility of (some of) the implications of ZFC in collapsing data into concise theory in the sciences provides evidence bolstering the adoption of these axioms and associated logic. This is an inductive justification for a deductive universe of discourse, and most mathematicians will not like, will shy away from, this kind of argument.

That aside, working mathematicians seem to be, by and large, utterly unconcerned about or even oblivious to issues of the foundations of mathematics. Nor is it of interest to most that far weaker systems than ZFC can yield the same economies in representing experimental data from the sciences without so much of the problematic "overhead."

Topics at the research level are extremely difficult. Perhaps many mathematicians shy away from questions about underpinnings as a worrisome distraction. When you are thinking about foundations you are *not* thinking about the theorem you are trying to prove. And dithering about foundational matters can easily get you branded as a crank or a heretic, not a career enhancing event. Doubt saps the will to bring forth the multi-month-or-year unyielding obsession characteristic of a successful research program. Doubt is the enemy of the graduate student, bent on proving that first theorem of his-or-her very own. One's eye must remain on the ball.

But that little worry about mathematics doesn't *really* go away, does it? Modern mathematics is undeniably elegant, and the beauty, depth and clarity of thought of our most successful mathematicians is a treasure. But peering from behind the leg of that Mathematical Colossus which stands athwart the fantastically successful scientific enterprise, founded on mathematical models, is a demon asking the annoying question.

"What the heck are you doing?"

The choices made by mathematicians of *which* objects to study is on esthetic (internal) grounds or utilitarian (external) grounds. Though mathematicians often refer to the latter when asked to explain what they do, and why, many find the former to more accurately reflect their true motivation.

As a social activity and art form, mathematics needs no other justification and I have nothing more to say about that.

My goal here will be to suggest a setting and an argument for the utilitarian aspects of mathematical studies, even those of the most abstract kind.

I will speculate on the nature of humans and the function of model building for us, and vote for eclecticism in mathematical thinking: not to drive anyone from

that paradise whose gates were cast open by Cantor with his "completed infinities" and whose contents have been studied with incredible vigor and style this hundred years past, but to suggest a purpose for using *many* approaches to mathematical foundations.

(i). **In The Beginning . . .**

I propose here a certain mythology, a story about how humans and consciousness came to be. It is similar in some ways to the tale of the earth resting on the backs of four turtles floating in an endless sea. It is founded in belief, and one is free to believe such stories or not.

But mythologies serve a purpose for humans and have meaning for us beyond superficial detail. They sometimes provide a paradigm for how to live and how to think about other matters. Let's begin our story with early life and a speculation on the genesis and nature of our internal models of reality.

An amoeba oozes toward greater concentration of a chemical in the water. Why?

In some form there is a model, internal to the amoeba, of a perceived external reality. That model incorporates a judgement: "This chemical is good." Coupled to this is volition: "Go toward good" and, of course, some method of implementing volition.

According to standard evolutionary doctrine, errors in the genetic material that encodes the information needed to assemble the amoeba occur regularly. Most of these changes are harmful or neutral in their affect on amoeba survival. Some are beneficial.

The part of the amoeba that constitutes and enables the internal model of the external is not immune from these changes. Contact with an external reality guides this process, and the whole assembly— internal model, judgement, a "tie-breaker" mechanism to avoid paralysis and volition—encodes a type of memory or imprint of every contact that animal's ancestors ever had with the external.

Any aspect of the model which hurt rather than helped the model-holder would preferentially kill its possessor, and that model would vanish from amoeba-kind. Any new feature which seriously helped the amoeba avoid danger or increase reproduction rate will propagate into the future, carrying with it to descendent species that memory, that reflection of *aspects of the actual external world.*

Now there is a huge random element here. There is no necessity or inevitability about most details of the internal model, so far as I can see. Many of these details *could*, most likely, have been different. Many features could be simply irrelevant, neither hurting nor harming the model-holder directly, though a vast amount of model-detritus would likely be a distraction and subject to advantageous pruning.

Inherent in the model is a distinction, for instance, of past from future. The model says, implicitly, that if the amoeba perceives *this* it will go *there.* So when it *actually* perceives this, it *does* go there. It seems that something akin to "modus ponens" is built into the amoeba. It is built into the very *idea of a model that causes its possessor to change itself or act.* These are the only models that influence survival.

I am not saying that the external universe is causal in all its aspects. I only speculate that *parts* of the world *are* like this, and living things have come to perceive aspects of the universe that are causal because that is advantageous. Perhaps models incorporating random or acausal or "global" elements either did not appear by chance or, if they did, failed to provide advantage. Perhaps they did appear and are present, and I simply don't recognize them.

