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History of Banach Spaces and Linear Operators

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Dedicated to my wife

and

to all who love Banach spaces

Contents

Preface

You can rewrite history and
make it look much more logical,
but actually it happens quite differently.
Atiyah [2004•]

The monographs

A.F. Monna: *Functional Analysis in Historical Perspective* (1973),

and

J. Dieudonné: *History of Functional Analysis* (1981),

as well as all articles devoted to the history of functional analysis deal only with the development before 1950. Now the time has come to cover the second half of the twentieth century too. I have undertaken this adventure.

Let me introduce myself by telling you that I received my M.Sc. degree in 1958, just at the time when the renaissance of Banach space theory started. Thus I have first-hand experience of the progress achieved during the past 50 years.

Due to the explosion of knowledge, writing about functional analysis as a whole seems to be no longer possible. Hence this book is focused on Banach spaces and (abstract bounded) linear operators. Other subjects such as topologies, measures and integrals, locally convex linear spaces, Banach lattices, and Banach algebras are treated only in so far as they turn out to be relevant for this purpose. The interplay with set theory is described carefully: Which axioms are needed in order to prove the Hahn–Banach theorem?

Results about non-self-adjoint operators on Hilbert spaces have been a source of inspiration for the theory of operators on Banach spaces. Such topics are discussed in great detail. However, I have omitted almost all operator-theoretic considerations that depend decisively on the existence of an inner product.

This book should be useful for readers who are interested in the question *Why and how something happened*. Furthermore, it may serve as a reference and guide for beginners who want to learn Banach space theory with some historical flavor. Helpful information is provided for those colleagues who prepare their own lectures on functional analysis.

Writing about the history of mathematical theories requires formulating precise definitions and statements. A few typical proofs are included. I hope that numerous examples and counterexamples will elucidate the scope of the underlying concepts. The text also contains many problems that could not be solved over the past years. Therefore it may become a stimulus for further research.

My monograph can be regarded as a historical supplement to the

Handbook of the Geometry of Banach Spaces (2001/2003),

edited by W.B. Johnson and J. Lindenstrauss. This two-volume treatise is almost exclusively addressed to present-day mathematics. Moreover, as expressed in the title, spaces play a dominant role. By way of contrast, it was my intention to give equal emphasis to both spaces *and* operators.

A main purpose is to present Banach space theory not as an isolated discipline but as a part of mathematics; see Section 7.1. We use tools from set theory, topology, algebra, combinatorics, probability theory, logic, etc. In return, Banach space theory has more to offer than just the triangle inequality. There are multifarious applications to integral and differential equations, approximation theory, harmonic analysis, convex geometry, numerical mathematics, analytic complexity, and, last but not least, to probability theory.

This book is supposed to be a history of mathematics and not a history of mathematicians. I have tried to write about ideas, not about people. From the scientific point of view, the main question is not *Who proved a theorem?* but *Why and how was a theorem proved?* Nevertheless, Chapter 8 pays homage to those who have contributed to Banach space theory; but it is not my intention to answer the question of Snow-white's stepmother (Grimm's fairy tales, original version on p. 680):

Looking-glass, looking-glass, on the wall
who in this land is the fairest of all?

Next, I give some hints: *How to use this book.*

The smallest units of the text are paragraphs (labeled by 3 or 4 numbers)

1.2.3 Chapter 1, Section 2, Paragraph 3

4.5.6.7 Chapter 4, Section 5, Subsection 6, Paragraph 7

Each section is–more or less–self-contained. Thus, as in the case of a handbook, everybody may decide for himself which parts are worth reading.

Just for the purpose of reference, some formulas are labeled (1.2.3.a) or (4.5.6.7.b). Such labels are not supposed to indicate any particular significance.

A thorough index contains about 2400 entries.

The organization of the bibliography is described on p. 683.

The bibliography consists of approximately 2600 items. Almost all items are supplied with attachments that tell the reader where they are quoted in this book. Hence an author index became superfluous. However, see p. 830.

I have carefully checked all references, and the reader may assume that they are correct (up to misprints). Many of the more than 4600 citations are supplemented by the page numbers of the original text on which the desired information can be found. In citations such as Lomonosov [1973, стр. 55] the Cyrillic letters стр. (short for страница=page) indicate that one should look at p. 55 of the original Russian version. When available (for me), I have then preferred to cite the English translation, for example, Lomonosov [1973, p. 213]; see p. 203.

While Pełczyński [1958] refers to a paper of Pełczyński published in 1958, Pełczyński (1932) means that he was born in 1932.

In the course of time, Russian names have been transliterated quite differently:

Кадец → Kadec, Kadets, Гавурин → Gavurin, Govurin, Gowurin,
Шмульян → Šmulian, Shmulyan, Тихонов → Tychonoff, Tikhonov.

In principle, the rules from *Mathematical Reviews* are used. A few exceptions are motivated by the fact that some classical authors like Alexandroff, Kolmogoroff, Khintchine, Tychonoff, and Urysohn published their fundamental works under names that were transliterated in the old-fashioned way.
In the case of Chinese, the last name is stressed:

(Ky) Fan, (Bor-Luh) Lin, (Pei-Kee) Lin.

It has become common that German authors whose names contain ß are using ss instead. Not without hesitation, I follow this convention also in the case of Gauß and Weierstraß.

Quotations are always set in *italic*.

Writing about modern history in which living mathematicians play an essential role is a sensitive task. It was my intention to avoid any comments that could be interpreted as a ranking of people. I hope that all those who are dissatisfied by the presentation of their own achievements will accept my sincere apology. This book does not contain the irrefutable truth; it just reflects my very personal views.

ACKNOWLEDGMENTS
My work became possible only because I received artificial lenses in my eyes. Further surgeries were successful as well. Hence, above all, my sincere gratitude is given to the Universitätsklinikum Jena. Though being an enthusiastic mathematician, I would exchange all my theorems for the ability to make one blind person seeing. My comfort: designing CAT scanners or computing the individual size of lenses needs mathematics.

The idea to write a historical book after my retirement dates back to the late 1980s. Since that time I have collected the required material. The main work started in autumn 1999. I am extremely grateful that my historical studies were financially supported from August 2000 to July 2003 by the DEUTSCHE FORSCHUNGSGEMEINSCHAFT (contracts CA 179/5–1 and CA 179/5–2). The letters CA stand for Bernd Carl, my former pupil and my formal boss during this period. Many thanks to him.

Since the library of the Jena University is quite poor in books and journals, writing a historical treatise would have been impossible without help from outside. Therefore my warmest thanks go to the Göttinger Universitätsbibliothek for an excellent copy service and to Oberwolfach, where I spent several weeks at the Forschungsinstitut. Furthermore, I am deeply indebted to my home university for providing me (even after my retirement) with office space and computer facilities.

Concerning LaTeX, I am a pupil of my pupil Jörg Wenzel. He wrote the underlying class file and, in particular, a program that collects all citations of any specific `bibitem`.

I am extremely thankful to all friends and colleagues who answered my persistent questions or who provided me with helpful comments. Representatively, I mention Joe Diestel, Aicke Hinrichs, Irmtraud Stephani, Dirk Werner, the copyeditor David Kramer, and an anonymous referee.

Last but not least, I owe my gratitude to BIRKHÄUSER (Boston) for publishing this voluminous book. According to my contract, a PDF file of the text was submitted before May 2005. In June 2005, a time-consuming process of reviewing began. The waiting for any feedback gave me an extra year for eliminating errors and misprints.
Now I am able to repeat the title of a comedy of William Shakespeare,

All's Well That Ends Well.

Jena, February 2007

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Notation and Terminology

In this text, I have used standard notation and terminology from Banach space theory. For additional information, the reader is advised to consult 2.1.1/3 and 2.2.1+3. The meaning of the different "zero"s is explained in 2.1.5. Since there are many cross-references, an index of symbols seems to be unnecessary. Nevertheless, the following comments may be helpful.

I have done my best to denote specific mathematical objects by specific letters. However, since the alphabets are rather small, this principle has been violated quite often. For example, X stands mostly for a Banach space but sometimes also for a topological (linear) space. Similarly, K may denote a compact Hausdorff space as well as a kernel of an integral operator. Hilbert spaces are labeled by H; the exception 2.2.14 proves this rule.

We let $\mathbb{N} := \{1,2,3,\dots\}$, $\mathbb{N}_0 := \{0,1,2,\dots\}$, and $\mathbb{Z} := \{0,\pm 1,\pm 2,\dots\}$, $\mathbb{R} := \{\text{real numbers}\}$, $\mathbb{C} := \{\text{complex numbers}\}$, while $\mathbb{K}$ stands for both $\mathbb{R}$ and $\mathbb{C}$.

In spectral theory, complex parameters are denoted by λ and ζ according to their placement: $\lambda I - T$ and $I - \zeta T$.

The dimension of a linear space X is denoted by $\dim(X)$, and $\operatorname{cod}(N) := \dim(X/N)$ stands for the codimension of any subspace N.

Operator ideals are denoted by gothic uppercase letters (Fraktur), while the corresponding sans serifs are used to denote the associated space ideals. Examples of the map $\mathfrak{A} \mapsto \mathsf{A}$ can be found in 6.3.13.2. It would be too cumbersome to present here a complete list of all symbols denoting specific operator ideals; see Section 6.3. In dependence on the underlying situation, norms or quasi-norms on an operator ideal $\mathfrak{A}$ are denoted either by $\|\cdot|\mathfrak{A}\|$ or α; see 6.3.2.4.

The cardinality of a set A is denoted by either $\operatorname{card}(A)$ or $|A|$, and χ_A stands for its characteristic function.

I hope that the fancy expressions $F^{2\pi}_{\text{our}} : L_2(\mathbb{R}) \to l_2(\mathbb{Z})$, $H_{\text{ilb}} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $\mathscr{B}_{\text{aire}}(M) \subseteq \mathscr{B}_{\text{orel}}(M)$, $\dots$ explain themselves.

The only new inventions are symbols such as $[l_p(\mathbb{I}),X]$, $[L_p(M,\mathscr{M},\mu),X]$, and $[C(K),X]$, which denote spaces of X-valued sequences and functions, respectively; see 4.8.3.7, 5.1.2.5, 5.7.2.10, and 6.9.7.2.

Final warning: When looking at another book or paper (referred to in the text) the reader is advised to take care of the (possibly different) notation and terminology. For example, one should bear in mind that the term *completely continuous* changed its meaning around 1950.

Introduction

First of all, the reader should look at p. 683

The *History of Banach Spaces and Linear Operators* may be subdivided as follows:

1900–1920: prenatal period
integral equations (Fredholm, Hilbert),
Hilbert spaces and Fischer–Riesz theorem,
Riesz representation theorem,
L_p and l_p (F. Riesz),

1920 : birth (Banach submitted his thesis in June 1920)
Banach, Hahn, and Wiener,

1920–1932: youth
principle of uniform boundedness,
Hahn–Banach theorem,
bounded inverse theorem,

1932 : maturity
Banach's monograph,

1932–1958: post-Banach period (interrupted by Holocaust and World War II)

1958 : classical books
Day, Dunford/Schwartz (Part I), Hille/Phillips, Taylor,

midlife crisis and big bang
Grothendieck's *Résumé*,
Mazur's school in Warszawa,
Dvoretzky's theorem,

1958↗ : modern period

The historical roots of **functional analysis**, which is a child of the early twentieth century, can be traced back to various problems of classical mathematics:

- trigonometric series,
- vibrating strings,
- Dirichlet principle (variational calculus),
- potential theory,
- spectral theory of matrices,
- infinite determinants.

The prehistory of functional analysis was described by several authors: [$\text{DIEU}_1^\bullet$, Chap. 8], [$\text{DIEU}_2^\bullet$, Chap. 1–4], [$\text{MON}^\bullet$], [$\text{KRA}^\bullet$, Chap. 23], Bernkopf [$1966^\bullet$, $1967^\bullet$], as well as Siegmund-Schultze [$1982^\bullet$], Birkhoff/Kreyszig [$1984^\bullet$], Kreyszig [$1986\text{a}^\bullet$], Fichera [$1994^\bullet$], and Smithies [$1997^\bullet$].

The reader should realize the great impact of the following schools:

Italy Ascoli (1843–1896), Arzelà (1847–1912), Pincherle (1853–1936), and in particular, Volterra (1860–1940),

France Hadamard (1865–1963), Borel (1871–1956), Baire (1874–1932), Lebesgue (1875–1941), and Fréchet (1878–1973),

Germany Weierstrass (1815–1897), Cantor (1845–1918), Hilbert (1862–1943), and Minkowski (1864–1909).

I hope that it was a good choice to place the starting point of this text at the turn of the nineteenth to the twentieth century. Landmarks are

1902 Lebesgue, *Intégrale, longueur, aire* (thesis),

1903 Fredholm, *Sur une classe d'équations fonctionnelles*,

1906 Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Vierte Mitteilung),

1906 Fréchet, *Sur quelques points du calcul fonctionnel* (thesis).

A main point in the subdivision above was proposed by Bourbaki, [BOU_{5b}, p. 173] or [$\text{BOU}^\bullet$, p. 217] (original version on p. 680):

The publication of Banach's treatise on "Opérations linéaires" marks, one could say, the beginning of adult age for the theory of normed spaces.

The following (incomplete) list of achievements provides a rough impression of the post-Banach period:

- weak and weak* topologies, Alaoglu's theorem,
- Kreĭn–Milman theorem,
- integration of vector-valued functions (Bochner, Gelfand, Dunford),
- vector measures and Radon–Nikodym theorems,
- Riesz–Dunford–Taylor operational (or functional) calculus,
- Hille–Yosida theorem for semi-groups of operators,
- Weierstrass–Stone approximation theorem,
- Kakutani's representation of abstract L-spaces and M-spaces,
- $C^\star$-algebras and $W^\star$-algebras.

I now comment on the **midlife crisis** of Banach space theory that finally led to a vehement revival.

The most important achievement of the early post-Banach period was the invention of the weak and weak* topologies. Both topologies are locally convex. Moreover, all linear spaces occurring in the theory of distributions carry a locally convex topology. These facts produced the impression that the concept of a locally convex linear space is the most crucial one. Following a proposal of L. Schwartz, one looked for conditions under which the basic theorems of Banach space theory remain true. Dieudonné [1953, p. 496] summarized the final outcome as follows:

Surprisingly enough, practically no important property is really special to Banach spaces, although the latter still constitute a very useful technical tool in the study of the general theory.

This development culminated in

N. Bourbaki: *Espaces vectoriels topologiques*, (1953/55).

Bourbaki's arguments against Banach spaces read as follows; [BOU_{5b}, p. 174] or [$\text{BOU}^{\bullet}$, p. 217] (original version on p. 680):

But in spite of the great number of researches on Banach spaces undertaken during the past 20 years (replaced by 40, in later editions)*, only little progress has been made in solving the problems that were left open by Banach himself; furthermore, if the theory of Banach algebras and its applications to harmonic analysis are excluded, then the almost complete absence of new applications of the theory to the great problems of classical analysis somewhat undermines the hopes based on it.*

Here is the opinion of Gelfand stated in [1956, pp. 315–316]:

Functional analysis in the nineteen-thirties was basically the theory of linear normed spaces and of linear operators in these spaces.

...

At the present time the development of analysis and the theory of generalized functions shows that the framework of linear normed spaces has become restrictive for functional analysis, and that it is necessary to study linear topological spaces, i.e. topological spaces with linear operations which are continuous in the given topology. The introduction of such spaces is dictated not by love of generality, but simply by the fact that the spaces which are the most interesting and necessary for concrete problems of analysis are not normed.

In his plenary lecture at the ICM 1983 in Warsaw, Pełczyński [1983, p. 237] described the situation as follows:

During the first 20 years after the war Banach spaces were treated like an old member of the academy, who deserves esteem for his merits but has nothing more to contribute.

I will not keep back my own opinion, which was stated in the foreword of *Nukleare lokalkonvexe Räume* [PIE_1] (original version on p. 680):

With a few exceptions the locally convex spaces encountered in analysis can be divided into two classes. First, there are the normed spaces, which belong to classical functional analysis, and whose theory can be considered essentially closed. The second class consists of the so-called nuclear locally convex spaces, which were introduced in 1951 by A. Grothendieck. The two classes have a trivial intersection, since it can be shown that only finite dimensional locally convex spaces are simultaneously normable and nuclear.

Of course, concerning the future of Banach space theory, this credo turned out to be a terrible misjudgment. There is only one excuse: around 1963, I was an enthusiastic follower of Bourbaki and I liked nuclear spaces very much.

Ironically, I refuted myself within a few years. Indeed, the theory of nuclear spaces gave the decisive impetus for inventing the concept of an operator ideal on Banach spaces. This development was predicted by Grothendieck [1956b, p. 1] (original version on p. 680):
In fact, almost all problems from the theory of topological tensor products of general locally convex spaces, including the theory of nuclear spaces, can finally be reduced to problems about Banach spaces.

At which date should one place the cut between classical and modern Banach space theory? In my opinion 1958 is a good choice, since in that year three monographs appeared:

M.M. Day: *Normed Linear Spaces*,
N. Dunford, J.T. Schwartz: *Linear Operators*, Part I,
A.E. Taylor: *Introduction to Functional Analysis*.

In 1948, Hille had published his treatise on *Functional Analysis and Semi-Groups*, with a second edition in 1957 (coauthor: R.S. Phillips).

These seminal surveys indicate that the post-Banach era had come to an end. Almost simultaneously, hopeful signs of a rebirth could be recognized:

1954 Grothendieck's *Résumé* was submitted (but not grasped),
1958 under the direction of Mazur, the Polish postwar generation appeared on the scene,
1960 Dvoretzky presented his theorem on spherical sections.

However, it took some time until the new ideas were fully appreciated. A historical milestone was

1969 J. Lindenstrauss/A. Pełczyński: *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications.*

One should not forget to mention some old masters who offered hope during the period of depression: R.C. James (1918), M.I. Kadets (1923), and V.L. Klee (1925).

Next, I describe some typical features of modern Banach space theory.

- Following ideas contained in Grothendieck's *Résumé*, Dvoretzky [1960, p. 123] stressed the fact that
  many problems in the theory of Banach spaces may be reduced to the finite-dimensional case, i.e. to problems concerning Minkowski spaces.
  The elaboration of this viewpoint led to the *local theory* of Banach spaces.

- A classical result of Riesz says that the eigenvalues of a compact operator tend to zero. On the other hand, according to Grothendieck, the eigenvalues of a nuclear operator are square summable. Hence *qualitative* statements were replaced by *quantitative* statements.
- New techniques from model theory (ultraproducts), probability theory, and combinatorics (Ramsey's theorem) were successfully employed.
- Interpolation theory.
- Clearly, the new spirit of Banach space theory also gave new inspiration for treating problems from the classical period. The theory concerned with the Radon–Nikodym property is an impressive example. Here we have a fascinating interplay between analytic, geometric, stochastic, and operator-theoretic aspects.
- Unfortunately, an unpleasant phenomenon has been observed again and again: though the definition of a Banach space is quite simple, the underlying axioms give room for all kinds of pathologies; see Section 7.4.

  Of course, *counterexamples* are needed to explore the limits of a theory; but what finally counts are *positive* results. Nevertheless, Enflo's space without the approximation property as well as the discoveries made by Gowers and Maurey have dramatically changed the philosophy of Banach space theory.

The two-volume treatise of Lindenstrauss/Tzafriri on *Classical Banach Spaces* has become the most important reference of the modern period. A rather complete list of monographs on Banach space theory and related fields is contained in the Chronology, pp. 673–679.

The final question is, *What can be said about the future of Banach space theory?* First of all, I stress that the following text contains many interesting problems waiting for their solutions. Hence many things remain to be done. On the other hand, there are new mathematical developments such as *asymptotic geometric analysis* and the *theory of operator spaces* in which Banach space techniques play an essential role; see Subsections 6.9.15 and 6.9.16. Further arguments may be found in Section 7.1.

Good friends warned me not to make any personal prediction. Therefore the reader is asked to form his own opinion.

1

The Birth of Banach Spaces

1.1 Complete normed linear spaces

1.1.1 As a tribute to STEFAN BANACH, we begin with a quotation from his thesis [1922, pp. 134–136] submitted to the University of Lwów in June 1920:

Soit E une classe composée tout au moins de deux éléments, d'ailleurs arbitraires, que nous désignerons p.e. par $X, Y, Z, \dots$

a, b, c, *désignant les nombres réels quelconques, nous définissons pour E deux opérations suivantes:*

1) *l'***addition** *des éléments de E*

$$X+Y, X+Z, \dots$$

2) *la* **multiplication** *des éléments de E par un nombre réel*

$$a \cdot X, b \cdot Y, \dots$$

Admettons que les propriétés suivantes sont réalisées:

$I_1]$ $X+Y$ *est un élément bien déterminé de la classe E,*

$I_2]$ $X+Y=Y+X$,

$I_3]$ $X+(Y+Z)=(X+Y)+Z$,

$I_4]$ $X+Y=X+Z$ *entraîne* $Y=Z$,

$I_5]$ *Il existe un élément de la classe E déterminé* θ
et tel qu'on ait toujours $X+\theta=X$,

$I_6]$ $a \cdot X$ *est un élément bien déterminé de la classe E,*

$I_7]$ $a \cdot X=\theta$ *équivaut à* $X=\theta$ *ou* $a=0$,

$I_8]$ $a \neq 0$ *et* $a \cdot X=a \cdot Y$ *entraînent* $X=Y$,

$I_9]$ $X \neq \theta$ *et* $a \cdot X=b \cdot X$ *entraînent* $a=b$,

$I_{10}]$ $a \cdot (X+Y)=a \cdot X+a \cdot Y$,

$I_{11}]$ $(a+b) \cdot X=a \cdot X+b \cdot X$,

$I_{12}]$ $1 \cdot X=X$,

$I_{13}]$ $a \cdot (b \cdot X)=(a \cdot b) \cdot X$.

Admettons ensuite que il existe une opération appelée **norme** (*nous la désignerons par le symbole* $\|X\|$), *définie dans le champ E, ayant pour contre-domaine l'ensemble de nombres réels et satisfaisant aux conditions suivantes:*

II_1] $\|X\| \geq 0$,

II_2] $\|X\| = 0$ *équivaut à* $X = \theta$,

II_3] $\|a \cdot X\| = |a| \cdot \|X\|$,

II_4] $\|X+Y\| \leq \|X\| + \|Y\|$,

III] *Si*

1° $\{X_n\}$ *est une suite d'éléments de E*

2° $\lim\limits_{\substack{r=\infty \\ p=\infty}} \|X_r - X_p\| = 0$,

il existe un élément X tel que $\lim_{n=\infty} \|X - X_n\| = 0$.

Banach's system of axioms is well-structured. The first group contains the concept of an abstract linear space, the properties of a norm follow, and finally, completeness is required. The sole imperfection is a certain redundancy in the algebraic part: axioms I_7], I_8], and I_9] are superfluous. But even nowadays we assume commutativity of addition though it could be obtained as a consequence; see **(F)** in 1.7.3.

On the suggestion of Fréchet [FRÉ, p. 141], a complete normed linear space is now referred to as a **Banach space**. In [BAN] the term *espace du type* (B) was used.

1.1.2 Hadamard [1903] was the first who considered the collection of *all* continuous real functions on a closed interval $[a,b]$, which is the most simple and most important Banach space: $C[a,b]$. Since he was interested in representations of continuous linear functionals, the topological and algebraic structure of $C[a,b]$ played a decisive role. Of course, sequences and sets of continuous functions occurred much earlier, in particular, in the work of Weierstrass, Ascoli, and Arzelà.

I cannot resist citing Riesz's beautiful presentation [1918, p. 72]:

Den folgenden Betrachtungen legen wir die Gesamtheit der auf der Strecke $a \leq x \leq b$ *erklärten, daselbst überall stetigen komplexen Funktionen* $f(x)$ *zu Grunde. Diese Gesamtheit werden wir der Kürze halber als* **Funktionalraum** *bezeichnen. Ferner nennen wir* **Norm** *von* $f(x)$ *und bezeichnen mit* $\|f\|$ *den Maximalwert von* $|f(x)|$*; die Grösse* $\|f\|$ *ist danach im Allgemeinen positiv und verschwindet nur dann, wenn* $f(x)$ *identisch verschwindet. Ferner bestehen für sie die Beziehungen*

$$\|cf(x)\| = |c|\|f(x)\|; \quad \|f_1 + f_2\| \leq \|f_1\| + \|f_2\|.$$

Die gleichmässige Konvergenz einer Funktionenfolge $\{f_n\}$ *gegen eine Grenzfunktion* f *ist gleichbedeutend damit, dass die Distanz* $\|f - f_n\|$ *gegen Null konvergiert. Eine notwendige und hinreichende Bedingung für die gleichmässige Konvergenz einer Funktionenfolge* $\{f_n\}$ *besteht nach dem sogenannten allgemeinen Konvergenzprinzip in der Beziehung* $\|f_m - f_n\| \to 0$ *für* $m \to \infty$, $n \to \infty$.

Nowadays, we carefully distinguish between a function f and its value at a point x. For a long time, however, the symbol $f(x)$ was used to denote the function itself.

1.1.3 Dieudonné [DIEU$_2^\bullet$, pp. 119–120] stressed the happy coincidence that

exactly at the beginning of Hilbert's work on integral equations, the horrible and useless so-called "Riemann integral" was replaced by a marvelous new tool, the "Lebesgue integral" (condensed quotation).

Less dramatically, Riesz [1909a, p. 450] says

daß der Begriff des Integrals durch Lebesgue jene glückliche und geistreiche Erweiterung gefunden hat, welcher nun manche, bisher gescheiterten Probleme ihre sinngemäße Erledigung verdanken.

Weyl [1944$^\bullet$, p. 649] did not share this enthusiasm:

I think Hilbert was wise to keep within the bounds of continuous functions when there was no actual need for introducing Lebesgue's general concepts.

As a counterpoint, I quote a remark of Garrett Birkhoff, [ALB$^\cup$, p. 10]:

It is my privat opinion that Hilbert never mastered the Lebesgue integral.

This claim is probably true, since Broggi, a Ph.D. student of Hilbert, *"proved"* in his thesis [1907, pp. 11–16] that every finitely additive probability is automatically countably additive.

1.1.4 After all, the triumphant advance of the Lebesgue integral was irresistible. Fischer [1907] and Riesz [1907a] invented the Hilbert space $L_2[a,b]$, which is often more elegant than $C[a,b]$. Subsequently, in a seminal paper, Riesz [1909a, p. 452] extended this definition to exponents $1 < p < \infty$:

In der vorliegenden Arbeit wird die Voraussetzung der quadratischen Integrierbarkeit durch jene der Integrierbarkeit von $|f(x)|^p$ ersetzt. Jede Zahl p bestimmt eine Funktionenklasse $[L^p]$.

Note that the concept of a norm was not yet in use, though Riesz proved Minkowski's inequality [1909a, pp. 456, 464]:

Wir werden im folgenden das Bestehen der Grenzgleichung

$$\lim_{i=\infty}\int_a^b |f(x)-f_i(x)|^p dx = 0,$$

wo $f(x)$ und die $f_n(x)$ der Klasse $[L^p]$ angehören, dadurch ausdrücken, daß wir sagen: die Folge $\{f_i(x)\}$ konvergiert in bezug auf den Exponenten p stark gegen die Funktion $f(x)$.

Furthermore [1909a, p. 468]:

Aus der Grenzgleichung

$$\lim_{i=\infty, j=\infty}\int_a^b |f_i(x)-f_j(x)|^p dx = 0,$$

wo $\{f_i(x)\}$ eine Folge von Funktionen der Klasse $[L^p]$ bedeutet, folgt die Existenz einer

Funktion $f(x)$ derselben Klasse, gegen welche $\{f_i(x)\}$ in bezug auf den Exponenten p stark konvergiert.

In summary: $[L^p]$ is a Banach space. Curiously enough, the simpler theory of the spaces l_p formed by all scalar sequences $x = (\xi_k)$ with $\sum_{k=1}^{\infty} |\xi_k|^p < \infty$ was treated only in 1913; see [RIE, Chap. III].

1.2 Linear spaces

In this section, we describe the algebraic background of Banach space theory. Further information about the history of linear spaces can be found in the articles of Dorier [1995•] and Moore [1995•].

1.2.1 The concept of a **linear space** or **vector space** (*sistema lineare*) was introduced by Peano [PEA, pp. 141–142] already in 1888. His definition could be used in any lecture today:

È definita la **somma** *di due enti $\boldsymbol{a}$ e $\boldsymbol{b}$, vale a dire è definito un ente, indicato con $\boldsymbol{a}+\boldsymbol{b}$, che appartiene pure al sistema dato, e che soddisfa alle condizioni:*

$$\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}, \quad \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}.$$

Essendo $\boldsymbol{a}$ un ente del sistema, ed m un numero intero e positivo, colla scrittura $m\boldsymbol{a}$ intenderemo la somma di m enti eguali ad $\boldsymbol{a}$. È facile riconoscere, essendo $\boldsymbol{a}, \boldsymbol{b}, \dots$ enti del sistema, $m, n, \dots$ numeri interi e positivi che

$$m(\boldsymbol{a}+\boldsymbol{b}) = m\boldsymbol{a}+m\boldsymbol{b}, \quad (m+n)\boldsymbol{a} = m\boldsymbol{a}+n\boldsymbol{a}, \quad m(n\boldsymbol{a}) = (mn)\boldsymbol{a}, \quad 1\boldsymbol{a} = \boldsymbol{a}.$$

Noi supporremo che sia attribuito un significato alla scrittura $m\boldsymbol{a}$, qualunque sia il numero reale m, in guisa che siano ancora soddisfatte le equazioni precedenti. L'ente $m\boldsymbol{a}$ si dirà **prodotto** *del numero (reale) m per l'ente $\boldsymbol{a}$.*

Infine supporemo che esista un ente del sistema, che diremo ente **nullo***, e che indicheremo con $\circ$, tale che, qualunque sia l'ente $\boldsymbol{a}$, il prodotto del numero 0 per l'ente $\boldsymbol{a}$ dia sempre l'ente $\circ$, ossia*

$$0\boldsymbol{a} = \circ.$$

Se alla scrittura $\boldsymbol{a}-\boldsymbol{b}$ si attribuisce il significato $\boldsymbol{a}+(-1)\boldsymbol{b}$, si deduce:

$$\boldsymbol{a}-\boldsymbol{a} = \circ, \quad \boldsymbol{a}+\circ = \boldsymbol{a}.$$

Linear mappings were defined as well [PEA, p. 145]:

Un'operazione $\boldsymbol{R}$, a eseguirsi su ogni ente $\boldsymbol{a}$ d'un sistema lineare A, dicesi **distributiva** *se il risultato dell'operazione $\boldsymbol{R}$ sull'ente $\boldsymbol{a}$, che indicheremo con $\boldsymbol{R}\boldsymbol{a}$, è pure un ente d'un sistema lineare, e sono verificate le identità*

$$\boldsymbol{R}(\boldsymbol{a}+\boldsymbol{a}') = \boldsymbol{R}\boldsymbol{a}+\boldsymbol{R}\boldsymbol{a}', \quad \boldsymbol{R}(m\boldsymbol{a}) = m(\boldsymbol{R}\boldsymbol{a}),$$

ove $\boldsymbol{a}$ e $\boldsymbol{a}'$ sono enti qualunque del sistema A, ed m un numero reale qualunque.

1.2.2 Using the well-ordering theorem, Hamel [1905] showed that $\mathbb{R}$, viewed as a linear space over the rationals, possesses a basis; see 7.5.18. This proof was extended by Hausdorff [1932, p. 295] to real and complex linear spaces. A family $(e_i)_{I\in\mathbb{I}}$ is called a **Hamel basis** of X if every element $x \in X$ admits a unique representation $x = \sum_{I\in\mathbb{I}} \xi_i e_i$ such that the number of coefficients $\xi_i \neq 0$ is finite. In a next step, Löwig [1934b] observed that all such bases have the same cardinality, which is said to be the **algebraic dimension**: $\dim(X)$.

Once more, we refer to Peano [PEA, p. 143]:

Numero delle dimensioni d'un sistema lineare è il massimo numero di enti fra loro indipendenti ce si possono prendere nel sistema. Un sistema lineare può anche avere infinite dimensioni.

1.2.3 Ironically, Peano's ideas got lost and were rediscovered by Weyl, almost 30 years later, [WEYL, pp. 15–16]:

Wir stellen folgendes einfache Axiomensystem der affinen Geometrie auf.

I. Vektoren

Je zwei Vektoren $\mathfrak{a}$ *und* $\mathfrak{b}$ *bestimmen eindeutig einen Vektor* $\mathfrak{a}+\mathfrak{b}$ *als ihre* **Summe***; eine Zahl* λ *und ein Vektor* $\mathfrak{a}$ *bestimmen eindeutig einen Vektor* $\lambda\mathfrak{a}$*, das* **λ-fache** *von* $\mathfrak{a}$*. Diese Operationen genügen folgenden Gesetzen.*

α) *Addition*

1. $\mathfrak{a}+\mathfrak{b} = \mathfrak{b}+\mathfrak{a}$ (*kommutatives Gesetz*).

2. $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c} = \mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$ (*assoziatives Gesetz*).

3. Sind $\mathfrak{a}$ *und* $\mathfrak{c}$ *irgend zwei Vektoren, so gibt es einen und nur einen Vektor* $\mathfrak{x}$*, für welchen die Gleichung* $\mathfrak{a}+\mathfrak{x} = \mathfrak{c}$ *gilt.*

β) *Multiplikation*

1. $(\lambda+\mu)\mathfrak{a} = (\lambda\mathfrak{a})+(\mu\mathfrak{a})$ (*erstes distributives Gesetz*).

2. $\lambda(\mu\mathfrak{a}) = (\lambda\mu)\mathfrak{a}$ (*assoziatives Gesetz*).

3. $1\mathfrak{a} = \mathfrak{a}$.

4. $\lambda(\mathfrak{a}+\mathfrak{b}) = (\lambda\mathfrak{a})+(\lambda\mathfrak{b})$ (*zweites distributives Gesetz*).

Aus den Additionsaxiomen läßt sich schließen, daß ein bestimmter Vektor $\mathfrak{o}$ *existiert, der für jeden Vektor* $\mathfrak{a}$ *die Gleichung* $\mathfrak{a}+\mathfrak{o} = \mathfrak{a}$ *erfüllt.*

Die Gesetze β) *folgen für rationale Multiplikatoren* λ, μ *aus den Additionsaxiomen. Gemäß dem Prinzip der Stetigkeit nehmen wir sie auch für beliebige reelle Zahlen in Anspruch, formulieren sie aber ausdrücklich als Axiome, da sie sich in dieser Allgemeinheit nicht aus den Additionsaxiomen herleiten lassen.*

II. Punkte und Vektoren

1. Je zwei Punkte A und B bestimmen einen Vektor $\mathfrak{a}$*; in Zeichen* $\overrightarrow{AB} = \mathfrak{a}$*. Ist A irgend ein Punkt,* $\mathfrak{a}$ *irgend ein Vektor, so gibt es einen und nur einen Punkt B, für welchen* $\overrightarrow{AB} = \mathfrak{a}$ *ist.*

2. Ist $\overrightarrow{AB} = \mathfrak{a}$ *und* $\overrightarrow{BC} = \mathfrak{b}$*, so ist* $\overrightarrow{AC} = \mathfrak{a} + \mathfrak{b}$.

In diesen Axiomen treten zwei Grundkategorien von Gegenständen auf, die Punkte und die Vektoren; drei Grundbeziehungen, nämlich diejenigen, die durch die Symbole

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{c}, \quad \mathfrak{b} = \lambda\mathfrak{a}, \quad \overrightarrow{AB} = \mathfrak{a}$$

ausgedrückt werden.

Alle Sätze, welche sich aus diesen Axiomen rein logisch folgern lassen, bilden das Lehrgebäude der affinen Geometrie, das somit auf der hier gelegten axiomatischen Basis deduktiv errichtet werden kann.

1.2.4 Without reference to anybody, Banach produced his own definition of a linear space, which has already been presented in 1.1.1.

1.2.5 Banach [1922, p. 135] or [BAN, p. 26] has used the symbols θ or Θ to denote the **null element** of a linear space; see also [WEYL, p. 16] and Wiener [1923a, p. 136]. This convention was heavily criticized by [STONE, p. 4]:

For the null element we have employed the familiar symbol 0*. The danger of confusing this meaning of the symbol with its ordinary meaning as the notation for the zero of the real or complex number system is slight if not wholly illusory: for in any formula or equation it is evident which symbols are to represent numbers and which symbols are to represent elements of Hilbert space. Accordingly, we have avoided the real disadvantages of employing a special symbol for the null element.*

Ironically, already on p. 36 he refuted himself by saying:

We find it convenient to denote by O the transformation which takes every element of $\mathfrak{H}$ *into the null element.*

Nevertheless, it seems to me that Stone's point of view is now generally accepted. The notation in this text, however, will be based on the old-fashioned strategy that 0 stands for the number zero, while all the other "zeros" are denoted as follows:

o : null element in a Banach space, $\quad O$: null operator, $\quad \emptyset$: empty set.

1.3 Metric spaces

In this section, we describe the elementary topological background of Banach space theory, which goes back to the fundamental work of Cantor *Über unendliche, lineare Punktmannichfaltigkeiten* [CAN]. For the present, we deal only with metric concepts. A historical sketch of *general topology* will be given in Section 3.2.

1.3.1 Our starting point is Fréchet's thesis [1906, p. 18]:

Considérons une classe (V) d'éléments de nature quelconque, mais tels qu'on sache discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'à deux quelconques d'entre eux A,B, on puisse faire correspondre un nombre $(A,B)=(B,A)\geq 0$ qui jouit des deux propriétés suivant:

1° *La condition nécessaire et suffisante pour que (A,B) soit nul est que A et B soient identiques.*

2° *Il existe une fonction positive bien déterminée $f(\varepsilon)$ tendant vers zéro avec ε, telle que les inégalités $(A,B)\leq\varepsilon$, $(B,C)\leq\varepsilon$ entraînent $(A,C)\leq f(\varepsilon)$, quels que soient les éléments A,B,C.*

Nous appellerons **voisinage** *de A et de B le nombre (A,B).*

1.3.2 Based on the above concept, Fréchet developed the theory of convergent sequences.

[1906, p. 18]: *Une suite d'éléments $A_1,A_2,\ldots$* **tend** *vers un élément A, si le voisinage (A_n,A) tend vers zéro avec $\frac{1}{n}$.*

[1906, p. 23]: *Nous dirons qu'une suite d'éléments $A_1,A_2,\ldots$ d'une classe (V) satisfait aux* **condition de Cauchy** *lorsqu'à tout nombre $\varepsilon>0$ on peut faire correspondre un entier n tel que l'inégalité $(A_n,A_{n+p})<\varepsilon$ soit vérifiée quel que soit p. Il est évident que si une suite tend vers une limite, elle satisfait aux condition de Cauchy.*

Nous dirons alors qu'une classe (V) admet une généralisation du théorème de Cauchy si toute suite d'éléments de cette classe, qui satisfait aux condition de Cauchy, a une élément limite.

[1906, p. 6]: *Nous dirons qu'un élément A est un* **élément limite** *d'un ensemble E lorsqu'il existe une suite infinie d'éléments de $E: A_1,A_2,\ldots$ qui sont distinct et tendent vers A.*

[1906, p. 6]: *Nous dirons qu'un ensemble E est* **compact** *lorsqu'il ne comprend qu'un nombre fini d'éléments ou lorsque toute infinité de ses éléments donne lieu à au moins un élément limite.*

Note that this limit point is not assumed to belong to the set E.

Finally, we stress that on p. 25 of his thesis, Fréchet characterized compactness via finite **ε-nets**.

The concept of **separability** is also due to him, [1906, p. 23].

1.3.3 So to speak, as a corollary, Fréchet treated a special case of (V) spaces, which has become most important, [1906, p. 30]:

a) *L'***écart** *(A,B) n'est nul que si A et B sont identiques.*
b) *Si A, B, C, sont trois éléments quelconques, on a toujours $(A,B)\leq(A,C)+(C,B)$.*

In the course of time it turned out that every *voisinage* can be replaced by an equivalent *écart*; Chittenden [1917], Frink [1937], and [WEIL$_1$, pp. 13–16]. Hence there is no reason why one should use the more complicated concept.

1.3.4 Giving no enthusiastic credit to Fréchet, Hausdorff presented the theory of **metric spaces** in a polished form.

[HAUS$_1$, p. 211]: *Unter einem* **metrischen Raum** *verstehen wir eine Menge E, in der je zwei Elementen (Punkten) x,y eine reelle nichtnegative Zahl, ihre Entfernung $\overline{xy} \geq 0$, zugeordnet ist; und zwar verlangen wir überdies die Gültigkeit der folgenden* **Entfernungsaxiome**:

(α) Symmetrieaxiom: *Es ist stets $\overline{yx} = \overline{xy}$.*

(β) Koinzidenzaxiom: *Es ist $\overline{xy} = 0$ dann und nur dann, wenn $x = y$.*

(γ) Dreiecksaxiom: *Es ist stets $\overline{xy} + \overline{yz} \geq \overline{xz}$.*

Hausdorff also established the modern terminology: *Vollständigkeit* (**completeness**, p. 315), *Fundamentalfolge* (now **Cauchy sequence**, p. 315) and *totale Beschränktheit* (**total boundedness** or **precompactness**, p. 311).

1.4 Minkowski spaces

1.4.1 The concept of a finite-dimensional **normed linear space** was introduced by Minkowski in his *Geometrie der Zahlen* (1896); see [MIN, p. 102]. As a credit for this contribution, such spaces are referred to as **Minkowski spaces**.

Es bedeute $f(x_1,\dots,x_n)$ irgend eine Function, welche folgende Bedingungen erfüllt:

$$\text{(A)} \qquad \begin{cases} f(x_1,\dots,x_n) > 0, \text{ wenn nicht } x_1 = 0,\dots,x_n = 0 \text{ ist,} \\ f(0,\dots,0) = 0, \\ f(tx_1,\dots,tx_n) = t\, f(x_1,\dots,x_n), \text{ wenn } t > 0 \text{ ist;} \end{cases}$$

$$\text{(B)} \qquad f(x_1+y_1,\dots,x_n+y_n) \leq f(x_1,\dots,x_n) + f(y_1,\dots,y_n);$$

$$\text{(C)} \qquad f(-x_1,\dots,-x_n) = f(x_1,\dots,x_n).$$

1.4.2 Minkowski also provided the most important examples of norms defined on $\mathbb{R}^n$, [MIN, p. 115]:

Nun sei p eine beliebige reelle Größe; und es werde

$$\left(\frac{(abs\, x_1)^p + \cdots + (abs\, x_n)^p}{n}\right)^{\frac{1}{p}} = f(x_1,\dots,x_n)$$

gesetzt. Es besteht nun bei beliebigen reellen Werthen $a_1,\dots,a_n$; $b_1,\dots,b_n$ für ein $p \geq 1$ immer die Ungleichung **[Minkowski inequality]**

$$f(a_1+b_1,\dots,a_n+b_n) \leq f(a_1,\dots,a_n) + f(b_1,\dots,b_n).$$

The n-dimensional (real or complex) Banach space so obtained will be denoted by l_p^n. Its norm is given by $\big(|x_1|^p + \cdots + |x_n|^p\big)^{1/p}$.

1.4.3 Let $e_1, \dots, e_n$ be a basis of a Minkowski space X. Then every element $x \in X$ admits a representation

$$x = \sum_{k=1}^{n} \xi_k e_k.$$

According to Riesz [1918, pp. 77–78], a compactness argument yields a constant $c > 0$ such that

$$\sum_{k=1}^{n} |\xi_k| \le c\|x\|. \tag{1.4.3.a}$$

Since

$$\|x\| \le \max_k \|e_k\| \sum_{k=1}^{n} |\xi_k|,$$

the spaces X and l_1^n are isomorphic; see 4.9.1.1. Therefore every bounded sequence in X has a convergent subsequence. Moreover, every finite-dimensional linear space is complete.

1.4.4 The famous **Riesz lemma** [1918, p. 75] says that given any proper closed subspace M of a Banach space X, there exists $x_0 \in X$ such that

$$\|x_0\| = 1 \quad \text{and} \quad \|x_0 - x\| \ \ge \tfrac{1}{2} \quad \text{for all } x \in M.$$

Hence the closed unit ball of every infinite-dimensional Banach space contains a sequence (x_k) such that $\|x_h - x_k\| \ge \frac{1}{2}$ whenever $h \ne k$.

The preceding considerations can be summarized as follows:

A Banach space is finite-dimensional if and only if every bounded sequence has a convergent subsequence.

1.5 Hilbert spaces

1.5.1 The closed unit ball of l_2 was used by Hilbert [1906a, p. 177] as a domain of linear, bilinear and quadratic forms:

Fortan ziehen wir durchweg nur solche Wertesysteme der unendlichvielen Variablen $\xi_1, \xi_2, \dots$ *in Betracht, die der Bedingung*

$$(x,x) = \xi_1^2 + \xi_2^2 + \cdots \le 1$$

genügen.

Hence, for a short while, the closed unit ball B_{l_2} was considered to be the decisive object for which Schoenflies coined the term *"Hilbertscher Raum"*; [SCHOE, p. 266]. In other words, one emphasized only the topological properties, and the significance of the linear structure was underestimated.

Young [YOU•, p. 312] tells the following story:

When Weyl presented a proof of the Fischer–Riesz theorem in a Göttingen Colloquium, Hilbert went up to the speaker afterward, to say "Weyl, you must just tell me one thing, whatever is a Hilbert space? That I could not understand."

Even if this anecdote is not true, it is at least nicely cooked up.

Riesz [RIE, p. 78] referred to the set of all square-summable sequences as **l'espace hilbertien**. To the best of my knowledge, the symbol l_2 first appeared in [BAN, p. 12]. Here is a collection of other notations:

Banach [1922, p. 134]: $(\mathscr{S}^2)$, [HEL$^+$, p. 1434]: R_∞, [STONE, p. 14]: $\mathfrak{H}_0$, [vNEU, p. 16]: F_Z, Köthe/Toeplitz [1934, p. 193]: σ_2.

1.5.2 The most decisive step in the early theory of Hilbert spaces was the **Fischer–Riesz theorem**, which states the completeness of $L_2[a,b]$. Both authors observed the significance of Lebesgue's integral as *the* basic ingredient. While Fischer [1907] gave a direct proof in terms of *convergence en moyenne*, Riesz [1907a] established the isometry between $L_2[a,b]$ and l_2 via an orthogonal expansion:

Soit $\varphi_1(x), \varphi_2(x), \dots$ un système normé de fonctions, définies sur un intervalle ab, orthogonales deux à deux, bornées ou non, sommables et de carrés sommables, c'est-à-dire tel que l'on ait

$$\int_a^b \varphi_i(x)\varphi_j(x)\,dx = 0 \quad (i \neq j); \qquad \int_a^b |\varphi_i(x)|^2\,dx = c^2$$

pour toutes les fonctions du système. Attribuons à chaque fonction $\varphi_i(x)$ du système un nombre a_i. Alors la convergence de $\sum_i a_i^2$ est la condition nécessaire et suffisante pour qu'il y ait une fonction $f(x)$ telle qu'on ait

$$\int_a^b f(x)\varphi_i(x)\,dx = a_i$$

pour chaque fonction $\varphi_i(x)$ et chaque nombre a_i.

For further information, the reader is referred to a recent paper of Horváth [2004•].

1.5.3 The geometric properties of the **Hilbert space** l_2 were stressed for the first time in the work of Schmidt [1908, pp. 56–57]:

Mit grossen Buchstaben, die vor das eingeklammerte x gesetzt werden, sollen durchweg Funktionen von folgenden Eigenschaften verstanden werden.

I. *Die Funktion ist nur für $x = 1, 2, 3, \dots$* ad inf. *definiert.*

II. *Die Quadratsumme der absoluten Beträge der von ihr durchlaufenen Werte convergiert.*

Wir definieren das Symbol $(A;B)$ durch die Gleichung

$$(A;B) = \sum_{x=1}^{x=\infty} A(x)B(x).$$

Mit $\|A\|$ bezeichnen wir die positive Grösse welche durch die Gleichung

$$\|A\|^2 = (A;\overline{A}) = \sum_{x=1}^{x=\infty} |A(x)|^2$$

definiert wird. Ist $(A;\overline{B}) = 0$, so bezeichnen wir $A(x)$ und $B(x)$ als zueinander **orthogonal**.

Referring to Kowalewski and Study, Schmidt [1908, footnote [8])] says:

Die geometrische Bedeutung der in diesem Kapitel entwickelten Begriffe und Theoreme tritt noch klarer hervor, wenn $A(x)$ statt als Funktion als Vector in einem Raume von unendlich vielen Dimensionen definiert wird.

Von Renteln [2007•] has discovered an interesting letter (from February 1907) in which Kowalewski explains his ideas to Engel.

1.5.4 The following chronology shows that the results described above were obtained independently.

Fischer	: Lecture at Brno on March 7;	submitted: April 29, 1907,
Riesz	: Lecture at Göttingen on February 26;	submitted: March 11, 1907,
Schmidt:	Lecture at Göttingen on February 12;	submitted: August 18, 1907.

1.5.5 Nowadays, by a **Hilbert space** we mean a real or complex linear space H that is complete with respect to the norm $\|x\| := \sqrt{(x|x)}$ induced by an **inner product** $(x|y)$.

This axiomatization had to wait for von Neumann [1927, pp. 15–17], [1930a, pp. 63–70], who additionally required H to be infinite-dimensional and separable. His system of axioms is categorical.

In contrast to Schmidt's definition of the inner product of the complex l_2, von Neumann assumed $\overline{(x|y)} = (y|x)$, which implies that $y \mapsto (x|y)$ is conjugate linear. This property guarantees that $(x|x)$ is always real, and the condition $(x|x) > 0$ for all $x \neq o$ makes sense.

1.5.6 The abstract **Cauchy–Schwarz inequality** $|(x|y)| \leq \|x\|\|y\|$ was proved by von Neumann [1930a, p. 64]. For the Euclidean space it goes back to Lagrange and Cauchy, while the version for integrals is due to Buniakowsky [1859, p. 4] and Schwarz [1885, p. 344]. The case of l_2 was treated by Schmidt [1908, p. 58].

1.5.7 A family $(e_i)_{i\in\mathbb{I}}$ is said to be **orthonormal** if $\|e_i\| = 1$ and $(e_i|e_j) = 0$ whenever $i \neq j$. Then we have

$$\sum_{i\in\mathbb{I}} |(x|e_i)|^2 \leq \|x\|^2 \quad \text{for } x \in H.$$

In the case of trigonometric functions a formula of this kind was used by Bessel [1828]. This is the reason why Schmidt [1908, p. 58] coined the name **Bessel's inequality**.

1.5.8 We now discuss the famous **orthonormalization process**, which goes back to Gram [1883] and Schmidt [1908, p. 61]. If $x_1,\dots,x_n$ are linearly independent, then an orthonormal system $(e_1,\dots,e_n)$ can be defined recursively:

$$e_h := \frac{x_h - \sum\limits_{k=1}^{h-1}(x_h|e_k)e_k}{\left\| x_h - \sum\limits_{k=1}^{h-1}(x_h|e_k)e_k \right\|} \quad \text{for } h=1,\dots,n.$$

The method above also works for infinite sequences. In the case of an uncountable family (x_i), however, difficulties occur; and this was the reason why Hilbert space theory remained restricted to the separable setting until 1934; see 1.5.10 and 7.5.18.

1.5.9 The main step in the orthonormalization process is the following.

Schmidt [1908, p. 64]: Let M be any closed subspace of a Hilbert space H.

Dann läßt sich ein beliebiges Element $x \in H$ stets auf eine und nur eine Weise in zwei Summanden zerlegen, deren einer in M liegt, während der andere zu M orthogonal ist.

The **orthogonal complement** is defined by

$$M^\perp := \{v \in H : (u,v) = 0 \text{ for all } u \in M\}.$$

We get $x=u+v$ with $u\in M$ and $v\in M^\perp$, and $P : x \mapsto u$ is said to be the **orthogonal projection** from H onto M. In other words, $H=M\oplus M^\perp$. In the important case that $M=\operatorname{span}\{x_1,\dots,x_n\}$ there is an explicit expression; Schmidt [1908, p. 67]:

$$v = \frac{\det\begin{pmatrix} x & (x|x_1) & \cdots & (x|x_n) \\ x_1 & (x_1|x_1) & \cdots & (x_1|x_n) \\ \vdots & \vdots & & \vdots \\ x_n & (x_n|x_1) & \cdots & (x_n|x_n) \end{pmatrix}}{\det\begin{pmatrix} (x_1|x_1) & \cdots & (x_1|x_n) \\ \vdots & & \vdots \\ (x_n|x_1) & \cdots & (x_n|x_n) \end{pmatrix}}.$$

The denominator, **Gram's determinant**, is different from 0 if and only if $x_1,\dots,x_n$ are linearly independent.

Originally, the infinite-dimensional and separable case was treated by letting

$$u := \sum_{n=1}^{\infty} (x|e_n)e_n,$$

where (e_n) is any orthonormal basis of M. Only Riesz [1934, pp. 36–37] found a proof that does not require separability; see also 2.2.6.

Here is a sketch of his reasoning, which he qualified as "anspruchslos":

Let $x \notin M$, put $\rho := \inf\{\|x-u\| : u \in M\} > 0$, and choose elements $u_n \in M$ such that $\|x-u_n\| \to \rho$. Then the parallelogram equation 1.5.12 applied to $x-x_m$ and $x-x_n$ yields

$$4\rho^2 + \|u_m-u_n\|^2 \leq 4\|x-\tfrac{u_m+u_n}{2}\|^2 + \|u_m-u_n\|^2 = 2\|x-u_m\|^2 + 2\|x-u_n\|^2 \to 4\rho^2.$$

Hence (u_n) is a Cauchy sequence, which has a limit $u_0 \in M$. In other words, there exists a best approximation u_0 of x.

Setting $\lambda = \frac{(x-u_0|u)}{(u|u)}$ for $u \in M$, we obtain

$$\rho^2 \leq \|x-(u_0+\lambda u)\|^2 = \|x-u_0\|^2 - \overline{\lambda}(x-u_0|u) - \lambda(u|x-u_0) + \lambda\overline{\lambda}(u|u) = \rho^2 - \frac{|(x-u_0|u)|^2}{(u|u)}.$$

Therefore $(x-u_0|u)=0$, which shows that $x-u_0 \in M^\perp$.

A similar proof is due to Rellich [1935, pp. 344–345].

1.5.10 Non-separable Hilbert spaces were first studied by Löwig [1934a]. With the help of well-ordering, he proved the existence of maximal orthonormal families $(e_i)_{i\in\mathbb{I}}$ and showed that all of these have the same cardinality. Löwig also observed that any element $x \in H$ can be written in the form

$$x = \sum_{i\in\mathbb{I}} (x|e_i)e_i,$$

and he proved **Parseval's equation**

$$\|x\|^2 = \sum_{i\in\mathbb{I}} |(x|e_i)|^2;$$

see Parseval [1799]. Since at most countably many Fourier coefficients $(x|e_i)$ differ from 0, the right-hand expressions make sense. More information on sums over arbitrary index sets will be given in 5.1.1.3.

Independently of Löwig, similar results were obtained at Göttingen by Rellich [1935] and Teichmüller [1935]. Most remarkably, Teichmüller [1939, p. 573] was the first to replace the well-ordering theorem by a maximal principle in a functional analytical proof.

1.5.11 Given any index set $\mathbb{I}$, we denote by $l_2(\mathbb{I})$ the Hilbert space of all scalar families $x = (\xi_i)$ for which

$$\|x|l_2\| := \left(\sum_{i\in\mathbb{I}} |\xi_i|^2\right)^{1/2}$$

is finite. The inner product of $x=(\xi_i)\in l_2(\mathbb{I})$ and $y=(\eta_i)\in l_2(\mathbb{I})$ is defined by

$$(x|y) := \sum_{i\in\mathbb{I}} \xi_i\overline{\eta_i}.$$

This construction yields (up to isomorphisms) all possible Hilbert spaces; Löwig [1934a, p. 27].

1.5.12 From the geometric point of view, Hilbert spaces are the most beautiful and the least interesting among all Banach spaces. Hence it is worthwhile to know which norms on a linear space X can be generated from an inner product. Amir has collected hundreds of such criteria; [AMIR]. The most important result along these lines is the **parallelogram equation** due to Jordan/von Neumann [1935, p. 721]:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|x\|^2 \quad \text{for all } x, y \in X.$$

1.5.13 The first monographs on Hilbert space theory were published in 1932 by von Neumann [vNEU] and Stone [STONE].

1.6 Albert A. Bennett and Kenneth W. Lamson

1.6.1 Shortly after being awarded the Ph.D. from Princeton University, Albert Bennett [1916] published a remarkable paper. His contribution was farsighted, but also inconsistent. The heuristic style gives room for various interpretations. I will always discuss the most promising version: *in dubio pro reo.*

The following text contains many quotations from Bennett's paper, which are set in *italic* without further reference.

1.6.2 First of all, the underlying algebraic structures do not consist of abstract elements but of vectors or functions:

By the term vector or function on a range [read: domain] *will be meant a correspondence from the elements of a range to a set of scalars. The scalars, which in all of the classical instances are real numbers, may be taken as Hensel p-adic numbers or elements in any perfekte bewertete Körper of Kürschák* [1913]; see p. 22

The sum and the difference of two vectors will be the usual sum and difference, respectively, and will be required to exist in the cases considered.

What results is a concrete **abelian group** G. However, since the nature of the elements does not play any role, from now on we take the abstract point of view.

1.6.3 *The notation $\|z\|$ will be used for the* **norm** *of $z \in G$. We shall require that $\|z\|$ is a uniquely defined real non-negative number such that*

$$\|z\| = \|-z\| \quad \textit{for } z \in G \quad \textit{and} \quad \|z_1 + z_2\| \leq \|z_1\| + \|z_2\| \quad \textit{for } z_1, z_2 \in G.$$

Moreover, if $z = o$, then $\|z\| = 0$. The converse implication is not assumed, but Bennett observes that one may pass to equivalence classes. Henceforth, we will follow this advice.

Completeness is obtained by the condition that every infinite series $\sum_{k=1}^{\infty} z_k$ with $\sum_{k=1}^{\infty} \|z_k\| < \infty$ converge to a limit. Using modern terminology, we may say that Bennett invented the concept of a **complete normed abelian group**. In order to make this property non-trivial, he requires the existence of non-zero elements with arbitrarily small norms; $0 < \|z_\varepsilon\| < \varepsilon$.

1.6.4 So far, so good! Now the adventure starts.

Bennett considers a family of complete normed abelian groups $G(m,n)$ indexed by so-called **signatures** $(m,n) \in \mathbb{Z}\times\mathbb{Z}$. In his philosophy, the underlying groups consist of vectors defined on different ranges. Let us suppose that (m,n) belongs to a given subset $\mathbb{S}$ of $\mathbb{Z}\times\mathbb{Z}$. Particular choices of $\mathbb{S}$ will be discussed below.

The product of two vectors will be required to exist uniquely in the cases considered, and to be associative, completely distributive with respect to addition, and such that the signature of the range of the product is the sum of the ranges of the factors.

