RICHARDSON EXTRAPOLATION

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1. Big O and Little O Notation

We introduce a handy notation used to discuss limiting behavior of a real valued function defined in a neighborhood of 0 on the real line.

For integer k and a functions f and g of this type we write

$$g \sim \mathcal{O}(h^k) \quad \text{and} \quad f \sim o(h^k)$$

(Pronounced: g is big owe of h^k and f is little owe of h^k)

provided

$$\lim_{h\to 0} \frac{g(h)}{h^k} = C \quad \text{for some constant } C \quad \text{and} \quad \lim_{h\to 0} \frac{f(h)}{h^k} = 0.$$

If $k = 0$ this means g has a bounded limit at 0 and f has limit 0 at 0.

Obviously $f \sim o(h^k) \Longrightarrow f \sim \mathcal{O}(h^k)$.

We assume $k \geq 0$. We write $f = g + \mathcal{O}(h^k)$ provided $f - g \sim \mathcal{O}(h^k)$ and intend $f = g + o(h^k)$ to mean that $f - g \sim o(h^k)$.

The following rules are also true with Big O replaced by Little O everywhere.

We presume $g \sim \mathcal{O}(h^k)$ and $f \sim \mathcal{O}(h^i)$ and M, N are real constants with $N \neq 0$.

(i) $Mf \sim \mathcal{O}(h^i)$.

(ii) $g + f \sim \mathcal{O}(h^{\min(\{k,i\})})$.

(iii) $gf \sim \mathcal{O}(h^{k+i})$.

(iv) $g/f \sim \mathcal{O}(h^{k-i})$ (for f nonzero in some neighborhood of 0.)

Finally we come to

(v) $\frac{M+f}{N+g} = \frac{M}{N} + \mathcal{O}(h^{\min(\{k,i\})})$.

In addition to these ten facts (five for Big O and five for Little O) we have

(xi) When $j \geq k$ and $g \sim \mathcal{O}(h^k)$ and $f \sim o(h^j)$ then $f + g \sim \mathcal{O}(h^k)$

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2. Richardson Extrapolation

Suppose we are working with an approximation method A to estimate a number I and the accuracy of the approximation is dependent on a small number h with

$$\lim_{h\to 0} A(h) = I.$$

We define error $E(h)$ by

$$I = A(h) + E(h).$$

We suppose here that $0 < k_1 < k_2$ and the error $E(h)$ is given by

$$E(h) = c_1\, h^{k_1} + c_2\, h^{k_2} + o(h^{k_2})$$

and it follows that for positive constant t

$$E(th) = c_1\, t^{k_1} h^{k_1} + c_2\, t^{k_2} h^{k_2} + o(h^{k_2})$$

So for positive constant t we have

$$\begin{aligned} E(th) - t^{k_1} E(h) &= c_1\, t^{k_1} h^{k_1} + c_2\, t^{k_2} h^{k_2} - t^{k_1}\, c_1\, h^{k_1} - t^{k_1}\, c_2\, h^{k_2} + o(h^{k_2}) \\ &= c_2\,(t^{k_2} - t^{k_1})\, h^{k_2} + o(h^{k_2}) \sim \mathcal{O}(h^{k_2}). \end{aligned}$$

This gives

$$I - A(th) - t^{k_1}(I - A(h)) = E(th) - t^{k_1} E(h) \sim \mathcal{O}(h^{k_2}).$$

Then

$$I(1 - t^{k_1}) - A(th) + t^{k_1} A(h) \sim \mathcal{O}(h^{k_2}) \Longrightarrow I = \frac{A(th) - t^{k_1} A(h)}{1 - t^{k_1}} + \mathcal{O}(h^{k_2}).$$

For instance, in the case $t = 1/2$ and $k_1 = 2$ and $k_3 = 3$ we have

$$I = \frac{A\left(\frac{h}{2}\right) - \frac{A(h)}{4}}{1 - \frac{1}{4}} + \mathcal{O}(h^3) = \frac{4\, A\left(\frac{h}{2}\right) - A(h)}{3} + \mathcal{O}(h^3).$$

This is the Richardson technique for enhancing the accuracy of an approximation method and is valid for *any* positive t *whenever* the error can be describes as indicated above.

Often k_1 and k_2 can be determined using power series methods, and k_1 is needed to apply the technique.

But k_1 can be guessed by applying the original approximation to a standard situation for a variety of h values. k_2 will, subsequently, be *revealed* by this same means applied to the Richardson improvement!