Also inherent in our example is the concept of space. "Go toward good" carries, implicitly, concepts of location and relative location.

Also there is an implied (possibly derivative) concept of "lumpiness" to the external. Good stuff is concentrated more in one place than in another. A bigger lump of good stuff is better than a smaller lump. "Two lumps are better than one" cannot be far from this.

It is extremely hard for me to imagine *any* substitute model of the external that could be advantageous to an amoeba that would not acknowledge, at least, "consecutiveness in time" and "more/less of this stimulator" and "relative location."

But then it *would* be difficult for me, wouldn't it? After all, I am this amoeba's descendent. Such a model probably wouldn't make any sense to this particular model builder. I wouldn't *recognize* it as a model, or anything at all except—possibly—a blob of non sequiturs. I could never *think* of it in the first place, and if it were presented to me by a computer program or an alien from Arcturus I might find it unsatisfying, even repulsive.

To appreciate the Parable of the Amoeba, and the remainder of the tale found below, there is no reason you must accept this mythology as whole cloth. You could postulate almost *any* system which *implies* the formation of our species via random mutation and natural selection.

(ii). **Roll Forward a Billion Years . . .**

Evolution has gradually added bells and whistles to the internal models. To reiterate, these bells and whistles are added by chance, but parsed in contact with an external reality by their ability to confer advantage.

First, individual rather than "programmed" memory is added. This is a record of interactions—and outcomes—of non-lethal contact with the external. This is part of a system that allows a creature to modify a model of the external during the lifetime of the creature, a tremendous advance. Cautious curiosity is rewarded. Still, any improvements are lost at death.

Another improvement would be the ability to transmit these ephemeral memories to related individuals. This would make curiosity more valuable. Even potentially very risky behavior, exploring the boundaries of a model or making erratic changes to a model, could have benefit. News of both beneficial and fatally bad ephemeral modification to a model would aid related individuals if that information could be communicated. Social structure appears.

(iii). **Another Few Hundred Million Years Passes . . .**

We arrive at . . . us: humanity.

The ideas produced by a human over a lifetime must number in the billions, and the specifics of their production depend on experiences of that individual. But the "idea space" from which these ideas are drawn is constrained by brain structure and the possible combinations of sensory input.

There is *"a priori"* knowledge. It is buried, with noise, in the propensities and limits of the idea-making machinery.

We have been shaped by the universe to make good models of the universe. Of course goodness is relative. Had different random genetic changes occurred, particularly early on in the evolutionary process, it is conceivable that the models we produce could have very different content. We are good at model building *in comparison to the current competition* and there is no telling how that ability would stand up to different competition. But we are good enough to survive, anyway, and our bodies and brains encode a vast amount of information about aspects of the external.

Humans take this model-building and run with it. We make models of everything, kaleidoscopes of models. We are compulsive model builders. We find patterns everywhere, *patterns where there are no patterns,* models of groups of models. We *must* find models for everything we perceive. It has paid off for all our ancestors.

We see a splash of color on our retina. We imagine dozens of potential models of what it could be. We settle on one "vision" as the right one: the flicker of an ear of a lion hidden in the grass. We are largely unaware of all the other potential matches. It *is* the ear of a lion, at least until proven otherwise, and we react.

Models of weather, models of the social structures within which we are embedded, models for the stars and the earth: many of the models are hallucinations, but "real" or not, we go on making them. It is what we are.

Models are built according to rules, the "logic" of the model. Humans are so complex that now only a fraction of our models are used for the ancient purely "external" purposes. The logic of human interaction and group membership, for instance, need bear little resemblance to the logic of any external model.

Experience tells me that much of human conversation consists of repeated assertions, and no attempt is made to justify these assertions. There is no necessity that these assertions make sense, no requirement that they even be understood by the speaker for their effect to be felt. They are merely repeated announcement of group affiliation: "I am a member of the group that says these things." They are like pheromones which allow two members of the same ant colony to recognize each other. This kind of communication is far more compelling to humans than mathematics. People can and will die for group affiliation, can easily be convinced to do so with the proper leverage. Very few humans are willing to die for the Axiom of Choice.

So the logic of group membership for this primate species is about leaders and family and proximity, and the words used are merely "passwords" to enter the group safe zone. A mother comforting her daughter is telling the child she is safe in the family. The words don't matter: it is the tone of voice. Listen to a brilliant

political speaker and watch him or her motivate the faithful. There is generally very little literal meaning or reality to the statements, but they are moving and effective nonetheless. We are built to understand the logic of the group.