That is, if $z_1 \in G(m_1,n_1)$, $(m_1,n_1) \in \mathbb{S}$, and $z_2 \in G(m_2,n_2)$, $(m_2,n_2) \in \mathbb{S}$, then there exists a *product* $z_1 \cdot z_2 \in G(m_1+m_2, n_1+n_2)$ provided that $(m_1+m_2, n_1+n_2) \in \mathbb{S}$. If $\mathbb{S}$ is a semi-group, then the direct sum of the $G(m,n)$'s becomes a *graded ring*. Continuity of multiplication is achieved by the condition

$$\|z_1 \cdot z_2\| \leq \|z_1\|\,\|z_2\|.$$

If $(0,0) \in \mathbb{S}$, then $G(0,0)$ is a complete normed ring. For simplicity, we will always put $G(0,0) := \mathbb{R}$.

Next, let $\mathbb{S} = \{(0,0),(1,0)\}$. In this case, $G(1,0)$ is a real **Banach space** under the additional assumption that $1 \cdot z = z \cdot 1 = z$, which does not appear in Bennett's paper but will be assumed henceforth. Algebraic manipulations yield that $\lambda \cdot z = z \cdot \lambda$ for rational numbers. By continuity, this formula remains true for all $\lambda \in \mathbb{R}$. Thus we have commutativity.

Letting $\mathbb{S} = \{(0,0),(1,0),(2,0)\}$ and $G(1,0) = G(2,0)$ leads to the concept of a **Banach algebra**.

1.6.5 Next, Bennett's basic example will be discussed:

Starting with an arbitrarily chosen range, which we shall refer to by the signature $(1,0)$, *we shall suppose that we may construct ranges of the signatures* $(0,0)$, $(0,1)$, $(-1,1)$, $(1,-1)$, $(-2,1)$, $(2,0)$.

Suppose that $X = G(1,0)$ and $Y = G(0,1)$ are Banach spaces. If $b \in G(-1,1)$ and $x \in X$, then $b \cdot x \in Y$ is defined, and $x \mapsto b \cdot x$ yields a linear map from X into Y such that $\|b \cdot x\| \leq \|b\|\|x\|$. Hence a possibility would be to put $G(-1,1) := \mathfrak{L}(X,Y)$, the Banach space of bounded linear operators $b : X \to Y$. Similarly, if $B \in G(-2,1)$ and $x_1, x_2 \in X$, then $(x_1,x_2) \mapsto B \cdot (x_1 \cdot x_2)$ is a bilinear map from $X \times X$ into Y. On the other hand, if $x \in G(2,0)$, then $x \mapsto B \cdot x$ defines a linear map from $G(2,0)$ into Y. Thus $G(2,0)$ can be regarded as an ancestor of the projective tensor product $X \widetilde{\otimes}_\pi X$.

1.6.6 For good reasons, Bennett did not consider $G(-1,0)$, though he used the terms *covariant* and *contravariant*. Since $x \cdot x^*$ is a scalar for $x \in G(1,0)$ and $x^* \in G(-1,0)$, the bilinear form $(x,x^*) \mapsto x \cdot x^*$ can be used to define a duality between $G(1,0)$ and $G(-1,0)$. If $G(1,0) = X$, then letting $G(-1,0) = X^*$ and $\|x^*\| = \sup\{|x \cdot x^*| : \|x\| \leq 1\}$ seems to be a good idea. But doing so, we may lose associativity of multiplication.

Indeed, suppose that X contains two linearly independent elements, say x_1 and x_2; choose x^* such that $x_1 \cdot x^* \neq 0$. Then, in view of $\lambda \cdot x_1 = x_1 \cdot \lambda$, the desired formula $(x_1 \cdot x^*) \cdot x_2 = x_1 \cdot (x^* \cdot x_2)$ cannot be true.

1.6.7 *A vector b with the signature* $(-1,1)$ *will be said to be non-singular, if and only if there exists a unique vector* b^{-1} *with the signature* $(1,-1)$ *such that the equation* $a = b \cdot x$*, where a is of signature* $(0,1)$ *and x is required to be of signature* $(1,0)$*, always has one and only one solution, given by* $x = b^{-1} \cdot a$.

This means that the operator b is continuously invertible. But what about the product $b^{-1} \cdot b$, which is supposed to be a scalar? Most naturally, one should take the trace. Unfortunately, this is impossible in the infinite-dimensional setting, since then no trace exists for identity maps such as the composition $b^{-1}b$.

1.6.8 Bennett's original goal was to find roots of $f(x) = o$, where f maps a subdomain of a Banach space X into a Banach space Y.

In describing Newton's Method, we presuppose that we are given initially a function $f(x)$*, where x is of signature* $(1,0)$ *and* $f(x)$ *of signature* $(0,1)$*, and such that we may expand* $f(x+h)$ *as follows:*

$$f(x+h) = f(x) + f'(x) \cdot h + \tfrac{1}{2} \cdot f''(\xi) \cdot h \cdot h,$$

where h and ξ *are of signature* $(1,0)$*,* $h \cdot h$ *of signature* $(2,0)$*,* $f'(x)$ *of signature* $(-1,1)$*, and* $f''(\xi)$ *of signature* $(-2,1)$.

In particular, $f'(x)$ may be viewed as the Fréchet derivative of f at the point x.

1.6.9 The basic examples considered by Bennett are the Banach spaces L_∞, L_2, and L_1 defined with respect to Lebesgue's measure on the unit interval. He always assumes that $G(1,0) = G(0,1)$; and $G(2,0)$ is taken to be L_∞, L_2, and L_1 over the unit square. Most interesting is the choice of $G(-1,1)$. All operators are supposed to be generated by a kernel F defined on the unit square. He considers the following examples:

$$\|F\| = \max_{0 \le s \le 1} \int_0^1 |F(s,t)|dt, \quad \|F\| = \Big(\int_0^1 \int_0^1 |F(s,t)|^2 ds\,dt \Big)^{1/2}, \quad \|F\| = \int_0^1 \max_{0 \le t \le 1} |F(s,t)|ds.$$

The left-hand expression is the usual norm of the induced operator from L_∞ into itself. Secondly, the Hilbert–Schmidt norm occurs, and finally, we have the nuclear norm of the induced operator from L_1 into itself. In none of these cases can the operator be invertible! But this is needed for applying Newton's method.

1.6.10 Bennett was perfectly right when making his final statement:

The method here used of obtaining a general result by a mere reiteration of the case of one variable offers several features of novelty and is suggested as, perhaps, of even more interest than the results obtained by its particular application to the present problem.

Although Bennett's paper contains some oddities, it presents a prophetic look into the future of Banach space theory. He tried to get everything at one blow, which seems to be impossible. His approach covers Banach spaces, Banach algebras, Banach spaces of linear and bilinear operators, as well as tensor products. It took more than 50 years to carry out this program. I am not able to check how often Bennett's paper has been cited. Certainly, it did not have great influence on the later development. Nevertheless, it is a fascinating piece of history.

1.6.11 In December 1917, Lamson [1920] submitted a thesis, which was written under Bliss (Chicago). The algebraic background of his approach is a linear space $\mathfrak{M}$ that consists of scalar-valued functions defined on an abstract set $\mathfrak{P}$:

To each element y of $\mathfrak{M}$ corresponds a positive or zero number, the "modulus" of y, which will be denoted by $\|y\|$.

(1) *$\mathfrak{M}$ is linear, that is, contains all functions of the form $c_1y_1+c_2y_2$, where c_1 and c_2 are real numbers, provided y_1 and y_2 are themselves in $\mathfrak{M}$.*

(2) $\|y_1+y_2\| \le \|y_1\|+\|y_2\|$.

(3) $\|cy\| = |c|\|y\|$, *for every real number c.*

(4) *If $\|y\| = 0$, then $y(p) = 0$ for every $p \in \mathfrak{P}$.*

(5) *For every Cauchy sequence in $\mathfrak{M}$ there exists a function in $\mathfrak{M}$ that is the limit of this sequence.*

Lamson's main goal was an implicit function theorem and its application to the calculus of variations. We state his result in modern terminology:

Let F be a continuous function from an open subset G of $X \times Y$ into Y such that $F(x_0,y_0) = y_0$ for some $(x_0,y_0) \in G$. If F satisfies a Lipschitz condition

$$\|F(x,y_1)-F(x,y_2)\| \le q\|y_1-y_2\| \quad \text{for } (x,y_1),(x,y_2) \in G$$

and fixed $0 < q < 1$, then there exist neighborhoods $U_\delta(x_0) = \{x \in X : \|x-x_0\| < \delta\}$ and $V_\varepsilon(y_0) = \{y \in Y : \|y-y_0\| < \varepsilon\}$ as well as a continuous function f from $U_\delta(x_0)$ into $V_\varepsilon(y_0)$ such that $U_\delta(x_0) \times V_\varepsilon(y_0) \subset G$, $f(x_0)=y_0$ and $F(x,f(x))=f(x)$ for all $x \in U_\delta(x_0)$.

Starting with $f_1(x) = y_0$ and putting $f_{i+1}(x) := F(x,f_i(x))$, one obtains the solution by successive approximations: $f(x) = \lim_{i\to\infty} f_i(x)$.

Lamson [1920, pp. 251–252] considered the following case: $\mathfrak{M}=X=Y$ *is the class of functions y_i which for each $i = 1,\dots,n$ are continuous with their first derivatives on the interval ab. The modulus is the maximum of the absolute values of $y_i(x)$ and $y_i{}'(x)$.*

The final version of the "soft" **implicit function theorem**, which yields solutions of $F(x,y) = o$, is due to Hildebrandt/Graves [1927, p. 150]; see 5.1.8.8.

1.7 Norbert Wiener

1.7.1 Formulated in modern terminology, Wiener was interested in a problem from analysis situs: is it possible to reconstruct the topology of some topological space X from the group of its homeomorphisms?

Denote this group by $\mathscr{H}(X)$, and let $x_0 \in X$ be a cluster point of a subset A. Then the following implication holds:

$$\text{if } f \in \mathscr{H}(X) \text{ and } f(x) = x \text{ for all } x \in A, \text{ then } f(x_0) = x_0.$$

Wiener arrived at the question: in which cases does this property characterize cluster points? The space $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$ provides a counterexample, since 0 is a fixed point of all homeomorphisms, but is no cluster point of any finite subset of $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. His main result says that every *normed linear space* X satisfies the required condition.

Suppose that $x_0 \in X$ is not a cluster point of A. Then there exists $\varepsilon > 0$ such that $\|x - x_0\| \leq 2\varepsilon$ implies $x \notin A$. By a suitable translation, we may arrange that $0 < \|x_0\| < \varepsilon$. Obviously,

$$f(x) := \begin{cases} x & \text{if } \|x\| \geq \varepsilon \\ \frac{\|x\|}{\varepsilon} x & \text{if } \|x\| \leq \varepsilon \end{cases}$$

defines a homeomorphism f with $f(x_0) \neq x_0$. On the other hand, for $x \in A$ we have $\|x\| \geq \varepsilon$ and hence $f(x) = x$. Note that completeness is irrelevant for this elementary construction.

1.7.2 We now present Wiener's axioms, [1920a, p. 313], [1920b, pp. 332–333], [1922, pp. 123–124]. The algebraic part should be compared with Weyl's elegant approach.

A **vector system** *is defined as a system* K *of elements correlated with a system* σ *of entities and the operations* $\oplus$, $\odot$, *and* $\|\cdot\|$ *in a manner indicated by the following propositions:*

(1) *If* ξ *and* η *belong to* σ, $\xi \oplus \eta$ *belongs to* σ,

(2) *If* ξ *belongs to* σ, *and n is a real number* ≥ 0, $n \odot \xi$ *belongs to* σ,

(3) *If* ξ *belongs to* σ, $\|\xi\|$ *is a real number* ≥ 0,

(4) $n \odot (\xi \oplus \eta) = (n \odot \xi) \oplus (n \odot \eta)$,

(5) $(m \odot \xi) \oplus (n \odot \xi) = (m + n) \odot \xi$,

(6) $\|n \odot \xi\| = n\|\xi\|$,

(7) $\|\xi \oplus \eta\| \leq \|\xi\| + \|\eta\|$,

(8) $m \odot (n \odot \xi) = mn \odot \xi$,

(9) *If A and B belong to* K, *there is associated with them a single member AB of* σ,

(10) $\|AB\| = \|BA\|$,

(11) *Given an element A of K and an element ξ of σ,*
there is an element B of K such that $AB = \xi$,

(12) $AC = AB \oplus BC$,

(13) $\|AB\| = 0$ *when and only when* $A = B$,

(14) *If* $AB = CD$, $BA = DC$.

As an additional condition, Wiener sometimes assumed that *the sum of two vectors is independent of their order*; [1920a, p. 315], [1920b, p. 333]. Therefore he was not aware of the fact that this is automatically the case.

1.7.3 First of all, we will show that Wiener's system σ is an abelian group and indeed a normed linear space.

(A) In order to check the associativity of $\oplus$, let $\xi, \eta, \zeta \in \sigma$ and fix $O \in \mathsf{K}$. By **(11)**, we successively find $A, B, C \in \mathsf{K}$ such that $\xi = OA$, $\eta = AB$, and $\zeta = BC$. Repeatedly using **(12)**, we infer from $\xi \oplus \eta = OA \oplus AB = OB$ and $\eta \oplus \zeta = AB \oplus BC = AC$ that
$$(\xi \oplus \eta) \oplus \zeta = OB \oplus BC = OC = OA \oplus AC = \xi \oplus (\eta \oplus \zeta).$$

(B) We now prove that the map $\xi \mapsto \alpha \oplus \xi$ is onto for every $\alpha \in \sigma$. Let $\beta \in \sigma$ and fix $O \in \mathsf{K}$. By **(11)**, we can choose $A, B \in \mathsf{K}$ such that $\alpha = OA$ and $\beta = OB$. Then **(9)** ensures the existence of $\xi := AB$. Since $OA \oplus AB = OB$ by **(12)**, it follows that $\alpha \oplus \xi = \beta$.

(C) Similarly, it turns out that the map $\xi \mapsto \xi \oplus \alpha$ is onto for every $\alpha \in \sigma$. The first step is as above. Additionally, we choose $C, D \in \mathsf{K}$ such that $OC = AO$ and $OD = BO$. Let $\xi := DC$. Then **(12)** and **(14)** imply that

$$\xi \oplus \alpha = DC \oplus OA = DC \oplus CO = DO = OB = \beta.$$

(D) If $A, B \in \mathsf{K}$, then $\omega := AA = BB$ is the null element of σ. Indeed, in view of **(B)** and **(C)**, we find $\alpha, \beta \in \sigma$ such that $\alpha \oplus BB = AA$ and $AA \oplus \beta = BB$. Hence

$$AA \oplus BB = (\alpha \oplus BB) \oplus BB = \alpha \oplus BB = AA,$$
$$AA \oplus BB = AA \oplus (AA \oplus \beta) = AA \oplus \beta = BB.$$

Summarizing the results obtained so far, we can state that σ is a group.

(E) To see that $1 \odot \xi = \xi$, we pick $\eta \in \sigma$ such that $\xi = (1 \odot \xi) \oplus \eta$. Then it follows from
$$1 \odot \xi = 1 \odot [(1 \odot \xi) \oplus \eta] = (1 \odot \xi) \oplus (1 \odot \eta)$$
that $1 \odot \eta = \omega$. Thus $\|\eta\| = \|1 \odot \eta\| = \|\omega\| = 0$, which in turn implies $\eta = \omega$.

(F) Observe that $2 \odot \xi = (1 \odot \xi) \oplus (1 \odot \xi) = \xi \oplus \xi$ for all $\xi \in \sigma$. Consequently, if $\xi, \eta \in \sigma$, then

$$(\xi \oplus \eta) \oplus (\xi \oplus \eta) = 2 \odot (\xi \oplus \eta) = (2 \odot \xi) \oplus (2 \odot \eta) = (\xi \oplus \xi) \oplus (\eta \oplus \eta).$$

By associativity, $\xi \oplus (\eta \oplus \xi) \oplus \eta = \xi \oplus (\xi \oplus \eta) \oplus \eta$, which implies $\eta \oplus \xi = \xi \oplus \eta$. Therefore σ is abelian.

(G) Finally, we note that $\|(-1)\odot\xi\| = \|\xi\|$ follows from **(10)**, since $\xi = AB$ gives $(-1)\odot\xi = BA$.

This proves that Wiener's higgledy-piggledy mixture of axioms really yields *normed linear spaces*.

1.7.4 Wiener's *Note on a paper of M. Banach* [1923a] concludes with the following footnote:

As a final comment on this paper, I wish to indicate the fact that postulates not unlike those of M. Banach have been given by me on several occasions. However, as my work dates only back to August and September, 1920, and M. Banach's work was already presented for the degree of doctor of philosophy in June, 1920, he has the priority of original composition. I have here employed M. Banach's postulates rather than my own because they are in a form more immediately adopted to the treatment of the problem in hand.

Finally we quote from Wiener's autobiography [WIE$^{\bullet}$, pp. 63–64]:

It is so in an aesthetic rather than in any strictly logical sense that, in those years after Strasbourg, Banach spaces did not seem to have the physical and mathematical texture I wanted for a theory on which I was to stake a large part of my future reputation. Nowadays it seems to me that some aspects of the theory of Banach spaces are taking on a sufficiently rich texture and have been endowed with a sufficiently unobvious body of theorems to come closer to satisfying me in this respect.

At that time, however, the theory seemed to me to contain for the immediate future nothing than some decades of rather formal and thin work. By this I do not mean to reproach the work of Banach himself but that of the many inferior writers, hungry for easy doctor's theses, who were drawn to it. As I foresaw, it was this class of writers that was first attracted to the theory of Banach spaces.

The chief factor which led me to abandon the theory of Banach spaces, after a few desultory papers, was that my work on the Brownian motion was now coming to a head. Differential space, the space of Brownian motion, is itself in fact a sort of vector space, very close to the Banach spaces, and it presented itself as a successful rival for my attentions because it had a physical character most gratifying to me. In addition, it was wholly mine in its purely mathematical aspects, whereas I was only a junior partner in the theory of Banach spaces. See Subsection 6.8.6.

In conclusion, Wiener did not realize the great significance of the concept of a Banach space for analysis and for mathematics as a whole. With the exception of some results about analytic X-valued functions (Wiener [1923a]), his contributions to the foundation of the theory are negligible. This opinion is inconsistent with that of Hille, who wrote in [WIE$^{\bowtie}$, Vol. III, p. 684]:

As one of the workers in this vineyard, I wish to pay homage to the genius and insight of Norbert Wiener.

1.8 Eduard Helly and Hans Hahn

1.8.1 In 1912, Helly published an extraordinary paper concerned with (bounded linear) functionals on the Banach space $C[a,b]$. Basically, he proved once more the Riesz representation theorem and the extension theorem (in terms of the moment problem). The point is that his techniques can easily be translated into the language of general Banach spaces. This remark applies in particular to the proof of the uniform boundedness principle. For a detailed discussion of his achievements, we refer to Sections 2.2, 2.3, 2.4, 3.3, and 3.4.

Due to World War I, Helly lost six irretrievable years. Back in Vienna, he submitted his *Habilitationsschrift*, which has become a milestone in the history of Banach space theory. Taking Minkowski's n-dimensional concept of an *Aichkörper* as a leitmotiv, he developed an infinite-dimensional theory; Helly [1921, pp. 60, 66–67]:

Als Punkt x eines Raumes von abzählbar unendlich viel Dimensionen sei jede Zahlenfolge $x_1, x_2, x_3, \dots$ bezeichnet, wobei die Größen x_k beliebige reelle oder komplexe Zahlen sein können.

Es sei nun eine **Abstandsfunktion** *$D(x)$ gegeben, die jedem Punkt x eines gewissen Bereiches eine reelle positive Zahl zuordnet und die folgenden Bedingungen genügt:*

I. Wenn x dem Definitionsbereich von $D(x)$ angehört, so soll auch λx ihm angehören und es soll

$$D(\lambda x) = |\lambda| D(x)$$

sein.

II. Wenn x und y dem Definitionsbereich von $D(x)$ angehören, so soll auch $x+y$ ihm angehören und es soll

$$D(x+y) \leq D(x) + D(y)$$

sein.

III. Aus $D(x) = 0$ folgt $x = 0$.

Mit Hilfe der Funktion $D(x)$ kann jetzt der Begriff der Grenze definiert werden. Die Begriffe Umgebung, Häufungsstelle, Ableitung einer Punktmenge ergeben sich in bekannter Weise.

Most important is his concept of duality.

Es sei $u = (u_1, u_2, u_3, \dots)$ so beschaffen, daß

$$(x,u) = \sum_{k=1}^{\infty} u_k x_k$$

für alle x mit endlichem $D(x)$ konvergiert, und daß die obere Grenze $\Delta(u)$ der Werte von (x,u), wenn x auf das Gebiet $D(x) = 1$ beschränkt wird, endlich ist. Dann ergibt sich die fundamentale Ungleichung $|(x,u)| \leq \Delta(u)\, D(x)$. Die so bestimmte Funktion $\Delta(u)$ genügt den Bedingungen I und II.

1.8.2 Acting as a referee of Helly's Habilitationsschrift (submitted in spring 1921), Hahn clearly realized the far-reaching background. Footnote [5]) of Hahn's subsequent paper [1922] says:

Die folgenden Überlegungen stammen im wesentlichen von E. Helly.

Though this acknowledgment concerns only Helly's definition of $\Delta(u)$ (*duale Abstandsfunktion*), it should be extended to all ideas described in the previous paragraph. In the introduction, Hahn declared that his investigations were initiated by a discussion with Schur during *der Versammlung der Deutschen Mathematikervereinigung*, which took place in Jena in September 1921. He defined (independently) the concept of a linear space and generalized Helly's *Abstandsfunktion* $D(x)$ to this abstract setting. Hahn's *Fundamentaloperation* $U(x,y)$ is an anticipation of the bilinear form, which determines a dual system. Curiously enough, he assumed linearity only in the first variable x. His main result is a principle of uniform boundedness, proved by a gliding hump method. No credit is given to Helly [1912, p. 268]. Hahn also provides a long list of concrete examples including L_p. However, papers of Riesz are not quoted. Of course, Hahn's major contribution to Banach space theory is his part in the shaping of the extension theorem, which will be discussed in 2.3.5.

1.9 Summary

1.9.1 The axiomatic method was developed at the turn of the century. The most important stimulus was Hilbert's treatise *Grundlagen der Geometrie* (1899).

Fréchet [1906] and Hausdorff [HAUS_1] built the theory of metric spaces, whose elements are points without any specific nature. Peano [PEA] had already done the same for linear spaces (1888). In Weber's *Lehrbuch der Algebra* (1899) we find the abstract concepts of a *group* and a *field*. Based on a seminal paper of Steinitz [1910, pp. 172–173], the Hungarian mathematician Kürschák [1913, pp. 211–215], a former teacher of Riesz, introduced a new concept: *bewertete Körper*.

Es sei jedem Element (jeder Größe) a eines Körpers $\mathfrak{K}$ eine reelle Zahl $\|a\|$ so zugeordnet, daß den folgenden Forderungen genügt wird:

1. *es ist $\|0\| = 0$, für jedes von Null verschiedene a ist $\|a\| > 0$.*
2. *für jedes Element a ist $\|1+a\| \le 1+\|a\|$.*
3. *für je zwei Elemente ist $\|ab\| = \|a\|\,\|b\|$.*
4. *es gibt in $\mathfrak{K}$ wenigstens ein solches Element, daß $\|a\|$ von Null und Eins verschieden ist.*

In jedem bewerteten Körper ist $\|a+b\| \le \|a\| + \|b\|$. Da die Bewertung von $a-b$ ein écart ist, so ist es naheliegend, die bekannten Begriffe des Limes und der Fundamentalreihe von den reellen und komplexen Zahlen auf die Größen eines beliebigen bewerteten Bereiches zu übertragen. Man braucht dabei nur in den gewöhnlichen Definitionen überall für den absoluten Wert den Begriff Bewertung zu setzen.

1.9.2 In 1913 at the latest, all **classical Banach spaces** had been discovered:

$$C[a,b],\ l_2,\ L_2[a,b],\ L_p[a,b], \text{ and } l_p \text{ (chronological order).}$$

The following comments of Riesz clearly indicate that he realized the common axiomatic background.

[1909a, p. 452]: *Die Untersuchung dieser Funktionenklassen* $[L^p]$ *wird auf die wirklichen und scheinbaren Vorteile des Exponenten* $p = 2$ *ein ganz besonderes Licht werfen; und man kann auch behaupten, daß sie für die axiomatische Untersuchung der Funktionenräume brauchbares Material liefert.*

[1918, p. 71]: *Die in der Arbeit gemachte Einschränkung auf stetige Funktionen ist nicht von Belang. Der in den neueren Untersuchungen über diverse Funktionalräume bewanderte Leser wird die allgemeine Verwendbarkeit der Methode sofort erkennen.*

Why did Riesz not cross the Rubicon? The answer to this question will remain a mystery forever! A possible reason is discussed in 2.6.4.4.

1.9.3 The decisive obstacle for developing Banach space theory at an earlier date was Peano's forgotten definition of a linear space.

Although there are many textbooks on linear algebra, the underlying theory is not rich enough for research of high standard, and linear structures became important only in tandem with topologies and orderings. For this reason, the algebraic concept of a linear space can be regarded a child of Banach space theory.

Due to the existence of Hamel bases, every abstract linear space admits concrete representations. Hence it would be enough to deal with Banach spaces formed by scalar families. But this yields no real simplification. On the contrary, it is just the advantage of an abstract theory to ignore the nature of elements under consideration.

1.9.4 The preceding shows that around 1913 the concept of a *complete normed linear space* was ripe for discovery. So to speak, *it must have been in the air*; see Bernkopf [1966•, p. 67]. This explains why several people almost simultaneously had almost the same idea.

	without completeness	with completeness
linear function spaces	Helly (May 1921)	Bennett (August 1916) Lamson (December 1917)
abstract linear spaces	Wiener (August 1920)	Banach (June 1920) Hahn (after September 1921)

Of course, finding the appropriate concept is only the first step. Subsequently, the foundation of the corresponding theory must be laid. This process will be described in the next chapter. It is my aim to show the dominating role of STEFAN BANACH, who is the **father** of the theory of *complete normed linear spaces*, which now bear his name with full right. But at the same time, I refer to FRIGYES RIESZ as the **grandfather** and to EDUARD HELLY as the (sometimes underestimated) **godfather**.

1.9.5 Further information about the early history of Banach spaces can be found in the following references:

[BOU$^\bullet$, Chap. 21], [DIEU$_1^\bullet$, Chap. 8], [DIEU$_2^\bullet$], [KAŁ$^\bullet$], [KRA$^\bullet$, Chap. 23], [MON$^\bullet$], [PIER$^\bullet$, Chap. 4];

Bernkopf [1966$^\bullet$], Birkhoff/Kreyszig [1984$^\bullet$], Dorier [1996$^\bullet$], Heuser [1995$^\bullet$], Köthe [1989$^\bullet$], Kreyszig [1986a$^\bullet$, 1986b$^\bullet$], Pietsch [1989$^\bullet$], Siegmund-Schultze [1982$^\bullet$], Smithies [1997$^\bullet$].

2

Historical Roots and Basic Results

2.1 Operators

2.1.1 First of all, we provide some basic notation.

The symbols $\subset$ and $\subseteq$ are used in the same way as $<$ and $\leq$; that is, $\subset$ indicates *proper* inclusion.

2.1.2 If not otherwise stated, X, Y, and Z (without or with subscripts) stand for (real or complex) **Banach spaces**. Elements are denoted by the corresponding lowercase letters: x, y, and z. We write $\mathbb{K}$ for the **scalar field**, be it $\mathbb{R}$ or $\mathbb{C}$. The **norm** of a Banach space X is denoted by $\|\cdot\|$, but sometimes we use the symbol $\|\cdot|X\|$ for better distinction. We refer to $B_X := \{x \in X : \|x\| \leq 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ as the **closed unit ball** and the **closed unit sphere** of X, respectively. The term **subspace** will be used to indicate that a subset of X is stable under the formation of finite linear combinations. Hence a subspace may or may not be closed.

2.1.3 By an **operator** $T : X \to Y$ we mean a continuous linear map from X (*domain, source*) into Y (*codomain, target*). As usual, $\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}$ or $\|T : X \to Y\|$ stands for the **operator norm**. The fact that

continuous linear operators = bounded linear operators

was observed in the early period of functional analysis; see the following quotations.

The collection of these operators is a Banach space $\mathfrak{L}(X,Y)$. In the case that $X = Y$, we simply write $\mathfrak{L}(X)$ instead of $\mathfrak{L}(X,X)$. The **identity operator** of a Banach space X will be denoted by I_X, or just by I.

With every operator $T \in \mathfrak{L}(X,Y)$ we associate its **null space** and its **range**:

$$N(T) := \{x : Tx = o\} \quad \text{and} \quad M(T) := \{Tx : x \in X\}.$$

2.1.4 In a conversation with Bernkopf [1967•, p. 346], Friedrichs recalled that at the end of the 1920s, Schmidt advised von Neumann:

Nein! Nein! Sagen Sie nicht Operator, sagen Sie Matrix!

This reflects the historical fact that the starting points were concrete operators

$$K_{\text{op}} : f(t) \mapsto \int_a^b K(s,t) f(t)\, dt \quad \text{and} \quad A_{\text{op}} : (\xi_k) \mapsto \left(\sum_{k=1}^{\infty} \alpha_{hk} \xi_k \right),$$

where K is a continuous kernel and $A = (\alpha_{hk})$ is an infinite matrix. The consequent distinction between A and A_{op} is often cumbersome. Thus A will mostly denote both the matrix and the operator.

We stress that Hilbert's approach to the *theory of equations in infinitely many unknowns* was based on bilinear forms. In their paper *Grundlagen für eine Theorie der unendlichen Matrizen*, Hellinger and Toeplitz took the same point of view. Moreover, they extended the matrix calculus from the finite to the infinite-dimensional setting, [1910, p. 309]:

Man kann die Hilbertschen Faltungssätze [1906a, pp. 179–180] *einfach dahin interpretieren, daß man für beschränkte unendliche Matrizen den für endliche Matrizen üblichen Kalkül aufstellen kann.*

2.1.5 In 1913, Riesz clearly demonstrated the advantages that can be achieved by thinking in terms of operators.

[RIE, pp. 78–79]: *Considérons l'***espace hilbertien***; nous y entendons l'ensemble des systèmes* (x_k) *tels que* $\sum |x_k|^2$ *converge. Nous étudierons les substitutions linéaires à une infinité de variables, portant sur l'espace hilbertien.*

A chaque élément (x_k) *de notre espace on fait correspondre (suivant une certaine loi) un élément bien déterminé* (x'_k). *On suppose que la correspondance soit* **distributive***, c'est-à-dire qu'elle fasse correspondre à élément* (cx_k) *l'élément* (cx'_k) *et à élément* $(x_k + y_k)$ *l'élément* $(x'_k + y'_k)$. *En Algèbre, où il ne s'agit d'un nombre fini de variables, la distributivité de la correspondance entraîne aussi sa continuité. Dans notre cas, il faut supposer explicitement la* **continuité***. Ce sont ces correspondances, distributives et continues à la fois, que nous appellerons* **substitution linéaire***.*

On démontre aisément que la continuité entraîne une autre propriété importante des substitutions linéaires, savoir celle d'être **bornée***. Il existe une constante positive M telle que*

$$\sum_{k=1}^{\infty} |x'_k|^2 \le M^2 \sum_{k=1}^{\infty} |x_k|^2.$$

Désignerons par M_A *la plus petite des valeurs M. Nous appellerons la constante* M_A *la* **borne de la substitution** *A.*

[RIE, p. 107]: *Étant donnée une suite indéfinie de substitutions* (A_n)*, nous dirons qu'elle* **tend uniformément** *vers la substitution A lorsque* $M_{A-A_n} \to 0$.

[RIE, p. 109]: *Pour que la suite de substitutions* (A_n) *converge uniformément, il faut et il suffit que, m et n tendant vers l'infini indépendamment l'un de l'autre, on ait* $M_{A_m - A_n} \to 0$.

[RIE, p. 85]: *Tout comme en Algèbre, on nomme* **produit** *de deux substitutions linéaires A, B et l'on désigne par AB la substitution évidemment linéaire qui résulte des deux premières, effectuées successivement:* $AB(x_k) = A(B(x_k))$.

Although Riesz did not state the inequalities

$$M_{A+B} \le M_A + M_B \quad \text{and} \quad M_{AB} \le M_A M_B,$$

he used them implicitly. Thus Riesz knew, in principle, that $\mathfrak{L}(l_2)$ is a Banach algebra.

2.1.6 In Stones's book [STONE, p. 65] from 1932, this result reads as follows:

The class of all bounded linear transformations with domain $\mathfrak{H}$ (Hilbert space) *is closed under the operations of addition, subtraction, multiplication, and scalar multiplication; it contains the transformations O and I. In this class, the distance between two transformations* T_1, T_2 *can be defined as the bound of their difference* $T_1 - T_2$*; for this quantity is symmetric, is positive except when* $T_1 \equiv T_2$*, and obeys the triangle inequality.* Note that completeness is missing.

2.1.7 The following quotations, in which $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{S}_3$ denote Banach spaces, are taken from Hildebrandt's address [1931, pp. 191, 193] delivered at a meeting of the American Mathematical Society in September 1930:

A transformation T *will be called a* **linear limited transformation** *on* $\mathfrak{S}_1$ *to* $\mathfrak{S}_2$ *if it satisfies the following conditions:*

(a) *To every element* ξ *of* $\mathfrak{S}_1$ *there corresponds an element* η *of* $\mathfrak{S}_2$.

(b) *For every* ξ_1 *and* ξ_2 *of* $\mathfrak{S}_1$ *and all complex numbers* a_1 *and* a_2

$$T(a_1\xi_1 + a_2\xi_2) = a_1 T(\xi_1) + a_2 T(\xi_2).$$

(c) *There exists a number* M *such that for all* ξ *of* $\mathfrak{S}_1$

$$\|T(\xi)\| \leq M\|\xi\|.$$

The smallest possible value of M *will be called the* **modulus** *of* T.

It is possible to set up an algebra of linear limited transformations. For instance, if T_1 *and* T_2 *are on* $\mathfrak{S}_1$ *to* $\mathfrak{S}_2$*, then* $c_1T_1 + c_2T_2$ *is defined as* $T_1(c_1\xi) + T_2(c_2\xi)$*. Obviously* $M(c_1T_1 + c_2T_2) \leq |c_1|M(T_1) + |c_2|M(T_2)$*. If we consider the totality of all linear limited transformations on* $\mathfrak{S}_1$ *to* $\mathfrak{S}_2$*, then this can be considered as a vector space of the same type as* $\mathfrak{S}$*, the norm being the modulus* $M(T)$.

If T_1 *is a linear limited transformation on* $\mathfrak{S}_1$ *to* $\mathfrak{S}_2$*, and* T_2 *on* $\mathfrak{S}_2$ *to* $\mathfrak{S}_3$*, then the succession* T_2T_1 *will define a linear limited transformation on* $\mathfrak{S}_1$ *to* $\mathfrak{S}_3$*, and it is obvious that the modulus* $M(T_2T_1)$ *satisfies the condition*

$$M(T_2T_1) \leq M(T_2)M(T_1).$$

2.1.8 Banach [BAN, p. 54] referred to $\|T\|$ as a norm; but for unknown reasons he never considered the collection of all operators $T : X \to Y$. Making propaganda for Banach algebras, Lorch [LOR, p. 122] wrote:

It is strange that over a period of 35 years or so, many Banach spaces which attracted most attention were also algebras but their study from this richer point of view was essentially never approached. A rapid examination of Banach's book seems to show that he multiplies for the first time on p. 153 where he considers the n^{th} *iterate of* T*. Students for the doctorate who sometimes feel that everything has been done should draw valuable conclusions from this circumstance.*

2.1.9 In Chapter 1, we have described the long and laborious process of passing from sequences and functions to abstract elements of a linear space. The preceding quotations show that there was a second barrier: one had to look at $\mathfrak{L}(X)$ as a ring without taking into account that its elements are operators.

Nagumo [1936, pp. 62–63] introduced the concept of a **Banach algebra** (vollständiger linearer metrischer Ring) and gave the following example:

Es sei $\mathfrak{B}$ ein linearer, normierter, vollständiger Raum. Die Menge $\mathfrak{R}$ aller stetigen linearen Abbildungen von $\mathfrak{B}$ auf $\mathfrak{B}$ selbst oder einen Teil von $\mathfrak{B}$ ist ein **linearer, metrischer Ring***, wenn man jeder Transformation $T \in \mathfrak{R}$ ihren Betrag durch*

$$|T| = \text{Obere Grenze von } |T\mathfrak{x}| \quad \textit{für} \quad |\mathfrak{x}| \leq 1 \quad (\mathfrak{x} \in \mathfrak{B})$$

definiert [read: zuordnet]. *Dann bedeutet die Relation $\lim_{n\to\infty} T_n = T$ nichts anderes als, dass $T_n\mathfrak{x}$ gegen $T\mathfrak{x}$ für alle $|\mathfrak{x}| \leq 1$ gleichmässig konvergiert. Man kann also leicht beweisen, dass $\mathfrak{R}$ vollständig ist.*

2.1.10 The following analogy shows the advantage of the abstract point of view. The geometric series

$$(1-\zeta)^{-1} = \sum_{n=0}^{\infty} \zeta^n$$

passes into the **Neumann series**

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

In order to ensure convergence, the classical condition $|\zeta| < 1$ must be replaced by $\|T\| < 1$. The proof is so obvious that following Nagumo [1936, p. 64], one may say, *Dem Leser überlassen*. But this is the very end of a long story. The beginning, which dates back at least to Beer's and Neumann's investigations in potential theory [NEU, Chap. 5, 6], was much more cumbersome. In modern terminology, Neumann looked at a concrete integral operator $T : X \to X$ with $\|T\| = 1$ whose special properties imply that the series $x + Tx + T^2x + \cdots$ converges for all $x \in X$.

2.1.11 The concept of a Neumann series was first used by Dixon [1901, p. 191], who considered operators of the form

$$A : (\xi_k) \mapsto \left(\sum_{k=1}^{\infty} \alpha_{hk} \xi_k \right).$$

Assuming that

$$q := \|A : l_\infty \to l_\infty\| = \sup_h \sum_{k=1}^{\infty} |\alpha_{hk}| < 1,$$

he verified the convergence of $x = y + Ay + A^2y + \cdots$ for all bounded scalar sequences $y = (\eta_k)$ and showed that x is the solution of $x - Ax = y$.

2.1.12 Geometric series of iterated kernels also appeared in papers of Plemelj [1904, p. 95] and Schmidt [1907b, pp. 162–164]. Given any continuous kernel K on the interval $a \le s,t \le b$, the iterated kernels K^n are defined by

$$K^1(s,t) := K(s,t) \quad \text{and} \quad K^{n+1}(s,t) := \int_a^b K^n(s,\xi)K(\xi,t)\,d\xi.$$

Induction yields

$$|K^{n+2}(s,t)| \le \left(\int_a^b |K(s,\sigma)|^2 d\sigma\right)^{1/2} \left(\int_a^b\int_a^b |K(\sigma,\tau)|^2 d\sigma\, d\tau\right)^{n/2} \left(\int_a^b |K(\tau,t)|^2 d\tau\right)^{1/2}.$$

Thus Schmidt's condition

$$q := \left(\int_a^b\int_a^b |K(s,t)|^2 ds\, dt\right)^{1/2} < 1$$

guarantees that the series

$$F(s,t) := \sum_{n=1}^{\infty} K^n(s,t)$$

is uniformly and absolutely convergent. The continuous kernel F satisfies the equation

$$F(s,t) - \int_a^b K(s,\xi)F(\xi,t)d\xi = K(s,t).$$

This means, in terms of operators, that $(I-K)(I+F)=I$. Similarly, $(I+F)(I-K)=I$. Hence $(I-K)^{-1}=I+F$.

We stress that Schmidt used the "wrong" norm, namely

$$\|K|\mathfrak{S}_2\| := \left(\int_a^b\int_a^b |K(s,t)|^2 ds\, dt\right)^{1/2}.$$

This quantity became extremely useful in the theory of operator ideals: see 4.10.1.2.

Subsequently, Hilb [1908, p. 86] proved a corresponding result for bilinear forms whose bound (in the sense of Hilbert) is less than 1.

We proceed with a quotation from [RIE, p. 93]:

Tant que $M_A < 1$, la série

$$E + A + A^2 + A^3 + \cdots$$

n'est qu'un développement formel de $(E-A)^{-1}$, analogue à celui de $(1-z)^{-1}$ en série entière.

Banach [1922, p. 160] observed that the **method of successive approximations** also works for non-linear operators. The upshot is **Banach's fixed point theorem** as formulated by Cacciopoli [1930, p. 799]:

Se in uno spazio funzionale metrico completo una trasformazione S converte due elementi aventi distanza d in due altri la cui distanza d' stia a d in un rapporto inferiore ad un numero fisso $\alpha < 1$, S ammette elemento unito ed uno sole, E; detto E' un altro elememto qualunque, si ha

$$E = \lim_{n\to\infty} S^n[E'],$$

essendo S^n la trasformazione risultante dall'applicare n volte consecutive la S.

2.2 Functionals and dual operators

2.2.1 We now deal with (bounded linear) **functionals** $\ell : X \to \mathbb{K}$. These are just the "operators" from a Banach space X into the underlying scalar field $\mathbb{K}$. The collection of these functionals, which will be denoted by X^*, becomes a Banach space under the **norm** $\|\ell\| := \sup\{|\ell(x)| : \|x\| \leq 1\}$. The symbols $\overline{X}$ (old-fashioned) and X' are also used. Here is a list of various names:

polarer Raum	:	Hahn [1927, p. 219],
transponierter Raum	:	Schauder [1930c, p.184],
espace conjugué	:	Schauder [1930c, p.184], [BAN, p. 188],
adjoint space	:	Alaoglu [1940, p. 252].

Nowadays, the terminology proposed by Bourbaki [1938, p. 1702] has been commonly accepted: **dual space**, or simply **dual**.

If $X^* = Y$, then X is called a **predual** of Y.

2.2.2 The main problem was to show that X^* is sufficiently large: for every non-zero element $x_0 \in X$ there exists a functional $\ell_0 \in X^*$ such that $\ell_0(x_0) \neq 0$. More precisely, it can be arranged that $\ell_0(x_0) = \|x_0\|$ and $\|\ell_0\| = 1$. This follows from the Hahn–Banach theorem, which will be discussed at full length in the next section.

2.2.3 Since X^* is a Banach space, the above construction can be repeated. This yields the **bidual** $X^{**} := (X^*)^*$.

Viewing $\ell(x)$ as a function of the variable ℓ, every element $x \in X$ induces a functional $K_X x : \ell \mapsto \ell(x)$ on X^*. The canonical operator $K_X : X \to X^{**}$, which was introduced by Hahn [1927, p. 219], is a metric injection: $\|K_X x\| = \|x\|$ for all $x \in X$.

Sometimes the use of the embedding K_X is rather cumbersome and unnecessary. For this reason, one frequently regards X just as a subspace of X^{**}.

2.2.4 Of particular significance is the case in which K_X maps X onto X^{**}. Then the relationship between X and X^* is symmetric. Banach spaces with this property were first exhibited by Hahn [1927, p. 220] under the name *reguläre Räume*. Nowadays, following Lorch [1939a], we call them **reflexive**. In the setting of sequence spaces, the phenomenon of non-reflexivity was discovered by Helly [1921, p. 80]:
Im allgemeinen kann nicht behauptet werden, daß sich $L(u)$ in der Form $L(u) = (u, q)$ darstellen läßt.
Details will be discussed in 2.2.8 and 2.3.3.

2.2.5 In order to emphasize the symmetry between X and X^*, functionals $\ell \in X^*$ are quite often denoted by x^*, and $\ell(x)$ is replaced by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. To the best of my knowledge, this notation goes back to [BOU$_{2a}$, p. 43] or [BOU$_{5b}$, Chap. IV, p. 48]. The intention is to emphasize the analogy with inner products; see 1.5.5.

2.2.6 Linear forms in infinitely many variables were already considered by Hilbert [1906a, p. 176]:
Eine lineare Form $L(x) = l_1x_1 + l_2x_2 + \cdots$ *ist dann und nur dann eine beschränkte Form, wenn die Summe der Quadrate ihrer Koeffizienten* $l_1^2 + l_2^2 + \cdots$ *endlich bleibt.*

Independently of each other, Fréchet [1907, p. 439] and Riesz [1907b, p. 1411] transferred this result to $L_2[a,b]$. In the words of the latter:
Pour chaque opération linéaire continue il existe une fonction k telle que la valeur de l'opération pour une fonction quelconque f est donnée par l'intégral du produit des fonctions f et k.

These results are usually expressed by writing

$$l_2^* = l_2 \quad \text{and} \quad L_2[a,b]^* = L_2[a,b].$$

The representation theorem for abstract non-separable Hilbert spaces had to wait for Löwig [1934a, p. 11] and Riesz [1934, p. 34]:
Für jede lineare Funktion $\ell(f)$ *gibt es ein eindeutig bestimmtes erzeugendes Element g, so daß* $\ell(f) = (f,g)$.

Riesz says: *Gewöhnlich stützt man den Beweis auf die Separabilität, d.i. auf das Vorhandensein einer abzählbaren überall dichten Teilmenge; von dieser Teilmenge geht man auf bekannte Weise zu einem vollständigen Orthogonalsystem und damit zur Darstellung des Raumes durch abzählbar viele Koordinaten über.*

The basic step in his new approach, which has already been described in 1.5.9, was adopted from the calculus of variations (Dirichlet principle).

Assigning to every element g the functional $\ell : f \mapsto (f|g)$ yields an isometric map R_H from the Hilbert space H onto its dual H^*, which is linear in the real case but conjugate linear in the complex case.

2.2.7 If $1 < p < \infty$, then the **dual exponent** p^* is defined by

$$1/p + 1/p^* = 1,$$

and **Hölder's inequality** [1889, p. 44, only an asymmetrical version] states that

$$\left| \int_a^b f(t)g(t)\,dt \right| \le \left(\int_a^b |f(t)|^p dt \right)^{1/p} \left(\int_a^b |g(t)|^{p^*} dt \right)^{1/p^*},$$

or in shorthand,

$$|\langle f,g\rangle| \le \|f|L_p\|\,\|g|L_{p^*}\| \quad \text{for all } f \in L_p[a,b] \text{ and } g \in L_{p^*}[a,b].$$

Maligranda [1998•] tells us *"Why Hölder's inequality should be called Roger's inequality."*

The formula $L_p[a,b]^* = L_{p^*}[a,b]$ was proved by Riesz [1909a, p. 475]:
Wird durch eine Vorschrift jeder Funktion $f(x)$ der Klasse $\left[L^{\frac{p}{p-1}}\right]$ eine Zahl A_f zugeordnet, und genügt diese Zuordnung den Forderungen:

$$A_{\mu_1 f_1 + \mu_2 f_2} = \mu_1 A_{f_1} + \mu_2 A_{f_2};$$
$$|A_f| \le M \left[\int_a^b |f(x)|^{\frac{p}{p-1}} dx \right]^{\frac{p-1}{p}}$$

wo die Schranke M nur von der Vorschrift abhängt, so gibt es eine bis auf eine Nullmenge eindeutig bestimmte Funktion $a(x)$ der Klasse $[L^p]$ derart, daß für alle $f(x)$

$$A_f = \int_a^b a(x) f(x)\, dx$$

ist.

The discrete counterpart of the preceding result goes back to Landau [1907]. For a proof of $l_p^* = l_{p^*}$ we refer to [BAN, pp. 67–68], where the symbol l_p (in fact: $l^{(p)}$) appeared for the first time.

The limit case $p = 1$ and $p^* = \infty$ is due to Steinhaus [1919, pp. 189, 193–194], who showed that $L_1[a,b]^* = L_\infty[a,b]$, where the right-hand Banach space consists of all essentially bounded scalar functions f and

$$\|f|L_\infty\| := \inf\{c \ge 0 : |f(t)| \le c \text{ almost everywhere}\}.$$

The right-hand expression is usually denoted by "ess-sup" or "vrai-sup."

As a discrete counterpart we have l_∞, the Banach space of all bounded scalar sequences with the sup-norm. The formula $l_1^* = l_\infty$ seems to be a part of the folklore.

2.2.8 The Banach space cs consists of all scalar sequences $x = (\xi_k)$ for which the series $\sum_{k=1}^\infty \xi_k$ is convergent. Its norm is given by

$$\|x|cs\| := \sup_n \left| \sum_{k=n}^\infty \xi_k \right|.$$

Via the bilinear form

$$\langle a, x\rangle = \sum_{k=1}^\infty \alpha_k \xi_k = \alpha_1 \sum_{k=1}^\infty \xi_k + \sum_{h=1}^\infty (\alpha_{h+1} - \alpha_h) \left(\sum_{k=h+1}^\infty \xi_k \right),$$

the dual cs^* can be identified with bv, the collection of all scalar sequences $a = (\alpha_k)$ such that the norm

$$\|a|bv\| := |\alpha_1| + \sum_{k=1}^\infty |\alpha_{k+1} - \alpha_k|$$

is finite. The letters *"cs"* and *"bv"* stand for *convergent series* and *bounded variation*, respectively. This example was constructed by Helly [1921, pp. 82–84] when he was looking for a non-reflexive Banach space. He observed that $f(a) := \lim_{k\to\infty} \alpha_k$ cannot be represented in the form $f(a) = \langle a,x\rangle$ with some $x \in cs$. Nowadays, this fact is obvious, since $(\xi_k) \leftrightarrow \left(\sum_{k=h}^{\infty} \xi_k\right)$ defines a metric isomorphism between cs and c_0, the Banach space of all null sequences. But in 1921 the elementary formula $c_0^{**} = l_\infty$ was still unknown.

2.2.9 Based on preliminary work of Hadamard [1903], the general form of functionals on $C[a,b]$ was discovered by Riesz [1909c]. Detailed proofs were given later; see [1911, p. 43].

Riesz's representation theorem:

Étant donnée l'opération linéaire $A[f(x)]$, on peut déterminer la fonction à variation bornée $\alpha(x)$ telle que pour toute fonction continue $f(x)$ on ait

$$A[f(x)] = \int_a^b f(x)\, d\alpha(x).$$

For the right-hand Stieltjes integral [1894], Riesz [1911, p. 37] showed that

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \le \text{maximum de } |f(x)| \times \text{variation totale de } \alpha(x).$$

This result has two decisive defects: the function $\alpha(x)$ is not uniquely determined, and extension to the multi-dimensional case turned out to be cumbersome. All trouble disappeared when Radon got the lucky idea to replace $\alpha(x)$ by a set function.

2.2.10 In the subsequent quotation from Radon [1913, pp. 1299, 1332–1333],

$$J := \{x = (\xi_k) : |\xi_1| \le M, \dots, |\xi_n| \le M\}$$

is a fixed interval of $\mathbb{R}^n$.

Es sei nun unter den in J enthaltenen Punktmengen eine Klasse T ausgezeichnet, welche folgenden Forderungen genügt:

a) Alle Intervalle in J gehören zu T.

b) Zugleich mit E_1 und E_2 gehören auch $E_1 \bullet E_2$ (Durchschnitt) *und $E_1 - E_2$* (Differenz) *zu T.*

c) Sind $E_1, E_2, \dots$ disjunkte Mengen von T in endlicher oder abzählbar unendlicher Anzahl, so gehört auch ihre Vereinigungsmenge $E_1 + E_2 + \cdots$ zu T.

Jeder Punktmenge E von T sei nun eine Zahl $f(E)$ zugeordnet. Dann heißt f eine **Mengenfunktion** *mit dem Definitionsbereiche T. Wir untersuchen insbesondere Mengenfunktionen, denen die Eigenschaft der* **absoluten Additivität** *zukommt.*

Darunter ist folgendes zu verstehen:

Sind $E_1, E_2, \dots$ disjunkte Mengen von T in endlicher oder abzählbar unendlicher Anzahl, so ist

$$f(E_1 + E_2 + \cdots) = f(E_1) + f(E_2) + \cdots,$$

wobei die rechts auftretende Reihe stets [absolut] *konvergent sein soll.*

Wir betrachten die Gesamtheit aller auf einer beschränkten, abgeschlossenen Menge E_0 stetigen Funktionen $F(P)$. Jeder dieser Funktionen sei eine Zahl $U(F)$ zugeordnet und es sein:

$$U(F_1+F_2) = U(F_1) + U(F_2),$$
$$\lim_{n=\infty} U(F_n) = U(F),$$

wenn F_n auf E_0 gleichmäßig gegen F konvergiert.

U heiße eine **stetige lineare Funktionaloperation** *und die kleinstmögliche Zahl N, so daß $|U(F)| \leq N$ sobald $|F| \leq 1$, ihre* **Maximalzahl**.

Jede lineare Funktionaloperation läßt sich durch eine Integraloperation darstellen, und zwar ist

$$U(F) = \int_{E_0} F\,df$$

und die Maximalzahl N von U hat den Wert

$$N = \int_{E_0} |df|.$$

Viewing E_0 as a subset of a suitable interval J, we have an absolutely additive set function f on a σ-ring T on J. Hence the preceding representation is not unique. However, uniqueness can easily be obtained by restriction to the Borel σ-algebra of E_0.

The n-dimensional set E_0 in Radon's approach can be replaced by a compact Hausdorff space or even by a completely regular topological space. These generalizations will be discussed in Section 4.6.

2.2.11 For the moment, we concentrate on a special case, qwhich was treated by Hildebrandt [1934]. He identified the dual of l_∞ with ba, the Banach space of all bounded additive scalar-valued functions μ defined on $\mathscr{P}(\mathbb{N})$, the power set of $\mathbb{N}$. The norm is given by the *total variation*

$$\|\mu|ab\| := \sup \sum_{k=1}^{n} |\mu(A_k)|,$$

where the right-hand supremum ranges over all finite families of mutually disjoint subsets $A_1, \ldots, A_n \in \mathscr{P}(\mathbb{N})$; see [DUN$_1^+$, p. 296].

The canonical map from l_1 into l_1^{**} associates with every sequence $(\mu_k) \in l_1$ the σ-additive set function

$$\mu(A) := \sum_{k \in A} \mu_k \quad \text{for all } A \in \mathscr{P}(\mathbb{N}).$$

The discovery of the relationship between set functions (measures) and linear functionals (integrals) is one of the most important achievements of Banach space theory.

2.2.12 We now deal with the concept of a **dual operator**, which goes back to Riesz [1909a, pp. 477–479]:
Über die Funktionaltransformation $T[f(x)]$, welche jeder Funktion der Klasse $[L^p]$ eine Funktion derselben Klasse zuordnet, setzen wir voraus, daß sie distributiv und in bezug auf den Exponenten p beschränkt sei.

Es bedeute nun $f(x)$ eine beliebige Funktion der Klasse $[L^p]$, $g(x)$ eine beliebige Funktion der Klasse $\left[L^{\frac{p}{p-1}}\right]$. Hält man in dem Integral

$$\int_a^b T[f(x)]g(x)\,dx$$

die Funktion $g(x)$ fest, so stellt es eine auf der Klasse $[L^p]$ lineare Funktionaloperation dar. Es gibt somit eine bis auf eine beliebige additive Nullfunktion wohlbestimmte Funktion $\psi(x)$ der Klasse $\left[L^{\frac{p}{p-1}}\right]$ derart, daß für jede Funktion $f(x)$

$$\int_a^b T[f(x)]g(x)\,dx = \int_a^b f(x)\psi(x)\,dx$$

ausfällt. Auf diese Art wird jeder Funktion $g(x)$ der Klasse $\left[L^{\frac{p}{p-1}}\right]$ eine Funktion $\mathfrak{T}[g(x)] = \psi(x)$ derselben Klasse zugeordnet. Die Transformation $\mathfrak{T}[g(x)]$ heißt die **Transponierte** *zur Transformation $T[f(x)]$.*

Riesz [1909a, p. 478] also showed that $\mathfrak{T}$ and T have the same norms: $M_{\mathfrak{T}} = M_T$.

Banach [1929, Part II, p. 235] and Schauder [1930c, p. 184] extended this definition to the abstract setting:
For every operator $T \in \mathfrak{L}(X,Y)$ there exists an operator $T^* \in \mathfrak{L}(Y^*,X^*)$ such that

$$\langle Tx, y^* \rangle = \langle x, T^*y^* \rangle \quad \text{whenever } x \in X \text{ and } y^* \in Y^*.$$

The formula $\|T^*\| = \|T\|$ is an immediate consequence of the Hahn–Banach theorem; see [BAN, p. 100].

As in the case of spaces, the naming was and is non-uniform:

Transponierte	: Riesz [1909a, p. 479], Schauder [1930c, p. 184],
opération adjointe	: Banach [1929, Part II, p. 235],
opération associée	: [BAN, p. 100],
opération conjuguée	: [BAN, p. 100], Schauder [1930c, p. 184]
dual operator	: rarely but in this book.

2.2.13 Independently of the Polish school, the concept of an adjoint transformation was also established by Hildebrandt [1931, p. 200]:
If T is a linear limited transformation on $\mathfrak{S}_1$ to $\mathfrak{S}_2$, and η_2 any element of $\mathfrak{T}_2$, the adjoint of $\mathfrak{S}_2$, then $(\eta_2, T(\xi_1))$ defines a linear limited operation of $\mathfrak{S}_1$, and is consequently expressible in the form (η_1, ξ_1), where η_1 belongs to $\mathfrak{T}_1$, the adjoint of $\mathfrak{S}_1$. We have so defined a transformation T^* on $\mathfrak{T}_2$ to $\mathfrak{T}_1$, which is obviously linear and limited and has the same modulus as T. The adjoint of T^* is T if $\mathfrak{S}_1$ is regular, otherwise it agrees with T on $\mathfrak{S}_1$, but may be on a space including $\mathfrak{S}_1$ to a space including $\mathfrak{S}_2$.

The very last observation concerning the **bidual** T^{**} was also made by Schauder [1930c, pp. 187–189]. In modern terminology, this fact can be expressed by a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle K_X}\downarrow & & \downarrow{\scriptstyle K_Y} \\ X^{**} & \xrightarrow[\quad T^{**} \quad]{} & Y^{**} \end{array} .$$

2.2.14 If $T:H\to K$ is an operator between Hilbert spaces, then instead of the dual operator $T^*:K^*\to H^*$ one considers the **adjoint operator** $T^\star:K\to H$, which is defined by the property that

$$(Tx|y) = (x|T^\star y) \quad \text{whenever } x\in H \text{ and } y\in K.$$

Of course, it always becomes clear from the context which of the two concepts is used. Thus my distinction between $*$ and $\star$ is just pedantry.

Using the isometries R_H and R_K from 2.2.6, we have

$$T^\star : K \xrightarrow{R_K} K^* \xrightarrow{T^*} H^* \xrightarrow{R_H^{-1}} H.$$

Hence T^* and $T^\star$ can be identified in the real case. In the complex case the formal difference looks as follows: if $T: l_2^n \to l_2^n$ is induced by the matrix (τ_{hk}), then T^* corresponds to the transposed (τ_{kh}), while $T^\star$ is generated by $(\overline{\tau}_{kh})$.

2.3 The moment problem and the Hahn–Banach theorem

2.3.1 Suppose that $g:[a,b]\to\mathbb{R}$ has bounded variation. Then

$$\gamma_h = \int_a^b t^h dg(t) \quad \text{with } h = 0,1,2,\dots$$

is called the h^{th} **moment**. This terminology stems from mechanics and is also used in probability theory. The Weierstrass approximation theorem implies that the associated measure μ is uniquely determined by its moments. The question, *What sequences* (γ_h) *are obtained in this way?* is referred to as the *algebraic moment problem.*

For $f\in L_2(\mathbb{T})$, the Fourier coefficients

$$\gamma_h(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}ht}dt \quad \text{with } h = 0,\pm1,\pm2,\dots$$

may be regarded as moments. Taking this point of view, the Fischer–Riesz theorem gives a complete answer to the *trigonometric moment problem.*

Last but not least, we consider a linear system in infinitely many unknowns:

$$\sum_{k=1}^{\infty} \alpha_{hk}\xi_k = \gamma_h \quad \text{for } h = 1,2,\ldots. \tag{2.3.1.a}$$

Choosing any Banach space X that contains all $a_h = (\alpha_{h1}, \alpha_{h2}, \ldots)$ and viewing $(\xi_1, \xi_2, \ldots)$ as a functional ℓ on X, we arrive at the abstract **moment problem**:

What conditions must be satisfied by the elements $x_1, x_2, \ldots \in X$ and the constants $\gamma_1, \gamma_2, \ldots$ in order to guarantee the existence of a functional $\ell \in X^*$ such that

$$\ell(x_h) = \gamma_h \quad \text{for } h = 1,2,\ldots\,? \tag{2.3.1.b}$$

It follows from

$$\Big|\sum_{h=1}^{n} \lambda_h \gamma_h\Big| = \Big|\ell\Big(\sum_{h=1}^{n} \lambda_h x_h\Big)\Big| \le \|\ell\| \Big\|\sum_{h=1}^{n} \lambda_h x_h\Big\|$$

that

$$\Big|\sum_{h=1}^{n} \lambda_h \gamma_h\Big| \le M \Big\|\sum_{h=1}^{n} \lambda_h x_h\Big\| \tag{2.3.1.c}$$

for $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, $n = 1,2,\ldots$, and some constant $M \ge 0$. In particular,

$$\sum_{h=1}^{n} \lambda_h x_h = o \quad \Rightarrow \quad \sum_{h=1}^{n} \lambda_h \gamma_h = 0. \tag{2.3.1.d}$$

The sufficiency of condition (2.3.1.c) was established step by step:

l_2	:	Schmidt [1908, p. 73],
$L_p[a,b]$, $1<p<\infty$	:	Riesz [1909a, p. 474],
$C[a,b]$	:	Riesz [1911, pp. 50–51], Helly [1912, p. 271],
l_p, $1<p<\infty$	:	[RIE, p. 61],
sequence spaces	:	Helly [1921, p. 81],
general case	:	Hahn [1927, p. 220], Banach [1929, Part I, p. 213].

In order to verify this result, one looks for functionals $\ell_n \in X^*$ that satisfy the equations

$$\ell_n(x_1) = \gamma_1, \ \ldots \ , \ell_n(x_n) = \gamma_n$$

and have minimal norms $\|\ell_n\| \le M$. In the separable case, the final solution is obtained by taking the limit of a weakly* convergent subsequence.

2.3.2 In the Hilbert space setting, Schmidt [1908, p. 73] proposed the following method. In view of (2.3.1.d), he assumed that the elements x_h are linearly independent. By the substitutions

$$x_i^\circ = \sum_{h=1}^{i} \tau_{ih} x_h \quad \text{and} \quad \gamma_i^\circ = \sum_{h=1}^{i} \tau_{ih} \gamma_h,$$

the equations $\ell(x_h) = \gamma_h$ pass into $\ell(x_i^\circ) = \gamma_i^\circ$. However, the invertible subdiagonal matrix (τ_{ih}) can be chosen such that the sequence (x_i°) becomes orthonormal. Thus the transformed as well as the original equations have a solution if and only if $(\gamma_i^\circ) \in l_2$.

2.3.3 The linear system (2.3.1.a),

$$\sum_{k=1}^{\infty} \alpha_{hk}\xi_k = \gamma_h \quad \text{for } h = 1,2,\dots,$$

admits a dual interpretation. Viewing the sequences $(\alpha_{h1}, \alpha_{h2}, \dots)$ as functionals ℓ_h on a Banach space X, we may look for some $x=(\xi_1, \xi_2, \dots)$ in X such that

$$\ell_h(x) = \gamma_h \quad \text{for } h = 1,2,\dots .$$

This requirement is stronger than the previous one, since condition (2.3.1.c) ensures only the existence of a solution in X^{**}. As an example, we consider the equations

$$\tfrac{1}{h}(\xi_1 + \dots + \xi_h) = 1 \quad \text{for } h = 1,2,\dots .$$

Then $x=(1,\dots,1,\dots)$ belongs to c_0^{**}, but not to c_0. This phenomenon was the reason for introducing the concept of reflexivity; see 2.2.4.

2.3.4 *Unsere Resultate bleiben auch für Systeme bestehen, die mehr als abzählbar viele Gleichungen enthalten.* This observation of Riesz [1909a, p. 475] leads to the **moment problem**

$$\ell(x_i) = \gamma_i \quad \text{for } i \in \mathbb{I}, \tag{2.3.4.a}$$

in which the index set $\mathbb{I}$ can be arbitrary. Suppose that

$$\left|\sum_{i\in\mathbb{F}} \lambda_i\gamma_i\right| \le M \left\|\sum_{i\in\mathbb{F}} \lambda_i x_i\right\|, \tag{2.3.4.b}$$

where $\mathbb{F}$ is any finite subset of $\mathbb{I}$ and $\lambda_i \in \mathbb{K}$. If X_0 denotes the linear span of the x_i's, then

$$\ell_0\left(\sum_{i\in\mathbb{F}} \lambda_i x_i\right) := \sum_{i\in\mathbb{F}} \lambda_i\gamma_i$$

defines a linear functional $\ell_0 : X_0 \to \mathbb{K}$ with $\|\ell_0\| \le M$. In this way, the moment problem is reduced to an extension problem.

Conversely, given any bounded linear functional $\ell_0 : X_0 \to \mathbb{K}$, then the requirement

$$\ell(x) = \ell_0(x) \quad \text{for } x \in X_0$$

is of the form (2.3.4.a) with the subspace X_0 as the underlying index set. Moreover, $\|\ell_0\| \le M$ passes into (2.3.4.b). Hence every solution of this moment problem is an extension ℓ of ℓ_0.

Conclusion: *moment problem = extension problem.*

2.3.5 Let X be any *real* linear space. A functional $p : X \to \mathbb{R}$ is said to be **sublinear** if

$$\begin{aligned} p(x_1 + x_2) &\le p(x_1) + p(x_2) && \text{for } x_1, x_2 \in X, \\ p(\lambda x) &= \lambda\, p(x) && \text{for } x \in X \text{ and } \lambda \ge 0. \end{aligned}$$

Hahn–Banach theorem (analytic version):

Every linear form $\ell_0 : X_0 \to \mathbb{R}$ that satisfies the inequality $\ell_0(x) \le p(x)$ on a subspace X_0 of X admits an extension $\ell : X \to \mathbb{R}$ with the same property:

$$\ell(x) \le p(x) \quad \text{for all } x \in X.$$

By means of Zorn's lemma, the proof can be reduced to the case $\operatorname{cod}(X_0) = 1$. Then there exists an element $x_0 \in X$ such that $X = \operatorname{span}\{X_0, x_0\}$, and it suffices to find a suitable value of $\ell(x_0)$. The required condition reads as follows:

$$\sup\{\ell_0(u) - p(u - x_0) : u \in X_0\} \le \ell(x_0) \le \inf\{p(v + x_0) - \ell_0(v) : v \in X_0\}.$$

In the special case of $C[a,b]$, these inequalities are due to Helly [1912, pp. 273–274]. Independently of each other, Hahn [1927, p. 217] and Banach [1929, Part I, p. 213] translated Helly's construction into the language of Banach spaces. Their major contribution is the generalization to the non-separable setting by transfinite induction. This technique had already been used by Banach [1923] in his solution of the measure problem for $\mathbb{R}$ and $\mathbb{R}^2$. Moreover, Banach [1929, Part II, p. 226] invented the sublinear functional p, whereas Helly and Hahn worked only with a norm.

Hahn [1927, p. 217]: *Sei $\mathfrak{R}_0$ ein vollständiger linearer Teilraum von $\mathfrak{R}$ und $f_0(x)$ eine Linearform in $\mathfrak{R}_0$ der Steigung M. Dann gibt es eine Linearform $f(x)$ in $\mathfrak{R}$ der Steigung M, die auf $\mathfrak{R}_0$ mit $f_0(x)$ übereinstimmt.*

2.3.6 Every *complex* normed linear space X can also be viewed as a **real** normed linear space; and the real part of every complex functional ℓ is a real functional u. Since $\ell(x) = u(x) + \mathrm{i}\, v(x)$ implies

$$u(\mathrm{i}x) + \mathrm{i}\, v(\mathrm{i}x) = \ell(\mathrm{i}x) = \mathrm{i}\,\ell(x) = -v(x) + \mathrm{i}\, u(x),$$

we have

$$\ell(x) = u(x) - \mathrm{i}\, u(\mathrm{i}x). \tag{2.3.6.a}$$

Thus the correspondence $\ell \leftrightarrow u$ is one-to-one. Moreover, $\|\ell\| = \|u\|$. Hence passing from ℓ_0 to u_0, extending u_0 to u, and returning from u to ℓ proves the Hahn–Banach theorem in the complex case:

If X_0 is a subspace of X, then every functional $\ell_0 \in X_0^*$ admits a norm-preserving extension $\ell \in X^*$.

This result is usually attributed to Murray [1936, p. 84], Bohnenblust/Sobczyk [1938, p. 91], and Sukhomlinov [1938, стр. 357]. However, formula (2.3.6.a) was already discovered by Löwig [1934a, p. 6].

2.3.7 It is admirable that Helly provided the basic tool that would have enabled Hahn and Banach to establish their extension theorem in the real and complex cases simultaneously.

Parallel to the reasoning described in 2.3.5, the domain of $\ell_0 : X_0 \to \mathbb{C}$ can be enlarged by adding one element x_0. In order to ensure the inequality

$$|\ell_0(x) + \lambda \ell(x_0)| \leq M \|x + \lambda x_0\| \quad \text{for } x \in X_0 \text{ and } \lambda \in \mathbb{C},$$

the required value $\ell(x_0)$ must belong to the intersection of all closed disks in the complex plane with centers $\ell_0(x)$ and radii $M \|x - x_0\|$, where x ranges over X_0.

This method works thanks to the famous **Helly theorem** [1921, pp. 76–77]:

Wenn in einem n-dimensionalen linearen Raum ein System konvexer Körper [convex bodies, 3.3.3.1] *gegeben ist, von welchen je* $n+1$ *mindestens einen Punkt gemein haben, so gibt es mindestens einen Punkt, der allen gegebenen Körpern angehört.*

Subsequently, Helly wrote:

Diesem Satz entsprechend ist im vorliegenden Fall bloß zu zeigen, daß je drei der betrachteten Kreise einen gemeinschaftlichen Punkt besitzen.

His final result [1921, p. 81] was a preliminary version of the complex Hahn–Banach theorem.

The preceding shows that Helly's theorem, which seems to be the best known of his results, grew out of functional analytic considerations. Though discovered in 1913, its publication was delayed by World War I; see Helly [1923]. In the meantime, Radon [1921] and König [1922] had presented their own proofs.

In recognition of Helly's merits it would be fair to speak of the **Helly**–Hahn–Banach theorem. But, I am afraid that this proposal comes too late. As an ersatz, let us pause for a standing ovation.

2.3.8 Further references: [BUS], Hochstadt [1980•], Narici/Beckenstein [1997•].

2.4 The uniform boundedness principle

2.4.1 The **principle of uniform boundedness** says the following.

Let $\mathscr{B}$ be a set of operators in $\mathfrak{L}(X,Y)$ such that $\|Tx\| \leq c(x)$ for all $T \in \mathscr{B}$ and $x \in X$, where the constants $c(x) > 0$ depend on x. Then $\mathscr{B}$ is uniformly bounded: $\|Tx\| \leq c\|x\|$ for all $T \in \mathscr{B}$ and $x \in X$. The constant $c > 0$ does not depend on x.

This fundamental theorem, which has various predecessors in classical analysis, is usually proved either by the **gliding hump method** 2.4.3 or with the help of **Baire's category theorem** 2.4.6.

2.4.2 In [1873, p. 578], du Bois-Reymond described a 2π-periodic continuous function $f \in C(\mathbb{T})$ that is not the uniform limit of its partial Fourier sums

$$f_n(s) := \sum_{|k| \leq n} \gamma_k(f) e^{iks}, \quad \text{where} \quad \gamma_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt.$$

Nowadays, this result is an immediate consequence of the principle of uniform boundedness. Indeed, we know from Lebesgue [LEB$_2$, pp. 86–87] that the constants

$$L_n := \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin\frac{(2n+1)t}{2}}{\sin\frac{t}{2}}\right| dt,$$

which bear his name, tend to ∞. Since L_n is the norm of the operator $S_n : f \mapsto f_n$, it follows that the sequence $(S_n f)$ cannot converge for all $f \in C(\mathbb{T})$.

For more information, the reader is referred to 5.6.4.4.

2.4.3 Here is a proof of the uniform boundedness principle based on the gliding hump method; see Lebesgue [1909]:

Suppose that $\mathscr{B}$ is unbounded. Then we alternately find $T_1, T_2, \ldots \in \mathscr{B}$ and $x_1, x_2, \ldots \in X$ such that

$$\tfrac{1}{4\cdot 3^n}\|T_n\| \geq \sum_{k<n} c(x_k) + n, \quad \|x_n\| \leq \tfrac{1}{3^n}, \quad \text{and} \quad \|T_n x_n\| \geq \tfrac{3}{4\cdot 3^n}\|T_n\|.$$

Let $x := \sum_{k=1}^{\infty} x_k$, and observe that

$$T_n x = \overbrace{T_n x_1 + \cdots + T_n x_{n-1}}^{\text{small}} + \overbrace{T_n x_n}^{\substack{\text{gliding}\\ \text{hump}}} + \overbrace{T_n x_{n+1} + \cdots}^{\text{small}} \ .$$

Actually, it follows from

$$\Big\|\sum_{k<n} T_n x_k\Big\| \leq \sum_{k<n}\|T_n x_k\| \leq \sum_{k<n} c(x_k)$$

and

$$\Big\|\sum_{k>n} T_n x_k\Big\| \leq \sum_{k>n}\|T_n x_k\| \leq \sum_{k>n}\tfrac{1}{3^k}\|T_n\| \leq \tfrac{1}{2\cdot 3^n}\|T_n\|$$

that

$$\begin{aligned}\|T_n x\| &\geq -\Big\|\sum_{k<n} T_n x_k\Big\| + \|T_n x_n\| - \Big\|\sum_{k>n} T_n x_k\Big\| \\ &\geq -\sum_{k<n} c(x_k) + \tfrac{3}{4\cdot 3^n}\|T_n\| - \tfrac{1}{2\cdot 3^n}\|T_n\| \geq n,\end{aligned}$$

which is a contradiction.

2.4.4 The uniform boundedness principle implies Landau's theorem [1907]:
If $\sum_{k=1}^{\infty}\alpha_k\xi_k$ converges for all sequences $(\xi_k)\in l_p$, then $(\alpha_k)\in l_{p^*}$.
Indeed, letting $\ell_n(x) := \sum_{k=1}^{n}\alpha_k\xi_k$ for $x = (\xi_k)\in l_p$, we get

$$\Big(\sum_{k=1}^{n}|\alpha_k|^{p^*}\Big)^{1/p^*} = \|\ell_n|l_{p^*}\| \leq c \quad \text{for } n = 1, 2, \ldots \ .$$

Another result along these lines goes back to Hellinger–Toeplitz [1910, pp. 321–322]: The infinite matrix (α_{hk}) defines an operator

$$A : (\xi_k) \mapsto \Big(\sum_{k=1}^{\infty} \alpha_{hk} \xi_k \Big)$$

on l_2 if and only if the double sequence $\Big(\sum_{h=1}^{m} \sum_{k=1}^{n} \eta_h \alpha_{hk} \xi_k \Big)$ is bounded for all $(\xi_k), (\eta_h) \in l_2$.

We also mention a criterion of Toeplitz [1911b] that characterizes *regular methods of summability*; see 6.9.11.3. Last but not least, the gliding hump method was applied by Schur [1921, pp. 88–90] when he proved that every weakly convergent sequence in l_1 is even norm convergent; see 3.5.3.

2.4.5 In [1912, p. 268], Helly established a uniform boundedness principle for $C[a,b]$ whose proof easily extends to general Banach spaces. Abstract versions were given by Banach [1922, p. 157], Hahn [1922, p. 6], and Hildebrandt [1923, p. 311]. When reading the original version of the Banach/Steinhaus paper [1927], Saks observed that the gliding hump method can be replaced by Baire's category theorem. As a consequence, Banach/Steinhaus substituted the simpler proof of Saks for their own.

2.4.6 We call a subset of a topological space **nowhere dense** or **rare** if its closure contains no inner point. Every countable union of such subsets is said to be of **first category** or **meager**. All other subsets are of **second category**.

Baire's category theorem says that every complete metric space is of second category.

More explicitly: if a complete metric space is the union of a sequence of closed subsets, then at least one of them contains an inner point. For the real line, this result is due to Baire [1899, p. 65], who used it in the classification of real-valued functions.

2.4.7 The basic idea in the proof of Baire's category theorem goes back to Osgood [1897, pp. 159–164], who discovered the following result.

Let $\mathscr{B}$ be a set of continuous scalar functions on $\mathbb{R}$ that is pointwise bounded: $|f(t)| \le c(t)$ for all $f \in \mathscr{B}$ and $t \in \mathbb{R}$. Then $\mathscr{B}$ is uniformly bounded on some interval: $|f(t)| \le c$ for all $f \in \mathscr{B}$ and $a < t < b$.

Assume the contrary and fix any interval (a_1, b_1). We find $f_1 \in \mathscr{B}$ and $t_1 \in (a_1, b_1)$ such that $|f_1(t_1)| > 1$. By continuity, there is an interval $[a_2, b_2] \subset (a_1, b_1)$ containing t_1 and such that $|f_1(t)| > 1$ for all $t \in [a_2, b_2]$. Proceeding in this way, we get

$$t_n \in (a_n, b_n) \quad \text{and} \quad |f_n(t)| > n \quad \text{for all} \quad t \in [a_{n+1}, b_{n+1}] \subset (a_n, b_n).$$

Additionally, it may be arranged that $(b_{n+1} - a_{n+1}) \le \frac{1}{2}(b_n - a_n)$. Then (t_n) becomes a Cauchy sequence whose limit t belongs to all intervals $[a_n, b_n]$. Hence $|f_n(t)| > n$ for $n = 1, 2, \dots$, which is a contradiction.

Replacing intervals by balls, the same proof works in complete metric spaces.

2.4.8 The Banach–Steinhaus–Saks approach (see 2.4.5) uses the absolutely convex and closed set

$$B := \{x \in X : \|Tx\| \le 1 \text{ for all } T \in \mathscr{B}\}.$$

Since $\mathscr{B}$ is pointwise bounded, B becomes absorbing: $X = \bigcup_{n=1}^{\infty} nB$. Consequently, by Baire's category theorem, B contains an inner point, and the principle of uniform boundedness follows immediately.

2.4.9 Gelfand [1936], [1938, p. 240] proved that lower semi-continuous semi-norms on a Banach space are even continuous; see also Orlicz [1929, Part I, footnote on p. 9]. Applying this result to

$$p(x) := \sup\{\|Tx\| : T \in \mathscr{B}\} \quad \text{for all } x \in X$$

yields another proof of the principle of uniform boundedness.

2.5 The closed graph theorem and the open mapping theorem

2.5.1 A linear map $T : D(T) \to Y$, defined on a (possibly proper) subspace of X, is said to be **closed** if its **graph**

$$G(T) := \{(x, Tx) : x \in D(T)\}$$

is closed in $X \times Y$ with respect to coordinatewise convergence. In this case, $D(T)$ becomes a Banach space under the new norm

$$\|x\|_T := \|x\| + \|Tx\|,$$

which makes T a bounded linear operator. This concept, which goes back to von Neumann [1930a, p. 70], plays only a minor role in our presentation. However, it is of great importance in the theory of differential operators and semi-groups.

2.5.2 We now discuss a fundamental result, which has many facets.

closed graph theorem, [BAN, p. 41]:
Every closed linear map T from *all* of a Banach space X into Y is continuous.

bounded inverse theorem, [BAN, p. 41]:
Every *one-to-one* operator T from X *onto* Y possesses a bounded inverse.

open mapping theorem, [BAN, pp. 38–40]:
Every operator T from X *onto* Y takes open sets into open sets.

Once we have proved one of the above conclusions, the others follow immediately.

Assume, for example, that $T : X \to Y$ is closed. Then $(x, Tx) \mapsto x$ is a one-to-one map from $G(T)$ onto X. Since the graph $G(T)$ is a Banach space under the norm $\|(x,y)\| := \|x\| + \|y\|$, applying the bounded inverse theorem yields a constant $c > 0$ such that $\|(x, Tx)\| := \|x\| + \|Tx\| \le c\|x\|$. Hence T is bounded.

2.5.3 The bounded inverse theorem is due to Banach [1929, Part II, p. 238]. However, his original proof, based on weak convergence, was quite involved. Soon afterward, Schauder [1930a] found a significant simplification. He combined Baire's category theorem 2.4.6 with the following observation:

If o is an inner point of $\overline{T(B_X)}$, then o is an inner point of $T(B_X)$.

Indeed, fix $\varepsilon > 0$ such that $\varepsilon B_Y \subseteq \overline{T(B_X)}$. Then $\frac{\varepsilon}{2^n} B_Y \subseteq \overline{T(\frac{1}{2^n} B_X)}$. Given $y \in \frac{\varepsilon}{2} B_Y$, we can find a sequence (x_n) in B_X such that

$$\Big\| y - \sum_{k<n} \tfrac{1}{2^k} T x_k \Big\| \leq \tfrac{\varepsilon}{2^n}.$$

Hence $y = \sum_{k=1}^{\infty} \frac{1}{2^k} T x_k \in T(B_X)$.

The open mapping theorem (Satz von der Gebietsinvarianz) is due to Schauder [1930a, p. 6].

2.5.4 The bounded inverse theorem reflects a fact that is highly important for applications in the theory of operator equations $Tx = y$. If T is one-to-one and onto, then it follows that small perturbations of y yield small perturbations of the solution x. The problem is *well-posed*.

2.5.5 Suppose that X is a Banach space under two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Banach [1929, Part II, p. 239] observed that these norms are equivalent precisely when they are comparable: if there is $c_1 > 0$ such that $\|\cdot\|_2 \leq c_1 \|\cdot\|_1$, then there is $c_2 > 0$ such that $\|\cdot\|_1 \leq c_2 \|\cdot\|_2$. The assumption about comparability is essential. Using a result of Löwig [1934b] and Lacey [1973], which says that every Hamel basis of an infinite-dimensional separable Banach space has cardinality $2^{\aleph_0}$, we may construct an algebraic but non-continuous isomorphism between, say, l_1 and l_2.

By the way, Baire's category theorem implies that an infinite-dimensional Banach space cannot have a countable Hamel basis, since it would be the union of a sequence of finite-dimensional subspaces without inner points.

2.5.6 We conclude this section by pointing out another interesting feature. Let $F_{\mathrm{our}}^{2\pi}$ be the map that takes every function f to the sequence of its Fourier coefficients

$$\gamma_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} dt.$$

Then the Riemann–Lebesgue lemma says that $F_{\mathrm{our}}^{2\pi} : L_1(\mathbb{T}) \to c_0(\mathbb{Z})$. Looking at the Dirichlet kernels 5.6.4.4, we have

$$\|D_n | L_1(\mathbb{T})\| \asymp \log n \quad \text{and} \quad \|\gamma_k(D_n) | c_0(\mathbb{Z})\| = 1.$$

Thus the periodic Fourier transform $F_{\mathrm{our}}^{2\pi}$ cannot be onto.

2.6 Riesz–Schauder theory

A substantial theory of equations $Sx = y$ is known only for "large" operators, which means that S should be close to an isomorphism. This is in particular true for operators of the form $I - T$, where T is "small." A main goal of functional analysis was to give the concepts *large* and *small* a precise sense. The Neumann series provides a first but trivial answer: $I - T$ is invertible whenever $\|T\| < 1$.

2.6.1 Completely continuous operators

2.6.1.1 Let $x_1 = (\xi_k^{(1)}), x_2 = (\xi_k^{(2)}), \dots$, and $x = (\xi_k)$ be sequences in the closed unit ball of l_2. Then $x_n \xrightarrow{w} x$ means that

$$\lim_{n\to\infty} \xi_k^{(n)} = \xi_k \quad \text{for } k = 1, 2, \dots ,$$

and we say that the sequence (x_n) tends *weakly* to x; see Subsection 3.1. This type of convergence was used by Hilbert [1906a, p. 200] in his definition of **Vollstetigkeit**:

Wir nennen eine Funktion $F(\xi_1, \xi_2, \dots)$ der unendlichvielen Variablen $\xi_1, \xi_2, \dots$ für ein bestimmtes Wertesystem derselben vollstetig, wenn die Werte von $F(\xi_1 + \varepsilon_1, \xi_2 + \varepsilon_2, \dots)$ gegen den Wert $F(\xi_1, \xi_2, \dots)$ konvergieren, wie man auch immer $\varepsilon_1, \varepsilon_2, \dots$ für sich zu Null werden läßt. Dabei sind die Variablen stets an die Ungleichung $|\xi_1|^2 + |\xi_2|^2 + \cdots \le 1$ gebunden.

Similarly, a bilinear form

$$A(x,y) = \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \eta_h \alpha_{hk} \xi_k \quad \text{with } x = (\xi_k) \text{ and } y = (\eta_h)$$

defined on $B_{l_2} \times B_{l_2}$ is said to be **completely continuous** if

$$x_n \xrightarrow{w} x \quad \text{and} \quad y_n \xrightarrow{w} y \quad \text{imply} \quad A(x_n, y_n) \to A(x,y).$$

2.6.1.2 Hilbert [1906a, p. 218] made the following observation:

Eine Bilinearform $A(x,y)$ ist stets vollstetig, wenn die Summe der Quadrate der Koeffizienten endlich bleibt.

In the later development, these so-called **Hilbert–Schmidt matrices**,

$$\sum_{h=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{hk}|^2 < \infty,$$

played an important role; see 4.10.1.2.

2.6.1.3 This is Hilbert's main result [1906a, pp. 219, 223]:

Wenn $A(x,y)$ eine vollstetige Bilinearform der unendlichvielen Variablen $\xi_1,\xi_2,\dots;$ $\eta_1,\eta_2,\dots$ ist, so haben gewiß entweder die unendlichvielen Gleichungen

$$\begin{array}{r} (1+\alpha_{11})\xi_1 + \qquad \alpha_{12}\,\xi_2 + \cdots = \alpha_1 \\ \alpha_{21}\,\xi_1 + (1+\alpha_{22})\xi_2 + \cdots = \alpha_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array} \tag{2.6.1.3.a}$$

für alle möglichen Größen $\alpha_1,\alpha_2,\dots$ mit konvergenter Quadratsumme eine eindeutig bestimmte Lösung $\xi_1,\xi_2,\dots$ mit konvergenter Quadratsumme –
oder die entsprechenden homogenen Gleichungen

$$\begin{array}{r} (1+\alpha_{11})\xi_1 + \qquad \alpha_{12}\,\xi_2 + \cdots = 0 \\ \alpha_{21}\,\xi_1 + (1+\alpha_{22})\xi_2 + \cdots = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

lassen eine Lösung mit der Quadratsumme 1 *zu.*

Es kommen also dem Gleichungssystem (2.6.1.3.*a*) *alle wesentlichen Eigenschaften eines Systems von endlichvielen Gleichungen mit ebensovielen Unbekannten zu.*

Hilbert's proof was laborious, and he used "Hilbert space techniques," which could not be extended to operators on Banach spaces.

2.6.2 Finite rank operators

2.6.2.1 We refer to $T\in\mathfrak{L}(X,Y)$ as a **finite rank operator** if $M(T) := \{Tx : x\in X\}$ is finite-dimensional:

$$\operatorname{rank}(T) := \dim\big[M(T)\big].$$

The ill-suited term **degenerate operator** is also in use; see [HIL$^+$, p. 48].

If $y_1,\dots,y_n$ is any basis of $M(T)$, then for every $x\in X$ there exist coefficients $\eta_1,\dots,\eta_n$ such that

$$Tx = \sum_{k=1}^{n} \eta_k y_k,$$

and inequality (1.4.3.a) implies that the linear forms $x_k^* : x \mapsto \eta_k$ are bounded. Hence an operator has finite rank if and only if it can be written in the form

$$Tx = \sum_{k=1}^{n} \langle x, x_k^*\rangle y_k \quad \text{for all } x\in X, \tag{2.6.2.1.a}$$

where $x_1^*,\dots,x_n^*\in X^*$ and $y_1,\dots,y_n\in Y$. I have no idea to whom this result should be attributed. Without proof such a representation was used by Schauder [1930c, p. 190].

It also appears in [BAN, p. 96]. However, kernels of the form

$$K(s,t) = \sum_{k=1}^{n} f_k(s)g_k(t)$$

were considered already by Goursat [1907, p. 163] and Schmidt [1907b, p. 164]. In the early 1940s, it became common to write formula (2.6.2.1.a) in the language of tensor products:

$$T = \sum_{k=1}^{n} x_k^* \otimes y_k. \tag{2.6.2.1.b}$$

The collection of all finite rank operators $T: X \to Y$ will be denoted by $\mathfrak{F}(X,Y)$. We write $\mathfrak{F}(X)$ instead of $\mathfrak{F}(X,X)$.

2.6.2.2 Let $T \in \mathfrak{F}(X)$ and consider any representation

$$T = \sum_{k=1}^{n} x_k^* \otimes x_k$$

with $x_1^*, \dots, x_n^* \in X^*$ and $x_1, \dots, x_n \in X$. Then, given $a \in X$, the equation $x - Tx = a$ passes into

$$x - \sum_{k=1}^{n} \langle x, x_k^* \rangle x_k = a.$$

Hence

$$\langle x, x_h^* \rangle - \sum_{k=1}^{n} \langle x, x_k^* \rangle \langle x_k, x_h^* \rangle = \langle a, x_h^* \rangle \quad \text{for } h = 1, \dots, n.$$

Putting $\xi_h := \langle x, x_h^* \rangle$, $\alpha_h := \langle a, x_h^* \rangle$, and $\tau_{hk} := \langle x_k, x_h^* \rangle$ yields

$$\xi_h - \sum_{k=1}^{n} \tau_{hk} \xi_k = \alpha_h \quad \text{for } h = 1, \dots, n.$$

Conversely, if $\xi_1, \dots, \xi_n$ solve the preceding equations, then

$$x := a + \sum_{k=1}^{n} \xi_k x_k$$

is a solution of $x - Tx = a$. In this way, we obtain a problem of linear algebra.

The preceding method of reduction was the starting point of the principle of related operators; see 6.4.3.2.

2.6.3 Approximable operators

2.6.3.1 An operator $T \in \mathfrak{L}(X,Y)$ is called **approximable** if it can be approximated by a sequence of finite rank operators $A_n \in \mathfrak{F}(X,Y)$. That is, $\lim_{n\to\infty} \|T - A_n\| = 0$.

The collection of all approximable operators $T: X \to Y$ will be denoted by $\overline{\mathfrak{F}}(X,Y)$. We write $\overline{\mathfrak{F}}(X)$ instead of $\overline{\mathfrak{F}}(X,X)$. In the abstract setting, such operators were mentioned for the first time in an address of Hildebrandt [1931]; see 2.6.6.4.

2.6.3.2 Let $T \in \overline{\mathfrak{F}}(X)$ and choose any $A \in \mathfrak{F}(X)$ such that the norm of $Q := T - A$ is less than 1. Then $I - Q$ is invertible by Neumann's series, and $x - Tx = a$ passes into $(x - Qx) - Ax = a$. Therefore

$$x - (I - Q)^{-1}Ax = (I - Q)^{-1}a.$$

Since $(I - Q)^{-1}A$ has finite rank, we can apply the process described in 2.6.2.2. Thus the original equation is equivalent to a finite linear system.

Schmidt [1907b, p. 165] developed this technique for integral operators induced by continuous kernels. This is the reason why Hellinger/Toeplitz [HEL$^+$, p. 1377] use the term *Schmidtsches Abspaltungsverfahren*.

2.6.3.3 In fact, the method should be attributed to Dixon [1901]. His main theorem deals with so-called *regular* matrices $A = (\alpha_{hk})$, which are defined by the property

$$\sum_{k=1}^{\infty} \sup_h |\alpha_{hk}| < \infty;$$

compare with 5.7.3.4 and 6.5.2.8. Choose some m such that

$$\sup_h \sum_{k=m+1}^{\infty} |\alpha_{hk}| \le \sum_{k=m+1}^{\infty} \sup_h |\alpha_{hk}| < 1,$$

and let

$$x = (\overbrace{\xi_1, \ldots, \xi_m}^{x_0}, \overbrace{\xi_{m+1}, \ldots}^{x_1}) \quad \text{and} \quad a = (\overbrace{\alpha_1, \ldots, \alpha_m}^{a_0}, \overbrace{\alpha_{m+1}, \ldots}^{a_1}).$$

Then, with the obvious meaning of

$$I = \begin{pmatrix} I_0 & O \\ O & I_1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

the equation $x - Ax = a$ splits into

$$(I_0 - A_{00})x_0 - A_{01}x_1 = a_0 \quad \text{and} \quad -A_{10}x_0 + (I_1 - A_{11})x_1 = a_1.$$

By 2.1.11, the inverse $(I_1 - A_{11})^{-1}$ exists as an operator on l_∞. Solving the right-hand equation for x_1 yields

$$x_1 = (I_1 - A_{11})^{-1}(a_1 + A_{10}x_0).$$

Hence the left-hand equation turns into

$$[I_0 - A_{00} - A_{01}(I_1 - A_{11})^{-1}A_{10}]x_0 = a_0 + A_{01}(I_1 - A_{11})^{-1}a_1,$$

which is a linear system in m unknowns. In this way, an infinite-dimensional problem is reduced to a finite-dimensional one.

2.6.3.4 Riesz [RIE, pp. 97–105] applied Dixon's method to operators in l_2. In a first step he translated the concept of *Vollstetigkeit* into the language of operators:
$T \in \mathfrak{L}(l_2)$ is called **completely continuous** if every weakly convergent sequence is mapped to a norm convergent sequence; [RIE, p. 96]. Most important is the fact that the completely continuous operators in Hilbert spaces are just the approximable ones; [RIE, p. 113, footnote]. The summary of his results reads as follows.

[RIE, p. 105]: *Lorsque la substitution A est complètement continue, les systèmes infinis*

$$\xi_h - \sum_{k=1}^{\infty} \alpha_{hk}\xi_k = \alpha_h \quad \textit{et} \quad \eta_k - \sum_{h=1}^{\infty} \eta_h \alpha_{hk} = \beta_k$$

conservent les propriétés essentielles des systèmes finis qui contiennent le même nombre d'équations que d'inconnues.

This is just Hilbert's formulation quoted in 2.6.1.3, but with the improvement that the transposed equation is considered simultaneously.

2.6.3.5 In order to define *Vollstetigkeit* for operators between Banach spaces, we first extend the concept of **weak convergence**; see Subsection 3.1:

$$x_n \xrightarrow{w} x \quad \Leftrightarrow \quad \lim_{n\to\infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle \quad \text{for all } x^* \in X^*.$$

Now it makes sense to say that an operator $T \in \mathfrak{L}(X,Y)$ is **completely continuous** if it carries every weakly convergent sequence to a norm convergent sequence. The collection of all completely continuous operators $T : X \to Y$ will be denoted by $\mathfrak{V}(X,Y)$. We write $\mathfrak{V}(X)$ instead of $\mathfrak{V}(X,X)$.

Schur [1921, pp. 88–90] showed that in l_1, weak convergence and norm convergence coincide; see 2.4.4. Hence $\mathfrak{V}(l_1) = \mathfrak{L}(l_1)$. Thus Hilbert's theory of operators in $\mathfrak{V}(l_2)$ does not hold for completely continuous operators on arbitrary Banach spaces. Consequently, a "better" generalization of complete continuity is required.

2.6.4 Compact operators

The most fascinating paper of Banach space theory was written by FRIGYES RIESZ and published in Acta Mathematica [1918]. He finished his studies on January 19, 1916. There is also a Hungarian version with the title *Linearis függvényegyenletekről*, which can be found in volume II of the collected works. Riesz gave a **determinant-free** approach to Fredholm's theory of integral equations. The latter will be discussed in Subsection 6.5.2.

Though presented in a classical style, the paper contains the spectral theory of compact operators in a perfect form. Only in [1930c] did Schauder add the missing part concerned with duality. This is the reason why we now use the name

Riesz–Schauder theory.

2.6.4.1 In this paragraph, A denotes a linear mapping of a linear space X into itself. All considerations are purely algebraic. Let

$$N_k(A) := \{x \in X : A^k x = o\} \quad \text{and} \quad M_h(A) := \{A^h x : x \in X\}.$$

Then

$$\{o\} = N_0(A) \subseteq N_1(A) \subseteq N_2(A) \subseteq \cdots \quad \text{and} \quad \cdots \subseteq M_2(A) \subseteq M_1(A) \subseteq M_0(A) = X.$$

If there is an index k such that $N_k(A) = N_{k+1}(A)$, then A is said to have **finite ascent**. Similarly, A has **finite descent** if $M_h(A) = M_{h+1}(A)$ for some h. In the case that this happens simultaneously, equality occurs at the same place: we find an index $d(A)$ such that

$$d(A) \le k \Leftrightarrow N_k(A) = N_{k+1}(A) \quad \text{and} \quad d(A) \le h \Leftrightarrow M_h(A) = M_{h+1}(A).$$

Moreover, X is the direct sum of the A-invariant subspaces $M_{d(A)}(A)$ and $N_{d(A)}(A)$. This is the **Riesz decomposition**

$$X = M_{d(A)}(A) \oplus N_{d(A)}(A),$$

which splits X into a regular and a singular part:

$$A \text{ is invertible on } M_{d(A)}(A) \text{ and nilpotent on } N_{d(A)}(A).$$

2.6.4.2 Taking into account the observation made in 2.6.3.5, Riesz [1918, p. 74] successfully modified Hilbert's definition of complete continuity:

Eine lineare Transformation heisse **vollstetig**, *wenn sie jede beschränkte Folge in eine kompakte überführt.*

This terminology was used for many years. Since 1950, on the suggestion of Hille [HIL, p. 14], "vollstetig" is mostly replaced by "compact." Later on, the old term *completely continuous* obtained a new meaning; see 2.6.3.5.

An operator $T \in \mathfrak{L}(X,Y)$ is called **compact** if every bounded sequence (x_n) in X has a subsequence (x_{n_i}) such that (Tx_{n_i}) is norm convergent in Y. The collection of all compact operators $T : X \to Y$ will be denoted by $\mathfrak{K}(X,Y)$. We write $\mathfrak{K}(X)$ instead of $\mathfrak{K}(X,X)$.

An operator $T \in \mathfrak{L}(X,Y)$ is compact if and only if it maps the closed unit ball B_X into a precompact subset of Y, which means that for every $\varepsilon > 0$, there exist elements $y_1, \dots, y_n \in Y$ such that

$$T(B_X) \subseteq \bigcup_{k=1}^{n} \{y_k + \varepsilon B_Y\}.$$

2.6.4.3 Letting $T \in \mathfrak{K}(X)$, we now present a summary of the *Riesz theory*.

- The operator $I - T$ has finite ascent and finite descent.
- The null spaces $N_k(I-T)$ are finite-dimensional.
- The ranges $M_h(I-T)$ are closed and finite-codimensional.
- The projections associated with the Riesz decomposition are bounded.
- The restriction of $I-T$ to $M_{d(I-T)}(I-T)$ is an isomorphism.

Hence all bad properties of $I-T$, if there exist any, are concentrated in the finite-dimensional subspace $N_{d(I-T)}(I-T)$, whose members are called **root vectors**.

This is a typical result of pure mathematics, which tells us that *in principle*, the equation $x-Tx=a$ can be reduced to a linear system in finitely many unknowns. But in contrast to the *Abspaltungsverfahren* 2.6.3.2, no concrete method is given.

I stress that the proofs are based mainly on the Riesz lemma 1.4.4 combined with compactness arguments. The theory of compact operators is a convincing example that deep and important mathematics can be–or should I say *must be*–elegant.

2.6.4.4 The main goal of Riesz was a determinant-free approach to Fredholm's theory of integral operators induced by continuous kernels on the Banach space $C[a,b]$. Most of his results and methods are independent of this special setting. However, there occurred an unpleasant exception, the **Fredholm alternative**.

Riesz [1918, pp. 95–96]: *Entweder besitzen die Gleichungen*

$$f(s)-\int_a^b K(s,t)f(t)\,dt=g(s),\quad \mathfrak{f}(t)-\int_a^b K(s,t)\mathfrak{f}(s)\,ds=\mathfrak{g}(t)$$

für alle g *und* $\mathfrak{g}$ *je eine eindeutig bestimmte Lösung, oder, wenn dies nicht der Fall ist, dann haben die entsprechenden homogenen Gleichungen ausser der identisch verschwindenden noch weitere Lösungen.*

Die weiteren Fredholm'schen Sätze, nämlich das Übereinstimmen der Anzahl der linear unabhängigen Lösungen der beiden homogenen Gleichungen und die durch diese Nullösungen ausgedrückten Bedingungen für die Lösbarkeit der inhomogenen Gleichungen erhält man durch einen sehr einfachen Kunstgriff (due to Hurwitz [1912]).

Riesz was not able to express the relationship between the transposed maps

$$f(t)\mapsto\int_a^b K(s,t)f(t)\,dt \quad\text{and}\quad \mathfrak{f}(s)\mapsto\int_a^b K(s,t)\mathfrak{f}(s)\,ds$$

in an abstract form. This could have been the reason why he continued working in $C[a,b]$. Only after the discovery of the Hahn–Banach theorem did duality provide the way out.

2.6.4.5 Hildebrand [1928] made the adventurous attempt to determine the dimension of the null space $\{x^*\in X^*: x^*-T^*x^*=o\}$ without knowing what X^* and T^* could be. Believe it or not, he succeeded somehow.

Indeed, the dimension of $\{x\in X: x-Tx=o\}$ is the maximum of rank(P), where P ranges over all projections such that $TP=P$. Hence a dual concept is obtained by taking the maximum of rank(Q), where Q ranges over all projections such that $QT=Q$. Obviously, this is just the codimension of $\{x-Tx: x\in X\}$. Considerations like these became important in the theory of Fredholm operators; see Subsection 5.2.2.

2.6.4.6 The concluding result is **Schauder's theorem**; see [1930c, pp. 185, 187]:

An operator $T : X \to Y$ is compact if and only if so is $T^* : Y^* \to X^*$.

Based on this fundamental result, Schauder [1930c, pp. 189–193] was able to complete the spectral theory of compact operators $T \in \mathfrak{K}(X)$.

- The homogeneous equations $x - Tx = \mathrm{o}$ and $x^* - T^*x^* = \mathrm{o}$ have the same finite number of linearly independent solutions.
- The inhomogeneous equation $x - Tx = a$ has a solution if and only if $\langle a, x^* \rangle = 0$ whenever $x^* - T^*x^* = \mathrm{o}$.
- The inhomogeneous equation $x^* - T^*x^* = a^*$ has a solution if and only if $\langle x, a^* \rangle = 0$ whenever $x - Tx = \mathrm{o}$.

2.6.5 Resolvents, spectra, and eigenvalues

2.6.5.1 Letting $T \in \mathfrak{L}(X)$, I contrast the following concepts, which are equivalent with respect to the substitution $\zeta = 1/\lambda$.

$\lambda I - T$	$I - \zeta T$
$R(\lambda, T) = (\lambda I - T)^{-1}$ resolvent	$F(\zeta, T) = T(I - \zeta T)^{-1}$ Fredholm resolvent
$\lambda x = Tx$ λ: eigenvalue $x \neq \mathrm{o}$: eigenvector	$x = \zeta Tx$ ζ: eigenvalue $x \neq \mathrm{o}$: eigenvector

The appropriate place of the complex parameter depends on the problem under discussion. In dealing with differential operators on Banach spaces or self-adjoint operators on Hilbert spaces the left-hand column is used. However, in Fredholm's theory of integral equations one would get a singularity at 0. Passing from $\lambda I - T$ to $I - \zeta T$ spirits away this unpleasant phenomenon: $0 \to \infty$; see also 6.5.6.4.

To explain the advantage of the Fredholm resolvent, I first stress that operators on $C[a, b]$ may fail to be induced by continuous kernels. This is, in particular, the case for the inverse of

$$f(t) \mapsto f(s) - \zeta \int_a^b K(s,t) f(t)\, dt.$$

On the other hand, $F(\zeta, K_{\mathrm{op}}) = K_{\mathrm{op}}(I - \zeta K_{\mathrm{op}})^{-1}$ can always be obtained from the resolvent kernel (*lösende Funktion*, Hilbert [1904b, p. 62]; Riesz [1918, p. 94]). The property of being finite-dimensional, completely continuous, approximable, or compact is carried over from T to $F(\zeta, T)$. Moreover, $(I - \zeta T)^{-1} = I + \zeta F(\zeta, T)$; see [RIE, p. 114].

The hybrid **eigenvalue** is derived from the German term “Eigenwert”; see Hilbert [1904b, p. 64]. The names *proper value* and *characteristic value* have also been used. When reading any paper, one should always check whether the terms above refer to $\lambda x = Tx$ or to $x = \zeta Tx$.

2.6.5.2 According to Riesz [1918, p. 90], the complex plane can be divided into two parts depending on whether $I-\zeta T$ is continuously invertible. In his terminology these are the sets of *regular* and *singular* points. With regard to the λ-plane, nowadays we use the terms **resolvent set** and **spectrum**, which go back to Hilbert [1906a, p. 160].

In 1913, Riesz made an observation that extends immediately from operators $A: l_2 \to l_2$ to the general case.

[RIE, p. 117]:

Les résultats que nous venons d'obtenir peuvent être résumés succinctement en disant que, pour les valeurs ordinaires de ζ, la substitution $A_\zeta = A(I-\zeta A)^{-1}$ et aussi les autres substitutions y liées, qui entraient dans le calcul, montrent les caractères d'une fonction holomorphe en ζ; see 5.2.1.1.

2.6.5.3 The non-zero spectrum of compact operators consists only of eigenvalues, and Riesz [1918, p. 90] proved the following *Satz*:

Die Eigenwerte ζ besitzen im Endlichen keinen Häufungspunkt.

In addition, Hildebrandt [1931, p. 198] stated without proof or reference that the Fredholm resolvent is meromorphic on the whole complex plane with poles at the eigenvalues. A discussion of this phenomenon can be found in 5.2.3.3.

2.6.6 Classical operator ideals

2.6.6.1 For a detailed presentation of the theory of operator ideals the reader is referred to Section 6.3. For the present, we need only one fundamental definition.

Suppose that in every $\mathfrak{L}(X,Y)$, we are given a subset $\mathfrak{A}(X,Y)$. Then

$$\mathfrak{A} := \bigcup_{X,Y} \mathfrak{A}(X,Y)$$

is said to be an **operator ideal**, or just an **ideal**, if the following conditions are satisfied:

(OI$_0$) $x^* \otimes y \in \mathfrak{A}(X,Y)$ for $x^* \in X^*$ and $y \in Y$.
(OI$_1$) $S+T \in \mathfrak{A}(X,Y)$ for $S,T \in \mathfrak{A}(X,Y)$.
(OI$_2$) $BTA \in \mathfrak{A}(X_0,Y_0)$ for $A \in \mathfrak{L}(X_0,X)$, $T \in \mathfrak{A}(X,Y)$, and $B \in \mathfrak{L}(Y,Y_0)$.

2.6.6.2 The collections

$\mathfrak{F}$: finite rank operators, 2.6.2.1 $\overline{\mathfrak{F}}$: approximable operators, 2.6.3.1

$\mathfrak{K}$: compact operators, 2.6.4.2 $\mathfrak{V}$: completely continuous operators, 2.6.3.5

are classical examples of ideals. These trivial observations are more or less folklore. In the case of $\mathfrak{K}$, Riesz [1918, p. 74] stated:

Es folgt unmittelbar aus der Definition, dass das Produkt T_1T_2 sicher vollstetig [read: compact] *wird, wenn wenigstens einer der beiden Faktoren vollstetig ist. Da ferner aus gleichmässig konvergenten, also auch aus kompakten Folgen durch Multiplikation* *mit einer Konstanten oder durch gliedweise Addition wieder gleichmässig konvergente resp. kompakte Folgen entstehen, so folgt, dass zugleich mit T, T_1, T_2 auch cT und $T_1 + T_2$ vollstetig sind.*

Concerning $\mathfrak{V}(l_2)$ we refer to [RIE, p. 97].

2.6.6.3 An operator ideal $\mathfrak{A}$ is said to be **closed** if all components $\mathfrak{A}(X,Y)$ are closed in $\mathfrak{L}(X,Y)$ with respect to the operator norm. The operator ideals $\overline{\mathfrak{F}}$, $\mathfrak{K}$, and $\mathfrak{V}$ have this property.

[BAN, p. 96]: *Etant donnée une suite $\{U_n(x)\}$ d'opérations linéaires totalement continues* [read: compact]*, toute opération linéaire $U(x)$ telle que* $\lim_{n\to\infty} |U_n - U| = 0$ *est aussi totalement continue.*

2.6.6.4 The following chain of inclusions is obvious: $\mathfrak{F} \subset \overline{\mathfrak{F}} \subseteq \mathfrak{K} \subset \mathfrak{V}$. Hildebrandt [1931, pp. 196–197] was the first who formulated the **approximation problem:**
The simplest completely continuous [read: compact] *transformations are those which transform* $\mathfrak{S}$ [read: a Banach space] *into a subset of finite dimension. It follows that if T is the strong limit of a sequence of such transformations, then T is completely continuous. Whether the theorem is reversible without additional limitation on the space considered seems to be still undetermined.*

In my opinion, this was the most important question ever asked in Banach space theory. Enflo's negative answer [1973] changed the philosophical background of the subject decisively.

The approximation problem has many facets, which will be discussed again and again; see Subsections 5.7.4 and 7.4.2.

2.7 Banach's monograph

Stefan Banach's

Théorie des opérations linéaires

appeared as the first volume in the famous series MONOGRAFJE MATEMATYCZNE, Warszawa 1932. Basically, it is a French translation of

Teorja operacyj, Tom I, Operacje liniowe (1931).

There are, however, several improvements. For example, Chapitre XII, with the title *Dimension linéaire* was added, and we have a substantially extended version of the valuable *Remarques* written in collaboration with Stanisław Mazur.

This excellent book became a milestone in the history of mathematics. In the opinion of Dieudonné [DIEU$_2^\bullet$, pp. 142–143] *it had on Functional Analysis* [or even on analysis] *the same impact that van der Waerden's book had on Algebra two years earlier.*

The following quotation from [BOU$_{5b}$, pp. 173–174] or [BOU$^\bullet$, p. 217] shows Bourbaki's high esteem for Banach's work (original version on p. 680):
The publication of Banach's treatise on "Opérations linéaires" marks, one could say, the beginning of adult age for the theory of normed spaces. All the results of which we have been speaking, as well as many others, are set out in this volume, in a way that is still a bit disorganized, but accompanied with many striking examples taken from varied domains of Analysis. ... As it happened, the work had considerable success, and one of its most immediate effects was the quasi-universal adoption of the language and notation used by Banach.

Banach's monograph is written in a modern style; large parts could be taken as a basis for today's lectures. However, due to the state of the art, two fundamental features are missing:
(1) Though Wiener [1923a] had stressed the importance of the complex case, Banach treats only real spaces. Hence, even in spectral theory [BAN, p. 157], the parameter h in $x-hU(x)$ is restricted to $\mathbb{R}$.
Heuser [1986$^\bullet$, p. 660] writes (original version on p. 681):
Despite the hints of Riesz and Wiener, Banach dealt only with real (B)-spaces which means that–without any necessity–he did not till a fruitful ground.
Maybe, Banach would have answered, *Sorry, I had no complex Hahn–(B) theorem at my disposal*; see 2.3.7.
(2) A more serious gap concerns weak topologies. Though all needed tools from general topology were available around 1930, Banach worked only with (transfinite) sequences. He had a good companion: in the preface to the revised version of his *Mengenlehre* from 1927, Hausdorff wrote (original version on p. 681):
Probably, some people will be sorry that–in order to save space–I have dropped the topological standpoint through which the first edition earned many friends. Instead I have restricted myself to the simpler theory of metric spaces.

Thus, in the words of Dieudonné [DIEU$_2^\bullet$, p. 212, footnote]:
it seems that Hausdorff lost faith in his ideas of 1914.

This statement is supported by the fact that Hausdorff [1932] did not use weak topologies when he studied *normale Auflösbarkeit*.

In retrospect, the "birth of weak topologies" was a tough and lengthy process.

3

Topological Concepts – Weak Topologies

3.1 Weakly convergent sequences

3.1.1 Coordinatewise convergence of sequences in the closed unit ball of l_2 was already used by Hilbert in his definition of *Vollstetigkeit*; see 2.6.1.1. The term **schwache Konvergenz** first appeared in Weyl's thesis [1908, p. 8]:

Jedes bestimmte Wertsystem $(x)=(x_1,x_2,\dots)$ *werden wir einen Punkt unseres Raumes von unendlichvielen Dimensionen nennen und* $x_1,x_2,\dots$ *bezw. seine* $1.,2.,\dots$ *Koordinate:* $x_1=\mathfrak{Co}_1(x)$, $x_2=\mathfrak{Co}_2(x),\dots$. *Haben wir eine unendliche Reihe solcher Punkte* $(x)^1,(x)^2,\dots$, *so sagen wir, sie* **konvergiere schwach** *gegen den Punkt* (x), *wenn für jeden Index i*

$$\underset{n=\infty}{L}\mathfrak{Co}_i(x)^n=\mathfrak{Co}_i(x)$$

ist.

We stress that Weyl did not assume boundedness, which means that in his sense (ne_n) would be a weak null sequence.

3.1.2 Riesz [1909a, p. 465] extended the concept of weak convergence to function spaces:

Die Folge $\{f_i(x)\}$ *von Funktionen der Klasse* $[L^p]$ *konvergiert in bezug auf den Exponenten p* **schwach** *gegen die Funktion* $f(x)$ *derselben Klasse, wenn a) die Integralwerte*

$$\int_a^b |f_i(x)|^p dx$$

insgesamt unterhalb einer endlichen Schranke liegen; b) für alle Stellen $a\le x\le b$

$$\lim_{i=\infty}\int_a^x f_i(x)\,dx=\int_a^x f(x)\,dx$$

ausfällt.

Das Bestehen der Grenzgleichung

$$\lim_{i=\infty}\int_a^b f_i(x)\gamma(x)\,dx=\int_a^b f(x)\gamma(x)\,dx$$

folgt, wenn $\gamma(x)$ *eine stückweise konstante Funktion ist, aus b). Auf Grund der Voraussetzung a), überträgt sich jene Grenzgleichung* [durch Approximation] *auf sämtliche Funktionen der Klasse* $\left[L^{\frac{p}{p-1}}\right]$.

Riesz [1909a, footnote on p. 466] added:

Es ist wohl nicht uninteressant zu bemerken, daß auch aus dem Bestehen der Grenzgleichung für alle $g(x)$ *aus* $\left[L^{\frac{p}{p-1}}\right]$ *die schwache Konvergenz von* $\{f_i(x)\}$ *gegen* $f(x)$ *folgt.*

This reformulation was the source of a main stream of Banach space theory.

3.1.3 We proceed with a quotation from [RIE, p. 55]:

Considérons la suite indéfinie de quantités $y_1^{(n)}, y_2^{(n)}, \dots$. *Nous supposons que ces quantités varient avec n, mais de telle façon que l'on ait constamment*

$$\sum_{k=1}^{\infty} |y_k^{(n)}|^p \leq G^p,$$

G désignant un nombre positif qui ne dépend pas des n. De plus, supposons que les $y_k^{(n)}$ *tendent, pour* $n \to \infty$, *vers une valeur déterminée* y_k^*. *Cela posé, nous dirons que la suite de systèmes* $(y_k^{(n)})$ **tend** *vers le système* (y_k^*).

Note that Riesz had omitted the term "faible," whereas he used the term "forte"; see [RIE, p. 79]. Probably, at this time, **weak** convergence was considered to be more important than convergence in norm. Hilbert [1906b, p. 439] also took such a point of view when he said, *ich will nunmehr an Stelle von „vollstetig" kurz das Wort „stetig" gebrauchen.*

3.1.4 The subsequent definitions are due to Banach, [1929, Part II, p. 231] and [BAN, pp. 122, 133]:

On dit qu'une suite $\{f_n\}$ *de fonctionnelles linéaires* **converge faiblement** *vers la fonctionnelle f, lorsqu'on a*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{pour tout } x \subset E.$$

Une suite $\{x_n\}$ *d'éléments de E s'appelle* **faiblement convergente** *vers l'élément* $x \subset E$, *lorsqu'on a*

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \textit{pour tout } f \subset \overline{E} \ \text{(dual).}$$

Banach [BAN, pp. 239–240] stressed the difference between both concepts, which are nowadays distinguished by referring to **weak*** and **weak convergence**:

Il est à remarquer que la convergence faible d'une suite de fonctionnelles linéaires définies dans un espace E du type (B) *n'est pas une condition suffisante pour la convergence faible de la même suite, lorsqu'on regarde cette dernière comme une suite d'éléments dans l'espace* $\overline{E}$. *Ainsi p. ex. dans* l_1 *la notion de la convergence faible varie suivant qu'on en considère les éléments comme des représentants des fonctionnelles linéaires ou non.*

3.1.5 Baire's work on real functions had already indicated that a satisfactory theory of weak topologies could not be obtained on the basis of sequences. He observed that the process of taking pointwise limits of continuous functions on $[0,1]$ does not stabilize after the first step. The Dirichlet function, for example, is of the second Baire class; see Baire [1899, pp. 97–98]. Von Neumann [1930b, p. 379] was the first to state a program of how to overcome these difficulties.

Nun ist es auch üblich, eine andere Topologie in $\mathfrak{H}$ [Hilbert space] *einzuführen, die sogenannte* **schwache**. *D.h. gewöhnlich definiert man nur die schwachen Limites und nicht die Umgebungen, und zwar so:* $f_1, f_2, \ldots$ *konvergiert schwach gegen* f, *wenn* $(f_n, \varphi) \to (f, \varphi)$ *für jedes feste* φ. *Es ist aber leicht, dies auf einen Umgebungsbegriff zurückzuführen: Sei* $\mathscr{U}(f_0; \varphi_1, \ldots, \varphi_s, \varepsilon)$ *die Menge aller* f *mit*

$$|(f - f_0, \varphi_1)| < \varepsilon, \ldots, |(f - f_0, \varphi_s)| < \varepsilon;$$

alle $\mathscr{U}(f_0; \varphi_1, \ldots, \varphi_s, \varepsilon)$ *mit beliebigen* $\varphi_1, \ldots, \varphi_s$ *(aus* $\mathfrak{H}$, *s beliebig) und* $\varepsilon > 0$ *sind die Umgebungen von* f_0.