Even mathematicians do this, when they promote (often with passionate conviction) a collection of axioms with which to do their work. However "logical" they may be when engaged in the work itself, they are one more band of primates (with respect to all concerned) making a home when they communicate in this way.

There is also a logic associated with art. Frequently, art connects different parts of the human experience or explores new ways of communicating aspects of that experience. Metaphors, connections, communication, emotion: all very important and often intensely pleasurable—or at least moving. And all about views of worlds and connections between them. Models, but usually *not* models of the physical world.

(iv). **What is Different About the Sciences?**

The practice of mathematics and science is different from the communications whose purpose is to announce group membership. And although there is a strong esthetic component, it is not purely art.

There is, obviously, connection between all our models. But when we create scientific models we have learned to gravitate toward models framed in "pristine" settings, in which the noise of pattern-upon-pattern is not so deafening, so far as possible.

With each specific mathematical topic or scientific theory mathematicians and scientists attempt to create *explicit or intentional* rather than *automatic* models of things external to themselves. *This is the primeval source and reason for model building.* We have the inbuilt skills to do this. We do it unconsciously and brilliantly every day. If we did not, we could not long survive.

Mathematics as a whole (and the sciences which spring from that mathematics) is a consciously created model of parts of the mind—a structure formed over eons and used by individuals to effectively interact with the external. It is an intentional mirror image of our model-building facility, which is itself a natural mirror image of the world.

Mathematics fits onto parts of the world which are important to us by the very nature of the structure which created it. We think those parts are important because the genes that form the mind that created mathematics tells us they are important. Those genes survived because they were tested and found to be "right," or at least "right enough," over billions of years.

When we create and argue about specific mathematical and scientific models we are *not* engaging in the kind of content-free vocalizations associated with affirmations of group membership, and the two should not be—but frequently are—conflated.

Instead we are attempting to consciously tap into the built-in facts about the external, encoded in our physical selves, similar to the automatic filtering that is done by the visual cortex before it presents an image match.

In that sense mathematics has *no* ontological or epistemological primacy over the sciences. To the contrary, our brains have been built by a multi-billion year experiment, and if our intuition tells us an argument must be valid or a fact true it is only because the form of this slowly assembled structure is organized to return that conclusion.

(v). **The Role of ZFC.**

When we announce group membership by stating we sail under the banner of ZFC we have made a choice about where we want to live. We might even, usefully, "believe" in ZFC, because it is hard for humans to summon up enormous effort without belief. But when we actually *do mathematics* we are *not* merely yammering "I am in this group" over and over. We are exploring the boundaries of a model whose *potential* utility is vastly more primitive than the social structure of a hierarchically organized primate species. We are trying to decode the information about the external encrypted in what we are. Some, or even most, of the things we discover will likely correspond to chance elements of our model-building machine, of no competitive value to our ancestor species but not dangerous either.

But we really are in no position to cast out the weirder products of our model machine unexamined, certainly not until we understand them as well as we can.

I posit that the currently fashionable ZFC set theory with its classical logic is nothing more (or less) than one among many potentially valuable tools to extract the information about the external encoded in our innate model-making facility through the scientific or other theories it engenders.

Introspection of certain types is nothing more than data-analysis of an ongoing survival-experiment run by (all of) our ancestors.

We are, as mathematicians, discovering patterns in this data. The framework provided by ZFC has been the subject of enormous and fruitful study this past century, and that single study is far from over. We know only some of the implications of a few foundational approaches.

A thousand years from now, will ZFC + Axiom XYZ have "won," philosophers and mathematicians having settled the issue? Extremely unlikely in my view, and the wrong question. Will the assumption of the existence of sets whose members cannot, even in principle, be described turn out to have compelling physical application? Perhaps.

Variant mathematics using different underlying logics will all find places. Different allowable rules of argument formation, different patterns of axioms and shockingly different ways of organizing models will abound, and over that length of time changing fashion will encourage groups to explore many avenues to the depth of ZFC, with its current hundred-year reign. Many of these are likely to be found by evolving computer programs, exploring paths of model formation which were not taken by our ancestors and which are, perhaps, literally unthinkable by humans.

Some of these will be found to be no better than ZFC at creating models for use in describing the external. Some will likely be better at certain things. Perhaps a "theory of everything," the holy grail of theoretical physics, awaits the development of new versions of mathematics.

(vi). **Summing Up For Mathematicians.**

So to the working ZFC mathematician who has come to doubt that he or she understands what the power set on the integers *really is* I suggest that you not worry about it.

From an esthetic standpoint the patterns you make can be, simply, beautiful—that might be reason enough for you to proceed. From a utilitarian standpoint, you are exploring the limits of the mathematics you have chosen, and the edge of ZFC is on the list of things humans must understand to properly evaluate the meaning and utility of ZFC. As a bonus, we are often—very often—surprised by what turns out to be useful. Much, obviously, remains to be done here!