This definition requires the concept of a *topological linear space*, whose history will be described in the next section.

3.2 Topological spaces and topological linear spaces

3.2.1 Topological spaces

3.2.1.1 The following quotations from Hausdorff's *Grundzüge der Mengenlehre* provide a crash course of general topology that could be presented in any modern lecture.

[HAUS$_1$, p. 213]: *Unter einem* **topologischen Raum** *verstehen wir eine Menge E, worin den Elementen (Punkten) x gewisse Teilmengen* U_x *zugeordnet sind, die wir Umgebungen von x nennen, und zwar nach Maßgabe der folgenden* **Umgebungsaxiome**:

(A) *Jedem Punkt x entspricht mindestens eine Umgebung* U_x*; jede Umgebung* U_x *enthält den Punkt x.*

(B) *Sind* U_x, V_x *zwei Umgebungen desselben Punktes x, so gibt es eine Umgebung* W_x*, die Teilmenge von beiden ist.*

(C) *Liegt der Punkt y in* U_x*, so gibt es eine Umgebung* U_y*, die Teilmenge von* U_x *ist.*

(D) *Für zwei verschiedene Punkte x, y gibt es zwei Umgebungen* U_x, U_y *ohne gemeinsamen Punkt.*

[HAUS$_1$, p. 214] *Ist jetzt A eine Punktmenge, d.h. eine Teilmenge von E, so nennen wir einen Punkt x einen* **inneren Punkt** *von A, wenn es eine zu A gehörige Umgebung* U_x *gibt.*

[HAUS$_1$, p. 215] *Eine Menge, deren sämtliche Punkte innere Punkte sind, nennen wir ein* **Gebiet**.

[HAUS_1, p. 219] *Wir definieren*: *x heißt ein* **Häufungspunkt** *von A, wenn in jeder Umgebung von x unendlich viele Punkte von A liegen.*

[HAUS_1, p. 221] *Die Menge A ist* **abgeschlossen***, wenn sie ihre Häufungspunkte enthält.*

[HAUS_1, p. 225] *Der Durchschnitt beliebig vieler und die Summe endlich vieler abgeschlossener Mengen ist wieder abgeschlossen.*

[HAUS_1, p. 228] *Das Komplement einer abgeschlossenen Menge ist ein Gebiet, und das Komplement eines Gebietes ist eine abgeschlossene Menge.*

Abgeschlossene Menge und Gebiete sind die wichtigsten und einfachsten Objekte der Theorie der Punktmengen.

3.2.1.2 The concept of a **closed subset** was introduced by Cantor [CAN, p. 137] when he studied *lineare Punktmannichfaltigkeiten* in $\mathbb{R}^n$, and Carathéodory [CARA, p. 40] reinvented the name **open subset** instead of Hausdorff's term *Gebiet*. However, we stress that the concept of a *domaine ouvert à n dimensions* had already appeared in the work of Baire [1899, p. 7]; see also Lebesgue [1902, p. 242].

Nowadays, following a proposal of Tietze [1923, p. 294], a topological space is mostly defined by prescribing the collection of its open subsets. A subfamily $\mathscr{B}$ of open sets is called a **base** if every open set can be obtained as a union of members of $\mathscr{B}$.

Kuratowski's approach [1922] was based on the mapping that assigns to every subset A its **closure** $\overline{A}$ (closed hull).

3.2.1.3 With the exception of Section 7.5, we deal only with **Hausdorff spaces**, which means that the separation axiom (D) from 3.2.1.1 is assumed to hold.

3.2.1.4 Let f be a function from a subset A of a topological space X to a subset B of a topological space Y.

[HAUS_1, pp. 359–361]: *Die Funktion $y=f(x)$ heißt im Punkt a stetig, wenn zu jeder Umgebung V_b des Punktes $b=f(a)$ eine Umgebung U_a des Punktes a existiert, deren Bild in V_b liegt.*

...

Man nennt die Funktion schlechthin **stetig***, wenn sie in allen Punkten stetig ist.*

...

Die Funktion $f(x)$ ist dann und nur dann stetig, wenn jedem Relativgebiet von B [read: open set] *als Urbild ein Relativgebiet von A, oder jeder in B abgeschlossenen Menge als Urbild eine in A abgeschlossene Menge entspricht.*

3.2.1.5 The term *homéomorphisme* was coined by Poincaré in his paper *Analysis situs* [1895, § 2] to denote a mapping on a domain in $\mathbb{R}^n$ that together with its inverse is differentiable. Moreover, the Jacobian determinants are supposed to be different from zero.

The modern concept of a **homeomorphism** appeared much later. In [KUR, p. 77] we find the following definition:

La fonction $y=f(x)$ transformant l'espace X en l'espace Y (tout entier) est dite bicontinue si elle est biunivoque et si la fonction $f(x)$, ainsi que la fonction inverse $f^{-1}(y)$, est continue. La transformation est dite alors une **homéomorphie** *et les espaces X et Y s'appellent* **homéomorphes**.

3.2.1.6 The classical treatise on general topology is Hausdorff's *Grundzüge der Mengenlehre* [HAUS_1], while Fréchet's monograph [FRÉ] had little impact. The first textbooks were written in the 1930s by Sierpiński [SIER, Polish version 1928], Kuratowski [KUR], and Alexandroff/Hopf [ALEX^+]. Concerning the next generation, I refer to Bourbaki's *Livre III* and [KEL]. The reader may also consult [$\text{KUN}^\cup$], the *Handbook of Set-Theoretic Topology*.

An excellent account of the early history of general topology was given by Taylor [1982/87$^\bullet$, Part II]. Among others, he discussed the concept of a neighborhood and made some comments on the (non-existent) relations between Fréchet and Hausdorff. The second volume of Hausdorff's *Gesammelte Werke* [$\text{HAUS}^\bowtie$] contains several historical articles that are highly recommended. For further information, the reader may also consult the *Handbook of the History of General Topology* [$\text{AULL}^\cup$] and [$\text{PIER}^\bullet$, Chap. 2].

3.2.2 Nets and filters

Many topological properties of metric spaces can be described in terms of sequences. However, this is not true in general, and it took a long time to find suitable substitutes. Finally, this problem was solved by introducing two competitive concepts: *nets* and *filters*.

3.2.2.1 A non-empty set $\mathbb{A}$ is said to be **directed** by a relation $\succeq$ if the following conditions are satisfied:

$(\mathbf{D_1})$ $\alpha \succeq \alpha$ whenever $\alpha \in \mathbb{A}$.

$(\mathbf{D_2})$ If $\alpha \succeq \beta$ and $\beta \succeq \gamma$, then $\alpha \succeq \gamma$.

$(\mathbf{D_3})$ For $\alpha_1 \in \mathbb{A}$ and $\alpha_2 \in \mathbb{A}$ there is an $\alpha \in \mathbb{A}$ such that $\alpha \succeq \alpha_1$ and $\alpha \succeq \alpha_2$.

A **Moore–Smith sequence**, **directed system**, or **net** is a family $(x_\alpha)_{\alpha\in\mathbb{A}}$ in a set X indexed by a directed set $\mathbb{A}$. To simplify notation, we will mostly write (x_α) instead of $(x_\alpha)_{\alpha\in\mathbb{A}}$. If X is a topological space, then (x_α) **converges** to x if for every neighborhood U_x we find an index $\alpha \in \mathbb{A}$ such that $x_\beta \in U_x$ whenever $\beta \succeq \alpha$. Based on this concept, closed subsets F are characterized by the property that $x_\alpha \in F$ and $x_\alpha \to x$ imply $x \in F$. Since we work in a Hausdorff space, the **limit** of a convergent net is unique.

3.2.2.2 The concept of a net goes back to Moore/Smith [1922, p. 103]. Important contributions are due to Birkhoff [1937], Tukey [TUK], and Kelley [1950a].

Sarason [1990•, pp. 18–19] tells the following story:

Steenrod is responsible for the term **net** *as it is now commonly used for generalized sequences (in the sense of Moore–Smith). Kelley had been planning to use the term "way"; that would have resulted in what we now call a "subnet" being referred to as a "subway." Steenrod, when informed by Kelley of his plan, apparently regarded Kelley's choice as frivolous, and after being prodded by Kelley, he suggested the term "net" as an alternative. His judgment prevailed.*

3.2.2.3 Let $\mathscr{F}(\mathbb{I})$ denote the collection of all finite subsets of any set $\mathbb{I}$. Obviously, $\mathscr{F}(\mathbb{I})$ is directed with respect to set-theoretic inclusion. Tukey [TUK, p. 17] referred to a directed system indexed by some $\mathscr{F}(\mathbb{I})$ as a **phalanx**. This martial name has not survived. A typical example of a phalanx is given by the partial sums

$$\sigma_F := \sum_{i\in F} \xi_i \quad \text{for } F \in \mathscr{F}(\mathbb{I})$$

of any scalar family $(\xi_i)_{i\in\mathbb{I}}$; see 5.1.1.3.

3.2.2.4 We refer to $x \in X$ as a **cluster point** or **accumulation point** of the net (x_α) if for every neighborhood U_x and every $\alpha \in \mathbb{A}$ there exists $\beta \in \mathbb{A}$ such that $x_\beta \in U_x$ and $\beta \succeq \alpha$. Putting $x_{(\alpha,U_x)} := x_\beta$ yields a **subnet**, which converges to x.

The concept of a subnet was introduced by Kelley [1950a, p. 277]. Unfortunately, subnets of a net $(x_\alpha)_{\alpha\in\mathbb{A}}$ cannot be defined on the same index set $\mathbb{A}$, and this defect is indeed unpleasant. Hence the French school proposed another tool: filters.

3.2.2.5 By a **filter base** $\mathscr{B}$ on a set X we mean a collection of subsets for which the following conditions are satisfied:

($\mathbf{F_1}$) $\mathscr{B}$ contains at least one subset, and all members of $\mathscr{B}$ are non-empty.

($\mathbf{F_2}$) For $B_1 \in \mathscr{B}$ and $B_2 \in \mathscr{B}$ there is a $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2$.

A filter base $\mathscr{B}$ with the following property is called a **filter**:

($\mathbf{F_3}$) If $A \supset B$ and $B \in \mathscr{B}$, then $A \in \mathscr{B}$.

Hausdorff's axioms (A) and (B) say that for every point x, the sets U_x form a filter base. Adding all supersets of these neighborhoods, we get the filter $\mathscr{U}(x)$. Every filter base that generates $\mathscr{U}(x)$ is said to be a **fundamental system** of neighborhoods of x. A filter $\mathscr{B}$ **converges** to x if $\mathscr{U}(x) \subseteq \mathscr{B}$. Since we work in a Hausdorff space, the **limit** of a convergent filter is unique.

3.2.2.6 We refer to $x \in X$ as a **cluster point** or **accumulation point** of the filter $\mathscr{B}$ if x is a cluster point of all $B \in \mathscr{B}$.

A filter $\mathscr{B}_0$ is said to be **finer** than a filter $\mathscr{B}$ if $\mathscr{B} \subseteq \mathscr{B}_0$. Then $\mathscr{B} \to x$ implies $\mathscr{B}_0 \to x$; and $x \in X$ is a cluster point of $\mathscr{B}$ if and only if there exists a finer filter $\mathscr{B}_0$ that converges to x. Hence *refining a filter* is just like passing from a sequence to a subsequence.

3.2.2.7 It follows from Zorn's lemma that the collection of all filters living on a fixed set contains maximal elements, the **ultrafilters**. This concept is the crux of filter theory. Of course, every sequence has genuine subsequences. Hence we can refine, refine, and refine ..., which is the basic idea of the classical diagonal process. Working with an ultrafilter means that we have a stable object.

3.2.2.8 Filters and ultrafilters were invented by Cartan [1937] and systematically applied in the *Éléments* of Bourbaki. However, limits and cluster points of filter bases were already studied in the thesis of Vietoris [1921, p. 185]; he used the "flowery" term *Kranz* (English: wreath, garland).

3.2.2.9 Every filter base is a directed set with respect to the partial ordering $A \succeq B$ defined by $A \subseteq B$. Conversely, given any net $(x_\alpha)_{\alpha\in\mathbb{A}}$, the collection of all *final sections* $B_\alpha := \{\beta\in\mathbb{A} : \beta\succeq\alpha\}$ with $\alpha\in\mathbb{A}$ is a filter base.

Hence it came as no surprise when Bartle and Bruns/Schmidt [1955] proved that both tools, nets and filters, can be replaced by each other. Nevertheless, as shown by the following quotations, patriotism plays a considerable role. Bartle [1955, p. 551] says that *nets are predominant in this country* [the USA]*, while the filter theory reigns supreme in France.* Typical examples are the books [KEL] and [BOU_{3a}].

The following pros and cons are stressed by Bartel [1955, p. 551]:

It is well-known that these theories are equivalent in the sense that there are no proofs attainable by one that cannot be reached by the other. However, it is undeniable that each of these approaches has advantages–however psychological–not immediately possessed by the other. The use of nets parallels very closely standard constructions involving sequences, and has the pronounced advantage that sequential arguments may quite readily be adapted to them. On the other hand, the fact that filters are dual ideals in the algebra of sets renders a certain symmetry to their use, and often an "algebraic elegance." In addition, they are admirably suited for certain transfinite arguments, and enjoy a uniqueness not possessed by nets.

Bourbaki's point of view is less modest, [BOU_{3a}, Chap. I, p. 129]:

L'introduction des filtres par H. Cartan, tout en apportant un instrument très précieux en vue de toute sorte d'applications (où il se substitue avantageusement à la notion de ≪convergence à la Moore–Smith≫), est venue, grâce au théorème des ultrafilters, achever d'éclaircir et de simplifier la théorie.

3.2.2.10 McShane [1962] proposed a useful combination of both concepts. He considered families $(x_\alpha)_{\alpha\in\mathbb{A}}$ together with a filter base $\mathscr{B}$ on the index set $\mathbb{A}$.

If the x_α's belong to a topological space X, then $x\in X$ is the limit of (x_α) provided that for every neighborhood U_x of x we can find a set $B\in\mathscr{B}$ such that $x_\alpha\in U_x$ whenever $\alpha\in B$. This is precisely what we know from our mathematical childhood modulo replacing $\alpha\geq\alpha_0$ by $\alpha\in B$. Simultaneously, the machinery of ultrafilters is at our disposal.

3.2.3 Compactness

3.2.3.1 Compactness is certainly the most important concept of general topology. In the context of metric spaces, it was introduced by Fréchet [1906, p. 6]. However, from our present point of view, his formulation in terms of sequences did not express the real essence, which is hidden in the **covering properties**. This observation was emphasized by Alexandroff/Urysohn [1924, pp. 259–260]:

Ist der Raum $\mathfrak{R}$ in der Summe eines Systems (beliebiger Mächtigkeit) von Gebieten enthalten, so ist er bereits in einer Summe von endlich vielen Gebieten dieses Systems enthalten.

Es liegt nahe, die Räume, in denen diese Eigenschaft stattfindet, besonders zu berücksichtigen und sie durch einen speziellen Namen auszuzeichnen; wir nennen sie also **bikompakt**.

The prefix *bi* was motivated as follows; see [ALEX, p. 223]:

Ein Raum heißt **initinal kompakt** *bis zu einer gegebenen Mächtigkeit $\mathfrak{a} \geq \aleph_0$, wenn jede offene Überdeckung von einer Mächtigkeit $\mathfrak{m} \leq \mathfrak{a}$ eine endliche Teilüberdeckung enthält. Andererseits wird ein Raum* **final kompakt** *von einer gegebenen Mächtigkeit $\mathfrak{a}$ genannt, wenn jede offene Überdeckung mit einer Mächtigkeit $> \mathfrak{a}$ eine Teilüberdeckung von einer Mächtigkeit $\leq \mathfrak{a}$ enthält.*

Bikompakte Räume sind gleichzeitig initial kompakt bis zu jeder beliebigen Mächtigkeit und final kompakt von jeder Mächtigkeit an (daher die Benennung „bikompakt“).

In the course of time, in particular under the influence of Bourbaki, the name “bicompact” was replaced by “compact”. In any case, when reading a text that is written before 1950, one should carefully check the underlying terminology.

3.2.3.2 Borel [1895, pp. 51–52] proved that every closed interval $[a,b]$ that is covered by countably many open intervals (a_n, b_n) can already be covered by finitely many of them. Subsequently, Lebesgue [LEB_1, p. 105] and others removed the assumption of countability; see Hildebrandt [1926, p. 425]. This fundamental result, which was established as a tool of measure theory, is referred to as the Borel–Lebesgue theorem. It can be traced back to a proof of Heine [1872] in which he showed that every continuous function on a closed interval is uniformly continuous, *ce qui a paru à quelques auteurs* [see, for example, [SCHOE, p. 76]] *une raison suffisante pour donner à théorème de Borel–Lebesgue le nom de ≪théorème de Heine–Borel≫*; see [BOU_{3a}, Chap. II, p. 177].

3.2.3.3 Originally, compactness of a set K was defined by requiring that every infinite subset have a cluster point that belongs to the underlying space but not necessarily to the set K. Thus K need not be closed. Fréchet [1906, pp. 6–7] said, *lorsqu'un ensemble est à la fois compact et fermé nous l'appellerons ensemble* **extrémal**, while Hausdorff [HAUS_2, p. 107] used the term **in sich kompakt** (*self-compact*).

From the modern point of view, all definitions of compactness included some kind of closedness, and the term “relatively” is used to indicate the contrary.

3.2.3.4 Nowadays, we distinguish the following concepts:

A subset K of a Hausdorff space is said to be

compact
if every open covering of K contains a finite subcovering,

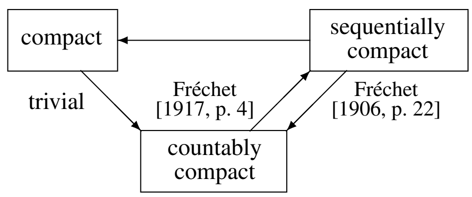
countably compact
if every countable open covering of K contains a finite subcovering,

sequentially compact
if every sequence in K has a subsequence converging to a limit in K.

Countable compactness is equivalent to the

Bolzano–Weierstrass property; see [JARN$^\bullet$, pp. 49–50]:
Every infinite subset of K has a cluster point in K.

Compact sets and sequentially compact sets are countably compact. No other implications hold in general. However, all concepts coincide in metric spaces:



The implication $\longleftarrow$ was proved by Fréchet [1906, p. 26] under some additional assumptions (separability), which could be removed by Hildebrandt [1912, pp. 278–281], Groß [1914, pp. 809–812], and Fréchet [1917, p. 5]. The first presentation in a textbook had to wait until 1932: [HAHN$_2$, p. 96]. Partial results can be found in [HAHN$_1$, pp. 89–93], [HAUS$_1$, pp. 230–231, 272–273], and [HAUS$_2$, pp. 129–130].

3.2.3.5 In the language of filters, compactness of a subset K can be characterized by modern Bolzano–Weierstrass properties:

- Every filter has a cluster point.
- Every filter is contained in a convergent filter.
- Every ultrafilter converges.

As usual, we have simplified the situation by looking at K as an autonomous Hausdorff space under the induced topology.

3.2.3.6 A collection of sets is said to have the **finite intersection property** if the intersection of any finite subcollection is non-void.

Using this concept, we get the following generalization of **Cantor's Durchschnittssatz:**

A Hausdorff space is compact if and only if every collection of closed sets with the finite intersection property has a non-void intersection.

3.2.3.7 An excellent account of the history of compactness and related covering properties was given by Hildebrandt [1926]; see also Pier [1980•].

3.2.4 Topological linear spaces

3.2.4.1 The concepts of *completeness* and *total boundedness* (*precompactness*) make no sense in topological spaces, since they express certain uniformity properties. In the words of von Neumann [1935, p. 1]:

The need of uniformity arises from the fact that the elements of a fundamental sequence are postulated to be "near to each other," and not near to any fixed point.

To close this gap, Weil [WEIL$_1$] introduced **uniform structures**. A slightly different approach is due to Tukey [TUK]. We will use this concept only in the special case of topological linear spaces.

3.2.4.2 Given a set X that is at the same time a linear space and a topological space, both structures should be compatible. Therefore Kolmogoroff [1934, p. 29] required *daß die Operationen der Addition und Multiplikation stetig sind.* Thinking in the tradition of Kuratowski [1922] and the Russian school, he introduced topologies via the correspondence $A \mapsto \overline{A}$, while neighborhoods were of only secondary importance. His main intention was to characterize the subclass of normed linear spaces:

Für die Normierbarkeit des Raumes X ist notwendig und hinreichend, daß in X mindestens eine beschränkte konvexe Umgebung der Null existiert.

Kolmogoroff's definition of a **bounded set** reads as follows; see [1934, p. 30]:

Eine Menge $A \subset X$ heißt **beschränkt**, *wenn für jede beliebige Folge $\{\lambda_n\}$ von reellen Zahlen und für jede beliebige Folge $\{x_n\}$ von Elementen, die A angehören, aus $\lambda_n \to 0$ die Beziehung $\lambda_n x_n \to o$ folgt.*

For linear spaces with a metrizable topology, the same definition was earlier given by Mazur/Orlicz [1933, p. 152].

Almost simultaneously, von Neumann [1935, p. 4] invented the concept of a **topological linear space** by using a filter base $\mathscr{U}$ whose members are neighborhoods of zero. In addition to the usual properties, which ensure the continuity of the maps

$$(x_1, x_2) \mapsto x_1 + x_2 \quad \text{and} \quad (\lambda, x) \mapsto \lambda x,$$

he assumed that there exist $U_1, U_2, \ldots \in \mathscr{U}$ such that $\bigcap_{n=1}^{\infty} U_n = \{o\}$. The necessity to require a certain weakening of *Hausdorff's first axiom of countability* was due to the fact that von Neumann used the “wrong” notion of compactness, namely countable compactness. In the subsequent development this superfluous condition was removed.

Von Neumann clearly realized the uniform structure of a topological linear space. He defined the concepts of **boundedness** and **total boundedness** (**precompactness**) as they are used today; [1935, pp. 7–8]. Moreover, **completeness** was understood as the property that every totally bounded and closed subset should be compact; [1935, p. 11]. This definition has not survived.

3.2.5 Locally bounded linear spaces

In the history of functional analysis there appeared various concepts of *pseudo-*, *quasi-* and *semi-norms*. The terminology was and is non-uniform.

3.2.5.1 Let X be a real or complex linear space. By a **quasi-norm** we mean a map that assigns to every element $x \in X$ a non-negative number $\|x\|$ such that the following conditions are satisfied:

(QN$_0$) $\|x\| = 0$ implies $x = o$.

(QN$_1$) $\|x+y\| \le Q\big[\|x\| + \|y\|\big]$, where $Q \ge 1$ does not depend on $x, y \in X$.

(QN$_2$) $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X$ and $\lambda \in \mathbb{K}$.

This concept was introduced by Hyers [1938, p. 77] and [1939, p. 562] under the names *pseudo-norm* and *absolute value*, respectively. Bourgin [1943, p. 651] proposed the label *quasi-norm*. Instead of **(QN$_1$)** the former required that $\|x\| \to 0$ and $\|y\| \to 0$ imply $\|x+y\| \to 0$, while the latter stated the quasi-triangle inequality explicitly.

Quasi-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be **equivalent** if there exist constants $c_1, c_2 > 0$ such that $\|x\|_1 \le c_2\|x\|_2$ and $\|x\|_2 \le c_1\|x\|_1$ for $x \in X$.

3.2.5.2 The **quasi-triangle inequality**

$$\|x+y\| \le Q\big[\|x\| + \|y\|\big] \tag{3.2.5.2.a}$$

generalizes the **triangle inequality** $\|x+y\| \le \|x\| + \|y\|$. A quasi-norm is referred to as a p**-norm** if it satisfies the p**-triangle inequality**

$$\|x+y\|^p \le \|x\|^p + \|y\|^p, \tag{3.2.5.2.b}$$

where $0 < p \le 1$.

3.2.5.3 Aoki [1942, p. 592] invented the concept of an F'-norm. Unfortunately, his presentation was quite sophisticated and, moreover, disfigured by misprints. Thus some decoding is needed. Finally, I arrived at the conclusion that by definition, an F'**-norm** satisfies property **(QN$_0$)**, the triangle inequality, and

(QN$_2^\circ$) $\|\lambda x\| = |\lambda|^p\|x\|$ for $x \in X$, $\lambda \in \mathbb{K}$ and some fixed $p > 0$.

Rolewicz [1957, p. 471] referred to **(QN_2°)** as ***p*-homogeneity**. Since the triangle inequality is assumed to hold, we have $n^p\|x\| = \|nx\| \leq \|x\| + \cdots + \|x\|$, which in turn implies that $0 < p \leq 1$. The formula $\|x\| = |||x|||^p$ establishes a one-to-one correspondence between p-homogeneous norms $\|x\|$ and p-norms $|||x|||$. More precisely, $\|\lambda x\| = |\lambda|^p \|x\|$ passes into $|||\lambda x||| = |\lambda| \, |||x|||$ and $\|x+y\| \leq \|x\| + \|y\|$ into $|||x+y|||^p \leq |||x|||^p + |||y|||^p$. The decision as to which concept should be preferred is a matter of taste.

3.2.5.4 Obviously, (3.2.5.2.b) implies (3.2.5.2.a) with $Q := 2^{1/p-1}$. Conversely, Aoki [1942, pp. 592–593] and Rolewicz [1957, p. 472] proved that every quasi-norm $\|\cdot\|$ is equivalent to the p-norm

$$|||x||| := \inf\left\{ \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p} : x = \sum_{k=1}^{n} x_k, \ n = 1, 2, \ldots \right\},$$

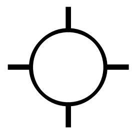
where Q and p are related to each other as above. The main idea of the proof was adopted from Birkhoff [1936] and Kakutani [1936], who independently showed that a topological group is metrizable if it satisfies Hausdorff's first axiom of countability.

3.2.5.5 Every quasi-norm on a linear space X generates a metrizable topology. We call X a **quasi-Banach space** if all Cauchy sequences are convergent; a ***p*-Banach space** is defined analogously.

The first example of a quasi-Banach space was given by Tychonoff [1935, p. 768] when he considered $l_{1/2}$. In particular, he proved that $\|x+y\| \leq 2\big[\|x\| + \|y\|\big]$.

3.2.5.6 A topological linear space X is said to be **locally bounded** if it has a bounded neighborhood of zero. Extending Kolmogoroff's criterion quoted in 3.2.4.2, Hyers [1938, p. 77] showed that X is locally bounded if and only if its topology can be deduced from a quasi-norm, which he obtained as the Minkowski functional of a bounded symmetric neighborhood B.

Every p-norm is continuous in its own topology. However, Hyers [1939, p. 566] observed that this may be false for quasi-norms. Suppose, for example, that the closed unit "ball" $B = \{x : \|x\| \leq 1\}$ of $\mathbb{R}^2$ has the form



3.2.5.7 The decisive defect of quasi-Banach spaces is the possible lack of continuous linear functionals. Day [1940] discovered that $L_p[0,1]$ with $0 < p < 1$ admits only the trivial functional, which vanishes everywhere.

3.2.5.8 Further information about locally bounded spaces can be found in [KAL$^+$] and [ROL$_1$].

3.3 Locally convex linear spaces and duality

3.3.1 Locally convex linear spaces

3.3.1.1 A subset A of a real or complex linear space X may have the following properties.

convex	:	$x_0, x_1 \in A$ and $0 < \lambda < 1$ imply $(1-\lambda)x_0 + \lambda x_1 \in A$,
absolutely convex	:	$x_0, x_1 \in A$ and $\lvert\lambda_0\rvert + \lvert\lambda_1\rvert \leq 1$ imply $\lambda_0 x_0 + \lambda_1 x_1 \in A$,
balanced or **circled**	:	$x \in A$ and $\lvert\lambda\rvert \leq 1$ imply $\lambda x \in A$,
absorbing	:	for $x \in X$ there is $\rho > 0$ such that $x \in \lambda A$ if $\lvert\lambda\rvert \geq \rho$.

The **convex hull** of a subset A is denoted by $\mathrm{conv}(A)$.

3.3.1.2 By a **semi-norm** p we mean a mapping that assigns to every element $x \in X$ a non-negative number $p(x)$ such that the following conditions are satisfied:

(SN$_1$) $p(x+y) \leq p(x) + p(y)$ for $x, y \in X$.

(SN$_2$) $p(\lambda x) = |\lambda| p(x)$ for $x \in X$ and $\lambda \in \mathbb{K}$.

Every semi-norm p is the **Minkowski functional** of its absolutely convex *"Aichkörper" B*:

$$p(x) = \inf\{\rho > 0 : x \in \rho B\} \quad \text{and} \quad B := \{x : p(x) \leq 1\}.$$

In the 3-dimensional case, this result goes back to Minkowski [≤1909, pp. 131–135]; see also Mazur [1933, p. 72] and [KEL$^+$, p. 15].

3.3.1.3 Topological linear spaces in which every point has a basis of convex neighborhoods were first considered by von Neumann [1935, p. 4]. In a paper containing his fixed point theorem, Tychonoff [1935, p. 768] coined the term *"lokal-konvexer Raum."* For any neighborhood of zero such that $U + U \subseteq 2U$ and $x \in X$, von Neumann [1935, p. 18] defined the expressions

$$\|x\|_U^+ = \inf\{\rho > 0 : x \in \rho U\} \quad \text{and} \quad \|x\|_U = \max\{\|+x\|_U^+, \|-x\|_U^+\}.$$

Wehausen [1938, p. 162] and Mackey [1946a, p. 520] referred to $\|x\|_U$ as a *pseudo-norm*.

Here is Dieudonné's definition of a **locally convex linear space**; [1942, p. 110]:

Une fonction p à valeurs finies et ≥ 0, définie dans un espace vectoriel E, est appelée une **semi-norm** *sur E, si elle satisfait aux conditions*

$$p(\lambda x) = |\lambda| p(x), \quad p(x+y) \leq p(x) + p(y).$$

Soit (p_α) une famille de semi-normes sur E; désignons par $U_{\alpha_1,\alpha_2,\ldots,\alpha_n,\varepsilon}$ l'ensemble des $x \in E$ satisfaisant la condition

$$\sup_{1\le i\le n} p_{\alpha_i}(x) \le \varepsilon;$$

*les $U_{\alpha_1,\alpha_2,\ldots,\alpha_n,\varepsilon}$ forment un système fondamental de voisinages de O dans une topologie d'***espace localement convexe** *sur E, lorsqu'on donne à n, aux indices α_i et au nombre $\varepsilon > 0$ toutes les valeurs possibles, à condition que, pour tout $x \neq O$, il existe un indice α tel que $p_\alpha(x) \neq o$. Réciproquement, la topologie de tout espace localement convexe peut être définie de cette manière.*

3.3.2 Weak topologies and dual systems

3.3.2.1 Von Neumann's definition of the weak topology (see 3.1.5) was carried over from Hilbert to Banach spaces by Wehausen [1938, p. 166]:

For each finite set of linear continuous functionals, $F_1,\ldots,F_n$, defined on a normed space L, and for each $\delta > 0$, let the neighborhood

$$U(F_1,\ldots,F_n;\delta) \equiv E_x\big[\,|F_i(x)| < \delta,\ i = 1,\ldots,n\,\big].$$

The set of all such neighborhoods defines the **weak topology** *for L.*

Two weeks earlier, Goldstine [1938, p. 128] showed that every element $x^{**} \in X^{**}$ is the "weak limit" of a net (x_α) in X such that $\|x_\alpha\| \le \|x^{**}\|$. However, no weak topology was involved.

3.3.2.2 Independently of the previous authors, weak topologies on Banach spaces and their duals were introduced by Alaoglu [1940, p. 254], Bourbaki [1938, p. 1702], and Kakutani [1939, p. 169]. The latter wrote, *It seems to me that too little attention has been paid to the weak topologies in Banach spaces, while on the contrary in the theory of Hilbert spaces weak topology plays an essential rôle.*

3.3.2.3 In my opinion, the most important contributions to the theory of weak topology are due to Shmulyan. Here is a quotation from [1940b, pp. 425–426]:

Im folgenden werden wir immer annehmen, dass der lineare topologische Raum E lokal konvex ist. Bezeichnen wir mit E^* die Gesamtheit aller stetigen und additiven Funktionale $f(x)$ in E. Es bezeichne ferner Γ einen beliebigen linearen Unterraum von E^*.

Im Raum E definieren wir mit Hilfe von Γ eine **schwache Topologie***, nämlich mit $U(x_0, f_1,\ldots,f_n;\delta)$ bezeichnen wir die Gesamtheit aller derjenigen Punkte $x \in E$, für welche*

$$|f_1(x-x_0)| < \delta,\ \ldots\ ,|f_n(x-x_0)| < \delta \quad (f_i \in \Gamma; \delta > 0).$$

Jede solche Menge U nennen wir eine Γ-Umgebung des Punktes x_0. Im folgenden werden wir immer voraussetzen, dass Γ die folgende Eigenschaft besitzt:

wenn $f(x) = 0$ für alle $f \in \Gamma$, dann $f = \theta$.

Such subsets Γ are said to be **total**; see [BAN, pp. 42, 58].

Letting $\Gamma := X^*$ yields the weak topology on a Banach space X. On the other hand, if Γ is the canonical image of X in X^{**}, then the **weak* topology** of X^* results.

3.3.2.4 In order to get a symmetric picture, Dieudonné [1940], [1942, p. 112] introduced the concept of a **dual system** $\langle X, Y\rangle$. This is a pair of linear spaces X and Y that are related to each other by a bilinear form $B : X \times Y \to \mathbb{K}$ such that

($\mathbf{B_1}$) if $B(x,y) = 0$ for all $x \in X$, then $y = o$.

($\mathbf{B_2}$) if $B(x,y) = 0$ for all $y \in Y$, then $x = o$.

The typical example of a dual system is a locally convex linear space X together with its dual: $\langle X, X^*\rangle$. For this reason, we mostly replace the symbol $B(x,x^*)$ by $\langle x,x^*\rangle$; see 2.2.5. Local convexity is needed in order to guarantee the existence of sufficiently many functionals, as required in ($\mathbf{B_2}$).

If the linear spaces X and Y form a dual system, then a base of zero neighborhoods of the $\sigma(X,Y)$-*topology* induced by Y on X is given by the absolutely convex sets

$$U(y_1,\dots,y_n,\varepsilon) := \{\, x \in X : |B(x,y_1)| < \varepsilon,\ \dots\ ,\ |B(x,y_n)| < \varepsilon \,\},$$

where $y_1,\dots,y_n \in Y$ and $\varepsilon > 0$. If $y \in Y$, then $B_y : x \mapsto B(x,y)$ defines a $\sigma(X,Y)$-continuous linear functional on X. Dieudonné [1942, p. 114] made the elementary but important observation that *all* $\sigma(X,Y)$-continuous linear functionals are obtained in this way.

The concept of a dual system was independently introduced by Mackey [1945, p. 160], who used the name *regular linear system*.

3.3.2.5 Dual systems have another important historical root, the theory of *perfect sequence spaces*; see [KÖT$_1$, § 30].

Let λ be a linear space consisting of scalar sequences (Stellen) $\mathfrak{x} = (x_1,x_2,\dots)$. The following definition is taken from Köthe/Toeplitz [1934, p. 197]:

Der zu λ **duale** *Raum* λ^* *ist die Menge aller Stellen* $\mathfrak{u} = (u_1,u_2,\dots)$*, für die* $\mathfrak{u}\mathfrak{x} := u_1x_1 + u_2x_2 + \cdots$ *stets absolut konvergiert, falls* $\mathfrak{x} = (x_1,x_2,\dots)$ *irgendeine Stelle aus* λ *ist.*

Eine Stelle $\mathfrak{x}$ *aus* λ *heißt* **Limes** *einer Folge* $\mathfrak{x}^{(1)},\mathfrak{x}^{(2)},\dots$ *von Stellen aus* λ*, wenn für jedes* $\mathfrak{u}$ *aus* λ^* *gilt:* $\lim\limits_{n\to\infty} \mathfrak{u}\mathfrak{x}^{(n)} = \mathfrak{u}\mathfrak{x}$.

footnote: *Man kann übrigens auch eine Topologie in* λ *einführen, die den obigen Konvergenzbegriff liefert. Sei* $\mathfrak{u}_1,\dots,\mathfrak{u}_n$ *eine endliche Zahl gegebener Stellen aus* λ^* *und* $\varepsilon > 0$*, so heiße* **Umgebung** $\mathfrak{U}(\mathfrak{u}_1,\dots,\mathfrak{u}_n;\varepsilon)$ *von* $\mathfrak{x}$ *die Gesamtheit aller Stellen* $\mathfrak{y}$ *aus* λ*, die die Ungleichungen*

$$|\mathfrak{u}_i\,(\mathfrak{x} - \mathfrak{y})| \le \varepsilon \quad (i = 1,\dots,n)$$

erfüllen.

I stress that this remark anticipated the concept of weak topology by some years.

3.3.2.6 The history of weak topologies is summarized in the following table:

von Neumann	February 1929	weak topology on Hilbert space
Köthe/Toeplitz	September 1934	perfect sequence spaces
Goldstine	October 1937	weakly convergent nets
Wehausen	November 1937	weak topology
Alaoglu	February 1938	weak and weak* topology
Bourbaki	June 1938	weak and weak* topology
Kakutani	June 1939	weak and weak* topology
Shmulyan	September 1939	weak Γ-topology
Dieudonné	August 1940	dual systems

3.3.2.7 Wehausen [1938, p. 168] proved that the weak topology of an infinite-dimensional Banach space can never be generated by a norm, and a category argument even shows that it is non-metrizable. The same statement holds for the weak* topology.

On the other hand, we have the following results of Shmulyan [1940b, pp. 435, 439] and Kreĭn/Shmulyan [1940, p. 578]:

The weak topology on the closed unit ball of a Banach space X is metrizable if and only if X^* is separable.

The weak* topology on the closed unit ball of a dual Banach space X^* is metrizable if and only if X is separable.

3.3.3 Separation of convex sets

3.3.3.1 This subsection deals with geometric consequences of the Hahn–Banach theorem. All linear spaces under consideration are supposed to be real. The first result along these lines was proved by Mazur [1933, pp. 71–73]:

Sei E ein linearer und normierter Raum. Ist $x_0 \in E$, so erklären wir die Abbildung $U(x) = x + x_0$ als Translation. Mengen, die man mittels Translation aus linearen Mengen erhält, nennen wir **lineare Mannigfaltigkeiten**. *Eine lineare abgeschlossene Mannigfaltigkeit $H \neq E$ bezeichnen wir als* **Hyperebene**, *wenn es keine lineare abgeschlossene Mannigfaltigkeit $H^* \neq E$ gibt, die H als echte Teilmenge enthält. Man kann leicht beweisen, daß wenn $F(x)$ ein lineares Funktional $\neq 0$ und c eine Zahl ist, so bildet die Menge der Punkte x, für die $F(x) - c = 0$ stattfindet, eine Hyperebene; umgekehrt gibt es für jede gegebene Hyperebene H ein lineares Funktional $F(x)$ sowie eine Zahl c derart, daß die Menge der Punkte x, für die $F(x) - c = 0$ gilt, mit H identisch ist.*

Wir sagen, daß die Menge A auf einer Seite der Hyperebene H liegt, wenn für beliebige $x, y \in A \setminus H$, $x \neq y$, die Strecke xy keinen Punkt der Menge H enthält.

Eine Menge soll ein **konvexer Körper** [convex body] *heißen, wenn sie abgeschlossen und konvex ist, sowie innere Punkte besitzt.*

Enthält die lineare Mannigfaltigkeit R keinen inneren Punkt des konvexen Körpers K, so gibt es eine Hyperebene H derart, daß $R \subseteq H$ *und K auf einer Seite von H liegt.*

Some authors use the term *linear manifold* as a synonym for (linear) *subspace*; see [DUN_1^+, p. 36].

3.3.3.2 The previous statement was generalized by Dieudonné [1941]. Its final form is due to Bourbaki, [BOU_{5a}, Chap. II, p. 69].

Hahn–Banach theorem (geometric version):

Soient E un espace vectoriel topologique, A un ensemble ouvert convexe non vide dans E, M une variété linéaire ne rencontrant pas A. Il existe alors un hyperplan fermé H contenant M et ne rencontrant pas A.

3.3.3.3 The next result is due to Eidelheit [1936]. Kakutani [1937] gave a much simpler proof, and Tukey [1942, pp. 95–97] observed that it easily follows by applying Mazur's theorem to the linear manifold $\{o\}$ and the convex body $\overline{A-B}$.

If A is an open convex set disjoint from the convex set B, then A and B can be separated by a plane, that is to say there exists a continuous linear functional f not identically zero and a real number c such that

$$f(x) \le c \text{ for } x \in A \quad \text{and} \quad f(x) \ge c \text{ for } x \in B.$$

This result was originally established in normed linear spaces. However, it turned out that its proof could easily be adapted to arbitrary topological linear spaces.

3.3.3.4 We proceed with a related result, [BOU_{5a}, Chap. II, p. 73]:

Dans un espace localement convexe tout ensemble convexe fermé est l'intersection des demi-espaces fermés qui le contiennent.

Hence, in a locally convex linear space X, the collection of closed convex subsets is determined by X^*. In other words, locally convex topologies that yield the same dual have the same closed convex subsets. This is, in particular, true for the norm topology and the weak topology of a Banach space; see Mazur [1933, p. 80], though in his terminology "schwach abgeschlossen" means weakly sequentially closed.

3.3.3.5 Let X and Y form a dual system. Given any subset A of X, the **annihilator** is defined by

$$A^\perp := \{\, y \in Y : \langle x, y \rangle = 0 \text{ for all } x \in A \,\}.$$

This concept, which first appeared in connection with the Fredholm alternative, was independently introduced by Dieudonné [1942, pp. 115–116] and Mackey [1945, p. 163]. Related sets are the **polar**

$$A^\circ := \{\, y \in Y : \langle x, y \rangle \le 1 \text{ for all } x \in A \,\} \quad \text{(real case)},$$

$$A^\circ := \{\, y \in Y : \operatorname{Re}\langle x, y \rangle \le 1 \text{ for all } x \in A \,\} \quad \text{(complex case)},$$

and the **absolute polar**

$$A^{\square} := \{\, y \in Y : |\langle x,y \rangle| \leq 1 \text{ for all } x \in A \,\}.$$

Dieudonné [1942, p. 116] and Mackey [1945, p. 163] observed that $A^{\perp\perp}$ is the $\sigma(X,Y)$-closed linear hull of A. The **absolute bipolar** $A^{\square\square}$ yields the $\sigma(X,Y)$-closed and absolutely convex hull of A; see Dieudonné [1950, p. 54] and Dieudonné/Schwartz [1950, p. 64]. The upshot is the **bipolar theorem** [BOU$_{5b}$, Chap. IV, p. 52], which says that $A^{\circ\circ}$ coincides with the $\sigma(X,Y)$-closed convex hull of A and $\{o\}$.

3.3.3.6 The results discussed in the preceding paragraph have their roots in the work of Banach, [BAN, pp. 116–117]:

Un ensemble vectoriel Γ de fonctionnelles linéaires définies dans un espace E du type (B) s'appelle **régulièrement fermé***, lorsqu'il existe pour toute fonctionnelle linéaire f_0 définie dans E, mais n'appartenant pas à Γ, un élément $x_0 \subset E$ qui remplisse les conditions*

$$f_0(x_0) = 1 \quad et \quad f(x_0) = 0 \;\; pour \;\; tout \;\; f \subset \Gamma.$$

The following generalization is due to Kreĭn/Shmulyan [1940, p. 556]:

A set $K \subseteq E^*$ will be called **regularly convex** *if for every $g \notin K$ $(g \in E^*)$ there exists an element $x_0 \in E$ such that*

$$\sup_{f \in K} f(x_0) < g(x_0) \quad (\text{real case}).$$

Clearly, it can always be arranged that $g(x_0) = 1$. Nowadays, we know that the above concepts are superfluous, since a subset of X^* is regularly convex if and only if it is weakly* closed and convex. This fact was originally proved by Shmulyan [1940b, p. 441] with a sophisticated method based on transfinite sequences; see Section 3.6.

3.3.3.7 Of course, every closed subspace M of a Banach space X is itself a Banach space. The dual concept of the **quotient space** X/M is less obvious. In Banach's monograph [BAN, p. 232] it appeared only in passing. I stress that Helly [1921, p. 71] defined a quantity $T(\xi)$, which is just a quotient norm. Hausdorff [1932, pp. 301–302] was the first to use *Quotientenräume* systematically. He also knew the following facts, which were explicitly stated by Dieudonné [1942, p. 124]:

$$M^* \text{ can be identified with } X^*/M^{\perp} \quad \text{and} \quad (X/M)^* \text{ can be identified with } M^{\perp}.$$

3.3.4 Topologies on $\mathfrak{L}(X,Y)$

3.3.4.1 Recall that $\mathfrak{L}(X,Y)$ denotes the collection of all (bounded linear) operators from the Banach space X into the Banach space Y. Based on preliminary work of Arens [1947], the subsequent definition was proposed by Dieudonné/Schwartz [1950, pp. 62–63].

Let $\mathscr{S}$ be a collection of non-empty bounded subsets of X that satisfies the following conditions.

(S_1) For $B_1 \in \mathscr{S}$ and $B_2 \in \mathscr{S}$ there is $B \in \mathscr{S}$ such that $B_1 \cup B_2 \subseteq B$.
(S_2) Every $x \in X$ belongs to some $B \in \mathscr{S}$.

Then the semi-norms
$$p_B(T) := \sup\{\, \|Tx\| : x \in B\} \quad \text{with } B \in \mathscr{B}$$
generate the locally convex **$\mathscr{S}$-topology** on $\mathfrak{L}(X,Y)$. Note that p_B does not change when B is replaced by its closed and convex hull: $\overline{\text{conv}}(B)$.

In the particular case that $Y = \mathbb{K}$, the absolute polars $B^{\square}$ with $B \in \mathscr{B}$ form a fundamental system of zero neighborhoods of the $\mathscr{S}$-topology on $X^* = \mathfrak{L}(X,\mathbb{K})$.

3.3.4.2 Most important are the limiting cases.

The collection of *all* bounded subsets yields the **uniform** operator topology on $\mathfrak{L}(X,Y)$, which is induced by $\|T\| := \sup\{\|Tx\| : \|x\| \le 1\}$.

The collection of all *finite* subsets yields the **strong** operator topology on $\mathfrak{L}(X,Y)$, which (a terminological dilemma) is the weak topology for $Y = \mathbb{K}$.

3.3.4.3 Another useful example is $\mathscr{K}$, the collection of all *compact* subsets. Mazur [1930a] proved that the closed and convex hull of every compact subset is compact as well. Moreover, every compact subset is contained in the closed and convex hull of a null sequence; see [LIND$_1^+$, p. 30]. This result has a lengthy history. For the first time, it was explicitly stated by Grothendieck [GRO$_1$, Chap. I, p. 112]. However, it can be traced back to the work of Banach [1929, Part II, pp. 229–230], [BAN, pp. 119–121], Dieudonné [1942, p. 129], [1950, p. 55], and Dieudonné/Schwartz [1950, p. 84]; see also 3.6.6. These authors showed that the **$\mathscr{K}$-topology** is the finest topology on X^* that induces the weak* topology on every bounded subset. Because of this property one also uses the name **bounded weak* topology**.

3.3.4.4 Finally, we mention the **weak operator topology** on $\mathfrak{L}(X,Y)$, which is generated by the semi-norms
$$p(T) = \sup\big\{|\langle Tx_h, y_k^*\rangle| : \ h = 1,\dots,m \ \text{and} \ k = 1,\dots,n\big\},$$
where $\{x_1,\dots,x_m\}$ and $\{y_1^*,\dots,y_n^*\}$ range over all finite subsets of X and Y^*, respectively.

The weak operator topology is just the $\sigma\big(\mathfrak{L}(X,Y),\mathfrak{F}(Y,X)\big)$-topology associated with the bilinear form
$$\langle S,T\rangle := \text{trace}(ST) \quad \text{for } T \in \mathfrak{L}(X,Y) \text{ and } S \in \mathfrak{F}(Y,X).$$
This **trace duality** will play a major role in Subsection 6.3.4.

3.3.4.5 For a separable Hilbert space H, most of the topologies described above were introduced and named by von Neumann [1930b, pp. 381–384]: *Gleichmäßige, starke und schwache Topologie in $\mathscr{B} = \mathfrak{L}(H)$.*

3.4 Weak* and weak compactness

3.4.1 Tychonoff's theorem

3.4.1.1 The most important example in Fréchet's theory of metric spaces was the set of all real sequences: (E_ω). He [1906, pp. 39–40] defined the écart by

$$d(x,y) := \sum_{k=1}^{\infty} \frac{1}{k!} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|} \quad \text{for } x = (\xi_k) \text{ and } y = (\eta_k)$$

and observed that the sequences $x_n = (\xi_k^{(n)})$ tend to $x = (\xi_k)$ if and only if for every fixed k, the k^{th} coordinates $\xi_k^{(n)}$ tend to ξ_k as $n \to \infty$.

3.4.1.2 Fréchet [1906, pp. 42–43] characterized (relatively!) compact subsets of (E_ω) by coordinatewise boundedness. The main step in his proof was a **diagonal process**: Consider any sequence (x_n) of sequences $x_n = (\xi_k^{(n)})$ such that $|\xi_k^{(n)}| \le c_k$. We begin by selecting a subsequence $(x_n^{(1)})$ of (x_n) such that the first coordinates $\xi_1^{(n,1)}$ of $x_n^{(1)} = (\xi_k^{(n,1)})$ converge to ξ_1 as $n \to \infty$. Suppose that sequences $x_n^{(h-1)} = (\xi_k^{(n,h-1)})$ have already been found for which the coordinates $\xi_1^{(n,h-1)}, \dots, \xi_{h-1}^{(n,h-1)}$ converge to $\xi_1, \dots, \xi_{h-1}$, respectively. By a further selection we pass to sequences $x_n^{(h)} = (\xi_k^{(n,h)})$ such that in addition, the h^{th} coordinates $\xi_h^{(n,h)}$ converge to ξ_h. Finally, it turns out that the sequences $x_n^{(n)} = (\xi_k^{(n,n)})$ converge to $x := (\xi_k)$.

The idea of diagonalization goes back to Ascoli's studies [1884, pp. 545–549] of compact subsets in $C[a,b]$. Without reference to these investigations, Hilbert [1904a, pp. 171–172] used the same method in his foundation of the Dirichlet principle.

Ironically, all techniques based on any diagonal process are commonly attributed to Cantor. This is due to his well-known proof in which he showed that the set of real numbers is uncountable; see Cantor [1892, p. 76].

3.4.1.3 Great efforts were required in order to extend Fréchet's theorem to the non-metrizable case. First of all, one had to find a substitute for (E_ω). Even more important was the question, *Which is the appropriate concept of compactness?* The latter problem has already been discussed in Subsection 3.2.3.

3.4.1.4 Let $(M_i)_{i \in \mathbb{I}}$ be any family of non-empty sets. The **Cartesian** (or direct) **product** $P := \prod_{i \in \mathbb{I}} M_i$ consists of all families $x = (x_i)$ with $x_i \in M_i$. We refer to $\pi_i : x \mapsto x_i$ as the projection of P onto the i^{th} coordinate set M_i. In the case that all factors coincide, $M_i = M$, we use the term **Cartesian power** and the symbol $M^{\mathbb{I}}$.

If the M_i's are topological spaces, then P is equipped with the weakest topology under which all projections π_i become continuous; see Čech [1937, pp. 829–830]. A fundamental system of neighborhoods of the point $x \in M$ is given by the products $\prod_{i \in \mathbb{I}} U_i$, where every U_i is a neighborhood of x_i, but only a finite number of U_i's are different from the whole space M_i. This implies that a net (x_α) in P converges to x if and only if $\pi_i(x_\alpha) \to \pi_i(x)$ for all $i \in \mathbb{I}$. The same criterion holds in terms of filters.

3.4.1.5 The famous **Tychonoff theorem** says that any product of compact spaces is compact as well. In fact, Tychonoff [1930, pp. 548–550] considered only the product of compact intervals. The general case was settled by Čech [1937, p. 830].

The original proofs were based on transfinite induction and on a special type of cluster points (*Konzentrationspunkte*). Thanks to Cartan [1937] and [BOU$_{3a}$, Chap. I, p. 96] this involved approach could be replaced by ultrafilter techniques.

Remember that Fréchet used a diagonal process. However, in the case of uncountably many factors, passing from sequences to subsequences, to subsubsequences, etc. does not make sense. But if we have an ultrafilter, then there is no need and even no possibility for any refinement. We are working with a stable object. Given an ultrafilter $\mathscr{U}$ in P, every projection $\pi_i(\mathscr{U})$ is an ultrafilter in M_i; and these ultrafilters converge, by assumption. If all M_i's are Hausdorff, then their limits x_i are unique. Otherwise, we have to use a choice function that fixes any limit x_i for every $i \in \mathbb{I}$; see also 7.5.12. Clearly, $\mathscr{U}$ converges to (x_i).

3.4.2 Weak* compactness theorem

3.4.2.1 The statement in question says that the closed unit ball of every dual Banach space is weakly* compact. This theorem is of equal significance for Banach space theory as the Hahn–Banach theorem, the principle of uniform boundedness, and the bounded inverse theorem (together with its relatives). Its history splits into a "separable" and a "non-separable" part.

3.4.2.2 If the underlying Banach space X is separable, then the weak* topology on B_{X^*} is metrizable. This observation makes life easier, since we may work with sequential compactness.

3.4.2.3 *Hat eine vollstetige Funktion F die Eigenschaft, absolut genommen für alle Werte der Variablen unterhalb einer endlichen Größe zu bleiben, so besitzt die Funktion F ein Maximum.*

The previous statement of Hilbert [1906a, p. 200] follows from the fact that the closed unit ball of l_2 is weakly compact. In a next step, Riesz [1909a, pp. 466–467], [RIE, p. 57] considered the spaces L_p and l_p with $1 < p < \infty$, respectively. Note that because of reflexivity there is no need to distinguish between weak and weak* compactness.

3.4.2.4 The first clear formulation of the weak* compactness theorem in $C[a,b]^*$ was given by Helly [1912, p. 267]. Referring to Riesz [1911, pp. 49–50], he states:
Aus jeder unendlichen Menge linearer Funktionaloperatoren, deren Maximalzahlen unter einer endlichen Grenze M liegen, läßt sich eine Teilreihe

$$U_1[f], U_2[f], U_3[f], \dots$$

herausgreifen, die gegen eine lineare Operation $U[f]$ konvergiert, deren Maximalzahl nicht größer als M ist.

In view of the Riesz representation theorem, Helly [1912, p. 283] obtained a *Hilfssatz*, namely his famous *Auswahlprinzip*:

Es sei irgend eine unendliche Menge von Funktionen beschränkter Schwankung gegeben, welche folgenden beiden Bedingungen genügen:

1. Es existiert eine Zahl G, so daß die Absolutwerte sämtlicher Funktionen der Menge die Zahl G nirgends übersteigen.

2. Es existiert eine Zahl M, so daß die Totalschwankungen sämtlicher Funktionen der Menge die Zahl M nirgends übersteigen.

Dann läßt sich aus der gegebenen Funktionenmenge eine Teilreihe herausgreifen, deren Funktionen im ganzen Intervall gegen eine Grenzfunktion konvergieren, die ebenfalls von beschränkter Schwankung ist und den Bedingungen 1. und 2. genügt.

3.4.2.5 We proceed with Banach's version, which was the final step in the separable setting, [1929, Part II, p. 232] and [BAN, p. 123]:

Si l'espace E est séparable, tout suite de fonctionnelles linéaires $\{f_n\}$ dont l'ensemble des normes est borné contient une suite partielle faiblement convergente.

Démonstration. Il suffit, en effet, d'extraire de la suite $\{f_n\}$ une suite partielle, convergente dans un ensemble dense dénombrable, ce qui est facile de faire par le procédé de la diagonale.

3.4.2.6 Identifying $x^* \in X^*$ with the scalar family $(\langle x, x^*\rangle)_{x\in X}$, the dual space X^* can be viewed as a subset of the Cartesian power $\mathbb{K}^X$. Moreover, the weak* topology of X^* is induced by the product topology of $\mathbb{K}^X$.

Due to this observation and the trivial fact that linearity is preserved under taking pointwise limits, the weak* compactness theorem is an elementary corollary of Tychonoff's theorem. Therefore priority discussions are rather superfluous. Nevertheless, here is a chronology:

(A$_1$) Alaoglu [1938, p. 196] (announcement, February 1, 1938):
The topology so defined in X^* is such that weakly closed and bounded subsets are bicompact.

(B$_1$) Bourbaki [1938, p. 1703] (announcement, June 8, 1938):
Toute sphère fermée dans $\overline{E}$ est faiblement complète et faiblement compacte.

(A$_2$) Alaoglu [1940, p. 255] (full proof, January 21, 1939).

(B$_2$) Dieudonné [1942, p. 128] (full proof *avec l'autorisation de Bourbaki*).

We also stress that the theorem was independently obtained by Shmulyan [1940b, pp. 427, 438] as well as by Kakutani [1940, p. 63].

To be historically complete, one should speak of the

the $\begin{array}{c}\text{Ascoli–Hilbert–Fréchet–Riesz–Helly–Banach–}\\ \text{Tychonoff–Alaoglu–Cartan–Bourbaki–Shmulyan–Kakutani}\end{array}$ theorem.

The term *weak* compactness theorem* would be more appropriate. However, to please my American colleagues, I will maintain their tradition: **Alaoglu's theorem**.

3.4.2.7 The following examples show that the concepts of **weak* compactness** and **weak* sequential compactness** are incomparable. Moreover, both properties are stronger than **weak* countable compactness**.

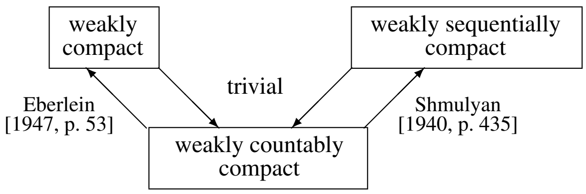
- The sequence of the coordinate functionals e_n^* in the closed unit ball of l_∞^* has no weakly* convergent subsequence.
- If $\mathbb{I}$ is uncountable, then all families (ξ_i) with countable support form a weakly* sequentially compact subset in the closed unit ball of $l_1(\mathbb{I})^*$, which fails to be weakly* compact.

3.4.2.8 Letting $f_x : x^* \mapsto \langle x, x^* \rangle$, we assign to every element $x \in X$ a continuous function f_x on the weakly* compact Hausdorff space B_{X^*}. Then it follows from $\|f_x\| := \sup\{|f_x(x^*)| : x^* \in B_{X^*}\}$ that $x \mapsto f_x$ yields an isometric embedding of X into $C(B_{X^*})$. Hence every Banach space can be obtained as a subspace of some $C(K)$, where K is a suitable compact Hausdorff space; see Alaoglu [1940, p. 258]. In this sense, the $C(K)$'s are *universal*.

The separable case was already treated by Banach and Mazur [BAN, pp. 185–187]. Using the fact that every compact metric space is the continuous image of Cantor's ternary set, they were able to embed X into $C[0,1]$.

3.4.3 Weak compactness and reflexivity

3.4.3.1 In contrast to the case of the weak* topology, the famous **Eberlein–Shmulyan theorem** states that the following concepts of weak compactness coincide:



For separable spaces, this result was obtained by Bourgin [1942, p. 604]. In the course of time, different and simplified proofs were given by Brace [1955], Pełczyński [1964b], and Whitley [1967].

Plainly, every weakly sequentially compact set is stable under passing to limits of weakly convergent sequences. However, it was unknown whether those sets were even weakly closed. Eberlein proved that weakly countably compact *and weakly closed* sets are weakly compact. Based on a verbal argument of Kaplansky, Day was able to remove the superfluous assumption about weak closedness; [DAY, 1st edition, p. 51].

3.4.3.2 The Eberlein–Shmulyan theorem is often phrased in a different form, which needs a slightly modified proof.
For every subset K of a Banach space X the following are equivalent:

- the weak closure of K is weakly compact,
- every sequence in K has a weakly convergent subsequence whose limit belongs to X,
- every sequence in K has a weak cluster point in X.

3.4.3.3 When dealing with the moment problem 2.3.3, Helly [1921, pp. 71–73] discovered the phenomenon of non-reflexivity, which implies that the *epsilon* in the following statement is unavoidable:

Für ein endliches Gleichungssytem

$$\langle x, x_h^* \rangle = \gamma_h \qquad h = 1, \dots, n$$

ist also die Bedingung

$$\Big| \sum_{h=1}^{n} \lambda_h \gamma_h \Big| \le M \Big\| \sum_{h=1}^{n} \lambda_h x_h^* \Big\| \quad \textit{mit beliebigen } \lambda_h$$

hinreichend für die Existenz einer Lösung $x \in X$*, die der Ungleichung* $\|x\| \le M + \varepsilon$ *genügt* ($\varepsilon > 0$ beliebig, aber fest).

Kakutani [1939, pp. 171–172] published a simplified proof (due to Mimura). He also stated a corollary, which has become known as **Helly's lemma**:

Let $X_0(f)$ *be an arbitrary bounded linear functional defined on the conjugate space* $\overline{E}$ *of* E*. Then there exists, for any system of bounded linear functionals* $\{f_i(x)\}$ $(i = 1, 2, \dots, n)$ *defined on* E*, and for any positive number* $\varepsilon > 0$ *a point* $x_0 \in E$ *such that*

$$\|x_0\| \le \|X_0\| + \varepsilon, \quad \textit{and} \quad X_0(f_i) = f_i(x_0) \quad \textit{for } i = 1, 2, \dots, n.$$

3.4.3.4 In this paragraph, the Banach space X is viewed as a subspace of its bidual via the canonical map $K_X : X \to X^{**}$; see 2.2.3. **Goldstine's theorem** says that $B_{X^{**}}$ is the $\sigma(X^{**}, X^*)$-closed hull of B_X. Nowadays, this result is obtained as an immediate consequence of $B_{X^{**}} = B_X^{\circ\circ}$ and the bipolar theorem.

We now sketch the original proof, which was more complicated. In a first step, X^* is isometrically embedded into $l_\infty(B_X)$ by assigning to x^* the family $\boldsymbol{x}^* = (\langle x, x^* \rangle)_{x \in B_X}$. Next, given $x^{**} \in B_{X^{**}}$, the Hahn–Banach theorem yields a norm-preserving extension $\ell \in l_\infty^*(B_X)$. By a straightforward generalization of Hildebrandt's representation theorem 2.2.11, we find an additive set function μ on the power set $\mathscr{P}(B_X)$ whose total variation equals $\|x^{**}\|$ such that

$$\langle x^{**}, x^* \rangle = \ell(\boldsymbol{x}^*) = \int_{B_X} \langle x, x^* \rangle d\mu(x) \quad \text{for } x^* \in X^*.$$

However, the right-hand integral can be expressed as the limit of finite sums

$$\sum_{i=1}^{m} \langle x_i, x^* \rangle \mu(A_i),$$

where $\{A_1,\ldots,A_m\}$ is any partition of B_X and $x_i \in A_i$. Hence

$$\sum_{i=1}^{m} x_i \mu(A_i) \xrightarrow{\sigma(X^{**},X^*)} x^{**}.$$

The index set is directed by the relation $\{B_1,\ldots,B_n\} \succeq \{A_1,\ldots,A_m\}$, which means that every B_j is contained in some A_i.

Another proof of Goldstine's theorem was given by Kakutani [1939], who used Helly's lemma: choose $x_{x_1^*,\ldots,x_n^*,\varepsilon}$ such that

$$\|x_{x_1^*,\ldots,x_n^*,\varepsilon}\| \le 1+\varepsilon \quad \text{and} \quad \langle x_{x_1^*,\ldots,x_n^*,\varepsilon}, x_i^* \rangle = \langle x^{**}, x_i^* \rangle \quad \text{for } i=1,\ldots,n.$$

3.4.3.5 A Banach space X is called **reflexive** if it can be identified with its bidual via the canonical map $K_X : X \to X^{**}$; see 2.2.4.

The first criterion of reflexivity was proved in [BAN, p. 189]:

Etant donné un espace E de type (B) séparable et tel que toute suite $\{x_i\}$ d'éléments de E à normes bornées dans leur ensemble contient une suite partielle faiblement convergente vers une élément de E, l'espace E est équivalent á l'espace $\overline{\overline{E}}$ (conjugué de $\overline{E}$).

Taylor [1940/42] wrote a series of papers that clearly indicate the state of the art concerning weak topologies in the pre-Alaoglu period. His "highlight" was a reformulation of Goldstine's observation [1938, p. 129] that reflexivity is equivalent to the weak completeness of B_X: every bounded weak Cauchy net has a limit.

3.4.3.6 The real breakthrough came with the weak* compactness theorem; see Bourbaki [1938, p. 1703], Kakutani [1940, p. 64], and Shmulyan [1939a, p. 473]:

A Banach space is reflexive if and only if its closed unit ball is weakly compact.

Thanks to the Eberlein–Shmulyan theorem, this criterion remains true when weak compactness is replaced by weak sequential compactness. One half of this observation had already been known to Gantmakher/Shmulyan [1937, p. 92].

3.4.3.7 Kreĭn/Shmulyan [1940, p. 575] proved the remarkable fact that reflexivity is a **three-space property**; see Subsection 6.9.6:

Let M be a closed subspace of a Banach space X. If two of the spaces X, M, and X/M are reflexive, then the third is reflexive as well.

Moreover, reflexivity passes from X to X^* and vice versa. The Russian school attributes this result to Plesner; see 3.6.8. However, there seems to be no written source.

3.4.3.8 A functional $x^* \in X^*$ is called **norm-attaining** if there exists $x_0 \in B_X$ such that $\langle x_0, x^* \rangle = \|x^*\|$.

Because of their weak continuity, all functionals on a reflexive space are norm-attaining. James proved that this property even characterizes reflexivity. In [1957, p. 167], he treated the separable case, while the general solution was given in [1964a, p. 215].

The following criterion of James [1964b, p. 139] is along the same lines:

A weakly closed subset S of a real Banach space X is weakly compact if and only if each continuous functional on X attains its sup on S.

3.4.3.9 Phelps [1957] weakened the concept of reflexivity by assuming that there exist sufficiently many norm-attaining functionals. More precisely, he called a Banach space X **subreflexive** if the set of those functionals is dense in X^* with respect to the norm topology. It came as a surprise when Bishop/Phelps [1961] were able to show that *all* Banach spaces have this property.

Nowadays, the following statement is referred to as the **Bishop–Phelps theorem**:

Let C be a closed bounded subset of a real Banach space X. Then the set of functionals that attain their maximum on C is dense in X^*.

3.4.3.10 Kreĭn/Shmulyan [1940, p. 581] proved that $\mathscr{W}$, the collection of all weakly sequentially compact subsets, is stable under the formation of closed and convex hulls. Of course, thanks to the Eberlein–Shmulyan theorem, we may omit "sequentially."

By an observation of Mackey [1946a, p. 523], there exists a finest locally convex topology on X^* for which the dual can be identified with X, and Arens [1947, p. 790] discovered that this is just the $\mathscr{W}$-topology; see 3.3.4.1.

3.5 Weak sequential completeness and the Schur property

3.5.1 A Banach space X is **weakly sequentially complete** if every weakly Cauchy sequence in X has a weak limit; see [BAN, p. 140]. Reflexive spaces are weakly sequentially complete, and the converse implication holds for spaces with a separable dual; Gantmakher/Shmulyan [1937] (implicit) and Goldstine [1938, pp. 125, 129].

3.5.2 A corollary of Rosenthal's l_1-theorem 5.6.3.8 states that a weakly sequentially complete Banach space is either reflexive or contains an isomorphic copy of l_1.

3.5.3 We say that a Banach space has the **Schur property** if weak convergence and norm convergence of sequences coincide. The naming is a tribute to Schur, who showed that l_1 enjoys the property in question; see 2.4.4. Banach spaces with the Schur property are weakly sequentially complete.

3.5.4 The most interesting result goes back to Steinhaus [1919, pp. 205–210], who implicitly showed that $L_1[a,b]$ is weakly sequentially complete. An explicit proof was given in [BAN, pp. 141–142]. Subsequently, Banach/Mazur [1933, p. 104] observed that $C[a,b]^*$ has this property as well.

3.5.5 Weak sequential completeness and the Schur property are preserved under the formation of closed subspaces. This is not true when we pass to quotients and duals.

The sequences $d_n = (\overbrace{1,\dots,1}^{n},0,\dots)$ form a weakly Cauchy sequence in c_0 without weak limit. Hence every Banach space that contains an isomorphic copy of c_0 fails to be weakly sequentially complete.

3.6 Transfinitely closed sets

3.6.1 Under the motto *"No transfinite numbers wanted,"* Tukey [TUK, p. 84] says, *I believe that transfinite numbers, particularly ordinals, have a proper place only in descriptive theories.* Kakutani [1939, p. 169, footnote] takes the same view: *It is our purpose to avoid, as far as possible, the use of transfinite method in the theory of Banach spaces.*

The following examples support this thesis. The first one is due to Birkhoff [1937, p. 46], and the second is a refinement of the well-known fact that the closed unit ball of l_∞^* fails to be weakly* sequentially compact, since (e_n^*) does not contain any weakly* convergent subsequence.

Throughout, λ denotes a *limit ordinal*.

3.6.2 First of all, we note that weak* convergence of nets in the closed unit ball of $l_\infty(\mathbb{I}) = l_1(\mathbb{I})^*$ means pointwise convergence.

Fix an uncountable index set $\mathbb{I}$, and let B_0, B_ω, and B be the collections of all scalar families $x = (\xi_i)$ with $|\xi_i| \le 1$ for which $\operatorname{supp}(x) := \{i \in \mathbb{I} : \xi_i \neq 0\}$ is finite, countable, and arbitrary, respectively. Then B is the weak* closure of B_0, the limits of weakly* convergent transfinite sequences in B_0 are contained in B_ω, but B_ω fails to be closed under transfinite limits.

Indeed, suppose that $x_\alpha \in B_0$ for $\alpha < \lambda$ and $x_\alpha \xrightarrow{w^*} x$. Let

$$\mathbb{S} := \operatorname{supp}(x) \quad \text{and} \quad \mathbb{S}_\xi := \bigcap_{\xi \le \alpha < \lambda} \operatorname{supp}(x_\alpha).$$

Then $\mathbb{S}$ is contained in the union of the finite sets $\mathbb{S}_\xi$, which form a monotone increasing transfinite sequence. If $|\mathbb{S}_\xi| \le N$ for all $\xi < \lambda$ and some fixed N, then $|\mathbb{S}| \le N$. Otherwise, for $n = 1,2,\dots$, we let ξ_n denote the least ordinal such that $|\mathbb{S}_{\xi_n}| \ge n$. Since $\mathbb{S} = \bigcup_{\xi<\lambda} \mathbb{S}_\xi = \bigcup_{n=1}^\infty \mathbb{S}_{\xi_n}$, the support $\mathbb{S}$ must be countable. Hence $x \in B_\omega$.

In order to prove the second property, we regard $\mathbb{I}$ as an ordinal. Define $d_\alpha = (\vartheta_\xi^{(\alpha)})$ by

$$\vartheta_\xi^{(\alpha)} = 1 \quad \text{if } \xi < \alpha \quad \text{and} \quad \vartheta_\xi^{(\alpha)} = 0 \quad \text{otherwise.}$$

Then $(d_\alpha)_{\alpha<\lambda}$ tends to d_λ. Taking $\lambda = \omega_1$ (the first uncountable ordinal) yields a transfinite sequence in B_ω whose limit is located outside.

3.6.3 The weak* closed hull of $N := \{e_1^*, e_2^*, \dots\}$ in l_∞^* coincides with its Stone–Čech compactification βN; see 4.5.4. More directly: if $\mathscr{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then the functional $x_\mathscr{U}^*$ defined by $\langle x, x_\mathscr{U}^* \rangle := \mathscr{U}\text{-}\lim\limits_n \xi_n$ for $x = (\xi_n) \in l_\infty$ belongs to $\overline{N}^{w^*}$. Thus N fails to be weakly* closed. On the other hand, it is closed under the formation of weakly* transfinite limits.

Let $(e_{k(\alpha)}^*)_{\alpha<\lambda}$ be weakly* convergent, and put $A_n := \{\alpha : k(\alpha) = n\}$. We need consider only two cases:
(1) If some A_n is cofinal, then $(e_{k(\alpha)}^*)_{\alpha<\lambda}$ tends to e_n^*.
(2) Otherwise, all A_n's are non-cofinal. Hence $A_1 \cup \dots \cup A_n$ is bounded above. Let α_n denote the least ordinal such that $\alpha > \alpha_n$ implies $k(\alpha) > n$. Then the sequence (α_n) is non-decreasing and cofinal: $\alpha_{k(\beta)} \ge \beta$. Starting with any β_1, we choose $\beta_{n+1} > \alpha_{k(\beta_n)}$. Obviously, $k(\beta_{n+1}) > k(\beta_n)$. Hence $k(\beta_n) \ge n$, which in turn yields $\beta_{n+1} > \alpha_n$. This proves that (β_n) is cofinal. If $x = (\xi_k) \in l_\infty$ is any sequence such that $\xi_{k(\beta_n)} = (-1)^n$, then $(\langle x, e_{k(\alpha)}^* \rangle) = (\xi_{k(\alpha)})$ does not converge. Contradiction!

3.6.4 In the rest of this section, all Banach spaces are supposed to be real. If the transfinite scalar sequence $(c_\alpha)_{\alpha<\lambda}$ is bounded above, then

$$\limsup_{\alpha<\lambda} c_\alpha := \inf_{\xi<\lambda} \sup_{\xi\le\alpha<\lambda} c_\alpha.$$

Given any bounded transfinite sequence $(x_\alpha^*)_{\alpha<\lambda}$ in X^*, Banach defined a subadditive and positively homogeneous functional by letting

$$p(x) := \limsup_{\alpha<\lambda} \langle x, x_\alpha^* \rangle \quad \text{for all } x \in X;$$

and he observed that there exists at least one $x_\lambda^* \in X^*$ such that

$$\langle x, x_\lambda^* \rangle \le \limsup_{\alpha<\lambda} \langle x, x_\alpha^* \rangle \quad \text{whenever } x \in X. \tag{3.6.4.a}$$

Replacing $+x$ by $-x$ yields

$$\liminf_{\alpha<\lambda} \langle x, x_\alpha^* \rangle \le \langle x, x_\lambda^* \rangle \quad \text{whenever } x \in X. \tag{3.6.4.b}$$

In order to elucidate Banach's idea, we stress that by the separation theorem, the weakly* closed and convex hull $\overline{\text{conv}}^{w^*}(B)$ of any bounded subset B of X^* consists of all $x_0^* \in X^*$ such that

$$\langle x, x_0^* \rangle \le \sup\big\{ \langle x, x^* \rangle : x^* \in B \big\} \quad \text{whenever } x \in X.$$

Therefore (3.6.4.a) is equivalent to

$$x_\lambda^* \in K := \bigcap_{\xi<\lambda} K_\xi, \quad \text{where} \quad K_\xi := \overline{\operatorname*{conv}_{\xi\le\alpha<\lambda}}^{\,w^*}(x_\alpha).$$

By Alaoglu's theorem, the subsets K_ξ are weakly* compact. Moreover, it easily turns out that K is the weakly* closed and convex hull of the set of all cluster points of the transfinite sequence $(x_\alpha^*)_{\alpha<\lambda}$.

3.6.5 A subset F of X^* is said to be **transfinitely closed** if for every weakly* bounded transfinite sequence $(x_\alpha^*)_{\alpha<\lambda}$ in F, there exists $x_\lambda^* \in F$ satisfying (3.6.4.a). For subspaces, this concept is due to Banach, [1929, Part II, p. 228], [BAN, p. 119].

3.6.6 The use of transfinite sequences was in the spirit of the late 1920s. Nevertheless, in view of Birkhoff's example the success of Banach's *ad hoc definition* looks like a mystery. The following observation shows the modern background: the transfinitely closed subsets are just the closed subsets of the $\mathscr{K}$-topology; see 3.3.4.3. Here is a streamlined version of Banach's original reasoning; see [1929, Part II, pp. 229–230], [BAN, pp. 119–121]:

Take any transfinitely closed subset F. Plainly, F is closed in the norm topology. Thus, given $x_0^* \in \complement F$, we may pick $c_1 > 0$ such that $\|x^* - x_0^*\| \le c_1$ implies $x^* \in \complement F$. Put $c_n := c_1 + n - 1$.

Suppose that we have already found *finite* subsets $A_1, \dots, A_{n-1}$ of X such that

$$\|x^* - x_0^*\| \le c_n \quad \text{and} \quad |\langle x, x^*\rangle - \langle x, x_0^*\rangle| \le c_k \|x\| \quad \text{for all } x \in A_k \text{ and } k = 1, \dots, n-1$$

imply $x^* \in \complement F$.

In the next step, we replace $\|x^* - x_0^*\| \le c_n$ by $\|x^* - x_0^*\| \le c_{n+1}$ and add

$$|\langle x, x^*\rangle - \langle x, x_0^*\rangle| \le c_n \|x\| \quad \text{for all } x \in A_n.$$

Again it should follow that $x^* \in \complement F$. Let $\mathscr{A}_n$ denote the collection of all (finite or infinite) subsets A_n for which the required implication holds. Clearly, we have $X \in \mathscr{A}_n$, since

$$|\langle x, x^*\rangle - \langle x, x_0^*\rangle| \le c_n \|x\| \quad \text{for all } x \in X$$

implies $\|x^* - x_0^*\| \le c_n$. Hence $x^* \in \complement F$.

Choose A_n as small as possible. In order to show that the cardinality $\lambda := |A_n|$ is finite, we assume the contrary. Then λ is a limit ordinal. Fix a well-ordering $A_n = \{x_\xi : \xi < \lambda\}$ and put $A_n^{(\alpha)} := \{x_\xi : \xi < \alpha\}$ for $\alpha < \lambda$. By $|A_n^{(\alpha)}| \le \alpha < \lambda$, we have $A_n^{(\alpha)} \notin \mathscr{A}_n$. Hence there exists $x_\alpha^* \in F$ such that

$$\|x_\alpha^* - x_0^*\| \le c_{n+1}, \tag{3.6.6.a}$$

$$|\langle x, x_\alpha^*\rangle - \langle x, x_0^*\rangle| \le c_k \|x\| \quad \text{for all } x \in A_k \text{ and } k = 1, \dots, n-1 \tag{3.6.6.b}$$

as well as

$$|\langle x, x^*\rangle - \langle x, x_0^*\rangle| \le c_n \|x\| \quad \text{for all } x \in A_n^{(\alpha)}.$$

The last formula means that

$$|\langle x_\xi, x_\alpha^*\rangle - \langle x_\xi, x_0^*\rangle| \le c_n \|x_\xi\| \quad \text{for all } \xi < \alpha. \tag{3.6.6.c}$$

Since F is transfinitely closed, we find $x_\lambda^* \in F$ such that

$$\langle x, x_\lambda^*\rangle \le \limsup_{\alpha<\lambda} \langle x, x_\alpha^*\rangle \quad \text{for all } x \in X.$$

Passing to the limit, (3.6.6.a) and (3.6.6.b) yield

$$\|x_\lambda^* - x_0^*\| \le c_{n+1} \quad \text{and} \quad |\langle x, x_\lambda^*\rangle - \langle x, x_0^*\rangle| \le c_k \|x\| \quad \text{for all } x \in A_k \text{ and } k = 1, \dots, n-1.$$

Moreover, (3.6.6.c) gives $|\langle x_\xi, x_\lambda^*\rangle - \langle x_\xi, x_0^*\rangle| \le c_n \|x_\xi\|$ for all $\xi < \lambda$ or

$$|\langle x, x_\lambda^*\rangle - \langle x, x_0^*\rangle| \le c_n \|x\| \quad \text{for all } x \in A_n.$$

Contradiction! Therefore A_n is indeed finite.

By induction, we get $A_1, A_2, \dots$. Since with respect to any enumeration,

$$A := \bigcup_{k=1}^{\infty} \Big\{ \tfrac{1}{c_k \|x\|} x : x \in A_k \Big\}$$

is a null sequence, $p_A(x^*) := \sup\{ |\langle x, x^*\rangle| : x \in A \}$ yields a semi-norm of the $\mathscr{K}$-topology. Let x^* be a functional on X such that $p_A(x^* - x_0^*) \le 1$. Then

$$|\langle x, x^*\rangle - \langle x, x_0^*\rangle| \le c_k \|x\| \quad \text{for all } x \in A_k \text{ and } k = 1, 2, \dots .$$

Choose n so large that $\|x^* - x_0^*\| \le c_n$. Now it follows that $x^* \in \complement F$. Consequently, the complement of any transfinitely closed subset F is open in the $\mathscr{K}$-topology.

Using Alaoglu's theorem and the fact that the weak* topology coincides with the $\mathscr{K}$-topology on every closed ball gives the converse implication.

3.6.7 By 3.3.3.4, the preceding result can be improved for convex subsets. Then transfinite closedness and weak* closedness coincide. As an important consequence, it follows that a convex subset is weakly* closed if and only if its intersections with all closed balls are weakly* closed. This is the famous **Kreĭn–Shmulyan theorem**, [1940, p. 564]. By the way, it suffices when the intersections with all balls rB_X, $r > 0$ centered at o are weakly* closed; see [MEG, p. 242].

3.6.8 According to Gantmakher/Shmulyan [1937, p. 92], the first criterion of reflexivity was discovered by Plesner; see 8.2.3.3:

A real Banach space X is reflexive if and only if for every transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in the closed unit ball B_X there exists $x_\lambda \in B_X$ such that

$$\langle x_\lambda, x^*\rangle \le \limsup_{\alpha<\lambda} \langle x_\alpha, x^*\rangle \quad \text{whenever } x^* \in X^*.$$

As a consequence, it follows that reflexivity is inherited not only from a Banach space to its dual but also from the dual to the original space.

4

Classical Banach Spaces

This book is devoted to the history of Banach spaces and their linear operators. Lattices, measures, and integrals will be treated only in so far as these concepts are needed for our purpose. The same holds for the theory of Banach algebras. Even under this restriction, more room has been spent than I like. On the other hand, these are fascinating subjects.

4.1 Banach lattices

Besides their algebraic and topological structure, the classical Banach spaces $C[a,b]$, $L_p[a,b]$, and l_p have another remarkable property: they are partially ordered. Therefore it is advisable to start with a historical sketch of the theory of Banach lattices.

4.1.1 By a **preorder** on a set M we mean a binary relation $a \le b$ (read: a is less than or equal to b) such that $a \le b$ and $b \le c$ imply $a \le c$ and such that $a \le a$ for all $a \in M$. If, moreover, it follows from $a \le b$ and $b \le a$ that $a = b$, then $\le$ is said to be a **partial order**.

In general, arbitrary elements a and b need not be comparable. If there always holds at least one of the relations $a \le b$ and $b \le a$, then the partial order is called **total** or **linear**.

Note that these concepts reflect properties of the real line. Hence, for a long time, only linear orders were of interest. Partial orderings first occurred in logic as well as in the work of Hausdorff, when he considered the asymptotic behavior of functions, [1909, p. 300]:

Der Fall liegt also formal folgendermaßen: in einer Menge bestehen zwischen je zwei Elementen f,g eine und nur eine der vier Relationen $f<g$, $f=g$, $f>g$, $f\|g$.

Here $f\|g$ means *“incomparable.”*

The term *teilweise geordnete Menge* was coined in [HAUS$_1$, p. 139].

4.1.2 Let A be any non-empty subset of a partially ordered set M. We refer to $b \in M$ as an **upper bound** of A if $a \le b$ for all $a \in A$. In this case, A is said to be **bounded above**. If A has a **least upper bound** or **supremum**, this element is denoted by $\sup A$. *Dual* notions are obtained by reversing the order: **lower bound**, **bounded below**, **greatest lower bound**, or **infimum**.

A partially ordered set M is called **Dedekind complete** or **reticulated** if every non-empty bounded above (below) subset has a supremum (infimum). In the case that these conclusions hold only for countable subsets, one says that M is **Dedekind countably complete** or **σ-reticulated**.

Dedekind completeness admits a symmetric characterization:

Non-empty subsets A and B such that $a \le b$ for $a \in A$ and $b \in B$ are separated by an element $x \in M$, which means that $a \le x \le b$ whenever $a \in A$ and $b \in B$. This property justifies the use of “Dedekind” in naming the preceding concepts, since we have a weak form of *Dedekind cuts*.

4.1.3 A partially ordered set is said to be a **lattice** if every two-element set $\{a,b\}$ has a supremum (**maximum**) and an infimum (**minimum**), which are denoted by

$$a \vee b = \sup\{a,b\} = \max\{a,b\} \quad \text{and} \quad a \wedge b = \inf\{a,b\} = \min\{a,b\},$$

respectively.

A **lattice** can also be defined as a set M with operations $\vee$ and $\wedge$ such that certain formulas, among others

$$a \wedge (a \vee b) = a \quad \text{and} \quad (a \wedge b) \vee b = b,$$

are satisfied. In this case, a partial order is obtained by letting $a \le b$ if and only if $a = a \wedge b$, which is equivalent to $a \vee b = b$.

A lattice M is called **distributive** if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for any choice of a, b, and c in M. It suffices to check one of these formulas; the other follows by duality. The important class of **Boolean lattices** will be treated in 4.8.1.1.

4.1.4 Lattice theory has various roots. On the one hand, it can be traced back to the work of Peirce [1880] and Schröder [SCHR, Band I] concerned with *symbolic logic*. On the other hand, Dedekind [1897] introduced the concept of a *Dualgruppe*, where the term *Gruppe* was not used in the modern sense. As an example, he considered $\mathbb{N}$ under the operations

$$a \wedge b := \text{greatest common divisor} \quad \text{and} \quad a \vee b := \text{least common multiple}.$$

The decisive breakthrough came only in the early 1930s. At least four approaches should be mentioned: Albert Bennett’s theory of **semi-serial order**, Fritz Klein’s **Verbands**theorie, Garrett Birkhoff’s **lattice** theory, and Ore’s **structure** theory. For details the reader is referred to the Enzyklopädie-Artikel of Hermes/Köthe [HERM$_1^+$] and Mehrten’s excellent historical account [MEH$^\bullet$].

I emphasize the fact that Bennett also contributed to the creation of Banach space theory; see Section 1.6. Here is his basic definition concerning partial order and lattice structure, [1930, pp. 418–419]:

The subjects of serial [read: linear] *order and of the calculus of classes have proved particularly attractive to postulationalists.*

A class K with at least two distinct elements will be said to have **semi-serial order** *if it satisfies postulates I–V given below.*

$\leq$ *is a dyadic relation among elements of K.*

I. (*Reflexiveness*) *$a \leq a$ whenever a belongs to the class.*
II. (*Restricted Symmetry*) *If $a \leq b$, and also $b \leq a$, then $a = b$.*
III. (*Transitivity*) *If $a \leq b$, and $b \leq c$, then $a \leq c$.*
IV. (*Predecessor*) *If $a \neq b$, and if neither $a \leq b$, nor $b \leq a$, then there is an element m, such that*
(1) *$m \leq a$, and $m \leq b$,*
(2) *if $x \neq m$, is such that $x \leq a$, and $x \leq b$, then $x \leq m$.*
V. (*Successor*) *If $a \neq b$, and if neither $a \leq b$, nor $b \leq a$, then there is an element n, such that*
(1) *$a \leq n$, and $b \leq n$,*
(2) *if $y \neq n$, is such that $a \leq y$, and $b \leq y$, then $n \leq y$.*

4.1.5 A partial ordering on a real linear space X is supposed to fulfill the following conditions of compatibility.

(PO$_1$) If $x,y,z \in X$, then $x \geq y$ implies $x+z \geq y+z$.

(PO$_2$) If $x,y \in X$ and $\lambda \in \mathbb{R}$, then $x \geq y$ and $\lambda \geq 0$ imply $\lambda x \geq \lambda y$.

The collection of all **positive** elements $x \geq \mathrm{o}$ is a convex **cone**.

A linear space consisting of real-valued functions has a natural order: $f \geq g$ if $f(t) \geq g(t)$ for all or for almost all points t.

4.1.6 A **linear lattice** or **vector lattice** or **Riesz space** is a partially ordered real linear space X in which every pair of elements x and y has a least upper bound, denoted by $\sup\{x,y\}$ or $x \vee y$. Then there also exists a greatest lower bound: $\inf\{x,y\}$ or $x \wedge y$. In particular,

$$\underset{\textit{positive part}}{x_+ := \sup\{+x,\mathrm{o}\}}, \quad \underset{\textit{negative part}}{x_- := \sup\{-x,\mathrm{o}\}} \quad \text{and} \quad \underset{\textit{absolute value}}{|x| := \sup\{+x,-x\}}.$$

We have

$$|x| \geq \mathrm{o}, \quad |\lambda x| = |\lambda||x|, \quad |x+y| \leq |x| + |y|$$

and $x = x_+ - x_-$. In a paper [1936, p. 642], which has become a milestone in the theory of Riesz spaces, Freudenthal showed that every linear lattice is distributive.

A linear lattice is **Archimedean** if $x \leq \mathrm{o}$ whenever the set of all nx with $n = 1,2,\ldots$ is bounded above.

4.1.7 By an **ideal** in a linear lattice X we mean a linear subset M such that $a \in M$, $x \in X$, and $|x| \le |a|$ imply $x \in M$. If, moreover, $\sup A$ belongs to M for all subsets A of M that have a supremum in X, then M is called a **band**.

Elements x and y are said to be **disjoint** if $|x| \wedge |y| = \mathsf{o}$. Then, according to [SCHAE, p. 52],

$$|x| \vee |y| = |x| + |y| = |x+y| = |x-y|. \tag{4.1.7.a}$$

One associates with every subset A its **disjoint complement**

$$A^{\perp} := \{ x \in X : |x| \wedge |y| = \mathsf{o} \text{ for all } y \in A \},$$

which is a band. In an Archimedean linear lattice, $A^{\perp\perp}$ yields the smallest band containing A. In particular, A is a band if and only if $A = A^{\perp\perp}$.

Clearly, $A \cap A^{\perp} = \{\mathsf{o}\}$. If $X = A + A^{\perp}$, then X is the direct sum of A and $A^{\perp}$. In this case, we refer to A as a **projection band**. This name is due to the fact that the decomposition $x = u + v$ with $u \in A$ and $v \in A^{\perp}$ defines a so-called **band projection** $P : x \mapsto u$.

In a Dedekind complete linear lattice every band is a projection band. This fundamental result goes back to Riesz [1928], [1940, p. 186]; its final version is due to Bochner/Phillips [1941, p. 318]. The historical root is a theorem of Lebesgue to be discussed in 4.2.5.5.

I stress the analogy with orthogonal decompositions of Hilbert spaces as described in 1.5.9. One even uses the same symbol, namely $\perp$; quite often disjoint elements are referred to as **orthogonal**, and $A^{\perp}$ is called the **orthogonal complement**; see [NAK, p. 11] and [SCHAE, p. 50]. Luckily, there is no danger of confusion.

4.1.8 A linear functional ℓ on a linear lattice X is **order bounded** if it is bounded on all intervals: $|\ell(x)| \le c_{a,b}$ whenever $a \le x \le b$. The collection of these functionals is referred to as the **order dual** $X^{\vee}$. We say that ℓ is **positive** if $\ell(x) \ge 0$ whenever $x \ge \mathsf{o}$. With respect to this natural ordering, $X^{\vee}$ becomes a Dedekind complete linear lattice. Indeed, let A be any subset of $X^{\vee}$ that is bounded above, and fix $x \ge \mathsf{o}$. Then, following Riesz [1940, pp. 178–180], we define the functional $\ell_0 := \sup A$ by

$$\ell_0(x) := \sup \left\{ \sum_{k=1}^{n} \ell_k(x_k) \,:\, \ell_k \in A,\ x_k \ge \mathsf{o},\ \sum_{k=1}^{n} x_k = x,\ n \in \mathbb{N} \right\}.$$

In particular,

$$\ell_+(x) := \sup \Big\{ \ell(y) \,:\, \mathsf{o} \le y \le x \Big\}. \tag{4.1.8.a}$$

4.1.9 *Wenn $f(x)$ eine Funktion beschränkter Schwankung ist, so kann man bekanntlich zwei im Intervall $[a,b]$ niemals abnehmende Funktionen $p(x)$ und $q(x)$ finden, so daß $f(x) = p(x) - q(x)$ ist, und außerdem $p(x)$ überall dort konstant bleibt, wo $q(x)$ wächst, und umgekehrt.*

This classical result, which is attributed to Jordan, was used by Helly [1912, p. 287] when he dealt with the Riesz representation theorem. At the ICM 1928 in Bologna, Riesz stated that functionals $\ell \in C[a,b]^*$ can be decomposed in the form $\ell = \ell_+ - \ell_-$, where $\ell_+, \ell_- \in C[a,b]^*$ are positive. Usually, Riesz's contribution [1928] is considered as the starting point of the theory of linear lattices. However, Riesz on his part referred to the work of Daniell, and so the invention of the concept of a linear lattice could as well be placed at an earlier date.

Daniell [1918, pp. 279–280]:

Two symbols have been taken over from symbolic logic:

$f(p)$ is said to be the **logical sum** *of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the greater of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically*

$$f(p) = f_1(p) \vee f_2(p).$$

$f(p)$ is said to be the **logical product** *of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the less of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically*

$$f(p) = f_1(p) \wedge f_2(p).$$

Daniell [1918, p. 282] suggested also the definition (4.1.8.a).

4.1.10 By a real **Banach lattice** we mean a real linear lattice X that is complete with respect to a norm. The different structures are related by the following condition, [BIRK, p. 116] and [NAK, p. 126]:

(M) $\|x\| \le \|y\|$ whenever $|x| \le |y|$.

It turns out that the operations $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ are continuous. Moreover, every Banach lattice is Archimedean.

Banach lattices were first considered by Kantorovich [1937, p. 153], who instead of **(M)** used the weaker property

(M$_+$) $\|x\| \le \|y\|$ whenever $\mathsf{o} \le x \le y$.

In his classical papers [1941a, p. 524] and [1941b, p. 995], Kakutani required the condition

(M$_0$) $\|x+y\| = \|x-y\|$ if $x \wedge y = \mathsf{o}$.

Obviously, **(M$_0$)** implies that $x = x_+ - x_-$ and $|x| = x_+ + x_-$ have the same norm. The latter property is even equivalent to **(M$_0$)**. Indeed, by

$$x \wedge y = \frac{x+y}{2} - \frac{|x-y|}{2},$$

it follows from $x \wedge y = \mathsf{o}$ that $x+y = |x-y|$.

We have **(M$_0$) + (M$_+$) = (M)**. Nowadays, **(M)** has been generally accepted as the most appropriate requirement, especially since it is satisfied for all concrete norms.

4.1.11 For every Banach lattice X, the *Banach* dual X^* and the *order* dual $X^\vee$ coincide. Thus it follows from 4.1.8 that X^* is Dedekind complete.

Obviously, $X^\vee \subseteq X^*$ is a consequence of **(M)**. The reverse inclusion can already be found in [BIRK, p. 118]. Unaware of this fact, Day attributed the result in question to Kaplansky (conversation about 1949); see [DAY, 1st edition, pp. 98–99].

As (more or less) observed in [NAK, pp. 139–140], the embedding $K_X : X \to X^{**}$ defined in 2.2.3 preserves the lattice operations; for example $|K_X x| = K_X|x|$. Hence every Banach lattice admits a Dedekind complete extension.

4.1.12 Birkhoff [BIRK, p. 117] characterized norm convergence in a Banach lattice exclusively in terms of the lattice structure. This implies the following result, which was explicitly stated by Goffman [1956, p. 537]:

Any two norms under which a linear lattice becomes a Banach lattice are equivalent. Compare with 2.5.5 and 4.10.2.8.

4.1.13 There is a "positive" **Hahn–Banach theorem;** [ZAA$_2$, p. 137]:

If X_0 is a linear sublattice of a Banach lattice X, then every positive functional $\ell_0 \in X_0^*$ admits a norm-preserving positive extension $\ell \in X^*$.

Note that $x \mapsto \|x_+\|$ is a sublinear functional on X. Since

$$\ell_0(x) = \ell_0(x_+) - \ell_0(x_-) \le \ell_0(x_+) \le \|\ell_0\|\|x_+\| \quad \text{for } x \in X_0,$$

the ordinary Hahn–Banach theorem 2.3.5 yields the required extension. Indeed, ℓ is positive because of

$$\ell(-x) \le \|\ell_0\|\|(-x)_+\| = 0 \quad \text{whenever } x \ge o,$$

and its norm can be estimated as follows:

$$|\ell(x)| \le \ell(|x|) \le \|\ell_0\|\||x|_+\| = \|\ell_0\|\|x\| \quad \text{for } x \in X.$$

A more general result is due to Mazur/Orlicz [1953, p. 158]:

Let ℓ_0 be a positive linear functional on a subspace X_0 of a partially ordered Banach space X. Suppose that

$$\ell_0(x) \le q(x) := \inf_{u \ge o} \|x + u\| \quad \text{for } x \in X_0,$$

where the right-hand infimum ranges over all positive elements $u \in X$. Then ℓ_0 admits a positive extension ℓ with $\|\ell\| \le 1$.

The trick "how to preserve positivity" was already discovered by Banach [BAN, pp. 29–34].

The preceding theorems imply that the functional $\ell_0(x) := \lim_{k\to\infty} \xi_k$, which is defined for convergent sequences $x = (\xi_k)$, can be extended to all of l_∞ such that positive sequences have a positive *Banach limit*; see 6.9.11.6.

4.1.14 Almost all classical Banach spaces are lattices. A counterexample is $C^m[a,b]$, the set of m-times continuously differentiable functions on $[a,b]$ with $m = 1,2,\dots$, which was introduced by Banach [1922, p. 134]; his notation: $\mathscr{C}^m\mathscr{C}$. Kantorovich [1937, pp. 155–156] showed that $L_p[a,b]$ with $1 \le p \le \infty$ is Dedekind complete, while $C[a,b]$ is not.

4.1.15 Kantorovich [1940] was among the first who studied operators between Banach lattices. Of particular significance is the concept of positivity: $x \ge o$ implies $Tx \ge o$. Sorry to say, this interesting subject is beyond the scope of my book.

4.1.16 Apart from the remarkable Chapter VII in Birkhoff's book [BIRK], the first monographs on partially ordered linear spaces and linear lattices were published by Kantorovich/Vulikh/Pinsker [KAN_2^+] as well as by Nakano [NAK], in 1950. Schaefer wrote the modern standard treatise, [SCHAE]. For further information, the reader is referred to [ALI^+], [FREM_1], [FREM_2, Vol. III], [LIND_2^+], [MEY-N], [VUL], [LUX^+], and [ZAA_2]. Historical comments are rare.

4.2 Measures and integrals on abstract sets

Borel [BOR_1], Lebesgue [1902], and Radon [1913] had founded the theory of measures and integrals on $\mathbb{R}^n$. In the 1920s, the most prominent textbooks about this subject were [CARA] and [HAHN_1]. Fréchet [1915] was the first to consider the basic concepts in an abstract setting.

Historical accounts are given in [$\text{DAL}^{+\bullet}$], [$\text{HAW}^\bullet$], [$\text{PES}^\bullet$], and [$\text{PIER}^\bullet$, Chap. 3]. I also refer to an article of Pier [$1994^\bullet$].

4.2.1 Set-theoretic operations

4.2.1.1 To begin with, I comment on a formal but nevertheless important aspect: terminology and notation.

Though the significance of a mathematical theory is determined by its subject, the scope of its basic concepts, and in particular, the depth of its theorems, the role of terminology and notation should not be underestimated.

Quite likely, the following quotation from Schröder's *Vorlesungen über die Algebra der Logik* will be appreciated only by native German speakers. Nevertheless, I cannot resist presenting some of his marvelous formulations.

[SCHR, p. 194]:

Neue Zeichen und Namen zu erfinden ist ja in der That nicht schwer, und was die Namen betrifft, so hat gerade die Philosophie hierin die Welt schon mit grossartigen Leistungen beglückt.

This was supposed to be a serious statement?

[SCHR, p. 128]:

Es kann daher das Zeichen $=$ hier als „einerlei mit“, oder, wenn man will als „identisch“ gelesen werden; indessen verschlägt es nicht, wenn wir uns bequemer der allgemeinen Übung anschliessen, dasselbe einfach als „gleich“ zu lesen.

[SCHR, p. 129]:

Das andere Zeichen $\subset$ lese man: „untergeordnet“, auch, wenn man will: „subordinirt“. Es ist ähnlich gestaltet, gewissermassen nachgebildet dem „Ungleichheitszeichen“ der Arithmetik, nämlich dem Zeichen $<$ für „kleiner (als)“.

[SCHR, p. 132]:

Die Kopula „ist“ wird bald die eine, bald die andere der beiden Beziehungen ausdrücken, die wir mittels der Zeichen $\subset$ und $=$ dargestellt haben. Zu ihrer Darstellung wird sich deshalb ein aus den beiden letzten zusammengesetztes Zeichen € als ein ohne weiteres, sozusagen nunmehr von selbst, verständliches und dem Gedächtnis sich einprägendes vor allen andern empfehlen. Ausführlichst wird dieses Zeichen als „untergeordnet oder gleich“ zu lesen sein.

Hence, more than hundred years ago, Schröder paved the way of the euro,

€

This fact supports the widespread hope that mathematical discoveries may become useful, sooner or later.

In the course of time the sign $=$ in € was shifted to the bottom: $\subseteqq$, now $\subseteq$. In contrast to the agreement made in this text, many authors use $\subset$ to denote not only proper inclusions but also equalities.

The well-known membership symbol ε was coined by Peano in 1889, while Russell created the stylized version $\in$.

[PEA$^{\bowtie}$, Vol. II, p. 27]:

Signo K significatur **classis**, *sive entium aggregatio. Signum ε significat* **est**. *Ita $a \varepsilon K$ significat a* **est quaedam classis** *K.*

It is remarkable that this notation was not used in functional analysis, general topology and measure theory before the mid 1930s. For example, in [ALEX$^+$, p. 24] we can read:

Es bezeichnet $p \subset A$, daß der Punkt p ein Element der Menge A ist; zwischen einem Punkt und der aus diesem einzigen Punkt bestehenden Punktmenge wird nicht unterschieden.

A similar point of view can be found in Banach's monograph [BAN, pp. 2, 13], where the sign $\subset$ also had a twofold meaning: $x(t) \subset (L^{(p)})$ and $G' \subset G$. On the other hand, in the presentation of the Banach–Tarski paradox [1924] expressions of the form $X \in K$ occurred. Doubtless, this was the influence of the logician.

4.2.1.2 The following quotation is taken from [CAN, p. 52]:

Gehören alle Punkte einer Menge P zu einer anderen Menge Q, so sagen wir: P sei in Q enthalten oder P sei ein Divisor von Q, Q ein Multiplum von P. Sind $P_1, P_2, P_3, \ldots$ *irgend welche Punktmengen in endlicher oder unendlicher Anzahl, so gehört zu ihnen sowohl ein kleinstes gemeinsames Multiplum, welches wir mit:* $\mathfrak{M}(P_1, P_2, P_3, \ldots)$ *bezeichnen und welches die Menge ist, die aus allen verschiedenen Punkten von* $P_1, P_2, P_3, \ldots$ *besteht und sonst keine anderen Punkte als Elemente besitzt, – wie auch ein grösster gemeinsamer Divisor, den wir* $\equiv \mathfrak{D}(P_1, P_2, P_3, \ldots)$ *setzen und welcher die Menge der Punkte ist, die allen* $P_1, P_2, P_3, \ldots$ *gemeinsam sind.*

For almost two decades, the notation was adopted from [CARA, pp. 22–23] and [HAHN_1, pp. 1–2]:

$A \dot{+} B$ stood for the *union* (Vereinigung) of A and B, while $A + B$ without dot meant that A and B were disjoint (Summe); the *intersection/product* was denoted by $A \bullet B$ or AB, and the *difference* by $A - B$. Finally, I mention the *symmetric difference* $A \underset{\bullet}{+} B := (A - B) + (B - A)$. The symbols $\dot{+}$ (koplus) and $\underset{\bullet}{+}$ (kontraplus) were not commonly used.

In the theory of linear spaces, the expressions $A + B$ and $A - B$ have a completely different meaning. Hence the set-theoretic operations are nowadays denoted by $A \cup B$ (union), $A \cap B$ (intersection), $A \setminus B$ (difference), and $A \triangle B$ (symmetric difference).

The following footnote from [WEIL_1, p. 6] indicates that the modern notation prevailed mainly through the influence of Bourbaki's *Éléments*:

Nous notons par le signe $\cap$ *l'intersection, par* $\cup$ *la réunion de deux ensembles; par* $\complement(A)$ *le complémentaire d'une ensemble A; les signes* $\subset$, $\in$ *signifient, comme d'habitude, ≪contenu dans≫, ≪élément de≫. Ces notations sont conformes à l'usage de N. Bourbaki et de ses collaborateurs.*

Moreover, [$\text{WEIL}^{\bullet}$, p. 114]:

Wisely, we had decided to publish an installment establishing the system of notation for set theory [BOU_{1a}]*, rather than wait for the detailed treatment that was to follow: it was high time to fix these notations once and for all, and indeed the ones we proposed, which introduced a number of modifications to the notations previously in use, met with general approval. Much later, my own part in these discussions earned me the respect of my daughter Nicolette, when she learnt the symbol* $\emptyset$ *for the* **empty set** *in school and I told her that I had been personally responsible for its adoption. The symbol came from the Norwegian alphabet, with which I alone among the Bourbaki group was familiar.*

The signs $\cap$ and $\cup$ were proposed by Peano [$\text{PEA}^{\bowtie}$, Vol. II, p. 4], in 1888:

Colla scrittura $A \cap B \cap C \cap \cdots$ *intenderemo la* **massima** *classe contenuta nelle classi* $A, B, C, \ldots$, *ossia la classe formata da tutti gli enti che sono ad un tempo A e B e C, ecc. Il segno* $\cap$ *si leggerà* **e**.

Colla scrittura $A \cup B \cup C \cup \cdots$ *intenderemo la* **minima** *classe che contiene le classi* $A, B, C, \ldots$*, ossia la classe formata dagli enti che sono o A o B o C, ecc. Il segno* $\cup$ *si leggerà* ***o****.*

4.2.2 Measures

4.2.2.1 The domain of a measure on a set M is a collection $\mathscr{M}$ of subsets stable under certain set-theoretic operations. The largest example is the **power set** $\mathscr{P}(M)$.

In the old-fashioned terminology of Hausdorff, a **Ring** was required to have the properties that $A, B \in \mathscr{M}$ implies $A \cup B \in \mathscr{M}$ and $A \cap B \in \mathscr{M}$. If $A \cap B \in \mathscr{M}$ is replaced by the stronger implication $A \setminus B \in \mathscr{M}$, then $\mathscr{M}$ was called a **Körper**.

Hausdorff added, [HAUS$_1$, p.14, footnote]:

Die Ausdrücke Ring und Körper sind der Theorie der algebraischen Zahlen entnommen, auf Grund einer ungefähren Analogie, an die man nicht zu weitgehende Ansprüche stellen möge.

A discovery of Stone, which will be discussed in 4.8.1.1, showed that Hausdorff made a wrong choice. Indeed, every "Körper" is a *Boolean ring* and every "Ring" is a *lattice*.

Here is the terminology that will be used from now on:

By a **ring** $\mathscr{M}$ on a set M we mean a non-empty collection of subsets that is stable under the formation of finite unions and differences. Hence $\emptyset \in \mathscr{M}$. If $M \in \mathscr{M}$, then we speak of an **algebra**. In this case, it is enough to know that $A, B \in \mathscr{M}$ implies $A \cup B \in \mathscr{M}$ and $\complement A \in \mathscr{M}$. A **$\sigma$-ring** or **$\sigma$-algebra** is also closed with respect to *countable* unions.

4.2.2.2 The name **measure** was and is given to various kinds of set functions, which take non-negative values including ∞. It will always be assumed that $\mu(\emptyset) = 0$.

The additivity may concern finite or countable unions of mutually disjoint sets:

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{whenever } A \cap B = \emptyset,$$
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k) \quad \text{whenever } A_h \cap A_k = \emptyset \text{ for } h \neq k.$$

The term *"countably additive"* is often replaced by *"σ-additive."* Roughly speaking, by a **measure space** we mean a triple $(M, \mathscr{M}, \mu)$. The following table shows some but not all possibilities and explains why the terminology is chaotic. Depending on his field of research and his taste, every author takes his personal view.

domain $\mathscr{M}$	
ring	algebra
σ-ring	σ-algebra

additivity
finite
countable

values in
$\{0,1\}$
$[0,1]$
$[0,\infty)$
$[0,\infty]$

For example, analysts use countably additive measures on σ-algebras with values in $\mathbb{R}_+$ or $\mathbb{R}_+ \cup \{\infty\}$, probabilists assume that $\mu(M) = 1$, while set theorists are mainly interested in finitely additive $\{0,1\}$-valued measures on algebras. To avoid any confusion, the particular properties of "measures" will be specified in each case.

4.2.2.3 In what follows, in working with $\mu(A)$, it is tacitly assumed that $A \in \mathscr{M}$. Of special interest are the **μ-null sets**: $\mu(A) = 0$. The term **μ-negligible** is also common.

4.2.2.4 The decisive breakthrough was achieved by Borel when he replaced the finitely additive *Peano–Jordan content* by a countably additive measure, which now bears his name.

[BOR$_1$, pp. 46–47]:
Lorsqu'un ensemble sera formé de tous les points compris dans une infinité dénombrable d'intervalles n'empiétant pas les uns sur les autres et ayant une longueur totale s, nous dirons que l'ensemble a pour mesure s.

Plus généralement, si l'on a une infinité dénombrable d'ensembles n'ayant deux à deux aucun point commun et ayant respectivement pour mesures $s_1, \dots, s_n, \dots$, *leur somme a pour mesure* $s_1 + \cdots + s_n + \cdots$.
$\cdots$
si un ensemble E a pour mesure s, et contient tous les points d'un ensemble E' *dont la mesure est* s', *l'ensemble* $E - E'$, *formé des points de E qui n'appartiennent pas à* E', *sera dit avoir pour mesure* $s - s'$.

4.2.2.5 Suppose that E is a bounded subset of the real line with measure $m(E)$, no matter how this quantity may be defined. Then Lebesgue [LEB$_1$, p. 104] argues as follows:
... *nous pouvons enfermer ses points* [that is, the points of E] *dans un nombre fini ou une infinité dénombrable d'intervalles; la mesure de l'ensemble des points de ces intervalles est la somme des longueurs des intervalles; cette somme est une limite supérieure de la mesure de E. L'ensemble de ces sommes a une limite inférieure* $m_e(E)$; *la* **mesure extérieure** *de E, et l'on a évidemment* $m(E) \le m_e(E)$.

Soit $C_{AB}(E)$ *le complémentaire de E par rapport à AB, c'est-à-dire l'ensemble des points ne faisant pas partie de E et faisant partie d'un segment AB de ox* [read: x-axis] *contenant E. On doit avoir*

$$m(E) = m(AB) - m(C_{AB}(E)) \ge m(AB) - m_e(C_{AB}(E));$$

la limite inférieure ainsi trouvée pour $m(E)$ *s'appelle la* **mesure intérieure** *de E,* $m_i(E)$.

Because the quantities $m_e(E)$ and $m_i(E)$ make sense for any bounded set E, Lebesgue was able to state his final definition, [LEB$_1$, p. 106]:
La mesure intérieure n'est jamais supérieure à la mesure extérieure. Les ensembles dont les deux mesures extérieure et intérieure sont égales sont dits **mesurables** *et leur mesure est la valeur commune de* m_e *et* m_i.

4.2.2.6 In this paragraph, I discuss the relation between Borel's and Lebesgue's approaches.

Starting from intervals, Borel formed

countable unions, then differences of two sets so obtained, again countable unions, etc.

Actually, he used transfinite induction on the set of all countable ordinals in order to enlarge the domain of the measure step by step. Apart from the fact that only sets with finite measure were considered, the final result was the smallest σ-algebra containing all intervals (open sets had not been used!). As a credit for his contribution, we now use the terms *Borel sets, Borel measures*, and *Borel σ-algebras*, even in the setting of topological spaces; see 4.6.2.

Lebesgue's construction is more direct. His outer and inner measures are obtained at one blow, and what remains is to decide whether these quantities coincide for a given set.

4.2.2.7 A measure space is said to be **complete** if every subset of a null set (4.2.2.3) belongs to $\mathscr{M}$. Any measure space can be completed by adding to $\mathscr{M}$ all subsets of the form $A \triangle N_0$, where $A \in \mathscr{M}$ and N_0 is a subset of a null set.

In this way, the non-complete Borel σ-algebra $\mathscr{B}_{\text{orel}}(\mathbb{R})$ extends to the σ-algebra of the Lebesgue measurable sets: $\mathscr{L}_{\text{eb}}(\mathbb{R})$. Jourdain [1905, pp. 177, 179] discovered a cardinality argument that shows that $\mathscr{B}_{\text{orel}}(\mathbb{R})$ is much smaller than $\mathscr{L}_{\text{eb}}(\mathbb{R})$. More precisely,

$$|\mathscr{B}_{\text{orel}}(\mathbb{R})| = 2^{\aleph_0} \quad \text{and} \quad |\mathscr{L}_{\text{eb}}(\mathbb{R})| = 2^{2^{\aleph_0}}.$$

The left-hand formula was already known to Baire [1899, p. 71], while the right-hand one follows from the fact that Cantor's ternary set is a (Peano–Jordan!) null set of cardinality $2^{\aleph_0}$. Hence it has $2^{2^{\aleph_0}}$ subsets.

4.2.2.8 Every function $\varphi : \mathscr{M} \to \mathbb{R}_+$ defined on any system of sets such that $\emptyset \in \mathscr{M}$ and $\varphi(\emptyset) = 0$ becomes finitely additive by passing to an appropriate subalgebra.

According to a proposal of Carathéodory [1914, p. 405], [CARA, p. 246], we refer to $A \in \mathscr{M}$ as **φ-measurable** if

$$\varphi(B) = \varphi(A \cap B) + \varphi(\complement A \cap B) \quad \text{for all } B \in \mathscr{M}.$$

Elementary manipulations show that the collection of these sets has indeed the required property.

This process is mainly applied to outer or inner measures, and the crucial point is to show that sufficiently many sets are measurable in this sense.

4.2.2.9 There is also a theory of **signed measures**. These are countably additive set functions φ defined on a σ-algebra $\mathscr{M}$ taking positive as well as negative values. Some authors, for example [HAL$_1$, p. 118] and [ROY, p. 203], even allow one, but only one, of the values $\pm\infty$, in my opinion, a mathematical monstrosity.

Thanks to the **Hahn decomposition theorem**, many considerations along these lines can be reduced to positive measures, [HAHN$_1$, p. 404] and [HAHN$^+$, p. 21]:

For every signed measure φ, the underlying set M admits a partition $M = M_+ \cup M_-$ such that $M_+ \cap M_- = \emptyset$,

$$\varphi(A) \geq 0 \quad \text{for } A \subseteq M_+ \quad \text{and} \quad \varphi(A) \leq 0 \quad \text{for } A \subseteq M_-.$$

Hahn published this theorem in his book from 1921. I stress that in the setting of abstract integrals, a corresponding result was obtained (independently and even a little bit earlier) by Daniell [1920, pp. 207–208]; see also 4.1.9 and 4.2.5.1.

4.2.3 From measures to integrals

4.2.3.1 For any bounded function $f : M \to \mathbb{R}$, the integral

$$\ell(f) = \int_M f(t)\, d\mu(t)$$

is obtained as the limit of the Lebesgue sums

$$\sum_{k=1}^{n} \alpha_k \mu\{\, t \in M : \alpha_{k-1} < f(t) \leq \alpha_k \} \tag{4.2.3.1.a}$$

as $\max\limits_{1 \leq k \leq n} (\alpha_k - \alpha_{k-1}) \to 0$, where

$$\alpha_0 < \cdots < \alpha_{k-1} < \alpha_k < \cdots < \alpha_n \quad \text{and} \quad \alpha_0 < \inf_{t \in M} f(t) \leq \sup_{t \in M} f(t) \leq \alpha_n.$$

Of course, the expression (4.2.3.1.a) makes sense only when μ is defined for all sets of the form $\{\, t \in M : \alpha < f(t) \leq \beta \}$. Functions with this property are called **μ-measurable**; see [LEB$_1$, p. 111].

4.2.3.2 The preceding definition can be generalized as follows. I quote Lebesgue's original version; [LEB$_1$, p. 115]:

Soit $f(x)$ une fonction mesurable non bornée. Choisissons des nombres $\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots$, en nombre infini, échelonnés de $-\infty$ à $+\infty$ et tel que $l_{i+1} - l_i$ soit toujours inférieur à ε. Nous pouvons former les deux séries

$$\sigma = \sum_{-\infty}^{+\infty} l_i m\big\{E[l_i \leq f(x) < l_{i+1}]\big\}, \quad \Sigma = \sum_{-\infty}^{+\infty} l_i m\big\{E[l_{i-1} \leq f(x) < l_i]\big\}.$$

On voit immédiatement que, si l'une d'elles est convergente, et par suite absolument convergente, l'autre l'est aussi et que, dans ces conditions, σ et Σ tendent vers une limite bien déterminée quand la maximum de $l_{i+1} - l_i$ tend vers zéro d'une manière quelconque. Cette limite est, par définition, l'intégrale de $f(x)$.

Nous appellerons **fonctions sommables** *les fonctions auxquelles s'applique la définition constructive de l'intégrale ainsi complétée.*

4.2.3.3 Riemann's integral of a function $f : [a,b] \to \mathbb{R}$ is obtained by dividing the domain $[a,b]$ into small subintervals, whereas Lebesgue decomposed the codomain. This "little" modification combined with the countable additivity of the underlying measure gave rise to a revolution.