And to those who choose variant logics, axiom systems or other approaches beyond my imagination: take heart and persevere. Maybe the path you pursue will be an attractive cul-de-sac. That is useful information for the rest of us and worth finding out. But maybe, *just maybe,* you are a pioneer in the mathematics of the next century.

(vii). **The Program for Understanding the External.**

The ultimate program: to glean facts about the true external.

The method:

(1) Study the model-making machinery (through the models it makes) of the species closest to hand. These models can be tested if they are candidates for use in a scientific theory. We may assume that it is impossible to know, for certain, what kinds of models will eventually prove to be revealing.

(2) Create or find and study new successful model builders to expose us to the unimaginable.

If we want a logic that deals naturally with quantum mechanics (perhaps more naturally than ours) we could, for example, set up a very complex "game of life" in which quantum effects determine success of an evolving program in competition with others. Let "subjective time" (in the game) roll forward a billion generations and see what types of mirrors the evolved model-builders use.

This might not quite do the job, however. If the environment we create challenges "life" forms with quantum scenarios, the scenarios reflect our current understanding of quantum mechanics which might not match well the true nature of these phenomena. After all, we did not evolve in a size-and-energy regime where effects we classify as "quantum" seem very useful. How can we create a universe in which the underlying phenomena and *not our thinking about it* are exposed? Removing ourselves from the design process might be awkward. Complete removal might not be possible or necessary. In any case, a hybrid "us-new" model-making tool might actually be more useful—for us.

As for the models we ourselves produce, it is likely that we have a lot of "junk" model-building facility analogous to the "junk DNA" whose purpose is mysterious to geneticists. But geneticists are changing their minds about the function of much of this genetic material, and we are certainly in no position to discard out of hand *any* of our model-building tools, *particularly* those that help us model the external.

How can we know in advance what type of information about the external might be encoded in the tools themselves except by using them?

Our "idea space" is constrained (was generated) by the external so every model, every fantasy, every wild association bears its imprint. If we are to sift this information about the external from these imaginings we must study them all, starting of course with the intentional models of the idealized external provided by mathematics and the physical theories enabled by mathematics.

By inventing we expose aspects of the invention-machine, whose properties reflect aspects of the external. There is no dichotomy of invention versus discovery. Invention *is* discovery.

The information density in our imaginings (and the imaginings of our potential creations such as programs) about external matters might be thin, and their contents difficult to interpret, but it is *something.* And it is our connection—likely our *only* connection—to the true external.

Consider the following analogy.

We collect a hundred samples each of a hundred different multi-function plastic kitchen tools, as advertised on a late-night infomercial and all manufactured at the same plant.

By detailed analysis of surface tool-marks, internal stress patterns, slight manufacturing variation, evidence of surface heating and so on, how much of the manufacturing machinery could you deduce? Surely not everything. But, just as surely, *a lot.*

(viii). **A Final Thought.**

As an amusing exercise in pure imagination, consider the following.

I take it as given that there is a real universe of some kind outside of myself. There is no reason that I can see to decide whether the universe is finite or not. Where would it end?

I find it to be just as likely that there are infinite chunks of the external that I find hard to think about as "completed wholes" as that there are none. After all, perhaps there is minimal selective advantage to being able to comprehend in entirety an infinite thing. Perhaps the process by which our model-making engine was created, starting with a "finite chunk" of the external, simply doesn't allow for that, would require infinite mutations to create an engine that could do that. That doesn't mean "infinite chunks" of reality aren't there, does it?

Perhaps, beyond the small piece we know, the universe abounds in completed infinities, even inhabited by sentient entities whose creation process started with an "infinite chunk" and who find the Natural Numbers as a completed whole to be as easy to visualize as three apples are to us. Why would the patterns of evolution and sentience be forbidden to such entities?

I remind the dubious reader that, as mathematicians, we speculate as part of unremarkable common daily practice about entities which we can, provably, never know much about beyond the assumptions which lead to existence. If we are guaranteed that there is no contradiction when we assume the existence of some

intriguing object we (many of us) find further study of the properties of these "things" to be worthwhile. "No contradiction" in this sense *almost*, for many of us, can substitute for an existence proof.

In this fashionable vein, and returning to our speculation, would not these hypothetical "transfinite" creatures have a problem remarkably similar to ours? Perhaps our own discoveries when we explore our model-making faculty, along with the efforts of an infinitude of other model-building species, could be of interest and use to an infinite scientist.