4.2.4 Integrals and the Banach spaces L_1 (the original symbol was L)

4.2.4.1 The concept of an abstract measure reflects the basic properties of Borel's or Lebesgue's measures. Similarly, the integrals of Lebesgue and Stieltjes led to the definition of an abstract integral. Based on preliminary work of Young [1914], an extremely satisfactory theory was accomplished by Daniell [1918], [1920]. I now present a polished version, which is due to Stone [1948/49, Note I].

The σ-ring $\mathscr{M}$ is replaced by a linear lattice $\mathscr{E}$ of real-valued "elementary functions" defined on a set M, and instead of a measure μ, a positive linear form $\ell : \mathscr{E} \to \mathbb{R}$ is given. We refer to ℓ as an **elementary integral** if it has the **Daniell property**:

$$\varphi_1(t) \geq \varphi_2(t) \geq \cdots \geq 0 \text{ and } \lim_{k\to\infty} \varphi_k(t) = 0 \text{ for all } t \in M \quad \text{imply} \quad \lim_{k\to\infty} \ell(\varphi_k) = 0.$$

Denote by $\mathscr{L}^{\text{big}}$ the collection of all functions $f : M \to \mathbb{R}$ for which there exist elementary functions $\varphi_1, \varphi_2, \ldots \in \mathscr{E}$ such that

$$|f(t)| \leq \sum_{k=1}^{\infty} |\varphi_k(t)| \leq \infty \quad \text{for all } t \in M \quad \text{and} \quad \sum_{k=1}^{\infty} \ell(|\varphi_k|) < \infty.$$

Put

$$N(f) := \inf \sum_{k=1}^{\infty} \ell(|\varphi_k|),$$

where the infimum extends over all possible sequences (φ_k). Note that N is a semi-norm on the linear lattice $\mathscr{L}^{\text{big}}$. Moreover, the positivity and the Daniell property ensure that

$$|\ell(\varphi)| \leq \ell(|\varphi|) = N(\varphi) \quad \text{for all } \varphi \in \mathscr{E}.$$

Hence the Hahn–Banach theorem yields an extension of ℓ, which can even be made positive; see 4.1.13. However, in order to ensure uniqueness, $\mathscr{L}^{\text{big}}$ must be reduced: one takes only the subset $\mathscr{L}$ consisting of those members that are approximable by elementary functions with respect to N.

Daniell [1920, p. 206] said that *two functions are* **nearly equal** *if the integral of their modular difference is zero*. Nowadays, we use the term **equal almost everywhere**. Moreover, he showed that the collection of the equivalence classes so obtained is complete. Hence one may say that the space $L(M, \mathscr{E}, \ell)$ of **integrable functions** on an abstract set was already invented in the pre-Banach era. The continuous extension of ℓ is referred to as the **Daniell integral**.

By the way, adapting Daniell's reasoning, Stone [1948/49, Note I, p. 339] observed that $L^{\text{big}}(M,\mathscr{E},\ell)$ is a Banach space as well; and in footnote[5] he added: *This result is believed to be new.*

The proof is based on a useful criterion:

A normed linear space is complete if and only if $\sum_{k=1}^{\infty}\|x_k\| < \infty$ implies that $\sum_{k=1}^{\infty} x_k$ converges.

4.2.4.2 Following Stone [1948/49, Note II, p. 448], we consider the function

$$\mathrm{mid}\{f,g,h\} := \max\big\{\min\{f,g\},\min\{g,h\},\min\{h,f\}\big\},$$

which takes the intermediate one of the values $f(t)$, $g(t)$, and $h(t)$. Using this notation, Stone called a function f **measurable** if $\mathrm{mid}\{f,g,h\}\in\mathscr{L}$ whenever $g,h\in\mathscr{L}$. His main contribution to Daniell's theory of integration is the additional requirement that

$$\varphi\in\mathscr{E} \quad \text{imply} \quad \min\{\varphi,1\}\in\mathscr{E},$$

which ensures the measurability of the constant function 1.

4.2.4.3 Stone [1948/49, Note II, p. 452] observed that $f\in\mathscr{L}^{\text{big}}(M,\mathscr{E},\ell)$ belongs to $\mathscr{L}(M,\mathscr{E},\ell)$ if and only if it is measurable. In other words, members of $\mathscr{L}(M,\mathscr{E},\ell)$ are characterized by a *quantitative* and a *qualitative* property, namely $f\in\mathscr{L}^{\text{big}}(M,\mathscr{E},\ell)$ together with measurability.

4.2.4.4 Bourbaki designed his theory of integration for locally compact spaces M. The starting point is a positive **Radon measure** (read: integral): a positive linear functional ℓ on the linear lattice of all continuous real functions φ with compact supports; see [BOU$_{6a}$, Chap. III, p. 54].

The definition of $N(f)$ reads as follows, [BOU$_{6a}$, pp. 104–109]:

$$N(f) := \inf\Big\{\ell^*(u) \,:\, |f|\le u\Big\},$$

where $u: M\to[0,+\infty]$ is lower semi-continuous and

$$\ell^*(u) := \sup\Big\{\ell(\varphi) \,:\, 0\le\varphi\le u\Big\}.$$

This idea goes back to Young [1911, p. 16]:

... the upper generalized integral is the lower bound of the integrals of lower semi-continuous functions not less than the function.

4.2.4.5 Though a computer knows only rational numbers, we need real numbers in order to ensure completeness. Otherwise, calculus would become a torso.

Similarly, in "real life" all functions are piecewise continuous, and as above, we have to look for "ideal elements." The role of *Dedekind cuts* is taken by equivalence classes of Lebesgue summable functions. Also in this case, the final result is *completeness*, now with respect to the distance $\int_a^b |f(t)-g(t)|\,dt$.

4.2.4.6 The following table shows the basic processes that were performed in the theory of measures and integrals:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{measures of elementary sets}\\ \text{such as intervals}\end{array}} \arrow[r, "(R)"'] \arrow[d, "\text{extension}"', "(E)"] & \boxed{\begin{array}{c}\text{integrals of elementary functions,}\\ \text{e.g., continuous or piecewise constant}\end{array}} \arrow[d, "(D)"', "\text{extension}"] \\
\boxed{\begin{array}{c}\text{measures on}\\ \sigma\text{-algebras}\end{array}} \arrow[r, "\text{integration}", "(L)"'] & \boxed{\begin{array}{c}\text{integrals on (somehow)}\\ \text{complete linear lattices}\end{array}}
\end{tikzcd}$$

(R) integration in the sense of Riemann and Stieltjes.

(E) extension of a content (Inhalt) via inner and outer measures; see [LEB$_1$], [CARA].

(L) Lebesgue's theory of integration; see [LEB$_1$].

(D) extension of an elementary integral by methods due to Young [1914], Daniell [1918], Stone [1948/49], and [BOU$_{6a}$].

Passing from integrals to measures is trivial: just put $\mu(A) := \ell(\chi_A)$, where χ_A denotes the **characteristic function** of the set A. However, sometimes the integral must be extended in order to ensure that $\ell(\chi_A)$ makes sense for sufficiently many A's. The most typical case will be treated in Section 4.6.

4.2.5 Banach spaces of additive set functions

4.2.5.1 Let $\mathscr{M}$ be any algebra on M. Then $ba(M,\mathscr{M})$ denotes the collection of all bounded and finitely additive set functions on $\mathscr{M}$, which is a Banach space under the norm $\|\mu|ba\|_0 := \sup\{|\mu(A)| : A \in \mathscr{M}\}$.

Alexandroff [1940/43, Part II, pp. 567–570] constructed a **Jordan decomposition** $\mu = \mu_+ - \mu_-$ of set functions $\mu \in ba(M,\mathscr{M})$:

$$\left.\begin{array}{l}\mu_+(B) := +\sup\{\mu(A) : A \subseteq B,\ A \in \mathscr{M}\}\\ \mu_-(B) := -\inf\{\mu(A) : A \subseteq B,\ A \in \mathscr{M}\}\end{array}\right. \quad \text{for } B \in \mathscr{M};$$

see also [DUN$_1^+$, p. 98] and 4.2.2.9. Actually, his considerations show that $ba(M,\mathscr{M})$ becomes a Banach lattice with respect to the **total variation**

$$\|\mu|ba\| := \sup\left\{\sum_{k=1}^{n} |\mu(A_k)| : A_1,\dots,A_n \in \mathscr{M},\ A_h \cap A_k = \emptyset \text{ if } h \neq k\right\}.$$

Hence this equivalent norm turns out to be more appropriate. Alexandroff referred to members of $ba(M,\mathscr{M})$ as **charges**; this name has survived in a comprehensive monograph: [BHA$^+$]. Denoted by V_1, the space $ba(M,\mathscr{M})$ was independently invented by Bochner/Phillips [1941, p. 319].

4.2.5.2 The Banach space $B(M,\mathscr{M})$, equipped with the sup-norm, consists of all uniform limits of $\mathscr{M}$**-simple functions**

$$f=\sum_{k=1}^{n}\alpha_k\chi_{A_k}\quad\text{with } \alpha_1,\dots,\alpha_n\in\mathbb{R} \text{ and } A_1,\dots,A_n\in\mathscr{M}.$$

If $\mathscr{M}$ is a σ-algebra, then $B(M,\mathscr{M})$ becomes just the collection of all $\mathscr{M}$**-measurable functions**: $\{t\in M : f(t)\le\alpha\}\in\mathscr{M}$ whenever $\alpha\in\mathbb{R}$.

Fix $\mu\in ba(M,\mathscr{M})$. Letting

$$\ell_0(f):=\sum_{k=1}^{n}\alpha_k\mu(A_k),$$

we obtain a well-defined linear form ℓ_0 on the space of μ-simple functions such that $|\ell_0(f)|\le\|\mu|ba\|\,\|f|B\|$. Hence there exists a unique continuous extension ℓ. Because of $\mu(A)=\ell(\chi_A)$, the correspondence $\mu\mapsto\ell$ is one-to-one and onto. This means that $ba(M,\mathscr{M})=B(M,\mathscr{M})^*$. The order is preserved as well.

Basically, this result goes back to Hildebrandt [1934, p. 870; see also 2.2.11] and Fichtenholz/Kantorovich [1934, p. 76], who treated special cases. Hildebrand was mainly concerned with l_∞^*, while Fichtenholz/Kantorovich took for $\mathscr{M}$ the σ-algebra of Lebesgue measurable subsets of an interval $[a,b]$. Nevertheless, the underlying method can easily be adapted to the general setting; see [DUN$_1^+$, pp. 240, 258] as well as [BHA$^+$, p. 135].

4.2.5.3 Since $B(M,\mathscr{M})$ is a Banach lattice with respect to the natural ordering, we know from the work of Riesz presented in 4.1.8 that $ba(M,\mathscr{M})=B(M,\mathscr{M})^*$ is Dedekind complete.

4.2.5.4 Let $ca(M,\mathscr{M})$ denote the collection of all bounded and countably additive set functions μ on $\mathscr{M}$. These are characterized by the property that

$$A_1\supseteq A_2\supseteq\cdots\quad\text{and}\quad\bigcap_{k=1}^{\infty}A_k=\emptyset\quad\text{imply}\quad\lim_{k\to\infty}\mu(A_k)=0.$$

Hewitt/Yosida [1952, pp. 48–53] showed that $ca(M,\mathscr{M})$ is a band in $ba(M,\mathscr{M})$. Members of $ca(M,\mathscr{M})^\perp$ are said to be *purely finitely additive*.

4.2.5.5 Fix any positive $\mu\in ba(M,\mathscr{M})$. Bochner/Phillips [1941, pp. 319–320] identified $\mu^{\perp\perp}$ as the collection of all μ**-continuous** $\nu\in ba(M,\mathscr{M})$: that is, for every $\varepsilon>0$ there exists a $\delta>0$ such that $\mu(A)\le\delta$ implies $|\nu(A)|\le\varepsilon$. One also says that ν is **absolutely continuous** with respect to μ.

This result provides a generalized **Lebesgue decomposition**:

Every $\lambda\in ba(M,\mathscr{M})$ can be written in the form $\lambda=\nu+\sigma$, where $\nu\in\mu^{\perp\perp}$ and $\sigma\in\mu^\perp$ are μ-continuous and μ-singular, respectively.

The original version reads as follows:

Every function of bounded variation is the sum $f=g+h$ of an absolutely continuous function g and a function h such that $h'(t)=0$ almost everywhere.

The crucial property that g is an indefinite integral will be discussed in the next section.

4.3 The duality between L_1 and L_∞

4.3.1 The **fundamental theorem of calculus** says that the *indefinite integral*

$$F(s) = \int_a^s f(t)\,dt \quad \text{with } a \le s \le b$$

is a *primitive* of the continuous function f.

However, Lebesgue [LEB$_1$, p. 129] observed that

il existe des fonctions continues à variation bornée qui ne sont pas des intégrales indéfinies. ... Pour qu'une fonction soit intégrale indéfinie, il faut de plus que sa variation totale dans une infinité dénombrable d'intervalles de longueur totale ℓ tende vers zéro avec ℓ.

Subsequently, Vitali [1905a] gave a complete characterization of indefinite integrals and coined the term **absolute continuity** (with respect to the Lebesgue measure); see also [VIT$^{\bowtie}$, p. 207]:

Se per ogni numero $\sigma > 0$ existe un numero $\mu > 0$ tale che sia minore di σ il modulo dell'incremento di $F(x)$ in ogni gruppo di ampiezza minore di μ di intervalli parziali di (a,b) distinti, si dirà che $F(x)$ è assolutamente continua.

4.3.2 The next step was again done by Lebesgue [1910, pp. 380–381], who defined **l'intégrale indéfinie de** f as a *fonction d'ensemble mesurable* by letting

$$f : E \mapsto \int_E f\,dP,$$

where f is supposed to be summable on a fixed measurable set in $\mathbb{R}^n$, and E ranges over all of its measurable subsets. Here is his main result, [1910, p. 399]:

Une fonction d'ensemble absolument continue et additive a une dérivée finie et déterminée presque partout et elle est l'intégrale indéfinie d'une fonction égale à cette dérivée.

4.3.3 Radon [1913, p. 1318] said that a non-negative countably additive set function $b(E)$ is a "Basis" of the countably additive set function $f(E)$, *wenn für alle Mengen, für welche $f(E)$ definiert und $b(E) = 0$ ist, auch $f(E) = 0$ folgt*; see 4.3.5.

Subject to this assumption, he showed [1913, p. 1351] the existence of a b-summable function Φ such that

$$f(E) = \int_E \Phi(P)\,db.$$

In Radon's considerations all measurable subsets E are contained in a fixed n-dimensional interval.

4.3.4 The generalization to the abstract case had to wait for almost 20 years. The concluding result was proved by Nikodym [1930, p. 168]. Consequently, the name *Radon–Nikodym theorem* has become common. The French school replaces *Radon* by *Lebesgue*, whereas the fans of Vitali seem to be in the minority.

4.3.5 In the rest of this section, we consider a measure space $(M, \mathscr{M}, \mu)$, where μ is countably additive on the σ-algebra $\mathscr{M}$. The following conditions can be used to restrict the size of μ.

finite measures: $\mu(M) < \infty$,

σ**-finite** measures: $M = \bigcup_{n=1}^{\infty} M_n$, where $\mu(M_n) < \infty$.

The more appropriate concept of *semi-finiteness* will be discussed in 4.3.8.

A countably additive set function $\nu : \mathscr{M} \to \mathbb{R}$ is called μ**-continuous** or **absolutely continuous** with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$.

In this terminology, the **Radon–Nikodym theorem** reads as follows:

If μ is finite, then every μ-continuous countably additive set function ν admits a representation

$$\nu(A) = \int_A f(t) d\mu(t) \quad \text{for all } A \in \mathscr{M}.$$

The unique function $f \in L_1(M, \mathscr{M}, \mu)$ is said to be the **Radon–Nikodym derivative** of ν with respect to μ. In shorthand, one writes $d\nu = f\, d\mu$.

Extending this result from the finite to the σ-finite case is straightforward. On the other hand, Saks [SAKS, p. 36] had earlier observed that the Radon–Nikodym theorem may fail for "big" measures.

4.3.6 As an immediate consequence of the preceding result, Nikodym [1931, p. 132] found the general form of a bounded linear functional ℓ on L_1, defined over an abstract measure space for which $\mu(M)$ is finite:

$$\ell(f) = \int f(t) g(t)\, d\mu(t) \quad \text{for all } f \in L_1 \quad \text{and} \quad \|\ell\| = \|g|L_\infty\| := \operatorname*{ess\text{-}sup}_{t \in M} |g(t)|.$$

Steinhaus–Nikodym theorem: $L_1(M, \mathscr{M}, \mu)^* = L_\infty(M, \mathscr{M}, \mu)$.

As mentioned in 2.2.7, the case $L_1[a,b]$ had been treated by Steinhaus who, however, based his proof on the formulas $L_2[a,b] \subset L_1[a,b]$ and $L_2[a,b]^* = L_2[a,b]$.

By the way, it seems that Nikodym did not know the Steinhaus paper [1919].

4.3.7 The following examples were essentially discovered by Botts; see McShane [1950, p. 406] and Schwartz [1951, p. 273].

Let $\mathbb{I}$ be any uncountable index set. Denote by $\mathscr{C}(\mathbb{I})$ the collection of all countable subsets as well as their complements. If γ is the counting measure, then it follows that $L_1(\mathbb{I}, \mathscr{C}(\mathbb{I}), \gamma) = l_1(\mathbb{I})$. Hence $L_1(\mathbb{I}, \mathscr{C}(\mathbb{I}), \gamma)^* = l_\infty(\mathbb{I})$ is strictly larger than $L_\infty(\mathbb{I}, \mathscr{C}(\mathbb{I}), \gamma)$, since a $\mathscr{C}(\mathbb{I})$-measurable function must be constant except for a countable subset.

Next, we fix a point $i_0 \in \mathbb{I}$. Then the following definition yields a countably additive measure on the power set $\mathscr{P}(\mathbb{I})$:

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is countable and } i_0 \notin A, \\ 1 & \text{if } A \text{ is countable and } i_0 \in A, \\ \infty & \text{if } A \text{ is uncountable.} \end{cases}$$

Note that the equivalence class associated with an integrable function f is uniquely determined by $f(i_0)$. Hence $L_1(\mathbb{I}, \mathscr{P}(\mathbb{I}), \mu)^* = \mathbb{K}$. On the other hand, $L_\infty(\mathbb{I}, \mathscr{P}(\mathbb{I}), \mu)$ is very big.

Conclusion: measures taking the value ∞ can be "dangerous."

4.3.8 A set $A \in \mathscr{M}$ of infinite measure is *purely infinite* if it contains no subset B such that $0 < \mu(B) < \infty$. This means that every integrable function vanishes on A. Hence those sets are rather superfluous, and we are entitled to concentrate our interest on **semi-finite** measure spaces, in which by definition, this phenomenon does not occur: Every set $A \in \mathscr{M}$ with $\mu(A) = \infty$ contains a subset $B \in \mathscr{M}$ with $0 < \mu(B) < \infty$. Then

$$\mu(A) = \sup\{\, \mu(A \cap L) : \mu(L) < \infty\} \quad \text{for all } A \in \mathscr{M}.$$

In the context of Bourbaki's theory of integration, semi-finiteness is achieved as follows: replace negligible subsets by locally negligible subsets, and integrable functions by essentially integrable functions.

Thanks to this convention, we have more sets and functions at our disposal, while the corresponding equivalence classes modulo null objects remain unchanged; see [BOU_{6b}, p. 13].

4.3.9 In this paragraph, I discuss the process of *localization*.

Denote by $\mathscr{M}_\bullet(\mu)$ the collection of all sets $L \in \mathscr{M}$ that have finite measure. Let μ_L be the restriction of μ to $\mathscr{M}_L := \{A \in \mathscr{M} : A \subseteq L\}$. If L ranges over $\mathscr{M}_\bullet(\mu)$, a family of finite measure spaces $(L, \mathscr{M}_L, \mu_L)$ is obtained.

A semi-finite measure space is said to be **localizable** if it satisfies the following property:

Let (f_L) be a family of $\mathscr{M}_L$-measurable functions such that $f_K = f_L$ almost everywhere on $K \cap L$ for any pair $K, L \in \mathscr{M}_\bullet(\mu)$. Then there exists an $\mathscr{M}$-measurable function f on the whole set M whose restrictions coincide with f_L almost everywhere on all sets $L \in \mathscr{M}_\bullet(\mu)$.

Note that because of semi-finiteness, the "global" function f is uniquely determined (as an equivalence class) by its "local" components. This is an abstract version of Bourbaki's *principe de localization*, see [BOU_{6a}, Chap. IV, p. 182].

Usually, localizability of a semi-finite measure space $(M, \mathscr{M}, \mu)$ is defined by requiring that the real lattice $L_\infty(M, \mathscr{M}, \mu)$ be Dedekind complete.

4.3.10 Suppose that $(M, \mathscr{M}, \mu)$ is localizable. By restriction, every functional ℓ on $L_1(M, \mathscr{M}, \mu)$ generates a family of functionals ℓ_L on the spaces $L_1(L, \mathscr{M}_L, \mu_L)$, which are "nice" because $\mu(L)$ is finite. Hence there exist g_L's in $L_\infty(L, \mathscr{M}_L, \mu_L)$ such that

$$\ell_L(f_L) = \int_L f_L(t) g_L(t) d\mu_L(t) \quad \text{for all } f_L \in L_1(L, \mathscr{M}_L, \mu_L).$$

Since the pieces g_L are compatible as required in the preceding paragraph, they can be glued together. The resulting function g represents ℓ:

$$\ell(f) = \int_L f(t) g(t) d\mu(t) \quad \text{for all } f \in L_1(M, \mathscr{M}, \mu). \tag{4.3.10.a}$$

The members of $L_\infty(M, \mathscr{M}, \mu)$ are equivalence classes of bounded measurable functions that coincide almost everywhere, and in view of semi-finiteness, it suffices to check this coincidence on every $L \in \mathscr{M}_\bullet(\mu)$. Thus g is uniquely determined by ℓ. Conversely, every $g \in L_\infty(M, \mathscr{M}, \mu)$ generates a functional $\ell \in L_1(M, \mathscr{M}, \mu)^*$ via (4.3.10.a). This proves that

$$L_1(M, \mathscr{M}, \mu)^* = L_\infty(M, \mathscr{M}, \mu). \tag{4.3.10.b}$$

Even more can be said:

A measure space is localizable precisely when (4.3.10.b) holds.

If $\mu(A) > 0$, then $\|\chi_A | L_\infty\| = 1$. Assuming $L_1^* = L_\infty$, we find a non-negative function $f \in L_1$ such that $\int_A f(t)\, dt > 0$. Hence A contains a subset B with $0 < \mu(B) < \infty$. This proves that $(M, \mathscr{M}, \mu)$ is semi-finite. Moreover, it follows from $L_1^* = L_\infty$ that the dual Banach lattice L_∞ is Dedekind complete; see 4.1.11 and 4.3.9.

The validity of the Radon–Nikodym theorem is another equivalent property.

4.3.11 The concept of localizability as well as its various characterizations are due to Segal [1951, pp. 279, 301]. Simplified proofs were given by Kelley [1966]. Both authors considered only semi-finite measure spaces such that

$$\underbrace{A \cap L \in \mathscr{M} \quad \text{for all } L \in \mathscr{M}_\bullet(\mu)}_{A \text{ is locally measurable}} \quad \text{implies} \quad A \in \mathscr{M}.$$

Measure spaces with these properties are nowadays called **locally determined**; see [FREM$_1$, p. 172].

4.4 The Banach spaces L_p

Throughout this section, it is assumed that $1 < p < \infty$ and $1/p + 1/p^* = 1$.

4.4.1 First of all, we recall from 1.1.4 that Riesz introduced the spaces $L_p[a,b]$. He also described how the interval $[a,b]$ can be replaced by any measurable subset of $\mathbb{R}^n$; see [1909a, pp. 496–497].

4.4.2 In a next step, Radon [1913, p. 1353] invented the space $L_p(b)$, where b is any non-negative σ-additive measure (*monotone absolut additive Mengenfunktion*, 2.2.10) on a cube J in $\mathbb{R}^n$ with finite total mass. From our present point of view, his definition was strange, since the elements of $L_p(b)$ were supposed to be real-valued *absolut additive Mengenfunktionen* f with the following property:

There exists $M \geq 0$ such that

$$\sum_{k=1}^{n} \frac{|f(E_k)|^p}{b(E_k)^{p-1}} \leq M^p \tag{4.4.2.a}$$

for all finite collections of mutually disjoint measurable sets $E_1, \ldots, E_n$ contained in J. *Diese Summenbildung ist so zu verstehen, daß dabei nur jene Teilmengen in Betracht kommen, für welche* $b(E_k) > 0$. The supremum of the left-hand expressions is denoted by

$$\int_J \frac{|df|^p}{db^{p-1}}.$$

Radon [pp. 1361–1362] defined weak convergence (*einfache Konvergenz*) and norm convergence (*starke Konvergenz*). He proved the weak compactness of the closed unit ball and the norm completeness of $L_p(b)$; see [pp. 1366, 1368].

Based on the bilinear form

$$\langle f, g \rangle = \int_J \frac{df \cdot dg}{db} = \lim \sum_{k=1}^{n} \frac{f(E_k) g(E_k)}{b(E_k)},$$

Radon [1913, p. 1364] built a duality between $L_p(b)$ and $L_{p^*}(b)$. This led him [p. 1371] to the important formula $L_p(b)^* = L_{p^*}(b)$. The set function $g \in L_{p^*}(b)$ associated with a functional $\ell \in L_p(b)^*$ was obtained by putting $g(A) := \ell(b_A)$, where $b_A : B \mapsto b(A \cap B)$; see [p. 1370].

Radon's approach has a remarkable advantage. Usually, the elements of $L_p(b)$ are *equivalence classes* of "point functions" Φ. In his philosophy the elements are *singletons*, namely "set functions" f. The relationship between the two concepts, which is based on the Radon[–Nikodym] theorem, was discussed only in passing (Einige Anwendungen) [p. 1412]:

$$f(E) = \int_E \Phi db. \tag{4.4.2.b}$$

Moreover, [p. 1414]: *Die Existenz jedes der beiden Integralausdrücke*

$$\int_E \frac{|df|^p}{db^{p-1}}, \quad \int_E |\Phi|^p db$$

zieht die des anderen nach sich und beide haben denselben Wert.

Radon's point of view will show its superiority when we represent operators from L_p into a general Banach space; see Subsection 5.1.5.

4.4.3 Forty years later and not being acquainted with Radon's work, Leader [1953] reproduced the preceding approach. In addition, he observed that the basic results remain true for finitely additive measures. In other words, countable additivity is exclusively required to pass from f to Φ in formula (4.4.2.b).

4.4.4 The Banach spaces L_p over $(M, \mathscr{M}, \mu)$ were invented only in the late 1930s. This step was more or less straightforward. First of all, we refer to Dunford [1938, pp. 335–338].

It should be mentioned that, under the subtitle "Mengenfunktionen," Hahn had prepared the second volume of his famous monograph *Reelle Funktionen* (*Punktfunktionen*). Due to his untimely death in 1934 and World War II, the German version was never published. However, Rosenthal edited an English translation, which appeared in 1948. Here we find a theory of L_p spaces [HAHN$^+$, §§ 14–15], presented in the spirit of Helly [1921] and Hahn [1922].

4.4.5 Starting from an elementary integral, Stone [1948/49, Note I] built the Banach space of integrable functions, denoted by L. In his Second Note he defined L_p as the collection of all functions f such that $|f|^{p-1}f \in L$. By 4.2.4.3, this is a "compact" form of the conditions that $|f|^p$ should be integrable and that f should be measurable.

Stone's approach is quick at first glance. However, one must prove that the simple functions are dense in L_p, and this requires some doing.

Finally, I stress the important fact that $f \leftrightarrow |f|^{p-1}f$ yields a homeomorphism between L_p and L.

4.4.6 For the moment, we let $0 < p < q < \infty$. Then a *Hilfssatz* of Pringsheim [1904, pp. 268–269] yields

$$l_p \subset l_q \quad \text{and} \quad \|x|l_q\| \le \|x|l_p\|;$$

see also Jensen [1906, pp. 192–193] and [HARD$^+$, pp. 4, 28]. On the other hand, Hölder's inequality implies that

$$L_q[0,1] \subset L_p[0,1] \quad \text{and} \quad \|f|L_p\| \le \|f|L_q\|;$$

see Radon [1913, pp. 1353–1354]. For $q/p = 2, 3, \dots$, *die eigenthümliche Ungleichung* $\|f|L_p\| \le \|f|L_q\|$ was already proved by Schlömilch [1858, pp. 305–306].

4.4.7 We know from 2.2.7 that $L_p[0,1]^* = L_{p^*}[0,1]$. This classical formula was extended by Dunford [1938, pp. 336–337] to L_p's defined over finite measure spaces $(M, \mathscr{M}, \mu)$. McShane [1950] and Schwartz [1951] treated the general case. Following Dunford's pattern, Schwartz based his proof on the Radon–Nikodym theorem, while McShane employed Banach space techniques. His idea has inspired a simple approach to be presented in the next paragraph.

Clearly, $L_2^* = L_2$ is a consequence of 2.2.6. I point out that von Neumann [1940, pp. 127–129] used this identity to give an ingenious proof of the Radon–Nikodym theorem; see also Stone [1948/49, Note IV].

4.4.8 A Banach space X is called **uniformly convex** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta \quad \text{whenever } \|x\|=\|y\|=1 \text{ and } \|x-y\| \geq \varepsilon.$$

This concept, which will be discussed extensively in Subsection 5.5.2, was introduced by Clarkson [1936]. In order to show that $L_p[a,b]$ is uniformly convex for $1<p<\infty$, he proved the following inequalities:

$$2\big(\|f\|^{p^*}+\|g\|^{p^*}\big)^{p-1} \leq \|f+g\|^p + \|f-g\|^p \leq 2\big(\|f\|^p+\|g\|^p\big) \quad \text{if } 1<p\leq 2,$$
$$2\big(\|f\|^{p^*}+\|g\|^{p^*}\big)^{p-1} \geq \|f+g\|^p + \|f-g\|^p \geq 2\big(\|f\|^p+\|g\|^p\big) \quad \text{if } 2\leq p<\infty.$$

Clarkson's reasoning [1936, pp. 400–401] automatically extends to L_p over any measure space. Replacing f by $\frac{f+g}{2}$ and g by $\frac{f-g}{2}$ yields further inequalities.

Independently from each other, Milman, Kakutani, and Pettis discovered that uniformly convex Banach spaces, and hence all L_p's with $1<p<\infty$, are reflexive; see 5.5.2.4. This fact can be used for an elegant proof of the formula $L_p^* = L_{p^*}$:

Identify L_{p^*} with a closed subspace of L_p^* by assigning to every function $g \in L_{p^*}$ the functional

$$\ell(f) := \int_M f(t)g(t)\,d\mu(t).$$

Because of reflexivity, L_{p^*} is even weakly closed. In addition, it is total. Consequently, L_{p^*} must be all of L_p^*.

Unfortunately, I do not know to whom this trick should be attributed. Of course, McShane [1950] was the initiator.

4.5 Banach spaces of continuous functions

4.5.1 Throughout this section, M denotes a topological space, which is always assumed to satisfy Hausdorff's separation axiom (D) stated in 3.2.1.1. The symbol $C(M)$ stands for the ring of all continuous real-valued functions f on M, and the subring of bounded functions will be denoted by $C_b(M)$. The latter is a Banach space under the norm $\|f\| := \sup\{|f(t)| : t \in M\}$.

A Hausdorff space M is said to be **pseudo-compact** if $C(M)$ and $C_b(M)$ coincide; see Hewitt [1948, p. 67].

The reader is warned that many topologists write $C^*(M)$ instead of $C_b(M)$, a sacrilege in the view of Banach space people.

4.5.2 There exist "big" (regular) Hausdorff spaces M on which all continuous functions are constant; Hewitt [1946, p. 503] and [STE$^+$, pp. 109–113, 222–224]. In order to avoid this pathology, Tychonoff [1930, p. 545] introduced the following concept:

Ein topologischer Raum heißt **vollständig regulär***, wenn zu jedem Punkt x_0 und zu jeder ihn nicht enthaltenden abgeschlossenen Menge A eine im ganzen Raum stetige Funktion $F(x)$ definiert werden kann, die im ganzen Raum der Bedingung $0 \le F(x) \le 1$ genügt und die überdies in x_0 gleich 0 und in sämtlichen Punkten von A gleich 1 ist.*

Satz. Ein Raum ist dann und nur dann einer Teilmenge eines bikompakten topologischen Raumes homöomorph, wenn er vollständig regulär ist.

4.5.3 The following construction was independently found by Čech [1937, pp. 826–828] and Stone [1937a, p. 460]:

For every topological space M there exists a continuous map ρ onto a completely regular space ρM such that $f(s) \mapsto f\big(\rho(t)\big)$ yields a one-to-one correspondence between $C(\rho M)$ and $C(M)$. Note that ρM is unique up to a homeomorphism.

In this way, considerations about $C(M)$ and $C_b(M)$ are reduced to the case that M is completely regular.

4.5.4 A further step of reduction is provided by the **Stone–Čech compactification**; Čech [1937, p. 831], Stone [1937a, pp. 461–463, 476]:

Every completely regular space M can be identified with a dense subset of a compact space βM such that every function $f \in C_b(M)$ admits a unique extension $\beta f \in C(\beta M)$.

Consequently, if we are interested in $C_b(M)$ as a Banach space, a Banach lattice, or a Banach algebra, then it is enough to deal with $C(K)$'s, where K is compact.

The Stone–Čech compactification is a highly theoretical tool; non-trivial explicit examples of βM are rare; 4.6.13. In particular, nobody will ever see a point of $\beta\mathbb{N} \setminus \mathbb{N}$. This trouble stems from the fact that the construction of βM is based on the axiom of choice; see Section 7.5.

A functional analytic approach to the Stone–Čech compactification works as follows:

Assigning to $t \in M$ the Dirac measure $\delta_t : f \mapsto f(t)$ yields a homeomorphic embedding of M into the weakly* compact unit ball of $C_b(M)$, and the required βM is obtained as the weak* closure of the δ_t's.

More information on the Stone–Čech compactification can be found in [WALK].

4.5.5 The preceding results show that a completely regular space M cannot be recovered from algebraic or functional analytic properties of $C_b(M)$. However, the situation improves drastically in the compact setting:

Banach–Stone theorem:

The Banach spaces $C(K_1)$ and $C(K_2)$ are isometric if and only if K_1 and K_2 are homeomorphic.

The compact and metric case was already treated by Banach [BAN, p. 170], while Stone [1937a, p. 469] contributed the rest; see also 5.4.1.9.

Gelfand/Kolmogoroff [1939] proved a counterpart of the preceding statement whose nature is purely algebraic:

The rings $C(K_1)$ and $C(K_2)$ are isomorphic if and only if K_1 and K_2 are homeomorphic.

4.5.6 The Weierstrass approximation theorem [1885] is one of the most profound results in analysis. It was generalized by Stone [1937a, pp. 467–468], who made the following observation:

The usual statement of the Weierstrass approximation theorem can be broken down into two propositions: (1) *every function continuous in a closed interval of* $\mathbb{R}$ *can be uniformly approximated by a "polynomial" constructed from the function* $f(\alpha)=\alpha$ *and the real numbers by the algebraic operations and the formation of absolute values;* (2) *the function* $f(\alpha)=|\alpha|$ *can be uniformly approximated by a polynomial* $p(\alpha)$ *where* $p(0)=0$*.*

Hence Stone had to find out what a "polynomial" in part (1) should be. Since his original formulation looks rather sophisticated, I state here an updated version:

Let P be a linear sublattice or a subring of $C(K)$ containing the constant functions and separating the points of K: given $s \neq t$ in K there is $p \in P$ for which $p(s) \neq p(t)$. Then the **Weierstrass–Stone theorem** says that every continuous function on K can be uniformly approximated by members of P.

The master himself presented a modern survey; see Stone [1962]. More historical information is given in [SEMA, pp. 118–119].

4.5.7 The preceding theorem has many and far-reaching consequences. For example, it follows that $C(K)$ over a compact metric space is separable; see also 6.9.1.1. This result goes back to Borsuk [1931, pp. 165–167], who used another approach.

4.5.8 In my opinion, [SEMA] is by far the best treatise on Banach spaces of continuous functions; detailed historical comments are of special value. The reader may also consult [LAC, Chap. 3]. The well-known monograph [GIL$^+$] treats $C(K)$ as a ring, while [KAP] stresses the lattice structure. The main goal of the latter consists in finding bands that sit between $C(K)$ and $C(K)^{**}$.

4.6 Measures and integrals on topological spaces

Throughout this section, M is supposed to be a completely regular Hausdorff space, and f always denotes a bounded continuous real-valued function on M. Moreover, F stands for closed and G for open subsets of M. All measures under consideration are finite.

4.6.1 We are interested in algebras $\mathscr{M}$ consisting of subsets of M for which $C_b(M)$ is contained in $B(M,\mathscr{M})$; see 4.2.5.2. The most natural choice seems to be the **Borel algebra** $\mathscr{B}^{\circ}_{\text{orel}}(M)$ generated by all closed sets or, equivalently, by all open sets. However, it turned out that $\mathscr{B}^{\circ}_{\text{orel}}(M)$ is sometimes too big.

Actually, we must ensure only that the Lebesgue sums

$$\sum_{k=1}^{n} \alpha_k \, \mu\{t \in M : \alpha_{k-1} < f(t) \leq \alpha_k\}$$

make sense. This is the case if

$$\{t \in M : \alpha < f(t) \leq \beta\} = \{t \in M : \alpha < f(t)\} \cap \{t \in M : f(t) \leq \beta\}$$

belongs to $\mathscr{M}$ whenever $\alpha < \beta$. Sets of the form

$$\{t \in M : \alpha < f(t)\} \quad \text{and} \quad \{t \in M : f(t) \leq \beta\}$$

were first considered by Alexandroff [1940/43, Part I, p. 319], who termed them **totally open** and **totally closed**, respectively. Since

$$\{t \in M : f(t) \leq \beta\} = \{t \in M : f_0(t) = 0\} \quad \text{with} \quad f_0(t) := \min\{\beta - f(t), 0\},$$

it is enough to use **Z-sets** in the sense of Hewitt [1948, p. 53]:

$$Z(f) := \{t \in M : f(t) = 0\} \quad \text{with} \quad f \in C_b(M).$$

Nowadays, the name **zero-set** is common, [GIL$^+$, p. 14]. The collection of all zero-sets is stable under the formation of finite unions and countable intersections; see Hewitt [1948, p. 70]. Complements of zero-sets are referred to as **cozero-sets**. By $\mathscr{B}^\circ_{\mathrm{aire}}(M)$ we will denote the **Baire algebra** generated by all zero-sets or, equivalently, by all cozero-sets.

It may happen that $\mathscr{B}^\circ_{\mathrm{aire}}(M)$ is strictly smaller than $\mathscr{B}^\circ_{\mathrm{orel}}(M)$; see 4.6.14. On the other hand, in metric spaces every closed set F can be obtained as the zero-set of d_F, where $d_F(t)$ is the distance of t from F.

4.6.2 The smallest σ-algebra containing all closed sets or, equivalently, all open sets, will be denoted by $\mathscr{B}_{\mathrm{orel}}(M)$. This is the famous **Borel σ-algebra**, and its members are referred to as **Borel sets**. Starting from the zero-sets or cozero-sets, we obtain the **Baire σ-algebra** $\mathscr{B}_{\mathrm{aire}}(M)$. The term **Baire set** was coined by Kakutani/Kodaira [1944, p. 444].

Feiste [1978, p. 162] showed that the algebras $\mathscr{B}^\circ_{\mathrm{aire}}[0,1] = \mathscr{B}^\circ_{\mathrm{orel}}[0,1]$ are strictly smaller than the σ-algebras $\mathscr{B}_{\mathrm{aire}}[0,1] = \mathscr{B}_{\mathrm{orel}}[0,1]$.

Finitely or countably additive **Borel** and **Baire measures** are defined on the following systems of subsets:

algebra	$\mathscr{B}^\circ_{\mathrm{aire}}(M) \longrightarrow \mathscr{B}^\circ_{\mathrm{orel}}(M)$
	$\downarrow \qquad\qquad \downarrow$
σ-algebra	$\mathscr{B}_{\mathrm{aire}}(M) \longrightarrow \mathscr{B}_{\mathrm{orel}}(M)$

4.6.3 Carathéodory [CARA, p. 258] discovered a remarkable property of set functions on topological spaces, which can be viewed as a kind of continuity.

A finitely additive and positive set function μ on $\mathscr{B}^\circ_{\mathrm{orel}}(M)$ or $\mathscr{B}_{\mathrm{orel}}(M)$ is called **inner regular** with respect to closed sets if

$$\mu(A) = \sup\{\, \mu(F) : F \subseteq A\,\}$$

and **outer regular** with respect to open sets if

$$\mu(A) = \inf\{\, \mu(G) : A \subseteq G\,\}.$$

Similarly, for finitely additive and positive set functions on $\mathscr{B}^\circ_{\mathrm{aire}}(M)$ or $\mathscr{B}_{\mathrm{aire}}(M)$, one can define **inner regularity** with respect to zero-sets and **outer regularity** with respect to cozero-sets.

In both cases the inner and outer definitions are equivalent by passing to complements. However, the concept of inner regularity with respect to compact sets has no outer counterpart.

Writing μ in the form $\mu = \mu_+ - \mu_-$, we can define the regularity of finitely additive μ's via that of μ_+ and μ_-. A direct characterization says that every Borel set A can be embedded between a closed set $F \subseteq A$ and an open set $G \supseteq A$ such that $G \setminus F$ becomes small. More precisely, given $\varepsilon > 0$ we may arrange that $|\mu(F_0)| \le \varepsilon$ for any closed subset F_0 in $G \setminus F$.

4.6.4 Next, I discuss various representation theorems for functionals on $C_b(M)$, which, together with their classical predecessors, are listed in the following table. The early history of this fascinating subject is described in an article of Gray [1984•]; see also 2.2.9 and 2.2.10.

Riesz	1909	interval $[a,b]$
Radon	1913	n-dimensional cube $[a,b]^n$
Banach	1937	compact metric space
Markoff	1938	normal space
Alexandroff	1940/43	completely regular space
Kakutani	1941	compact space

The results above are commonly summarized under the catchword

Riesz representation theorem.

4.6.5 The first approach beyond the compact case is due to Markoff (junior) [1938]. He assumed the underlying space to be **normal**:

If F_0 and F_1 are disjoint closed sets, then there exist disjoint open sets $G_0 \supseteq F_0$ and $G_1 \supseteq F_1$.

Let $\ell \in C_b(M)^*$ be a positive functional. Given any set A in M, we define an **inner measure**

$$\mu_*(A) := \sup\{\, \mu(F) : F \subseteq A \,\} \quad \text{with} \quad \mu(F) := \inf\{\, \ell(f) : \chi_F \le f \le 1 \,\}$$

and an **outer measure**

$$\mu^*(A) := \inf\{\, \mu(G) : A \subseteq G \,\} \quad \text{with} \quad \mu(G) := \sup\{\, \ell(f) : 0 \le f \le \chi_G \,\}.$$

The Carathéodory process described in 4.2.2.8 yields algebras of measurable sets on which μ_* and μ^*, respectively, become finitely additive. The main point is to show that these algebras include $\mathscr{B}^\circ_{\text{orel}}(M)$. **Urysohn's lemma** [1925, pp. 290–291] turned out to be the crucial tool:

If F_0 and F_1 are disjoint closed sets in a normal space, then there exists a continuous function $0 \le f \le 1$ such that $f(t) = 0$ on F_0 and $f(t) = 1$ on F_1.

Restricting μ_* and μ^* to $\mathscr{B}^\circ_{\text{orel}}(M)$, we get one and the same measure μ such that

$$\ell(f) = \int_M f(t)\, d\mu(t). \tag{4.6.5.a}$$

Markoff used the "outer" way, which goes back to von Neumann's construction [1934] of Haar measures on locally compact groups.

Markoff's representation theorem; see [DUN$_1^+$, p. 262]:

Let M be a normal Hausdorff space. Then the formula (4.6.5.a) yields a metric isomorphism $\ell \leftrightarrow \mu$ between $C_b(M)^*$ and the closed subspace of $ba(M, \mathscr{B}^\circ_{\text{orel}})$ whose members are the regular, bounded, and finitely additive set functions on the Borel algebra of M.
In addition: this correspondence preserves order.

4.6.6 In a series of fundamental papers, Alexandroff [1940/43] removed Markoff's restriction to normal spaces. His basic idea was to replace closed sets by zero-sets, open sets by cozero-sets, and therefore $\mathscr{B}^\circ_{\text{orel}}(M)$ by $\mathscr{B}^\circ_{\text{aire}}(M)$. In this setting, Urysohn's lemma becomes a triviality:

If Z_0 and Z_1 are disjoint zero-sets in any topological space, then there exists a continuous function $0 \le f \le 1$ such that $f(t) = 0$ on Z_0 and $f(t) = 1$ on Z_1.

Indeed, if $Z_0 = Z(f_0)$ and $Z_1 = Z(f_1)$, then $f := \frac{|f_0|}{|f_0|+|f_1|}$ works.

Alexandroff's representation theorem; see Wheeler [1983, pp. 115–117]:

Let M be a completely regular Hausdorff space. Then the formula (4.6.5.a) yields a metric isomorphism $\ell \leftrightarrow \mu$ between $C_b(M)^*$ and the closed subspace of $ba(M, \mathscr{B}^\circ_{\text{aire}})$ whose members are the regular, bounded, and finitely additive set functions on the Baire algebra of M.
In addition: this correspondence preserves order.

4.6.7 We now compare the theorems of Markoff and Alexandroff, both proved in Leningrad. First of all, note that even in compact spaces $\mathscr{B}^{\circ}_{\text{aire}}(M)$ need not coincide with $\mathscr{B}^{\circ}_{\text{orel}}(M)$. However, every regular finitely additive measure on $\mathscr{B}^{\circ}_{\text{aire}}(M)$ extends to a regular finitely additive measure on $\mathscr{B}^{\circ}_{\text{orel}}(M)$; see Lembcke [1970, p. 68] and Bachman/Sultan [1980, p. 390]. In normal spaces, the Borel extension is regular with respect to zero-sets and therefore unique. In this way, *Alexandroff's measure* passes into *Markoff's measure*.

Note that there are examples of completely regular spaces in which even regular countably additive Baire measures may have different regular Borel extensions; see Wheeler [1983, pp. 132–133].

Supported by the preceding results, Wheeler [1983, p. 135] believes that

the elegance and orderliness of the theory of Baire measures make it the proper setting for measure theory on completely regular spaces.

4.6.8 Functionals $\ell \in C_b(M)^*$ may have various additional properties, which lead to a useful and detailed classification. The first step in this direction was done by LeCam [1957, p. 212], who introduced the concept of **tightness**. In what follows, we will take only a brief look at the tip of the iceberg. For more information, the reader is referred to Varadarajan's thesis [1961, pp. 165, 174] as well as to the surveys of Gardner/Pfeffer [1984] and Wheeler [1983].

4.6.9 A corollary of the Daniell–Stone theory 4.2.4.1 says that whenever $\ell \leftrightarrow \mu$, the following are equivalent:

$f_1(t) \ge f_2(t) \ge \cdots \ge 0$ and $\lim\limits_{k\to\infty} f_k(t) = 0$ for all $t \in M$ imply $\lim\limits_{k\to\infty} \ell(f_k) = 0$.

$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq \emptyset$ and $\bigcap\limits_{k=1}^{\infty} Z_k = \emptyset$ imply $\lim\limits_{k\to\infty} \mu(Z_k) = 0$.

In this case, μ admits a countably additive extension to $\mathscr{B}_{\text{aire}}(M)$. Since every cozero-set is the union of an increasing sequence of zero-sets, countably additive Baire measures are always inner regular with respect to zero-sets; see Varadarajan [1961, p. 171]. However, a regular countably additive extension to $\mathscr{B}_{\text{orel}}(M)$ need not exist.

4.6.10 A functional $\ell \in C_b(M)^*$ is said to be **tight** if its restriction to the unit ball is continuous under the topology of uniform convergence on compact subsets. This happens if and only if for every $\varepsilon > 0$ there exist a constant $c > 0$ and a compact set K such that

$$|\ell(f)| \le c \sup_{t\in K} |f(t)| + \varepsilon \|f\|.$$

Tightness of ℓ can be characterized by the condition that the corresponding set function μ admit a countably additive extension to $\mathscr{B}_{\text{orel}}(M)$ that is inner regular with respect to compact sets; see Knowles [1967, p. 144]. This extension turns out to be unique.

The school of Schwartz [SCHW$_2$, p. 13] uses the name **Radon measure**.

4.6.11 We now discuss the setting of compact Hausdorff spaces, which is the most important one. Compact metric spaces were treated by Banach [1937, p. 326], while Kakutani [1941b, pp. 1009–1012] settled the general case. This line of development was independent of the Markoff–Alexandroff approach.

Kakutani's representation theorem; see [DUN$_1^+$, p. 265]:
Let K be a compact Hausdorff space. Then the formula (4.6.5.a) yields a metric isomorphism $\ell \leftrightarrow \mu$ between $C(K)^*$ and the closed subspace of $ba(K,\mathscr{B}_{\text{orel}})$ whose members are the regular and countably additive set functions on $\mathscr{B}_{\text{orel}}(K)$.
In addition: this correspondence preserves order.

4.6.12 In this paragraph, the completely regular Hausdorff space M is viewed as a subspace of its Stone–Čech compactification βM. Since every function $f \in C_b(M)$ admits a unique extension $\beta f \in C(\beta M)$, the spaces $C_b(M)$ and $C(\beta M)$ can be identified. In this way, every $\ell \in C_b(M)^*$ induces a unique $\beta\ell \in C(\beta M)^*$. Unfortunately, in the general case, there is no simple connection between the corresponding set function μ and $\beta\mu$, which are defined on $\mathscr{B}^\circ_{\text{aire}}(M)$ and $\mathscr{B}_{\text{orel}}(\beta M)$, respectively. However, if $\ell \ge 0$ is tight, then it follows that $(\beta\mu)^*(M) = \mu(M)$, where $(\beta\mu)^*$ denotes the outer measure associated with $\beta\mu$; see Knowles [1967, p. 144] and [SCHW$_2$, pp. 60–61]. Moreover, we have $\mu(M \cap B) = \beta\mu(B)$ for $B \in \mathscr{B}_{\text{orel}}(\beta M)$. Note that *all* Borel sets of M are obtained as intersections $M \cap B$ with $B \in \mathscr{B}_{\text{orel}}(\beta M)$.

4.6.13 In concluding this section, I present a remarkable class of topological spaces that serve mainly as counterexamples.

As usual, an ordinal γ is identified with the collection of all ordinals $\xi < \gamma$. In particular, $\gamma+1 = \{\,\xi \in \text{Ord} : \xi \le \gamma\,\}$. For $\alpha, \beta \in \text{Ord}$, the intersections

$$\{\,\xi \in \text{Ord} : \xi < \alpha\,\} \cap \gamma, \quad \{\,\xi \in \text{Ord} : \alpha < \xi < \beta\,\} \cap \gamma \ \text{ and } \ \{\,\xi \in \text{Ord} : \beta < \xi\,\} \cap \gamma$$

yield a base of open sets for the **interval topology** on γ; see Frink [1942, pp. 569–570].

The real line was the first instance in which intervals were used to produce a topology. Haar/König [1911] generalized Cantor's approach to the setting of linearly ordered sets. Topological spaces constituted by ordinals occurred in the work of Mazurkiewicz/Sierpiński [1920], Tychonoff [1930, p. 553], Dieudonné [1939, p. 145], and Hewitt [1948, p. 63]. The latter observed that $\omega_1 + 1$ is the Stone–Čech compactification of ω_1, the first uncountable ordinal; see also [GIL$^+$, pp. 72–76] and [STE$^+$, pp. 68–70].

4.6.14 Spaces of ordinals can be used in measure theory. According to [HAL$_1$, acknowledgments]: *The example 52.10* [of his book] *was discovered by Dieudonné.* Since I could not find a written source, it seems that Halmos obtained this information by oral communication.

Every continuous function $f_0 : \omega_1 \to \mathbb{R}$ is bounded and even constant on some tail $\xi \ge \alpha$. Thus it admits a unique continuous extension $f : \omega_1 + 1 \to \mathbb{R}$. Hence $C_b(\omega_1)$ and $C(\omega_1 + 1)$ can be identified. $\mathscr{B}_{\text{aire}}(\omega_1 + 1)$ consists of all countable subsets of ω_1 as well as their complements. The singleton $\{\omega_1\}$ is Borel, but not Baire.

Of course, the point evaluation $\delta : f \mapsto f(\omega_1)$ can be obtained from a regular Borel measure on ω_1+1, which is given by $\delta(B):=1$ if $\omega_1 \in B$, and $\delta(B):=0$ if $\omega_1 \notin B$.

On the other hand, viewing δ as a functional on $C_b(\omega_1)$, we may look for a representing additive set function δ_0 in the sense of Markoff–Alexandroff:

First of all, observe that a closed subset of ω_1 is uncountable if and only if it is cofinal. Denote the collection of these sets by $\mathscr{C}(\omega_1)$. Next, let $\mathscr{A}(\omega_1)$ consist of all subsets A of ω_1 such that A or $\complement A$ contains some $C \in \mathscr{C}(\omega_1)$. Put $\delta_0(A):=1$ or $\delta_0(A):=0$, accordingly. Since $\mathscr{C}(\omega_1)$ has the countable intersection property, it follows that δ_0 is a countably additive measure on the σ-algebra $\mathscr{A}(\omega_1)$. In addition, its restriction to $\mathscr{B}_{\mathrm{orel}}(\omega_1)$ is regular. Hence, letting $\delta_0(\{\omega_1\}):=0$, we get a second (non-regular!) Borel measure on ω_1+1 that also represents $\delta : f \mapsto f(\omega_1)$. For details, the reader is referred to [FLO, pp. 350–351].

4.7 Measures versus integrals

The following statement of Bauer is taken from [RADO$^{\bowtie}$, Vol. I, pp. 30–31] (original version on p. 681):

Radon measures are closely related to topology and, therefore, to the geometry of the $\mathbb{R}^n$. Due to the study of abstract measures this coupling had been lost in oblivion. At the beginning of the 1940s, people became interested in measures on locally compact and Polish spaces. Subsequently, these considerations were extended to general topological spaces. This process led to a revival of Radon's ideas.

For a time, the representatives of the "abstract" and the "concrete" standpoint were opposed to each other in a warlike situation.

Bull. Amer. Math. Soc. **59** (1953), 249–255; Review of Bourbaki's *Intégration*, Chap. I–IV, by Halmos:

My conclusion on the evidence so far at hand is that the authors have performed a tremendous tour de force*; I am inclined to doubt whether their point of view will have a lasting influence.*

Bull. Amer. Math. Soc. **59** (1953), 479–480; Review of Mayrhofer's *Inhalt und Maß* by Dieudonné:

Finally, the reviewer wants to take exception to the author's statement that measure theory (as understood in this book) is the foundation of the theory of integration. This was *undoubtedly true some years ago; but is fortunately no longer so, as more and more mathematicians are shifting to the "functional approach" to integration. It is always rash to make predictions, but the reviewer cannot help thinking that, despite its intrinsic merits, this book, as well as its brethren of the same tendency, will in a few years have joined many an other obsolete theory on the shelves of the Old Curiosity Shop of mathematics.*

Luckily enough, both reviewers were wrong.

While Bauer stressed the problem whether we need topology, it seems to me that the main controversy was between set functions and linear functionals. This opinion is supported by a quotation from [BOU$_{6b}$, p. 123] or [BOU$^{\bullet}$, p. 228] (original version on p. 681):
But it is with [CARA] *also that the notion of integral yields the field for the first time to that of a measure, which had been for Lebesgue an auxiliary technical means. Since then, the authors that have treated integration are divided between these two points of view, not without involving themselves in debates which have caused a lot of ink to flow if not a lot of blood.*

4.8 Abstract L_p- and M-spaces

4.8.1 Boolean algebras

4.8.1.1 A commutative ring $\mathscr{R}$ is said to be **Boolean** if $x^2 = x$ for all elements $x \in \mathscr{R}$. Then $x + x = \mathrm{o}$.

A distributive lattice $\mathscr{B}$ containing a smallest element o and a largest element $\mathbf{1}$ is called **Boolean** if every $x \in \mathscr{B}$ has a complement $x' \in \mathscr{B}$, which is characterized by the properties $x \wedge x' = \mathrm{o}$ and $x \vee x' = \mathbf{1}$.

Stone [1936, pp. 43–44] realized that every Boolean lattice becomes a Boolean ring with unit under the algebraic operations

$$a + b := \overset{\text{symmetric difference}}{(a \wedge b') \vee (a' \wedge b)} \quad \text{and} \quad a \cdot b := a \wedge b$$

and that conversely, every Boolean ring with unit can be made a Boolean lattice. In this sense, the concept of an algebraic **ideal** $\mathscr{I}$ has a lattice-theoretic counterpart:

(1) If $a, b \in \mathscr{I}$, then $a \vee b \in \mathscr{I}$. (2) If $a \in \mathscr{I}$ and $x \in \mathscr{B}$, then $a \wedge x \in \mathscr{I}$. Usually, (2) is replaced by the equivalent condition that $a \in \mathscr{I}$, $x \in \mathscr{B}$, and $x \leq a$ imply $x \in \mathscr{I}$.

A Boolean lattice is commonly called a **Boolean algebra**, and the same name is used for a Boolean ring with unit.

Huntington [1933, p. 278, footnote]:

The name Boolean algebra for the calculus originated by Boole, extended by Schröder, and perfected by Whitehead seems to have been first suggested by Sheffer, in 1913.

4.8.1.2 After some preliminary announcements in the Proceedings of the National Academy of Sciences, Stone's representation theory of Boolean algebras was published in [1936] and [1937a]. In a summary of his results [1938, p. 814] Stone says:

A much deeper insight into the structure of Boolean rings is made possible by the introduction of topological concepts. A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize."

4.8.1.3 The *representation space* K of a Boolean algebra $\mathscr{B}$ consists of all prime ideals $\mathscr{P}$ in $\mathscr{B}$, and the subsets $\pi(a) := \{\mathscr{P} \in K : a \notin \mathscr{P}\}$ with $a \in \mathscr{B}$ form a concrete Boolean algebra under the set-theoretic operations.

Next, a topology is obtained on K by taking the $\pi(a)$'s as a base of open sets. The resulting Hausdorff space turns out to be compact. Since $\pi(a)$ and $\pi(a')$ are complements of each other, the $\pi(a)$'s are closed as well. Hence every open set of K is the union of its **clo(sed-and-o)pen** subsets. A topological space with this property is called **0-dimensional**. Because of compactness, we can also say that K is **totally disconnected**: points $a \neq b$ are contained in complementary open sets.

For every topological space M, the collection of all clopen subsets is a Boolean algebra, which will be denoted by $\mathscr{C}_{\rm lop}(M)$. A subset is clopen if and only if its characteristic function is continuous. Thus clopen sets are simultaneously zero-sets and cozero-sets. Moreover, $\mathscr{B}_{\rm aire}(M)$ is the smallest σ-algebra containing $\mathscr{C}_{\rm lop}(M)$.

Representation theorem; Stone [1936, p. 106], [1937a, pp. 378–379]:

Every abstract Boolean algebra is isomorphic to a concrete Boolean algebra $\mathscr{C}_{\rm lop}(K)$. The totally disconnected and compact representation space K is unique up to a homeomorphism.

Stone's construction extends to Boolean rings without unit. In this case, however, one gets only a locally compact space.

4.8.1.4 Cantor's ternary set and the one-point compactification of $\mathbb{N}$ are the most typical examples of totally disconnected and compact spaces.

4.8.1.5 In further papers [1937b, p. 266] and [1949, pp. 180, 185], Stone answered the question, *How can Dedekind completeness of a Boolean algebra be characterized via its representation space?* Following Hewitt [1943, p. 326], we refer to Hausdorff spaces with this property as **extremally disconnected**: the closure of any open set is open. Extremally disconnected and compact Hausdorff spaces are called **Stonean**.

Similarly, $C(K)$ is Dedekind countably complete if and only if the closure of any cozero-set of K is open. In [GIL$^+$, p. 22] and [SEMA, p. 431] such spaces are said to be **basically disconnected** and **ω-extremally disconnected**, respectively.

The preceding results were independently obtained by Nakano [1941a].

4.8.1.6 The Stone–Čech compactification of a discrete space $\mathbb{I}$ (all sets are clopen) turns out to be extremally disconnected. Based on this fact, it follows that K is extremally disconnected if and only if its identity map factors continuously through some $\beta\mathbb{I}$. That is, $I_K : K \to \beta\mathbb{I} \to K$. In view of $l_\infty(\mathbb{I}) = C(\beta\mathbb{I})$, passing to the associated Banach spaces yields $I_{C(K)} : C(K) \to l_\infty(\mathbb{I}) \to C(K)$; see 4.9.2.2.

The history of this result is described in [SEMA, p. 445].

4.8.1.7 In 4.8.1.3 we have defined a lattice isomorphism π between $\mathscr{B}$ and $\mathscr{C}_{\mathrm{lop}}(K)$:

$$\pi\Big(\bigvee_{i\in\mathbb{I}} a_i\Big) = \bigcup_{i\in\mathbb{I}} \pi(a_i) \quad \text{for every } \textit{finite} \text{ index set } \mathbb{I}.$$

However, the reader should keep in mind that in the infinite case, the right-hand side is of course open, but need not be closed. If $\mathscr{B}$ is Dedekind complete, we have

$$\pi\Big(\bigvee_{i\in\mathbb{I}} a_i\Big) = \overline{\bigcup_{i\in\mathbb{I}} \pi(a_i)}.$$

I stress that in the theory of Boolean algebras, it is often more suggestive to write $\bigvee_{i\in\mathbb{I}} a_i$ instead of $\sup_{i\in\mathbb{I}} a_i$.

4.8.1.8 Let K be any totally disconnected compact Hausdorff space. Kakutani [1941a, p. 533] showed that a clopen subset A of K cannot be the union of infinitely many pairwise disjoint and non-empty clopen sets $A_1, A_2, \ldots$. Otherwise, finitely many of the open sets A_k would already cover the compact set A. Contradiction! Thus every finitely additive measure on $\mathscr{C}_{\mathrm{lop}}(K)$ is automatically countably additive.

The Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ may be identified with the collection of all ultrafilters on $\mathbb{N}$. In particular, with every natural number n we associate the ultrafilter $\mathscr{U}_n := \{A \in \mathscr{P}(\mathbb{N}) : n \in A\}$. It turns out that $\pi : A \mapsto \{\mathscr{U} \in \beta\mathbb{N} : A \in \mathscr{U}\}$ yields an isomorphism between the Boolean algebras $\mathscr{P}(\mathbb{N})$ and $\mathscr{C}_{\mathrm{lop}}(\beta\mathbb{N})$.

Recall that all finitely additive measures on $\mathscr{C}_{\mathrm{lop}}(\beta\mathbb{N})$ are countably additive. Strangely enough, this is not so on $\mathscr{P}(\mathbb{N})$. Why? The answer is given by the fact that the isomorphism π does not preserve countable unions:

$$\bigcup_{n=1}^{\infty} \pi(\{n\}) = \big\{\mathscr{U}_n : n \in \mathbb{N}\big\} \neq \beta\mathbb{N} = \pi(\mathbb{N}) = \pi\Big(\bigcup_{n=1}^{\infty}\{n\}\Big).$$

4.8.1.9 The pathology just described can be circumvented by a theorem that was independently proved by Loomis [1947, p. 757], and Sikorski [1948, pp. 255–256]; see also [HAL$_2$, § 23], [SEMA, p. 286], and [SIK, § 29].

Let $\mathscr{B}$ be any Boolean σ-algebra. Form Stone's representation space K, and denote the collection of all meager subsets (sets of first category) by $\mathscr{M}_{\mathrm{eag}}(K)$; see 2.4.6. Then $\mathscr{B}$ is isomorphic to the quotient $\mathscr{B}_{\mathrm{aire}}(K)/\mathscr{B}_{\mathrm{aire}}(K) \cap \mathscr{M}_{\mathrm{eag}}(K)$, and most importantly, the supremum of countably many equivalence classes in $\mathscr{B}_{\mathrm{aire}}(K)/\mathscr{B}_{\mathrm{aire}}(K) \cap \mathscr{M}_{\mathrm{eag}}(K)$ is obtained by taking the set-theoretic union of arbitrary representatives. In the case of Dedekind completeness, $\mathscr{B}_{\mathrm{aire}}(K)$ can be replaced by $\mathscr{B}_{\mathrm{orel}}(K)$, since $\overline{G}$ is clopen and $\overline{G}\backslash G$ is rare for every open subset G.

4.8.2 Measure algebras

4.8.2.1 Given any countably additive measure μ on a σ-algebra $\mathscr{M}$, we denote by $\mathscr{M}_0(\mu)$ the collection of all μ-null sets: $\mu(A) = 0$. Nikodym [1930, pp. 137–140] defined the **μ-distance**

$$\|A, B\|_\mu := \mu(A \triangle B) = \mu(A \setminus B) + \mu(B \setminus A) \quad \text{for } A, B \in \mathscr{M}.$$

Passing to equivalence classes, he showed that $\mathscr{M}/\mathscr{M}_0(\mu)$ becomes a complete metric space. Actually, Nikodym treated only the case of $\mu(M)$ finite. Otherwise, one has to use a modified metric, for example $\min\{\mu(A\triangle B),1\}$.

4.8.2.2 Wecken [1939, pp. 378–379] discovered another remarkable property of $\mathscr{M}/\mathscr{M}_0(\mu)$: it turns out to be a Boolean σ-algebra. Under the name Σ-*System* he introduced the important concept of a **measure algebra**. This is a Boolean σ-algebra $\mathscr{B}$ on which there exists a countably additive measure,

$$\tilde{\mu}\Big(\bigvee_{k=1}^{\infty} a_k\Big) = \sum_{k=1}^{\infty} \tilde{\mu}(a_k) \quad \text{whenever } a_h \wedge a_k = \mathsf{o} \text{ for } h \neq k.$$

An extra condition requires that $\tilde{\mu}$ be **strictly positive**: $\tilde{\mu}(a) > 0$ if $a > \mathsf{o}$.

Every Boolean σ-algebra $\mathscr{B}$ carries a useless measure: $\tilde{\mu}(a) := \infty$ for $a > \mathsf{o}$. In order to avoid such trivialities, one assumes that there are sufficiently many members $a \in \mathscr{B}$ with $\tilde{\mu}(a) < \infty$.

More precisely, $\tilde{\mu}$ is called **semi-finite** if for every $a \in \mathscr{B}$ with $\tilde{\mu}(a) = \infty$ we can find $b \in \mathscr{B}$ such that $b < a$ and $0 < \tilde{\mu}(b) < \infty$; see 4.3.8.

Following Fremlin [FREM$_1$, p. 136], we refer to $(\mathscr{B},\tilde{\mu})$ as a **Maharam algebra** if $\mathscr{B}$ is Dedekind complete and $\tilde{\mu}$ is strictly positive as well as semi-finite. Then every element a is the supremum of all elements $b \leq a$ with finite measure; see Segal [1951, p. 278]. This condition guarantees that many considerations can be carried out in the subring $\mathscr{B}_\bullet(\tilde{\mu}) := \{a \in \mathscr{B} : \tilde{\mu}(a) < \infty\}$.

Wecken [1939, p. 380] had earlier observed that the measure algebra $\mathscr{M}/\mathscr{M}_0(\mu)$ is Maharam if the underlying measure space $(M,\mathscr{M},\mu)$ has finite total mass. In the general case, we obtain a Maharam algebra precisely when $(M,\mathscr{M},\mu)$ is localizable; see 4.3.9 and [KEL$^+$, p. 129].

4.8.2.3 An element $a > \mathsf{o}$ in a Boolean algebra $\mathscr{B}$ is said to be an **atom** if $b \leq a$ implies either $b = a$ or $b = \mathsf{o}$.

Denote the collection of all atoms by A. Assuming Dedekind completeness, there exists $a := \sup A$. Then the algebra $\mathscr{B}(a) := \{x \in \mathscr{B} : x \leq a\}$ is isomorphic to $\mathscr{P}(A)$, while $\mathscr{B}(a') := \{x \in \mathscr{B} : x \leq a'\}$ becomes atomless; a' denotes the complement of a.

4.8.2.4 The next result is usually attributed to Carathéodory [1939, p. 129]. However, it was already known to Bochner/von Neumann [1935, p. 264, footnote 17]; see also Halmos/von Neumann [1942, p. 335]. An exposition of the proof can be found in [ROY, pp. 264–266].

Every atomless and separable measure algebra with total mass 1 is isomorphic to Lebesgue's measure algebra of the interval $[0,1]$. This correspondence preserves the measure.

Note that separability is meant with respect to the metric $\tilde{\mu}(a\triangle b)$.

4.8.2.5 For every index set $\mathbb{I}$, there exists a canonical measure on $[0,1]^{\mathbb{I}}$ that is the $\mathbb{I}^{\text{th}}$ power of Lebesgue's measure on $[0,1]$. The associated measure algebra will be denoted by $\mathscr{L}([0,1]^{\mathbb{I}})$.

A further example is given by the Cantor group $\{0,1\}^{\mathbb{I}}$ equipped with its Haar measure. Since the map

$$(\xi_1,\xi_2,\dots)\mapsto\sum_{k=1}^{\infty}2^{-k}\xi_k$$

induces an isomorphism between the measure algebra of $\{0,1\}^{\mathbb{N}}$ and Lebesgue's measure algebra $\mathscr{L}([0,1])$, one may subsequently replace $[0,1]$ by $\{0,1\}$.

4.8.2.6 A Boolean algebra is called **homogeneous** if it has the following equivalent properties; see [MONK$^{\cup}$, Vol. I, p. 135]:

For every non-zero element $a\in\mathscr{B}$, the algebra $\mathscr{B}(a):=\{x\in\mathscr{B}:x\le a\}$ is isomorphic to the whole algebra.

Every non-empty clopen subset of Stone's representation space is homeomorphic to the whole space.

The original definition of Maharam [1942, p. 108] is different. However, if $\mathscr{B}$ admits a finite measure, then both concepts coincide; see Fremlin [1989, p. 912].

Maharam's key result [1942, p. 109], which generalizes 4.8.2.4, says that every homogeneous measure algebra with total mass 1 is isomorphic to some $\mathscr{L}([0,1]^{\mathbb{I}})$. This correspondence preserves the measure.

In order to include the only finite homogeneous Boolean algebra, namely $\{\mathsf{o},\mathbf{1}\}$, we agree that $[0,1]^0$ is a singleton.

4.8.2.7 The upshot is the famous **Maharam theorem**:
Every Maharam algebra is isomorphic to a direct sum of concrete measure algebras $\mathscr{L}([0,1]^{\mathbb{I}})$, and it is uniquely determined by a family of cardinals.

First of all, one cuts off the atoms as described in 4.8.2.3. Using Zorn's lemma, we can decompose the remaining part into a direct sum of measure algebras with finite total mass.

Next, and this was the starting point of Maharam's work, every measure algebra with finite total mass splits into a finite or countably infinite number of homogeneous summands. Finally, as stated in 4.8.2.6, each of these summands can be identified with some $\mathscr{L}([0,1]^{\mathbb{I}})$.

In terms of the Stone representation space K, the invariants above are obtained as follows:

Form the set of all isolated points and determine its cardinality: $\gamma(0)$.

Fix any infinite cardinal $\aleph_\alpha$, and let K_α denote the Stone representation space of $\mathscr{L}([0,1]^{\aleph_\alpha})$. Count how many pairwise disjoint copies of K_α can be placed into K, and let $\gamma(\aleph_\alpha)$ be this cardinality.

The result is a transfinite sequence $\gamma(0), \gamma(\aleph_0), \gamma(\aleph_1), \ldots$ such that $\gamma(\aleph_\alpha) = 0$ for $\alpha \geq \alpha_0$. Details of the proof and further historical comments can be found in [SEMA, pp. 466–477]; see also Fremlin [1989, § 3].

4.8.2.8 Originally, measure algebras were obtained from measure spaces by working with equivalence classes modulo null sets: $\mathscr{M}/\mathscr{M}_0(\mu)$. On the other hand, it follows from the Loomis-Sikorski theorem 4.8.1.9 that every measure algebra can be produced in this way. However, the mapping $(M, \mathscr{M}, \mu) \mapsto (\mathscr{B}, \tilde{\mu})$ is not one-to-one. On the contrary, every measure algebra $(\mathscr{B}, \tilde{\mu})$ has a vast variety of generating measure spaces $(M, \mathscr{M}, \mu)$ of different quality. Thus we are free to make a good choice; see 5.1.3.14.

Segal [1951, p. 308] described this phenomenon as follows:

In various situations it is desirable to replace a given measure space by a metrically equivalent one in which measurability difficulties are minimized, even tho the new space may, roughly speaking, be much larger than the original one.

4.8.2.9 Until now, the important question, *Which Dedekind complete Boolean algebras carry a strictly positive measure?* still waits for a convincing answer.

The problem may be considered on two levels. In the strong form, one allows only measures with finite values. Then the *countable chain condition* turns out to be necessary: every family of pairwise disjoint non-zero elements is countable; Tarski [1945, pp. 51, 55]. Further contributions along these lines are due to Maharam [1947, p. 160], Horn/Tarski [1948, p. 481], Kelley [1959, p. 1170], and Gaifman [1964, p. 67]. However, for the purpose of Banach space theory, this point of view would be too restrictive. We need, for example, counting measures on uncountable sets.

Luckily, the semi-finite case can be reduced to the previous one. In order to ensure the existence of a reasonable measure, it suffices to know that for every non-zero element $a \in \mathscr{B}$, there exists a non-zero element $b \leq a$ such that the Boolean algebra $\mathscr{B}(b) := \{x \in \mathscr{B} : x \leq b\}$ carries a strictly positive and finite measure. For a proof of this result and much more, I refer to a readable survey of Fremlin [1989, § 5].

4.8.2.10 The preceding problem can be rephrased in topological terms.

A positive $\ell \in C(K)^*$ is called **normal** if $\inf_{\alpha \in \mathbb{A}} \ell(f_\alpha) = 0$ for all downward directed systems $(f_\alpha)_{\alpha \in \mathbb{A}}$ in $C(K)$ such that $\inf_{\alpha \in \mathbb{A}} f_\alpha = o$. Here *downward directed* means that $f_\alpha \leq f_\beta$ whenever $\alpha \geq \beta$. It turns out that the linear span of these ℓ's is a band in $C(K)^*$.

4.8.2.11 According to Dixmier [1951a, pp. 160–165], a Stonean space K is referred to as **hyper-Stonean** if for every continuous function $f > o$ there exists a normal functional $\ell > o$ such that $\ell(f) > 0$. This happens if and only if $C(K)$ has a predual, which is just the band described above.

Kelley [1959, footnote on p. 1169]:
The term "hyperstonean" seems unfortunate. In spite of my affection and admiration for Marshall Stone, I find the notion of a Hyper-Stone downright appalling.

4.8.2.12 For every topological space M, we denote the collection of all rare subsets by $\mathscr{R}_{\text{are}}(M)$; see 2.4.6. Note that $\mathscr{R}_{\text{are}}(M)$ is an ideal in $\mathscr{P}(M)$; see [BOU$_{3b}$, p. 73].

Dixmier [1951a, p. 157] showed that in every Stonean space K, a positive $\ell \in C(K)^*$ is normal if and only if all rare subsets are negligible with respect to the completion of the associated Borel measure.

In a hyper-Stonean space K all meager subsets are rare. Hence 4.8.1.9 implies that $\mathscr{C}_{\text{lop}}(K)$ is isomorphic to $\mathscr{B}_{\text{orel}}(K)/\mathscr{B}_{\text{orel}}(K) \cap \mathscr{R}_{\text{are}}(K)$. Most importantly, $\mathscr{C}_{\text{lop}}(K)$ carries a semi-finite and strictly positive measure μ. Since every equivalence class of bounded $\mathscr{B}_{\text{orel}}$-measurable functions on K contains one and only one continuous member, we get $C(K) = L_\infty(K, \mathscr{B}_{\text{orel}}, \mu) = L_1(K, \mathscr{B}_{\text{orel}}, \mu)^*$; see 4.10.4.24/25.

Summary: A Stonean space K is hyper-Stonean if and only if the Dedekind complete Boolean algebra $\mathscr{C}_{\text{lop}}(K)$ can be made Maharam.

4.8.2.13 The concluding example shows that there exist Dedekind complete Boolean algebras without any "good" measure.

For every topological space M, we denote by $\mathscr{R}_{\text{op}}(M)$ the collection of all *regular open* subsets: G coincides with the interior of its closure. $\mathscr{R}_{\text{op}}(M)$ is a Dedekind complete Boolean algebra with respect to the set-theoretic ordering. However, care is recommended concerning Boolean operations. For example, the Boolean complement of a regular open set is the set-theoretic complement of its closure.

If this construction is applied to the real line, then $\mathscr{R}_{\text{op}}(\mathbb{R})$ is isomorphic to the quotient $\mathscr{B}_{\text{orel}}(\mathbb{R})/\mathscr{B}_{\text{orel}}(\mathbb{R}) \cap \mathscr{M}_{\text{eag}}(\mathbb{R})$; see [BIRK, p. 103] and 4.8.1.9. Szpilrajn [1934, p. 305] had earlier observed that every finite and countably additive measure vanishes identically on $\mathscr{B}_{\text{orel}}(\mathbb{R})/\mathscr{B}_{\text{orel}}(\mathbb{R}) \cap \mathscr{M}_{\text{eag}}(\mathbb{R})$. The sole alternative is that the measure takes the value ∞ except for o; see [FREM$_1$, pp. 122–124].

4.8.2.14 A monumental presentation of measure theory is given in [FREM$_2$]; measure algebras are treated in Volume III.

4.8.3 Abstract L_p-spaces

4.8.3.1 Let $1 \le p < \infty$. A Banach lattice X is said to be an **abstract L_p-space** if

$$\|x+y\|^p = \|x\|^p + \|y\|^p \quad \text{whenever } |x| \wedge |y| = o. \tag{4.8.3.1.a}$$

We know from (4.1.7.a) that $|x+y| = |x| + |y|$ if $|x| \wedge |y| = o$. Thus it suffices to verify the preceding identity only for positive elements. The significance of (4.8.3.1.a) was discovered by Bohnenblust [1940, p. 633].

The concept of an **abstract L-space** goes back to Garrett Birkhoff [1938, p. 155], who required that

$$\|x+y\| = \|x\| + \|y\| \quad \text{whenever } x, y \ge o,$$

which implies $(\mathbf{M}_+)$, but not $(\mathbf{M}_0)$; see 4.1.10. Kakutani [1941a, p. 525] observed that $(\mathbf{M}_0)$ can always be achieved by a renorming. Then one gets (4.8.3.1.a) with $p = 1$, and this is the present definition of an abstract L-space.

4.8.3.2 Bernau [1973, p. 285] inferred from (4.8.3.1.a) that

$$\|x+y\|^p \geq \|x\|^p + \|y\|^p \quad \text{whenever } x, y \geq \mathrm{o}.$$

This inequality is basic for the proof that abstract L_p-spaces are Dedekind complete.

4.8.3.3 Of course, (4.8.3.1.a) holds in any $L_p(M, \mathscr{M}, \mu)$. However, more is true:

Every abstract L_p-space is isometric and lattice isomorphic to some concrete function space $L_p(M, \mathscr{M}, \mu)$.

The desired measure space $(M, \mathscr{M}, \mu)$ is created in three steps.

(1) Assume that the abstract L_p-space X contains a **Freudenthal order unit**. This is a positive member $u \in X$ such that $|x| \wedge u = \mathrm{o}$ implies $x = \mathrm{o}$. Then the subset $\mathscr{B}_u := \{e \in X : e \wedge (u-e) = \mathrm{o}\}$ simulates a Boolean algebra of characteristic functions. Since X is Dedekind complete, so is $\mathscr{B}_u$. Letting $\tilde{\mu}_u(e) := \|e\|^p$ yields a countably additive and finite measure on $\mathscr{B}_u$.

(2) Choose a measure space $(M_u, \mathscr{M}_u, \mu_u)$ that generates the Maharam algebra $(\mathscr{B}_u, \tilde{\mu}_u)$.

(3) Zorn's lemma gives a maximal family $(u_i)_{i \in \mathbb{I}}$ of positive and mutually orthogonal elements. Since each u_i is a Freudenthal unit in the band $\{u_i\}^{\perp\perp}$, the construction above produces a family of measure spaces $(M_{u_i}, \mathscr{M}_{u_i}, \mu_{u_i})$, and $(M, \mathscr{M}, \mu)$ is obtained as their direct sum.

The preceding representation theorem is due to the joint efforts of Bohnenblust [1940], Kakutani [1941a], Nakano [1941b], and others; see also Bretagnolle/Dacunha-Castelle/Krivine [1966, p. 241]. For a historical account, I refer to [LAC, p. 135].

4.8.3.4 Given any Maharam algebra $(\mathscr{B}, \tilde{\mu})$, then–in spite of the fact that we may choose different generating measure spaces $(M, \mathscr{M}, \mu)$–all $L_p(M, \mathscr{M}, \mu)$'s are isometric and lattice isomorphic; $1 \leq p < \infty$ fixed. Even more is true:

Suppose that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are semi-finite and strictly positive measures on $\mathscr{B}$. Fix generating measure spaces $(M_1, \mathscr{M}_1, \mu_1)$ and $(M_2, \mathscr{M}_2, \mu_2)$. Since the Radon–Nikodym theorem is at our disposal, we may conclude that $L_p(M_1, \mathscr{M}_1, \mu_1)$ and $L_p(M_2, \mathscr{M}_2, \mu_2)$ are "fully" isomorphic: identify the characteristic function of $B_1 \in \mathscr{M}_1$ and the characteristic function of $B_2 \in \mathscr{M}_2$ provided that B_1 and B_2 have the same image in $\mathscr{B}$. Thus every Dedekind complete Boolean algebra $\mathscr{B}$ that can be made *"Maharam"* gives rise to a well-defined space $L_p(\mathscr{B})$, where the choice of the underlying measure spaces is irrelevant.

The collection of all band projections on a linear lattice X is a Boolean algebra, denoted by $\mathscr{B}(X)$; see [SCHAE, p. 61]. If X is Dedekind complete, then so is $\mathscr{B}(X)$.

Every band projection on $L_p(M, \mathscr{M}, \mu)$ has the form $P_A : f \mapsto \chi_A f$ with $A \in \mathscr{M}$. For μ's with finite total mass, this result is due to Douglas [1965, p. 455], and the general case follows by localization. Therefore the Boolean algebras $\mathscr{B}\big(L_p(M, \mathscr{M}, \mu)\big)$ and $\mathscr{M}/\mathscr{M}_0(\mu)$ are isomorphic, which means that $\mathscr{B}$ can be recaptured from $L_p(\mathscr{B})$.

Summary: For fixed $1 \leq p < \infty$, letting $X = L_p(\mathscr{B})$ yields a one-to-one correspondence $X \Leftrightarrow \mathscr{B}$ between abstract L_p-spaces X and Dedekind complete Boolean algebras $\mathscr{B}$ on which there exist semi-finite and strictly positive measures. Note that isometric and lattice isomorphic linear spaces X are identified.

4.8.3.5 If X is an abstract L_p-space, then the Boolean algebra of its band projections can be used for a direct proof of the representation theorem 4.8.3.3. Indeed, we obtain a countably additive measure on $\mathscr{B}(X)$ by letting

$$\tilde{\mu}(P) := \sum_{i\in\mathbb{I}} \|Pu_i\|^p \quad \text{for } P\in\mathscr{B}(X).$$

Here $(u_i)_{i\in\mathbb{I}}$ is any maximal family of positive and mutually disjoint elements of X. This construction gives a Maharam algebra $(\mathscr{B}(X),\tilde{\mu})$ such that $L_p(\mathscr{B}(X))$ is isometric and lattice isomorphic to X.

4.8.3.6 Note that a considerable amount of information gets lost in passing from $(M,\mathscr{M},\mu)$ to $L_p(M,\mathscr{M},\mu)$. However, if we are interested only in functional analytical properties of the Banach space $L_p(M,\mathscr{M},\mu)$ and not in the fact that its elements are equivalence classes of functions on a specific set M, then this degree of freedom turns out to be advantageous. For example, according to 4.3.7, it may happen that $L_1(M,\mathscr{M},\mu)^* \neq L_\infty(M,\mathscr{M},\mu)$. Luckily enough, there exists another measure space $(N,\mathscr{N},\nu)$ such that $L_1(M,\mathscr{M},\mu)$ and $L_1(N,\mathscr{N},\nu)$ can be identified and, moreover, $L_1(N,\mathscr{N},\nu)^* = L_\infty(N,\mathscr{N},\nu)$. Wonderful!

4.8.3.7 The **l_p-sum** of a finite family of Banach spaces was introduced in [BAN, p. 182], where the norm of an element $z=(x,y)\in X\times Y$ is defined by

$$\|z\| = \big(\|x\|^p + \|y\|^p\big)^{1/p} \quad \text{with } 1\leq p<\infty.$$

The resulting space will be denoted by $X\oplus_p Y$.

In the case of an infinite family $(X_i)_{i\in\mathbb{I}}$, the elements (x_i) of $[l_p(\mathbb{I}),X_i]$ with $x_i\in X_i$ have the norm

$$\|(x_i)|l_p\| = \Big(\sum_{i\in\mathbb{I}} \|x_i\|^p\Big)^{1/p},$$

which is supposed to be finite. The right-hand limit is taken in the sense described in 3.2.2.3.

For the index set $\mathbb{N}$, the preceding construction goes back to Day [1941, p. 314]. Not necessarily countable index sets occurred in the work of Kakutani [1941a, p. 528] when he decomposed abstract L-spaces into a direct sum of bands generated by singletons.

There is disagreement concerning terminology. Some people like the name **direct products**, while others prefer **direct sums**. The notation is even more multifarious:

$$(X_1\times\cdots\times X_n)_{l_p}, \quad \bigoplus_{n=1}^{\infty}{}_{l_p} X_n, \quad \Big(\sum_{i\in\mathbb{I}} \oplus X_i\Big)_p,$$

and I could not resist contributing my own symbol, namely $[l_p(\mathbb{I}),X_i]$.

4.8.3.8 Transferring the structure theory of Maharam algebras, described in 4.8.2.7, from $\mathscr{B}$ to $L_p(\mathscr{B})$ yields an important representation theorem; see [LAC, p. 136]: Every abstract L_p-space is isometric and lattice isomorphic to an l_p-sum

$$\Big[l_p(\mathbb{I}), L_p\big([0,1]^{\aleph(i)}\big)\Big],$$

where either $\aleph(i) > \aleph_0$, or $\aleph(i) = 1$, or $\aleph(i) = 0$. In the latter case, we agree that $L_p\big([0,1]^0\big)$ is the scalar field. Note that $L_p\big([0,1]\big) = L_p\big([0,1]^2\big) = \cdots = L_p\big([0,1]^{\aleph_0}\big)$.

4.8.3.9 As a corollary of the previous result, we get the duality $L_p^* = L_{p^*}$ for abstract L_p-spaces and $1 < p < \infty$. The limiting cases $p = 1$ and $p = \infty$ are treated in 4.8.4.2.

4.8.4 Abstract M-spaces

4.8.4.1 A Banach lattice X is said to be an **abstract M-space** if

$$\|x+y\| = \max\{\|x\|, \|y\|\} \quad \text{whenever } |x|\wedge|y| = o.$$

Originally, Kakutani [1941b, p. 994] required a seemingly stronger property:

$$\|x\vee y\| = \max\{\|x\|, \|y\|\} \quad \text{whenever } x, y \geq o.$$

Subsequently, in a joint paper [1941] with Bohnenblust, it was shown that the two conditions are equivalent.

4.8.4.2 We know from Kakutani [1941b, p. 1021] that the dual of an abstract M-space is an abstract L-space and that the dual of an abstract L-space is an abstract M-space. Hence it would be justified to speak of abstract L_∞-spaces instead of abstract M-spaces. Kakutani's notation was suggested by the fact that in [BAN, pp. 10–11] the letters M and m were used to denote L_∞ and l_∞, respectively.

4.8.4.3 An **order unit** is a positive element u such that the closed unit ball of X has the form $\{x \in X : |x| \leq u\}$. Obviously, every Banach lattice with an order unit is an abstract M-space.

4.8.4.4 Every Banach lattice $C(K)$ defined on a compact Hausdorff space K is an abstract M-space with order unit: $u(t) = 1$ for all $t \in K$. Most remarkably, the converse holds as well. Kakutani [1941b, p. 998] and the Kreĭn brothers [1940] proved independently that every abstract M-space with order unit is isometric and lattice isomorphic to some $C(K)$. Moreover, K is unique up to a homeomorphism.

The required K is obtained as the collection of all functionals $\ell \in X^*$ such that $\ell(u) = 1$ and $\ell(x \wedge y) = \ell(x) \wedge \ell(y)$. By Alaoglu's theorem, K becomes a compact Hausdorff space with respect to the weak* topology.

In contrast to the theory of abstract L_p-spaces, it may happen that a Banach space $C(K)$ cannot be recaptured from the Boolean algebra of its band projections: $\mathscr{B}\big(C(K)\big)$. For example, on $C[a,b]$ there exist only the trivial band projections, which means that $\mathscr{B}\big(C[a,b]\big) = \{o, \mathbf{1}\}$.

4.8.4.5 Eilenberg [1942, p. 568] posed the problem *to relate "interesting" topological properties of K with "interesting" metric properties of* $C(K)$.

The following table provides a small sample of such results. In the right-hand column, the non-commutative counterparts are listed; see also Section 4.10.

Boolean algebra $\mathscr{C}_{\text{lop}}(K)$	compact space K	M-space with unit $C(K)$	Banach algebra
	arbitrary	arbitrary	$B^\star$-algebra
forms a base of open sets	totally disconnected	property $(*)$	
Dedekind countably complete	basically disconnected	Dedekind countably complete	$B^\star_p$-algebra
Dedekind complete	extremally disconnected / Stonean	Dedekind complete	$AW^\star$-algebra
Maharam algebra	hyper-Stonean	dual Banach lattice	$W^\star$-algebra

Property $(*)$ means that the simple continuous functions are dense in $C(K)$.

4.8.5 The Dunford–Pettis property

Finally, I discuss a remarkable property of abstract M- and L-spaces, which is in a sense a counterpart of reflexivity.

4.8.5.1 Recall from 2.6.3.5 that an operator $T \in \mathfrak{L}(X,Y)$ is **completely continuous** if it carries every weakly convergent sequence to a norm convergent sequence. We denote the collection of all completely continuous operators $T: X \to Y$ by $\mathfrak{V}(X,Y)$.

4.8.5.2 An operator $T \in \mathfrak{L}(X,Y)$ is said to be **weakly compact** if every bounded sequence (x_n) in X has a subsequence (x_{n_i}) such that (Tx_{n_i}) is weakly convergent in Y. The collection of all weakly compact operators $T: X \to Y$ will be denoted by $\mathfrak{W}(X,Y)$.

By the Eberlein–Shmulyan theorem 3.4.3.1, an operator $T \in \mathfrak{L}(X,Y)$ is weakly compact if and only if it maps the closed unit ball B_X into a relatively weakly compact subset of Y.

Referred to as *weakly completely continuous*, such operators were introduced by Kakutani and Yosida for the purpose of ergodic theory, 5.3.5.4.

4.8.5.3 Grothendieck [1953, pp. 135–137] was the first to define a subclass of spaces using operator ideals.

A Banach space X is said to have the **Dunford–Pettis property** if every weakly compact operator from X into any Banach space Y is completely continuous: $\mathfrak{W}(X,Y) \subseteq \mathfrak{V}(X,Y)$.

4.8.5.4 Next, I state a powerful criterion that was discovered by Grothendieck [1953, p. 138]:

A Banach space X enjoys the Dunford–Pettis property if and only if given weakly null sequences (x_n) in X and (x_n^*) in X^*, then $\langle x_n, x_n^* \rangle \to 0$.

Consequently, if X^* has the Dunford–Pettis property, then so does X. The converse implication fails: Stegall showed that $[l_1, l_2^n]$ has the Dunford–Pettis property, while $[l_\infty, l_2^n] = [l_1, l_2^n]^*$ does not; see Diestel [1980, p. 22].

4.8.5.5 The preceding considerations were initiated by the following corollary of Dunford/Pettis [1940, p. 370]:

When S is Euclidean and α is Lebesgue measure, then any weakly completely continuous operation U from $L(S)$ to X has the property of taking weakly compact sets into compact sets.

In the next volume of the Transactions, Phillips [1940, p. 535] proved the same result for L_1 over any measure space $(M, \mathscr{M}, \mu)$.

More than 10 years later, Grothendieck [1953, p. 139] verified the Dunford–Pettis property for $C(K)$. He even showed in [1953, p. 153] that $\mathfrak{V}(C(K),Y) = \mathfrak{W}(C(K),Y)$ for all Y.

I stress that the proof of Dunford–Pettis–Phillips is based on their representation theorem for weakly compact operators $T : L_1 \to X$, which will be discussed in 5.1.3.3. On the other hand, Grothendieck used purely measure theoretic techniques.

4.8.5.6 Obviously, $T \in \mathfrak{W}(X,Y)$ and $S \in \mathfrak{V}(Y,Z)$ imply $ST \in \mathfrak{K}(X,Z)$. Hence, if X has the Dunford–Pettis property, then the square of every weakly compact operator $T : X \to X$ is compact; see Dunford/Pettis [1940, p. 370].

4.8.5.7 Further information on the Dunford–Pettis property as well as the relevant proofs can be found in [DIE$_2^+$, pp. 154, 176–178]. I refer also to Diestel's survey [1980] and to [LIN].

4.9 Structure theory

4.9.1 Isomorphisms, injections, surjections, and projections

4.9.1.1 In the theory of Banach spaces, we have two different concepts of isomorphy. Many properties such as separability and reflexivity are preserved under the passage to an equivalent norm. On the other hand, the parallelogram identity may get lost.

Consequently, we must distinguish between **isomorphisms** and **isometries** (metric isomorphisms). In the first case, $U : X \to Y$ is supposed to be linear, one-to-one, and onto as well as continuous in both directions; see 2.5.2. In other words, $U \in \mathfrak{L}(X,Y)$ admits an inverse $U^{-1} \in \mathfrak{L}(Y,X)$, or

$$a\|x\| \le \|Ux\| \le b\|x\| \quad \text{for all } x \in X,$$

where a and b are positive constants. Banach spaces X and Y are called **isomorphic** if there exists an isomorphism $U \in \mathfrak{L}(X,Y)$. Then we write $X \cong Y$.

One refers to a linear map U from X onto Y as an **isometry** if it preserves the norm: $\|Ux\| = \|x\|$ for all $x \in X$. Certainly, the most important example of an isometry appears in the Fischer–Riesz theorem: $L_2[a,b] = l_2$.

The concepts of **isomorphy** and **isometry** (metric isomorphy, equivalence) can be found in [BAN, p. 180].

4.9.1.2 A remarkable result of Mazur/Ulam [1932, p. 947] says that in the real case, a map U from X onto Y such that $U\mathsf{o} = \mathsf{o}$ and $\|Ux - Ux_0\| = \|x - x_0\|$ whenever $x, x_0 \in X$ is automatically linear; see also [BAN, p. 166].

4.9.1.3 The difference between isometry and isomorphy can most obviously be seen in the finite-dimensional case: while all n-dimensional Banach spaces are isomorphic, they may have unit balls of multifarious shapes.

4.9.1.4 The **Banach–Mazur distance** of isomorphic Banach spaces X and Y is defined by

$$d(X,Y) = \inf\Big\{ \|U\|\|U^{-1}\| : E_n \underset{U^{-1}}{\overset{U}{\rightleftarrows}} F_n \Big\}. \tag{4.9.1.4.a}$$

For isomorphic spaces X, Y, and Z we have a multiplicative triangle inequality: $d(X,Z) \le d(X,Y)d(Y,Z)$. In order to get the usual triangle inequality, Banach [BAN, p. 242] originally used the quantity $\log d(X,Y)$.

The infimum in (4.9.1.4.a) need not be attained: there exist non-isometric Banach spaces X and Y for which $d(X,Y) = 1$; see Pełczyński [1979, p. 230].

4.9.1.5 For $\lambda > 0$, the map

$$U_\lambda : (\xi_1, \xi_2, \xi_3, \dots) \mapsto \big(\lambda\xi, \xi_1 - \xi, \xi_2 - \xi, \dots\big), \quad \text{where } \xi := \lim_{k\to\infty} \xi_k,$$

defines an isomorphism from c (space of convergent sequences) onto c_0 (space of null sequences) such that

$$\|U_\lambda : c \to c_0\| \le \max\{\lambda, 2\} \quad \text{and} \quad \|U_\lambda^{-1} : c_0 \to c\| \le \frac{1+\lambda}{\lambda}.$$

Letting $\lambda = 1$, Banach [BAN, p. 181] proved that $d(c, c_0) \le 4$. A better estimate, namely $d(c, c_0) \le 3$, is obtained for $\lambda = 2$. Finally, Cambern [1968] showed by sophisticated computation that $d(c, c_0) = 3$.

4.9.1.6 The concept of a (metric) isomorphism can be weakened in two directions.

An operator J from X into Y is said to be an **injection** if there exists a constant $c > 0$ such that $\|Jx\| \geq c\|x\|$ for all $x \in X$. In the case that $\|Jx\| = \|x\|$, we use the term **metric** injection.

For every closed subspace M of X, the canonical embedding J_M^X from M into X has this property. If there exists an injection from X into Y, then we say that Y contains an isomorphic copy of X.

An operator Q from X onto Y is called a **surjection**. In the case that the open unit ball of X is mapped onto the open unit ball of Y, we use the term **metric** surjection.

For every closed subspace N of Y, the quotient map Q_N^Y from Y onto Y/N has this property. If there exists a surjection from X onto Y, then we say that Y is a continuous image of X.

4.9.1.7 If $(x_i^*)_{i \in \mathbb{I}}$ is weakly* dense in the closed unit ball of X^*, then $J_X : x \mapsto (\langle x, x_i^* \rangle)$ yields a metric injection from X into $l_\infty(\mathbb{I})$. Taking B_{X^*} as its own index set, we also can view J_X as a map from X into $C(B_{X^*})$, where B_{X^*} carries the weak* topology. The fact that every separable Banach space admits an isometric embedding into $C[0,1]$ was known to Banach and Mazur; [BAN, p. 185]. For the non-separable case, I refer the reader to Alaoglu [1940, p. 258] and Phillips [1940, footnote ([12]) on p. 538].

A dual counterpart of the preceding construction is obtained as follows; see Banach/Mazur [1933, p. 111]. We fix any family $(x_i)_{i \in \mathbb{I}}$ that is norm dense in the unit ball of X and put $Q_X : (\xi_i) \mapsto \sum_{i \in \mathbb{I}} \xi_i x_i$. Then Q_X is a metric surjection.

4.9.1.8 In the Remarks of his monograph [BAN, p. 244], Banach considered, inter alia, the following property of a Banach space:

(7) *Existence pour tout sous-ensemble linéaire fermé S d'un sous-ensemble linéaire fermé T tel que tout élément x se laisse représenter d'une seule manière dans la forme $x = s + t$ où $s \subset S$ et $t \subset T$.*

In modern terminology, a closed subspace M is said to be **complemented** if there exists a closed subspace N such that every $x \in X$ admits a unique representation $x = u + v$ with $u \in M$ and $v \in N$. This fact is expressed by writing $X = M \oplus N$. The closed graph theorem implies that the linear map $P : x \mapsto u$ is continuous. In view of $P^2 = P$, one refers to P as the **projection** of X onto M along N. If $\|P\| \leq c$, then M is called c**-complemented**. Conversely, every $P \in \mathfrak{L}(X)$ with $P^2 = P$ yields a decomposition $X = M(P) \oplus N(P)$, where

$$M(P) := \{Px : x \in X\} \quad \text{and} \quad N(P) := \{x \in X : Px = o\}.$$

In summary, we get Murray's lemma [1937, pp. 138–139]:

Let $\mathfrak{M}$ be a closed linear manifold in Λ. The existence of a complementary manifold $\mathfrak{N}$ to $\mathfrak{M}$ is equivalent to the existence of a projection E of Λ on $\mathfrak{M}$.

4.9.1.9 In order to show that every n-dimensional subspace M is complemented, we choose a basis $(e_1,\dots,e_n)$ of M. The Hahn–Banach theorem yields functionals $e_1^*,\dots,e_n^*\in X^*$ such that

$$x=\sum_{k=1}^{n}\langle x,e_k^*\rangle e_k \quad \text{for } x\in M.$$

Then

$$P:=\sum_{k=1}^{n} e_k^*\otimes e_k$$

is the required projection. Using an Auerbach basis 5.6.2.4, we may arrange that $\|e_k\|=1$ and $\|e_k^*\|=1$. Hence $\|P\|\le n$. Much later, it was proved that there even exists a projection P with $\|P\|\le\sqrt{n}$; see 6.1.1.7.

All finite-codimensional closed subspaces are complemented as well.

4.9.1.10 The first example of a non-complemented closed subspace was found by Banach/Mazur [1933, footnote $(^{19})$ on p. 111].

Assume that $C[a,b]=M\oplus N$, where M is an isometric copy of l_1. Note that $C[a,b]^*$ is weakly sequentially complete, while M^* is not. Hence $C[a,b]^*=M^*\oplus N^*$ yields a contradiction.

For $p\neq 2$, Murray [1937] constructed non-complemented closed subspaces of $L_p[a,b]$. However, his method was quite sophisticated.

An elegant approach is due to Sobczyk [1941b, p. 81]. He proved that a closed subspace M is the range of a projection P if and only if there exists an involution, $U^2=I$, such that $M:=\{x\in X : Ux=x\}$. In this case, P and U are related as follows: $2P=U+I$.

Both Murray and Sobczyk used local techniques. Finite-dimensional spaces are glued together, and it is shown that a certain numerical parameter tends to infinity. Sobczyk's construction is of particular interest, since Walsh matrices are used in Banach space theory for the first time.

Phillips [1940, p. 539] and Sobczyk [1941a] discovered the first natural examples: c and c_0 are non-complemented in l_∞.

The latter result implies that a Banach space X, viewed as a subspace of X^{**}, need not be complemented in its bidual. On the other hand, Dixmier [1948, p. 1066] observed that X^* is always complemented in X^{***}. The required projection is given by K_X^*.

The Gowers–Maurey space has the strange property that every complemented subspace is either finite-dimensional or finite-codimensional; see 7.4.5.1.

4.9.1.11 We recall from 1.5.9 that every closed subspace M of a Hilbert space H is complemented: $H=M\oplus M^\perp$. This orthogonal decomposition yields a self-adjoint projection P, and the correspondence $M\leftrightarrow P$ is one-to-one.

4.9.1.12 The previous result has a converse. Lindenstrauss/Tzafriri [1971] solved the famous **complemented subspace problem**:

Every Banach space in which all closed subspaces are complemented is isomorphic to a Hilbert space.

4.9.1.13 Let $1 \le p < \infty$. We know from Pełczyński [1960, p. 213] that every complemented infinite-dimensional subspace of l_p is isomorphic to l_p.

Another theorem of Pełczyński [1960, p. 216] says that conversely, an infinite-dimensional subspace of l_p is complemented if it is isometric to l_p.

Assuming isometry is essential for $p \neq 2$. Indeed, there exist non-complemented subspaces of l_p that are isomorphic to l_p; see Rosenthal [1970, pp. 285–286] for $2 < p < \infty$, Bennett/Dor/Goodman/Johnson/Newman [1977, p. 184] for $1 < p < 2$, and Bourgain [1981a, p. 139] for $p = 1$.

4.9.1.14 Excellent surveys on "complemented subspaces" were given by Kadets/Mityagin [1973] and Mascioni [1989].

4.9.2 Extensions and liftings

4.9.2.1 Extension and lifting processes are "almost" dual to each other.

A Banach space Y has the

extension property

if every operator $T_0 : X_0 \to Y$ defined on a closed subspace X_0 of an arbitrary Banach space X admits an extension T such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "T_0"] \arrow[d, "J"'] & Y \\ X \arrow[ur, "T"'] & \end{tikzcd}.$$

Here J denotes the canonical injection from X_0 into X.
We use the term **metric** extension property if the operator T can be chosen such that $\|T\| = \|T_0\|$.

A Banach space X has the

lifting property

if every operator $T_0 : X \to Y_0$ taking values in a quotient Y_0 of an arbitrary Banach space Y admits a lifting T such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T_0"] \arrow[dr, "T"'] & Y_0 \\ & Y \arrow[u, "Q"'] \end{tikzcd}.$$

Here Q denotes the canonical surjection from Y onto Y_0.
We use the term **metric** lifting property if for any $\varepsilon > 0$, the operator T can be chosen such that $\|T\| \le (1+\varepsilon)\|T_0\|$.

Every functional y_0^* on Y with $\|y_0^*\| = 1$ induces a metric surjection $Q : y \mapsto \langle y, y_0^* \rangle$ from Y onto $\mathbb{K}$. The liftings of the identity map $I_{\mathbb{K}} = 1 \otimes 1$ are given by $1 \otimes y_0$ provided that $\langle y_0, y_0^* \rangle = 1$. Thus we can achieve $\|1 \otimes y_0\| = \|y_0\| = 1$ precisely when y_0^* attains its norm. This example shows that in the right-hand case, the $\varepsilon > 0$ is unavoidable.

4.9.2.2 Phillips [1940, p. 538] represented operators $T : X \to l_\infty(\mathbb{I})$ in the form $T : x \mapsto (\langle x, x_i^* \rangle)$, where (x_i^*) are bounded families in X^*. Moreover, $\|T\| = \sup_{i \in \mathbb{I}} \|x_i^*\|$. Hence those operators can be extended by applying the classical Hahn–Banach theorem coordinatewise. Conclusion: $l_\infty(\mathbb{I})$ has the metric extension property.

Viewing an arbitrary Banach space Y as a subspace of a suitable $l_\infty(\mathbb{I})$, Phillips further observed that Y has the extension property if and only if it is complemented in some $l_\infty(\mathbb{I})$; see 4.8.1.6.

4.9.2.3 Based on preliminary work of Akilov, it finally turned out that a Banach space X has the metric extension property if and only if it is isometric to some $C(K)$ associated with a Stonean space K. In other words, X must be a Dedekind complete M-space with unit. First of all, Nachbin [1950, pp. 30, 43–44] and Goodner [1950, pp. 102–103] obtained this result by assuming that the closed unit ball of X has an extreme point. This superfluous condition was removed by Kelley [1952, p. 323]. Hasumi [1958] treated the complex case.

In the real setting, the metric extension property is also equivalent to the **binary intersection property**; Nachbin [1950, p. 30]:

Any collection of balls $\{x \in X : \|x - x_i\| \leq r_i\}$ such that every pair has a non-empty intersection contains a common element.

The sufficiency of this condition can be seen by adapting Helly's reasoning; see 2.3.7.

4.9.2.4 Grothendieck [1953, p. 169] proved that a Banach space with the extension property is either finite-dimensional or non-separable.

4.9.2.5 Even nowadays, no convincing characterization of Banach spaces with the extension property seems to be known.

4.9.2.6 The dual situation is completely solved:

A Banach space has the (metric) lifting property if and only if it is (metrically) isomorphic to some $l_1(\mathbb{I})$.

The metric case, which is more complicated, was treated by Grothendieck [1955, p. 558], and Köthe [1966a, p. 188] contributed the remaining part. Surveys were given by Köthe [1966b] and Nachbin [1961].

4.9.3 Isometric and isomorphic classification

4.9.3.1 The spaces $l_p, L_p[0,1]$, $l_q, L_q[0,1]$, c_0, and $C[0,1]$ with $1 \leq p \neq q < \infty$ are pairwise non-isomorphic. Exception: $l_2 \cong L_2[0,1]$.

These results are explicitly or implicitly contained in [BAN, Chap. XII and p. 245]. A posthumous note of Paley [1936] deals with some cases that were left unanswered by Banach; see also Kadets [1958a].

In the following, I sketch some optimized proofs that have been accomplished in the course of time. The basic idea in proving $X \ncong Y$ is to find a certain property $\mathbb{P}$, invariant under isomorphisms, such that X has $\mathbb{P}$, whereas Y does not.

$c_0 \ncong C[0,1]$: Note that c_0^* is separable, while $C[0,1]^*$ is not.

$l_1 \ncong L_1[0,1]$: Recall from 3.5.3 that l_1 has the Schur property. Looking at the sequence $(\sin k\pi t)$ shows that this is not the case for $L_1[0,1]$.

$l_p \ncong L_p[0,1]$: It follows from [BAN, p. 194] that l_p cannot contain any isomorphic copy of l_2 unless $p = 2$. Another proof of this fact uses a result of Pełczyński [1960, p. 214], which says that every infinite-dimensional subspace of l_p contains some l_p. Hence, if there were an injection $A : l_2 \to l_p$, then we could find an injection $B : l_p \to l_2$. But this is impossible, since by a theorem of Pitt [1936] either A or B must be compact; see also Rosenthal [1969, p. 206]. On the other hand, there is an isomorphic embedding of l_2 into $L_p[0,1]$ that can be obtained with the help of a lacunary trigonometric system; [BAN, p. 204]. This is the first occurrence of a Khintchine inequality in Banach space theory. Many others were to follow.

Nowadays, the relations $l_p \ncong l_q$, $l_q \ncong L_p[0,1]$, and $L_p[0,1] \ncong L_q[0,1]$ are obtained by comparing Rademacher type and cotype indices; see Subsection 6.1.7. The original proof, due to Banach and Mazur, is mainly based on the following fact, [BAN, p. 197]:

Toute suite de fonctions $\{x_i(t)\}$ appartenant à $(L^{(p)})$, faiblement convergent vers 0, contient une suite partielle $\{x_{i_k}(t)\}$ telle que l'on a

$$\Big\| \sum_{k=1}^{n} x_{i_k} \Big\| = \begin{cases} O(n^{\frac{1}{p}}) & \textit{pour } 1 < p \le 2 \\ O(n^{\frac{1}{2}}) & \textit{pour } \quad p > 2. \end{cases}$$

A similar result holds for l_p. But in this case, the behavior does not change at $p = 2$, and we always get $O(n^{\frac{1}{p}})$; see [BAN, p. 200].

4.9.3.2 Clarkson's inequalities [1936, p. 400] (see also 4.4.8) say that

$$\begin{aligned} \|f+g\|^p + \|f-g\|^p &\le 2\big(\|f\|^p + \|g\|^p\big) \quad \text{if } 1 < p \le 2, \\ \|f+g\|^p + \|f-g\|^p &\ge 2\big(\|f\|^p + \|g\|^p\big) \quad \text{if } 2 \le p < \infty, \end{aligned} \qquad f, g \in L_p.$$

In the limiting case $p = 2$, the parallelogram identity emerges. However, for $p \ne 2$, equality holds if and only if $|f| \wedge |g| = o$. This simple result [ROY, p. 275], which is due to Lamperti [1958, p. 461], has proved to be extremely useful. I stress that the following consequence is already contained in [BAN, p. 175]:

Every isometry U from $L_p(M, \mathscr{M}, \mu)$ onto $L_p(N, \mathscr{N}, \nu)$ preserves orthogonality: $f \perp g \;\Rightarrow\; Uf \perp Ug$. Hence there is a one-to-one correspondence between projection bands in both spaces, and the measure algebras $\mathscr{M}/\mathscr{M}_0(\mu)$ and $\mathscr{N}/\mathscr{N}_0(\nu)$ are isomorphic. This in turn implies that $L_p(M, \mathscr{M}, \mu)$ and $L_p(N, \mathscr{N}, \nu)$ are not only isometric with respect to their Banach space structure but also isomorphic as Banach lattices. Consequently, for $p \ne 2$, the *isometric* classification of L_p spaces coincides with their isometric *and lattice-theoretic* classification; see 4.8.3.3.

4.9.3.3 The case of separable infinite-dimensional L_p spaces is summarized in a table, which shows all possible non-isometric and non-isomorphic prototypes.

	isometric	isomorphic
with lattice structure	A l_p, $L_p[0,1]$ $L_p[0,1] \oplus_p l_p^n$ $L_p[0,1] \oplus_p l_p$	B the same as in the left-hand box
without lattice structure	C the same as in the above box, except for $p=2$	D l_p, $L_p[0,1]$, except for $p=2$

Comments:

(A) This result is due to Bohnenblust [1940, p. 636].

(B) In view of **(A)**, it is enough to show that l_p, $L_p[0,1]$, $L_p[0,1] \oplus_p l_p^n$, and $L_p[0,1] \oplus_p l_p$ are non-isomorphic as linear lattices. This can be achieved by counting the **atoms** a: if $|x| \le |a|$, then $x = \lambda a$; see [LUX$^+$, pp. 145–147].

(C) We know from the preceding paragraph that isometry of L_p spaces implies order isomorphy. Thus the problem is reduced to **(A)**.

(D) The formula $L_p[0,1] \cong L_p[0,1] \oplus_p l_p$ is obtained by the decomposition method, which will be explained in the next paragraph. That $L_p[0,1]$ also absorbs the l_p^n's can be inferred as follows:

$$L_p[0,1] \oplus_p l_p^n \cong \big(L_p[0,1] \oplus_p l_p\big) \oplus_p l_p^n \cong L_p[0,1] \oplus_p \big(l_p \oplus_p l_p^n\big) \cong L_p[0,1] \oplus_p l_p.$$

References: [LAC, p. 128] and [LIND$_0^+$, p. 124].

4.9.3.4 Next, I present the **decomposition method**, which was developed in order to show that l_∞ and $L_\infty[a,b]$ are isomorphic; Pełczyński [1958].

Let X and Y be Banach spaces isomorphic to their Cartesian squares:

$$X \cong X \oplus X \quad \text{and} \quad Y \cong Y \oplus Y.$$

Assume, in addition, that each of these spaces has a complemented copy in the other:

$$X \cong Y \oplus X_0 \quad \text{and} \quad Y \cong X \oplus Y_0.$$

Then

$$X \cong Y \oplus X_0 \cong (Y \oplus Y) \oplus X_0 \cong Y \oplus (Y \oplus X_0) \cong Y \oplus X$$

and

$$Y \cong X \oplus Y_0 \cong (X \oplus X) \oplus Y_0 \cong X \oplus (X \oplus Y_0) \cong X \oplus Y.$$

Hence $X \cong Y$. Ingenious proofs are simple!

4.9.3.5 A theorem of Mazurkiewicz/Sierpiński [1920, p. 21] says that every compact metric space of cardinality $\aleph_0$ is homeomorphic to an ordinal space $\omega_0{}^{\xi} m+1$, where $\xi < \omega_1$ and $m = 1,2,\ldots$ are topological invariants; see 4.6.13. Thanks to the Banach–Stone theorem 4.5.5, this yields an isometric classification of all $C(K)$'s with $|K| = \aleph_0$. Here ω_0 and ω_1 denote the first countable and uncountable ordinal, respectively.

The isomorphic counterpart is due to Bessaga/Pełczyński [1960a, pp. 59–61]. They proved that $C(\alpha+1)$ and $C(\beta+1)$ are isomorphic if and only if either $\alpha \le \beta < \alpha^{\omega_0}$ or $\beta \le \alpha < \beta^{\omega_0}$. In particular, the spaces $C(\omega_0+1)$, $C({\omega_0}^{\omega_0}+1)$, $C({\omega_0}^{{\omega_0}^{\omega_0}}+1), \dots$ are non-isomorphic. Therefore $l_1 = C(K)^*$ has a vast variety of different preduals; see also 5.4.1.6. By the way, Benyamini/Lindenstrauss [1972] constructed a predual of l_1 that is not isomorphic to any $C(K)$.

4.9.3.6 The remaining case was settled by **Milyutin's theorem**:

$C(K)$ is isomorphic to $C[0,1]$ for every *uncountable* compact metric space K.

This result has a curious history; see [SEMA, p. 380]:

Banach [BAN, p. 185] raised the question whether $C[0,1] \cong C\big([0,1]^2\big)$. This problem was considered to be open for a long time. However, in 1964, one of the Gurariĭ's (Владимир Ильич) discovered that a solution was already contained in Milyutin's thesis from 1952. Milyutin published the proof in [1966].

A completely different story is told by Vitushkin [2003, p. 16].

4.10 Operator ideals and operator algebras

4.10.1 Schatten–von Neumann ideals

4.10.1.1 The following quotations are taken from [vNEU, pp. 93, 95]:

Sei A ein linearer Operator, wir nehmen irgendein normiertes vollständiges Orthogonalsytem $\varphi_1, \varphi_2, \dots$ [note that von Neumann's Hilbert spaces are separable!] *für welches alle* $A\varphi_\mu$ *Sinn haben und setzen*

$$\mathrm{Spur}\,(A) = \sum_{1}^{\infty}{}^{\mu}\,(A\varphi_\mu, \varphi_\mu). \tag{4.10.1.1.a}$$

Es ist zu zeigen, daß dies wirklich nur von A (*und nicht von den* φ_μ!) *abhängt.*

...

Bei aller Kürze und Einfachheit unserer auf die Spur bezüglichen Betrachtungen sind dieselben mathematisch nicht einwandfrei. Wir haben nämlich Reihen $\sum_{1}^{\infty}{}^{\mu}\,(A\varphi_\mu, \varphi_\mu)$ *ohne Rücksicht auf ihre Konvergenz betrachtet, ineinander umgeformt* (*umsummiert*) *– kurzum alles getan, was man korrekterweise nicht tun soll. Zwar kommen derartige Nachlässigkeiten in der theoretischen Physik auch sonst vor, und die vorliegende wird in unseren quantenmechanischen Anwendungen kein Unheil anrichten – es muß aber doch festgestellt werden, daß es sich um eine Nachlässigkeit handelt.*

Umso wesentlicher ist es zu betonen, daß dieser Begriff auch ganz exakt begründet werden kann. Im Rest dieses Paragraphen werden wir deshalb diejenigen Tatsachen über die Spur zusammenstellen, die in absoluter mathematischer Strenge beweisbar sind.

In a first step, von Neumann showed that the quantity

$$\Sigma(A) = \sum_{1}^{\infty}{}^{\mu,\nu} \, |(A\varphi_\mu, \psi_\nu)|^2$$

does not depend on the special choice of the complete orthonormal systems $\varphi_1, \varphi_2, \dots$ and $\psi_1, \psi_2, \dots$. I stress that up to the missing square root, $\Sigma(A)$ is the **Hilbert–Schmidt norm** of the operator A.

Subject to the condition $\Sigma(A) < \infty$, von Neumann observed that the "Spur" of a positive Hermitian operator A, which may be ∞, is the sum of its eigenvalues.

4.10.1.2 In his monograph [STONE, pp. 66–67], Stone referred to

$$N(A) := \left[\sum_{k=1}^{\infty} \|A\varphi_k\|^2 \right]^{1/2}$$

as *the* norm of A, and he showed that *the class of all bounded linear transformations of finite norm is a Hilbert space* under the inner product

$$(A,B) = \sum_{k=1}^{\infty} (A\varphi_k, B\varphi_k).$$

In the concrete Hilbert space l_2, we just get the operators that are induced by a Hilbert–Schmidt matrix; see 2.6.1.2. The first abstract version is due to von Neumann [1927, pp. 37–41]. I also refer to Schatten/von Neumann [1946, pp. 616–617], where the **Schmidt class** is denoted by (sc). Nowadays, the term **Hilbert–Schmidt ideal** is used, and its members are called **Hilbert–Schmidt operators**.

4.10.1.3 The natural domain of a **trace** was identified by Schatten/von Neumann [1946, p. 617] as the collection of all operators that can be written in the form $A = C^\star B$ such that $N(B)$ and $N(C)$ are finite, trace(A) being defined by (4.10.1.1.a).

Moreover, they proved that the **trace class** (tc) is a Banach space under the norm $M(A) = \text{trace}\big((A^\star A)^{1/2}\big)$; see [1946, p. 624].

4.10.1.4 Next, I discuss the **Schmidt representation** of a compact operator $T \in \mathfrak{K}(H)$ in a Hilbert space:

There exist *finite* or *countably infinite* orthonormal families $(u_i)_{i\in\mathbb{I}}$ and $(v_i)_{i\in\mathbb{I}}$ as well as a family $t = (\tau_i)_{i\in\mathbb{I}}$ of strictly positive numbers such that

$$Tx = \sum_{i\in\mathbb{I}} \tau_i (x|u_i) v_i \quad \text{for } x \in H.$$

The u_i's and v_i's are eigenelements of the operators $T^\star T$ and $T T^\star$, respectively. The family $(\tau_i)_{i\in\mathbb{I}}$ is unique up to a permutation, and in the case of an infinite index set $\mathbb{I}$ we have $\tau_i \to 0$. Letting

$$Ux = \sum_{i\in\mathbb{I}} (x|u_i) u_i \quad \text{and} \quad Vx = \sum_{i\in\mathbb{I}} (x|v_i) v_i \quad \text{for } x \in H$$

yields the commutative diagram

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, shift right, "U"'] & H \arrow[d, shift right, "V"'] \\ l_2(\mathbb{I}) \arrow[r, "D_t"'] \arrow[u, shift right, "U^\star"'] & l_2(\mathbb{I}) \arrow[u, shift right, "V^\star"'] \end{tikzcd} .$$

Hence, instead of T, one can often work with the **diagonal operator** $D_t : (\xi_i) \mapsto (\tau_i \xi_i)$.

This representation was obtained by Schmidt [1907a, pp. 461–466] for operators induced by continuous, but not necessarily symmetric, kernels. The extension to the abstract setting is straightforward. More information can be found in 6.2.1.1.

4.10.1.5 In terms of the Schmidt representation, operators T in (sc) and (tc) can be characterized by the conditions $(\tau_i) \in l_2(\mathbb{I})$ and $(\tau_i) \in l_1(\mathbb{I})$, respectively. Therefore it is reasonable to introduce a real parameter $0 < p < \infty$. In modern terminology, an operator $T \in \mathfrak{K}(H)$ is said to be of **Schatten–von Neumann class** $\mathfrak{S}_p(H)$, often denoted by $\mathfrak{C}_p(H)$, if $(\tau_i) \in l_p(\mathbb{I})$. For $1 \le p < \infty$, von Neumann [1937] (finite-dimensional case) and Schatten/von Neumann [1948, p. 580] showed that $\mathfrak{S}_p(H)$ is a Banach space under the norm

$$\|T|\mathfrak{S}_p\| := \Big(\sum_{i \in \mathbb{I}} \tau_i^p\Big)^{1/p};$$

see [SCHA$_2$, p. 72], [DUN$_2^+$, Chap. XI.9], [GOH$_3^+$, p. 92], and [RIN, p. 93]. The original proof of this fact is based on trace duality:

The linear spaces $\mathfrak{S}_p(H)$ and $\mathfrak{S}_{p^*}(H)$ constitute a dual system with respect to the bilinear form $\langle S, T \rangle := \mathrm{trace}(ST)$. In this sense, $\mathfrak{S}_p(H)^*$ can be identified with $\mathfrak{S}_{p^*}(H)$. In particular,

$$\|S|\mathfrak{S}_p\| = \sup\big\{\, |\mathrm{trace}(ST)| \,:\, \|T|\mathfrak{S}_{p^*}\| \le 1 \,\big\}.$$

In the limiting cases $p = 1$ and $p = \infty$, we have $\mathfrak{S}_1(H)^* = \mathfrak{L}(H)$ and $\mathfrak{K}(H)^* = \mathfrak{S}_1(H)$. The latter formulas were obtained by Schatten [1946, p. 78] and Schatten/von Neumann [1946, pp. 623–625]. Dixmier [1950, pp. 390, 394] discovered the same result; but he did not know the right norm for trace class operators.

4.10.1.6 Though the Banach spaces $\mathfrak{S}_p(H)$ with $1 \le p < \infty$ are only semi-classical, they have proved to be quite important. Their main significance, however, stems from the fact that they are even Banach ideals over H as described in 6.3.1.6, 6.3.2.1, and 6.3.2.4.

4.10.2 Banach algebras

For convenience, I consider only complex Banach algebras with a unit element **1**.

4.10.2.1 The concept of a **ring of operators** goes back to von Neumann [1930b, Einleitung, p. 372]:

Die Menge $\mathscr{B}$ aller beschränkten Operatoren [on a complex separable Hilbert space] *ist also ein Ring – mit Nullteilern und nicht kommutativ.*

Bei der definitorischen Festlegung dessen, welche Teilmengen $\mathscr{M}$ von $\mathscr{B}$ auch Ringe sind, würden wir uns in ein unentwirrbares Dickicht von pathologischen Bildungen verlieren, wenn wir nur verlangen, daß $\mathscr{M}$ mit A, B auch αA, $A \pm B$, AB enthält. Es muß nämlich die Forderung gestellt werden, daß $\mathscr{M}$ im Sinne einer (noch zu formulierenden) Topologie von $\mathscr{B}$ abgeschlossen sei.

Übrigens werden wir noch eine vereinfachende Forderung an die Ringe $\mathscr{M}$ stellen: mit A soll auch $A^\star$ dazugehören. Dies ist eine bedeutsame Einschränkung, die es ermöglicht, den Komplikationen der Elementarteilertheorie aus dem Wege zu gehen.

4.10.2.2 The preceding shows that rings of operators occurred already in 1930. In the following years von Neumann continued his studies in collaboration with Murray. The result was a celebrated series of papers that laid the ground for the theory of **von Neumann algebras**. However, the fact that the members of these rings are operators on a Hilbert space was basic for all considerations. After a feeble attempt by Nagumo [1936], the abstract point of view prevailed only thanks to the efforts of the Russian school: Gelfand, Naĭmark, Raĭkov, and Shilov.

4.10.2.3 In what follows, we consider a Banach space $\mathscr{A}$ that is at the same time a ring (with unit $\mathbf{1}$) such that $\|xy\| \le \|x\|\|y\|$ and $\|\mathbf{1}\| = 1$.

In the USA, the Russian terminology "нормированное кольцо" was replaced by **Banach algebra**, or simply by *B*-**algebra**. This name first appeared in a paper of Ambrose [1945, p. 364]. Hille [HIL, p. 12], and Rickart [1946, footnote [1])] claim that it was suggested by Zorn.

Besides a small section in [HIL], the first monographs on Banach algebras are [GEL$^+$], [LOO], [NAI], [DIX$_1$], [DIX$_2$], and [RIC] . The reader is also referred to [BONS$^+$], [BOU$_7$], [TOP], [SAKAI], and [TAK]. Historical surveys were given by Lorch [1951$^\bullet$], Bonsall [1970$^\bullet$], Kadison [1982$^\bullet$, 1990$^\bullet$, 1994$^\bullet$], and Murray [1990$^\bullet$]; see also Blackadar [1994].

4.10.2.4 The **spectrum** $\sigma(x)$ of an element x in a Banach algebra $\mathscr{A}$ consists of all complex numbers λ for which $\lambda - x$ (more precisely $\lambda \cdot \mathbf{1} - x$) fails to be invertible. Basic theorems about analytic $\mathscr{A}$-valued functions imply that $\sigma(x)$ is a non-empty and compact subset of $\mathbb{C}$; see 5.2.1.1. We refer to

$$r(x) := \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

as the **spectral radius** of x. In the commutative case, the famous formula

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \tag{4.10.2.4.a}$$

goes back to Gelfand [1941, p. 11]. The generalization to non-commutative Banach algebras was straightforward. Curiously, the equality

$$\lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\to\infty} \|x^n\|^{1/n},$$

which can be derived from Problem 98 in [POL$^+$, p. 17], was discovered only around 1952; see [RIE$^+$, pp. 420–421] and [RUST, p. 222].

4.10.2.5 The main tools in Gelfand's theory of commutative Banach algebras are maximal ideals. He proved that an ideal $\mathscr{I}$ in $\mathscr{A}$ is maximal if and only if $\mathscr{A}/\mathscr{I}$ is isomorphic to the complex field. This observation yields a one-to-one correspondence between maximal ideals $\mathscr{I}$ and multiplicative linear functionals ℓ such that $\ell(\mathbf{1}) = 1$:

$$\mathscr{I} = \{x \in \mathscr{A} : \ell(x) = 0\} \quad \text{and} \quad \ell : \mathscr{A} \to \mathscr{A}/\mathscr{I} = \mathbb{C}.$$

Denoting maximal ideals by M, Gelfand wrote $x(M)$ instead of $\ell(x)$.

I point out that in Gelfand's original approach the concept of the spectrum did not occur. However, he used to following fact, [1941, p. 9]:

Dafür dass ein Element x ein inverses Element besitze, ist notwendig und hinreichend, dass x keinem maximalen Ideal angehöre.

His "spectral radius" of an element x is defined to be the maximum of all $|\ell(x)|$, where ℓ is as above.

Gelfand [1941, p. 8] proved a crucial *Satz*, which was independently obtained by Mazur [1938]:

If every non-zero element of a Banach algebra $\mathscr{A}$ admits an inverse, then $\mathscr{A}$ is isomorphic to the complex field.

The history of this result is described in [ŻEL, p. 18].

4.10.2.6 Let $\mathscr{R}$ be a ring with unit $\mathbf{1}$. The **radical** $\mathfrak{R}_{\mathrm{ad}}$ consists of all elements $x \in \mathscr{R}$ such that $\mathbf{1} + ax$, or equivalently $\mathbf{1} + xa$, is invertible whenever $a \in \mathscr{R}$. It turns out that $\mathfrak{R}_{\mathrm{ad}}$ is a two-sided ideal; see [HIL$^+$, p. 699]. The ring $\mathscr{R}$ is said to be **semi-simple** if $\mathfrak{R}_{\mathrm{ad}} = \{\mathrm{o}\}$. For arbitrary rings, these purely algebraic concepts were introduced by Jacobson [1945, p. 303].

An element x in a Banach algebra is called **quasi-nilpotent** if $\lim_{n\to\infty} \|x^n\|^{1/n} = 0$. Such elements belong to the radical and, in the commutative case, they even form the radical; see Gelfand [1941, p. 10].

4.10.2.7 For every Banach space X, the operator algebra $\mathfrak{L}(X)$ is semi-simple. Indeed, given any non-zero operator $T \in \mathfrak{L}(X)$, we choose $x_0 \in X$ and $x_0^* \in X^*$ such that $\langle Tx_0, x_0^* \rangle = -1$. Then $I + (x_0^* \otimes x_0)T$ is non-invertible, since x_0 belongs to its null space.

4.10.2.8 Based on preliminary work of Rickart [RIC, pp. 70–76], Johnson [1967] proved the following result:

Any two norms under which a semi-simple algebra becomes a Banach algebra are equivalent; see also 2.5.5 and 4.1.12.

4.10.2.9 I conclude this subsection with a result of Eidelheit [1940, pp. 100–101] that should be compared with 4.5.5.

If $\mathfrak{L}(X_1)$ and $\mathfrak{L}(X_2)$ are isomorphic as algebras, then X_1 and X_2 are isomorphic as Banach spaces; see [RIC, p. 76].

4.10.3 $B^\star$-algebras = $C^\star$-algebras

4.10.3.1 A map $x \mapsto x^\star$ from an algebra $\mathscr{A}$ into itself is said to be an **involution** or **$\star$-operation** if

$$(x^\star)^\star = x, \quad (x+y)^\star = x^\star + y^\star, \quad (xy)^\star = y^\star x^\star, \quad (\lambda x)^\star = \overline{\lambda} x^\star$$

for $x, y \in \mathscr{A}$ and $\lambda \in \mathbb{C}$. The above is an axiomatic formulation of the basic properties that are satisfied when one passes from an operator $T \in \mathfrak{L}(H)$ to its adjoint $T^\star \in \mathfrak{L}(H)$.

The important identity $\|T^\star T\| = \|T\|^2$ gave rise to the definition of a **$B^\star$-algebra**: This is a B-algebra $\mathscr{A}$ with a $\star$-operation such that

$$\|x^\star x\| = \|x\|^2 \quad \text{for all } x \in \mathscr{A}. \tag{4.10.3.1.a}$$

The name was coined by Rickart [1946, p. 528].

4.10.3.2 The theory of $B^\star$-algebras, originally called **normed $\star$-rings**, goes back to Gelfand/Naĭmark. In a first step [1943, p. 199], they proved a famous *representation theorem*, which was referred to as **Lemma 1:**

Every commutative $B^\star$-algebra is isometric and algebraically $\star$-isomorphic to some function algebra $C(K)$ with the involution $f \mapsto \overline{f}$.

4.10.3.3 In the non-commutative case, the goal was an axiomatic characterization of subalgebras of $\mathfrak{L}(H)$ that are closed with respect to the uniform operator topology and stable under passage to adjoint operators. The main result of Gelfand/Naĭmark [1943, p. 198] asserts that these are just those Banach $\star$-algebras satisfying (4.10.3.1.a) and such that

$$1 + x^\star x \quad \text{is invertible for all } x \in \mathscr{A}. \tag{4.10.3.3.a}$$

However, they conjectured the second condition to be redundant; [1943, footnote **) on p. 198]. An affirmative solution was given only *after a decade of mystery*; [SAKAI, p. 9]. Details will be discussed in 4.10.3.5.

Taking into account a possibly negative answer, Segal [1947, p. 73] referred to *concrete* subalgebras of $\mathfrak{L}(H)$ as **$C^\star$-algebras**. The same name was given to *abstract* $\star$-algebras for which (4.10.3.1.a) and (4.10.3.3.a) hold. The following comment from [DOR$^+$, p. 6] motivates the choice of *"C"*; see also 4.10.4.1.

*The "C" stood for "***closed***" in the uniform topology of $B(H)$. It has been speculated by some authors that the "C" was meant to indicate that a $C^\star$-algebra is a noncommutative analogue of $C(X)$; however, Professor Segal has assured the first named author* [read: Doran] *that he didn't have this in mind–also he agreed that it was certainly a reasonable supposition.*

Ironically, when it turned out that

$$B^\star\text{-}algebras = C^\star\text{-}algebras,$$

the latter term survived.

4.10.3.4 An element a of a $B^\star$-algebra $\mathscr{A}$ is said to be **Hermitian** or **self-adjoint** if $a = a^\star$. Such elements have a real spectrum; see Gelfand/Naĭmark [1943, p. 201]. The same authors found that $x = a + \mathrm{i}b$ and $x^\star = a - \mathrm{i}b$, where

$$a = \frac{x + x^\star}{2} \quad \text{and} \quad b = \frac{x - x^\star}{2\mathrm{i}}$$

are Hermitian.

In view of $\|x^\star x\| = \|x\|^2$, the norm of a $B^\star$-algebra is uniquely determined by its values on the Hermitian part. However, for a Hermitian element a, we may conclude from (4.10.2.4.a) that

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Thus the norm is unique altogether; see Gelfand/Naĭmark [1943, p. 203].

4.10.3.5 **Positive** elements $x \in \mathscr{A}$ are defined by various equivalent properties:

(1) x is Hermitian and has a non-negative spectrum.
(2) $x = a^2$ for some Hermitian $a \in \mathscr{A}$.
(3) $x = a^\star a$ for some $a \in \mathscr{A}$.
(4) x is Hermitian and $\big\|\|x\| \cdot \mathbf{1} - x\big\| \leq \|x\|$.

As stated by Gelfand/Naĭmark [1943, p. 201], the implication **(1)**$\Rightarrow$**(2)** follows from the operational calculus, which is based on the representation theorem 4.10.3.2. Plainly, **(2)**$\Rightarrow$**(3)**. The equivalence **(2)**$\Leftrightarrow$**(4)** was proved by Kelley/Vaught [1953, p. 52]. They also observed that these conditions define a closed convex cone, which will be denoted by $\mathscr{A}_+$. On the other hand, Fukamiya [1952, p. 19] showed that condition **(1)** yields a convex cone as well. But still the final step for closing the circle was missing, namely **(3)**$\Rightarrow$**(1)**. The happy end, based on a "little" remark of Kaplansky, can be found in Math. Reviews, where Schatz [1954] reported on Fukamiya's paper.

In view of the above, every element $x^\star x$ has a non-negative spectrum. Hence $\mathbf{1} + x^\star x$ must be invertible, which proves that (4.10.3.3.a) is indeed redundant.

4.10.3.6 Among others, Fukamiya [1952, p. 18] discovered that in a $B^\star$-algebra, (4.10.3.3.a) is equivalent to the property that

$$x^\star x + y^\star y = \mathsf{o} \quad \text{implies} \quad x = y = \mathsf{o}.$$

In other words, $\mathscr{A}_+ \cap (-\mathscr{A}_+) = \{\mathsf{o}\}$. Hence the cone $\mathscr{A}_+$ induces a partial ordering on $\mathscr{A}_{\mathrm{herm}}$, the real Banach space of all Hermitian elements. Sherman [1951, p. 227] proved that $\mathscr{A}_{\mathrm{herm}}$ becomes a lattice if and only if $\mathscr{A}$ is commutative.

4.10.3.7 The **absolute value** of an element x in a $B^\star$-algebra $\mathscr{A}$ is defined by $|x| := (x^\star x)^{1/2}$. I stress that choosing $(x^\star x)^{1/2}$ instead of $(xx^\star)^{1/2}$ is just a matter of convention.

In order to show the existence of square roots, one does not need the full operational calculus. If $o \le a \le \mathbf{1}$, then $\sqrt{a} = \lim_{n\to\infty} b_n$ can be obtained by successive approximation, Visser [1937]:

$$b_{n+1} := b_n + \tfrac{1}{2}\big(a - b_n^2\big) \quad \text{and} \quad b_0 := o.$$

4.10.3.8 A member p of a $B^\star$-algebra $\mathscr{A}$ is said to be a **projection** if $p^2 = p$ and $p^\star = p$. It follows from $\|p\|^2 = \|p^\star p\| = \|p\|$ that $\|p\| = 1$ whenever $p \neq o$. Every projection is positive, and for two projections p and q we have $p \le q$ if and only if $p = pq$ and/or $p = qp$. Projections p and q with $pq = qp = o$ are said to be **orthogonal**.

The collection of all projections on $\mathscr{A}$ will be denoted by $\mathscr{A}_{\mathrm{pro}}$. Kaplansky [1951, p. 236] observed that *if p and q commute, they have $p+q-pq$ as a least upper bound; this is apparently all that can be said in general concerning the lattice properties of this partially ordered set.*

4.10.3.9 A projection z in a $B^\star$-algebra $\mathscr{A}$ is called **central** if it commutes with all members of $\mathscr{A}$. Then

$$\mathscr{A} = \mathscr{A} z \oplus \mathscr{A}(\mathbf{1} - z) \quad \text{and} \quad \|a\| = \max\{\|az\|, \|a(\mathbf{1}-z)\|\} \quad \text{for } a \in \mathscr{A}.$$

In other words, central projections can be used to decompose $B^\star$-algebras into smaller pieces.

4.10.4 $W^\star$-algebras

4.10.4.1 This subsection deals with subalgebras of $\mathfrak{L}(H)$ that are closed with respect to the weak operator topology 3.3.4.4 and stable under passage to adjoint operators. Because this concept goes back to von Neumann [1930b, p. 388], Dieudonné suggested the name **von Neumann algebras**, which was used by Dixmier in the subtitle of his book [DIX$_1$]. Nowadays, such algebras are often referred to as $W^\star$**-algebras**. The latter term was coined by Segal [1950, p. 295].

Kaplansky [BAS$^\cup$, p. xii–xiii] made the following comment:

The French school likes to name mathematical objects for people and called them von Neumann algebras. Segal proposed the name $W^\star$-algebras, as a companion for his $C^\star$-algebras. Here C stands for uniformly closed, W for weakly closed.

The verdict in this terminology battle is in and it is mixed. Von Neumann algebras have won out, but the French attempt to replace $C^\star$-algebras by Gelfand–Neumark algebras failed.

4.10.4.2 The following algebraic approach is due to von Neumann [1930b, p. 388]:

Sei $\mathscr{W}$ eine Teilmenge von $\mathscr{B}=\mathfrak{L}(H)$. Die Menge aller A von $\mathscr{B}$ für die A und $A^\star$ mit jedem B von $\mathscr{W}$ vertauschbar sind, heiße $\mathscr{W}'$, **commutant***. So können auch Iterierte $\mathscr{W}'', \mathscr{W}''', \ldots$ gebildet werden. $\mathscr{W}'$ ist stets ein Ring, ..., aber es ist auch schwach abgeschlossen.*

Finally, he stated the famous **bicommutant theorem**, [1930b, p. 397]:

Die **1** *enthaltenden* [schwach abgeschlossenen] *Ringe und die $\mathscr{W}$ mit $\mathscr{W}=\mathscr{W}''$ sind untereinander identisch.*

4.10.4.3 Fundamental results of Dixmier [1951a, pp. 169–171] imply that $C(K)$ is a $W^\star$-algebra if and only if K is hyper-Stonean; see 4.8.2.11/12 and 4.10.4.14. In particular, the real part of $C(K)$ must be Dedekind complete.

Every commutative $W^\star$-algebra can be identified with some concrete algebra $L_\infty(M,\mathscr{M},\mu)$, where the measure space $(M,\mathscr{M},\mu)$ is localizable; see 4.3.9 and [SAKAI, p. 45]. Assigning to every bounded measurable function f the multiplication operator $\varphi \mapsto f\varphi$ on the Hilbert space $L_2(M,\mathscr{M},\mu)$, one gets a representation as an operator algebra.

4.10.4.4 Next, I stress the difference between $C^\star$-algebras and $W^\star$-algebras.

	commutative	non-commutative
$C^\star$-algebra	$C(K)$	uniformly closed $\star$-subalgebra of $\mathfrak{L}(H)$
$W^\star$-algebra	$L_\infty(M,\mathscr{M},\mu)$ $(M,\mathscr{M},\mu)$ localizable	weakly closed $\star$-subalgebra of $\mathfrak{L}(H)$

The main disadvantage of $C^\star$-algebras is their possible lack of projections. Moreover, experts know examples for which the collection of all projections fails to be a lattice. Ironically, nobody was able to tell me a printed source; it seems to be just folklore.

4.10.4.5 The **center** of an algebra $\mathscr{A}$ is defined by

$$\mathscr{Z} := \{z\in\mathscr{A} : zx=xz \text{ whenever } x\in\mathscr{A}\}.$$

For commutative algebras, the center is as large as possible: $\mathscr{Z}=\mathscr{A}$. The other extreme case is that of a **factor**: $\mathscr{Z}=\{\lambda\cdot\mathbf{1} : \lambda\in\mathbb{C}\}$. A $W^\star$-algebra $\mathscr{W}$ is a factor if and only if O and $\mathbf{1}$ are its only central projections. This means that $\mathscr{W}$ cannot be decomposed in the sense of 4.10.3.9. Hence factors are the smallest non-commutative building blocks of $W^\star$-algebras. Because of this fact, Murray and von Neumann concentrated their interest on factors. The development of the "global" theory started only at the end of the forties.

An abstract $W^\star$-algebra can be represented as follows:

In a first step, we identify its center with some $L_\infty(M,\mathscr{M},\mu)$. Next, a factor $\mathscr{W}(t)$ is placed at every point $t \in M$. Finally, we form the **direct integral**

$$\int_M^{\oplus} \mathscr{W}(t)\, d\mu(t),$$

whose elements are (equivalence classes of) bounded "measurable" functions $\boldsymbol{f}$ such that $\boldsymbol{f}(t) \in \mathscr{W}(t)$.

The critical aspect of this construction is described in [TAK, p. 269]:

We must have a proper definition of measurability for functions with values in different spaces. Since we handle spaces indexed by points of a measure space, we have to specify how spaces sitting in distinct points are bound together.

Von Neumann had already proved in 1938 that every $W^\star$-algebra on a separable Hilbert space is a direct integral of factors. This result was published only after World War II; see von Neumann [1949]. For detailed expositions the reader is referred to [SCHW, p. 53] and [TAK, p. 275]. Based on the work of Tomita [1954], the general case was treated in [NAI, Chap. VIII]. However, I was informed by Ringrose that Tomita's reasoning contains a subtle error. Thus the situation in the non-separable setting is still unclear.

4.10.4.6 Rickart [1946, pp. 533–536] was the first to look for $B^\star$-algebras with "sufficiently many projections." He defined $B^\star_p$**-algebras** by the following property.

For every element $x \in \mathscr{A}$ there exists a projection $p \in \mathscr{A}$ such that $x = xp$ and $xy = o$ imply $py = o$. Since $w^\star w = o$ and $w = o$ are equivalent, it suffices that the above condition hold for Hermitian elements x and y. If $p_0 \in \mathscr{A}$ is any projection with $xp_0 = x$, then $p \le p_0$. Hence p is minimal and unique. Replacing x by $x^\star$ yields a projection q such that $x = qx$ and $yx = o$ imply $yq = o$. The projections of every $B^\star_p$-algebra form a Dedekind countably complete lattice; [BERB, pp. 14, 45].

The concept of a $B^\star_p$-algebra was suggested by the observation that every operator $T \in \mathfrak{L}(H)$ has a **right projection**, or **right support** $P \in \mathfrak{L}(H)$, whose null space coincides with that of T. As will be explained in 4.10.4.9, this is a basic ingredient in proving the spectral theorem for Hermitian operators. In other words, the properties of a $B^\star_p$-algebra just ensure the validity of an "abstract" spectral decomposition. I wonder why Rickart mentioned this fact only in passing, [1946, footnote [12]) on p. 535].

4.10.4.7 Hilbert's spectral theorem of Hermitian operators [1906a, p. 175] is the most important result of Hilbert space theory. Simplified proofs were given by Hellinger [1909] and Riesz [1910]. Summaries can be found in [RIE, Chap. V], [HEL$^+$, pp. 1575–1583], [SZ-N, pp. 23–25], and [WIN]. For a historical account I refer to Steen [1973$^\bullet$]. The original versions were formulated in terms of quadratic forms. This is certainly the reason why Wintner's monograph had almost no impact on the further development. For many years, [STONE] became the main reference.

4.10.4.8 Nowadays, the **spectral theorem** reads as follows:

For every Hermitian operator $A \in \mathfrak{L}(H)$ there exists a **resolution of the identity** (translation of Hilbert's term *Zerlegung der Einheit*). This is a family of orthogonal projections E_λ depending on a real parameter λ and having the following properties.

(1) $E_\lambda \le E_\mu$ if $\lambda \le \mu$.

(2) $E_\lambda = O$ if $\lambda < a := \inf \sigma(A) = \inf\{(Ax|x) : \|x\| = 1\}$.

(3) $E_\lambda = I$ if $\lambda > b := \sup \sigma(A) = \sup\{(Ax|x) : \|x\| = 1\}$.

(4) $E_\lambda T = TE_\lambda$ for all $T \in \mathfrak{L}(H)$ such that $AT = TA$.

(5) $\lim\limits_{\varepsilon \searrow 0} E_{\lambda-\varepsilon} x = E_\lambda x$ for all $x \in H$ and $\lambda \in \mathbb{R}$.

(6)
$$A = \int_a^b \lambda \, dE_\lambda .$$

The right-hand Stieltjes integral is the limit (with respect to the operator norm) of the sums
$$\sum_{k=1}^{n} \lambda_k \Big(E_{\lambda_k} - E_{\lambda_{k-1}} \Big) \quad \text{as} \quad \max\{\lambda_k - \lambda_{k-1}\} \to 0,$$
where $\lambda_0 < a < \lambda_1 < \cdots < \lambda_{n-1} < b < \lambda_n$. Note that the only purpose of **(5)** is to ensure uniqueness.

According to [RIE, pp. 128–137], the E_λ's can be constructed via a functional calculus. First, $p(A)$ is defined for all polynomials viewed as members of $C[a,b]$. The correspondence $p \mapsto p(A)$ preserves the order and the algebraic operations. In particular, $p(t) \ge 0$ on $[a,b]$ implies $p(A) \ge O$. Next, $f(A)$ is obtained for all $f \in C[a,b]$ by continuous extension. But this is not enough. We need to find $E_\lambda := \chi_\lambda(A)$, where χ_λ denotes the characteristic function of the interval $[a, \lambda)$. This can be achieved by using continuous functions $f_1 \le f_2 \le \cdots$ that converge pointwise to χ_λ.

4.10.4.9 Riesz discovered another and quite elementary approach to the spectral theorem. First of all, he proved an analogue of the Hahn decomposition of set functions; see 4.2.2.9.

Riesz [1930, p. 37]:

Zu jeder selbstadjungierten beschränkten Transformation A gibt es eine Zerlegung der identischen Transformation E in zwei Einzeltransformationen: $E = E_- + E_+$ derart dass

1º diese Transformationen mit A und mit jeder mit A vertauschbaren beschränkten linearen Transformation vertauschbar sind;

2º die Transformationen AE_- und AE_+ negativ resp. positiv definit sind und ihre Summe gleich A ist;

3º dass für alle f, für welche $Af = 0$ ist, auch $E_- f = 0$, $E_+ f = f$ gelten.

This means that Hermitian operators are decomposed into their positive and negative parts:

$$A_+ := \frac{|A|+A}{2} \quad \text{and} \quad A_- := \frac{|A|-A}{2}.$$

In modern terminology, E_- is the right projection of A_-; see 4.10.4.6.

Applying the preceding construction to the operators $\lambda E - A$ yields the desired resolution of the identity; see also [LYUS$^+$, § 36] and [RIE$^+$, pp. 275–278]. The sole tools are the formation of the absolute value (square root) 4.10.3.7 and the existence of right projections. Thanks to this observation, the spectral theorem as well as its proof can easily be translated into the language of $B_p^\star$-algebras.

4.10.4.10 An element u is **unitary** if $u^\star u = uu^\star = \mathbf{1}$. Following Murray/von Neumann [1936, pp. 141–142], we use the term **partial isometry** if $u = uu^\star u$. In view of

$$(u - uu^\star u)^\star (u - uu^\star u) = (\mathbf{1} - u^\star u)(u^\star uu^\star u - u^\star u),$$

this is equivalent to the properties that $u^\star u$ and $uu^\star$ are projections.

4.10.4.11 The famous **polar decomposition** was obtained by von Neumann [1932a, p. 307]:

For every element a in a $W^\star$-algebra there exists a partial isometry u such that $a = u|a|$ and $|a| = u^\star a$. Moreover, it can be arranged that $u^\star u$ is the least projection p with $|a| = p|a|$. Subject to this extra condition, u is unique.

Any real-valued continuous function f, viewed as an element of $L_\infty[0,1]$, has the polar decomposition $f(t) = \operatorname{sgn} f(t)|f(t)|$, which is not available in the $C^\star$-algebra $C[0,1]$.

4.10.4.12 The projections of every $W^\star$-algebra $\mathscr{W}$ form a Dedekind complete lattice $\mathscr{W}_{\text{pro}}$ with o as the minimal and $\mathbf{1}$ as the maximal element. In particular, if $(p_i)_{i \in \mathbb{I}}$ is any family of mutually orthogonal projections, then $\sum_{i \in \mathbb{I}} p_i$ is defined to be the supremum of the upward directed system formed by all finite partial sums.

In view of $\mathscr{W} = \mathscr{W}_{\text{pro}}''$, every $W^\star$-algebra is uniquely determined by its projections. This conclusion can also be derived from the fact that the linear span of $\mathscr{W}_{\text{pro}}$ is norm dense in $\mathscr{W}$.

4.10.4.13 A fundamental defect of the theory of $W^\star$-algebras was described by Kaplansky [1951, p. 235]:

A notable advantage of the $C^\star$-case is the existence of an elegant system of intrinsic postulates due to Gelfand and Neumark; so one can, and does, study $C^\star$-algebras in an abstract fashion that pays no attention to any particular representation. A corresponding characterization of $W^\star$-algebras is not known.

Further, he proposed the following definition; see [1951, p. 236]:

*An $AW^\star$-**algebra*** [the "A" suggesting "abstract"] *is a $C^\star$-algebra satisfying:*

(A) *In the partially ordered set of projections, any set of orthogonal projections has a LUB* [read: least upper bound]*,*

(B) *Any maximal self-adjoint commutative subalgebra is generated by its projections.*

In assessing the strength of these postulates, it is helpful to examine the commutative case. Then (A) *and* (B) *are precisely equivalent to the assertion that the space of maximal ideals is a complete Boolean algebra. On the other hand, for a commutative $W^\star$-algebra the space of maximal ideals is a measure algebra. Thus in the commutative case, $AW^\star$-algebras generalize $W^\star$-algebras to the same extent that complete Boolean algebras generalize measure algebras.*

One may add that the $B_p^\star$-algebras correspond to the Boolean σ-algebras.

It seems that Kaplansky did not know the results of Szpilrajn [1934, p. 305] and Horn/Tarski [1948, p. 490], who discovered Dedekind complete Boolean algebras without reasonable measures; see 4.8.2.13. Therefore A(bstract)$W^\star$-algebras need not admit realizations as "concrete" $W^\star$-algebras, and the name is somewhat misleading.

4.10.4.14 Sakai [1956] found a *"space-free"* characterization:

A $C^\star$-algebra is a $W^\star$-algebra if and only if it is an adjoint space, when considered as a Banach space.

The necessity of this condition had been already discovered by Dixmier [1953, p. 15], who identified the **predual** $\mathscr{W}_*$ of a $W^\star$-algebra $\mathscr{W}$.

More precisely, $\mathscr{W}_*$ consists of all members of $\mathscr{W}^*$ that are linear combinations of **normal** positive linear functionals; see 4.8.2.10:

$$\inf_{\alpha\in\mathbb{A}} \ell(x_\alpha) = 0 \quad \text{for all downward directed systems with } \inf_{\alpha\in\mathbb{A}} x_\alpha = \mathrm{o}.$$

As a consequence, it follows that the predual of a $W^\star$-algebra is uniquely determined. I recall from 4.9.3.5 that l_1 is the dual of uncountably many non-isomorphic Banach spaces. More information about preduals can be found in a survey of Godefroy [1989].

Dixmier [1953, p. 13] also obtained a direct description of $\mathscr{W}_*$:

By definition, $\mathscr{W}$ is closed in some $\mathfrak{L}(H)$ with respect to the weak operator topology. Moreover, it is weakly* closed in $\mathfrak{L}(H) = \mathfrak{S}_1(H)^*$. Hence $\mathscr{W} = (\mathfrak{S}_1(H)/\mathscr{W}^\perp)^*$, in view of 3.3.3.7.

The preceding and the subsequent considerations show that Banach space techniques had a strong impact on the theory of $W^\star$-algebras.

4.10.4.15 Dixmier [1950, p. 398] showed that functionals on $\mathfrak{L}(H)$, continuous in the strong operator topology, can be represented in the form

$$\ell(T) = \sum_{k=1}^{n} (Tx_k|y_k) \quad \text{for } T \in \mathfrak{L}(H),$$

where $x_1, \ldots, x_n; y_1, \ldots, y_n \in H$. Hence they are continuous even in the weak operator topology. Thus both topologies of $\mathfrak{L}(H)$ yield the same dual and therefore the same closed convex subsets; see 3.3.3.4.

Dixmier [1950, p. 399]:

Von Neumann [1930b, p. 396] *a démontré qu'un sous-espace vectoriel $V \subset \mathscr{B}$ fortement fermé est aussi faiblement fermé, lorsque V est une sous-algèbra de $\mathscr{B}$. On voit que cette restriction est inutile.*

In the late 1920s, duality theory was not available to von Neumann. Consequently, when defining **rings of operators**, he stressed [1930b, footnote [40])]:

Man beachte: schwach abgeschlossen ist mehr wie stark abgeschlossen!

Hence something non-obvious had to be shown, and he obtained the desired conclusion as a by-product of his proof of the bicommutant theorem.

4.10.4.16 According to Murray/von Neumann [1936, p. 151], projections p and q are called **equivalent**, $p \sim q$, if there exists a partial isometry $u \in \mathscr{W}$ such that $p = u^\star u$ and $q = uu^\star$. In the language of operators this condition means that p and q have isomorphic ranges.

Finiteness of an abstract set can be characterized by the property that there is no proper subset of the same cardinality. Similarly, a projection p is **finite** if $p \sim q$ and $p \geq q$ imply $p = q$; see Murray/von Neumann [1936, p. 155]. This concept has proved to be quite useful in the non-commutative case. On the other hand, in a commutative algebra, $p \sim q$ implies equality. Hence we are faced with the strange fact that even projections of infinite rank are called *finite*. For example, take any $\{0,1\}$-sequence in the $W^\star$-algebra l_∞.

4.10.4.17 A projection p is **abelian** if the subalgebra $p\mathscr{W}p$ is commutative. Abelian projections are finite.

4.10.4.18 A **trace** on a $W^\star$-algebra $\mathscr{W}$ is a function $\tau : \mathscr{W}_+ \to [0,\infty]$ such that

$$\tau(x+y) = \tau(x) + \tau(y), \quad \tau(\mathrm{o}) = 0, \quad \tau(\lambda x) = \lambda\,\tau(x), \quad \tau(x^\star x) = \tau(xx^\star)$$

for $x, y \in \mathscr{W}$ and $\lambda > 0$. Instead of $\tau(x^\star x) = \tau(xx^\star)$ one may require that $\tau(u^\star xu) = \tau(x)$ for every unitary element u.

A trace τ that takes only finite values is referred to as **finite**, and τ is called **semi-finite** if for every $a \in \mathscr{W}_+$ with $\tau(a) = \infty$, we can find $b \in \mathscr{W}_+$ such that $b \le a$ and $0 < \tau(b) < \infty$. Moreover, a trace is **normal** (see 4.10.4.14) if

$$\sup_{\alpha \in \mathbb{A}} \tau(x_\alpha) = \tau(x) \quad \text{for all upward directed systems with } \sup_{\alpha \in \mathbb{A}} x_\alpha = x.$$

A *relative trace* for Hermitian elements of a factor was first constructed by Murray/von Neumann [1936, p. 219]. Originally they proved only that $\tau(x+y) = \tau(x) + \tau(y)$ whenever x and y commute. This restriction was removed in [1937, pp. 216–219].

4.10.4.19 A $W^\star$-algebra $\mathscr{W}$ may have the following properties formulated in terms of projections (right-hand row) or traces (left-hand row):

finite:

All projections in $\mathscr{W}$ are finite.	For every non-zero element $x \in \mathscr{W}_+$ there is a finite normal trace τ such that $\tau(x) > 0$.

semi-finite:

For every non-zero projection $p \in \mathscr{W}$ there is a finite non-zero projection $q \in \mathscr{W}$ with $q \le p$. It suffices if this condition is satisfied for every non-zero central projection p.	For every non-zero element $x \in \mathscr{W}_+$ there is a semi-finite normal trace τ such that $\tau(x) > 0$.

properly infinite:

The only finite central projection in $\mathscr{W}$ is o.	The only finite normal trace on $\mathscr{W}_+$ is o.

purely infinite:

The only finite projection in $\mathscr{W}$ is o.	The only semi-finite normal trace on $\mathscr{W}_+$ is o.

discrete:

For every non-zero projection $p \in \mathscr{W}$ there exists a non-zero abelian projection $q \in \mathscr{W}$ with $q \le p$. It suffices if this condition is satisfied for every non-zero central projection p.

continuous:

The only abelian projection in $\mathscr{W}$ is o.

The equivalence of both sides is by no means trivial; see [SAKAI, p. 97]. First results along these lines are due to Dixmier [1949, p. 249], [1951b, p. 192], [1952, p. 5]. Commonly, the left-hand properties are used as starting points; see [BERB, § 15], [SAKAI, pp. 83–86], [STR$^+$, p. 100], and [TAK, pp. 296–297]. However, Dixmier [DIX$_1$, pp. 96–102] presented an approach that is based on traces.

In the case of factors, the classification above goes back to Murray/von Neumann [1936, p. 172]; the global theory is due to Dixmier [1951b] and Kaplansky [1951].

4.10.4.20 The preceding properties can be arranged in pairs:

Every $W^\star$-algebra $\mathscr{W}$ admits decompositions such that the direct summands are

$$\begin{array}{rcl} \text{finite} & \oplus & \text{properly infinite,} \\ \text{semi-finite} & \oplus & \text{purely infinite,} \\ \text{discrete} & \oplus & \text{continuous.} \end{array}$$

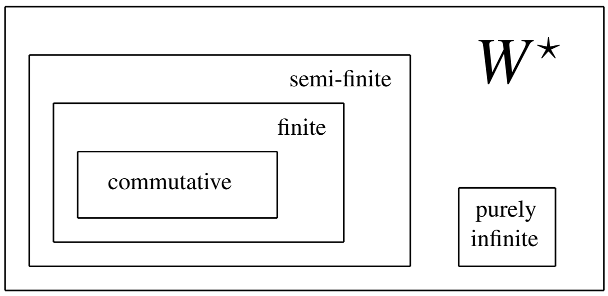
Cutting across these decompositions yields

$$\mathscr{W} = \overbrace{\mathscr{W}_{\mathrm{I_{fin}}} \oplus \mathscr{W}_{\mathrm{I}_\infty} \oplus \mathscr{W}_{\mathrm{II_{fin}}} \oplus \mathscr{W}_{\mathrm{II}_\infty}}^{\text{semi}-\text{finite}} \oplus \mathscr{W}_{\mathrm{III}}.$$

More precisely, we obtain the following **types**:

	finite	properly infinite	
			purely infinite
discrete	type $\mathrm{I_{fin}}$	type I_∞	
continuous	type $\mathrm{II_{fin}}$	type II_∞	
purely infinite			type III

Commutative $W^\star$-algebras are finite and hence semi-finite. This means that the classification of $W^\star$-algebras is tailored to the non-commutative case.



4.10.4.21 The full operator algebras $\mathfrak{L}(l_2^n)$ with $n = 1, 2, \ldots$ are of type $\mathrm{I_{fin}}$ (finite and discrete), whereas $\mathfrak{L}(l_2)$ is of type I_∞ (properly infinite and discrete).

Factors of type $\mathrm{II_{fin}}$ and II_∞ had already been constructed by Murray/von Neumann [1936, p. 208], whereas type III followed in [1940, p. 158]. This was by no means an easy task. Takesaki [1994$^\bullet$] gave an account of the history of type III von Neumann algebras, and for a modern presentation of various examples, the reader is referred to his book [TAK, pp. 362–371].

4.10.4.22 Next, I describe a concept that was an important tool in the Murray–von Neumann theory, [1936, p. 165]. They proved that every factor $\mathscr{W}$ carries a **dimensional function** δ, which assigns to every projection $p \in \mathscr{W}$ a non-negative number or ∞ such that the following conditions are satisfied:

(1) $\delta(p) = 0$ if and only if $p = \mathsf{o}$.

(2) $\delta(p)$ is finite if and only if the projection p is finite.

(3) For any pair of projections, $p \sim q$ implies $\delta(p) = \delta(q)$.

(4) $\delta(p+q) = \delta(p) + \delta(q)$ whenever the projections p and q are orthogonal.

Recall that $\mathscr{W}_{\text{pro}}$, the domain of δ, is a Dedekind complete Boolean algebra.

The dimensional function δ is unique up to a positive coefficient, and after normalization, its range has the form

$$\underbrace{\overbrace{\{0,1,\ldots,n\}}^{\text{type I}_{\text{fin}}}, \quad \overbrace{\{0,1,\ldots,\infty\}}^{\text{type I}_\infty}}_{\text{discrete}}, \quad \underbrace{\underbrace{\overbrace{[0,1]}^{\text{type II}_{\text{fin}}}, \quad \overbrace{[0,\infty]}^{\text{type II}_\infty}}_{\text{continuous (old fashioned)}} \quad \text{and} \quad \overbrace{\{0,\infty\}}^{\text{type III}}}_{\text{continuous (now)}}.$$

This motivates the terms “discrete” and “continuous” as used in 4.10.4.19.

Roughly speaking, the relationship between dimensional functions and traces is the same as that between finitely additive measures and integrals.

It may be convenient to replace **(4)** by the requirement

$$\delta\Big(\sum_{i\in\mathbb{I}} p_i\Big) = \sum_{i\in\mathbb{I}} \delta(p_i)$$

for any family of mutually orthogonal projections. The “gages” considered by Segal [1953, p. 401] are supposed to have this property.

4.10.4.23 A trace τ is called **faithful** (fidèle) if $\tau(x^\star x) = 0$ implies $x = \mathsf{o}$.

Next, I quote a famous result of Dixmier [1952, p. 46]:

Pour qu’il existe des pseudo-traces normales fidèles et essentielles, il faut et il suffit que $H^{pi} = 0$.

Concerning the definition of the subspace H^{pi}, Dixmier refers to his paper [1951b, p. 192]; the superscript ^{pi} comes from *purement infinie*.

Four years later, in [DIX$_1$, p. 98], the preceding statement reads as follows:

Soit $\mathscr{W}$ une algèbre de von Neumann. Les conditions suivantes sont équivalentes:

(1) *Il existe sur $\mathscr{W}_+$ une trace normale semi-finie fidèle.*

(2) *$\mathscr{W}$ est semi-finie.*

Here is the translation: the name “pseudo-trace” was previously used for traces that may take the value ∞, “essentielle” passed into “semi-finie,” and $H^{pi} = 0$ or $\mathscr{W}_{\text{III}} = \{\mathsf{o}\}$ meant that $\mathscr{W}$ is semi-finite.

Dixmier's upshot was a formal analogy:

A $W^\star$-algebra $\mathscr{W}$ is semi-finite if and only if for every non-zero projection $p \in \mathscr{W}$ we can find a finite non-zero projection $q \in \mathscr{W}$ with $q \le p$.	A trace τ is semi-finite if and only if for every non-zero element $a \in \mathscr{W}_+$, we can find $b \in \mathscr{W}_+$ such that $b \le a$ and $0 < \tau(b) < \infty$.

4.10.4.24 Since the term "semi-finite" occurs in the theory of measures as well as in the theory of $W^\star$-algebras, some comments may be useful:

- Commutative $W^\star$-algebras are semi-finite, and even finite.
- A complex $L_\infty(M,\mathscr{M},\mu)$ is a $W^\star$-algebra if and only if $(M,\mathscr{M},\mu)$ is localizable; all commutative $W^\star$-algebras are of this form; see 4.8.2.12 and 4.10.4.3.
- There exist semi-finite measure spaces $(M,\mathscr{M},\mu)$ for which $L_\infty(M,\mathscr{M},\mu)$ fails to be a $W^\star$-algebra.
- Non-commutative semi-finite $W^\star$-algebras have nothing to do with classical measure theory, but they give raise to a *non-commutative* generalization.

4.10.4.25 Every semi-finite $W^\star$-algebra $\mathscr{W}$ carries a normal semi-finite and faithful trace τ. Then

$$\mathscr{S} := \{x \in \mathscr{W} : \tau(x^\star x) < \infty\} \quad \text{and} \quad \mathscr{T} := \Big\{ \sum_{i=1}^{n} x_i y_i : x_i, y_i \in \mathscr{S} \Big\}$$

are two-sided ideals, and $\mathscr{T}_+ := \mathscr{T} \cap \mathscr{W}_+$ consists of all members of $\mathscr{W}_+$ whose trace is finite. Obviously, τ uniquely extends to $\mathscr{T} = \mathscr{T}_+ - \mathscr{T}_+$. This yields a positive linear functional, which will also be denoted by τ. We have

$$\tau(x^\star) = \overline{\tau(x)} \quad \text{and} \quad \tau(ax) = \tau(xa) \quad \text{for } x \in \mathscr{T} \text{ and } a \in \mathscr{W},$$

$$\tau(|x|) = \sup\{|\tau(ax)| : a \in \mathscr{W},\ \|a\| \le 1\} \quad \text{whenever } x \in \mathscr{T},$$

and

$$\|a\| = \sup\{|\tau(ax)| : x \in \mathscr{T},\ \tau(|x|) \le 1\} \quad \text{whenever } a \in \mathscr{W}.$$

Completing $\mathscr{T}$ with respect to the norm $x \mapsto \tau(|x|)$, we get a Banach space, which by analogy with the commutative case is denoted by $L_1(\mathscr{W},\tau)$. It turns out that $\langle \mathscr{W}, L_1(\mathscr{W},\tau)\rangle$ becomes a dual system under the bilinear form $B(a,x) := \tau(ax)$. In particular, $L_1(\mathscr{W},\tau)$ is a "concrete" predual of $\mathscr{W}$; see Dixmier [1953, pp. 21–23].

4.10.4.26 The standard trace on $\mathfrak{L}(H)_+$ is defined by

$$\operatorname{trace}(T) := \sum_{i \in \mathbb{I}} (Te_i|e_i).$$

As stated in 4.10.1.1, von Neumann had already proved that the right-hand expression does not depend on the special choice of the orthonormal basis $(e_i)_{i\in\mathbb{I}}$.

We know from [DIX_1, p. 94] that every normal trace on $\mathfrak{L}(H)_+$ is either proportional to the standard trace above or takes the value ∞ for all strictly positive operators.

In the fist case, the ideal $\mathscr{S}$ (defined in the preceding paragraph) consists of all Hilbert–Schmidt operators, while $\mathscr{T}$ is the trace class.

Operators belonging to the Schatten–von Neumann class $\mathfrak{S}_p(H)$ are characterized by the condition that

$$\|T|\mathfrak{S}_p\| = \text{trace}\big(|T|^p\big)^{1/p}$$

is finite. This fact suggested the definition of "non-commutative analogues" of the classical L_p spaces: given any (sufficiently nice) trace τ on a $W^\star$-algebra $\mathscr{W}$, a Banach space $L_p(\mathscr{W},\tau)$ is obtained by completing $\mathscr{T}$ with respect to the norm $x \mapsto \tau\big(|x|^p\big)^{1/p}$.

Based on preliminary work of Dye [1952], the results concerning "non-commutative integration" are mainly due to Dixmier [1953, pp. 23–34] and Segal [1953]. Modern presentations can be found in [DIX$_1$, Chap, I, § 6], [TAK, pp. 309–323], and a survey of Pisier/Xu [2003].

4.10.4.27 Only for completeness, I mention that the fourth paper, *On rings of operators*, dealt with the following problem; Murray/von Neumann [1943].

Let $\mathscr{W}_1$ and $\mathscr{W}_2$ be $W^\star$-algebras consisting of operators on Hilbert spaces H_1 and H_2, respectively. Which $\star$-isomorphisms $\mathscr{W}_1 \xrightarrow{\varphi} \mathscr{W}_2$ are **spatial**? That is, one looks for a unitary operator $H_1 \xrightarrow{U} H_2$ such that $\varphi(A_1) = UA_1U^\star$ whenever $A_1 \in \mathscr{W}_1$; see [STR$^+$, Chap. 8].

4.11 Complexification

4.11.1 Complex eigenvalues are unavoidable, even in the spectral theory of real matrices. The study of analytic functions requires complex scalars as well. Hence complexification of a real structure is an important process.

4.11.2 The **complexification** $X_\mathbb{C}$ of a real Banach space $X_\mathbb{R}$ consists of all formal expressions $x+\mathrm{i}y$ with $x,y\in X_\mathbb{R}$. A "complex" norm $\|\cdot\|_\mathbb{C}$ on $X_\mathbb{C} = X_\mathbb{R} + \mathrm{i}X_\mathbb{R}$ is supposed to satisfy the conditions

$$\|x+\mathrm{i}y\|_\mathbb{C} = \|x-\mathrm{i}y\|_\mathbb{C} \quad \text{and} \quad \|x\|_\mathbb{R} = \|x+\mathrm{i}\,o\|_\mathbb{C}.$$

The *least* admissible norm is given by

$$\|x+\mathrm{i}y\|_\mathbb{C}^{\min} := \sup_{0\le s<2\pi} \|x\cos s + y\sin s\|_\mathbb{R}. \tag{4.11.2.a}$$

There also exists a *greatest* counterpart, whose form is more complicated:

$$\|x+\mathrm{i}y\|_\mathbb{C}^{\max} := \inf \int_0^{2\pi} \|\boldsymbol{f}(t)\|_\mathbb{R}\,dt, \tag{4.11.2.b}$$

where the infimum ranges over all continuous $X_\mathbb{R}$-valued functions $\boldsymbol{f}$ such that

$$x = \int_0^{2\pi} \boldsymbol{f}(t)\cos t\,dt \quad \text{and} \quad y = \int_0^{2\pi} \boldsymbol{f}(t)\sin t\,dt.$$

Note that

$$\langle x, x^* \rangle + \langle y, y^* \rangle \leq \|x+\mathrm{i}y\|_{\mathbb{C}}^{\max} \|x^* + \mathrm{i}y^*\|_{\mathbb{C}}^{\min} \quad \text{for } x, y \in X_{\mathbb{R}} \text{ and } x^*, y^* \in X_{\mathbb{R}}^*.$$

Moreover, we have

$$\max\{\|x\|, \|y\|\} \leq \|x+\mathrm{i}y\|_{\mathbb{C}}^{\min} \leq \|x+\mathrm{i}y\|_{\mathbb{C}}^{\max} \leq \|x\| + \|y\|.$$

This chain of inequalities cannot be improved: if x and y are orthonormal elements in a Hilbert space, then

$$\max\{\|x\|_{\mathbb{R}}, \|y\|_{\mathbb{R}}\} = \|x+\mathrm{i}y\|_{\mathbb{C}}^{\min} = 1 \quad \text{and} \quad \|x+\mathrm{i}y\|_{\mathbb{C}}^{\max} = \|x\|_{\mathbb{R}} + \|y\|_{\mathbb{R}} = 2.$$

The norms $\|\cdot\|_{\mathbb{C}}^{\min}$ and $\|\cdot\|_{\mathbb{C}}^{\max}$ coincide only in the 1-dimensional case.

If either $\|\cdot\|_{\mathbb{C}}^{\min}$ or $\|\cdot\|_{\mathbb{C}}^{\max}$ is simultaneously used on $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, then “real” linear operators $T_{\mathbb{R}} : X_{\mathbb{R}} \to Y_{\mathbb{R}}$ extend to “complex” linear operators $T_{\mathbb{C}} : x+\mathrm{i}y \mapsto T_{\mathbb{R}}x+\mathrm{i}T_{\mathbb{R}}y$ such that $\|T_{\mathbb{C}}\| = \|T_{\mathbb{R}}\|$.

The process of complexification was first studied by Michal/Wyman [1941]. Following a verbal proposal of Taylor, they defined the norm

$$\|x+\mathrm{i}y\|_{\mathbb{C}} := \sup\Big\{ \sqrt{|\ell(x)|^2 + |\ell(y)|^2} \,:\, \|\ell\| \leq 1 \Big\},$$

where ℓ denotes a “real” linear functional. This formula yields just $\|x+\mathrm{i}y\|_{\mathbb{C}}^{\min}$. The reader is also referred to Taylor [1943, p. 665] and [RUST, pp. 7–16].

4.11.3 Taylor [1943, p. 665] asked, *Can every complex Banach space be decomposed into a real and an imaginary part?*

More than forty years later, a negative answer was given; see below.

With every complex Banach space Z we associate its **complex conjugate** $\overline{Z}$ consisting of all “formal” elements $\overline{z}$, where z ranges over Z. The algebraic operations are defined by $\overline{\alpha}\,\overline{z} := \overline{\alpha z}$ and $\overline{z_1} + \overline{z_2} := \overline{z_1 + z_2}$. Moreover, $\|\overline{z}\| = \|z\|$. The map $C : z \mapsto \overline{z}$ yields a **conjugate linear** isometry between Z and $\overline{Z}$:

$$C(\alpha z) = \overline{\alpha} Cz \quad \text{and} \quad C(z_1 + z_2) = Cz_1 + Cz_2 \quad \text{for } z, z_1, z_2 \in Z \text{ and } \alpha \in \mathbb{C}.$$

If Z is obtained by complexification, then the element conjugate to $z = x+\mathrm{i}y$ can be identified with $\overline{z} = x - \mathrm{i}y$. In this case, $Id : x+\mathrm{i}y \mapsto x+\mathrm{i}y$ yields a *complex* linear isometry between Z and $\overline{Z}$.

Bourgain [1986b] and Kalton [1995] found examples of complex Banach spaces that are not isomorphic to their complex conjugates. Of course, such spaces disprove Taylor’s conjecture. I mention that Bourgain’s construction uses random techniques; see 7.3.1.7. The example of Kalton is simple and explicit.

A finite-dimensional version of the preceding result was independently developed by Szarek [1986].

4.11.4 The Riesz representation theorem 2.2.6 for functionals on a complex Hilbert space H says that H^* and $\overline{H}$ can be identified: $\ell(f) = (f|g)$.

4.11.5 The term **complex Banach lattice** is contradictory, since we have no lattice operations $\wedge$ and $\vee$. What remains is the formation of absolute values. An axiomatic approach was accomplished by Mittelmeyer/Wolff [1974].

Let $x \mapsto |x|$ be a map from a complex linear space X into itself such that $\big||x|\big| = |x|$ and $|\alpha x| = |\alpha||x|$ for $x \in X$ and $\alpha \in \mathbb{C}$. Here $\big||x|\big|$ stands for the absolute value of $|x|$.

In order to make the formula $|x+y| \le |x| + |y|$ meaningful, a relation $\le$ is required: $x \le y$ if and only if $|y-x| = y-x$. Now the triangle inequality can be coded as

$$\Big|\big||x|+|y|\big| - |x+y|\Big| = |x|+|y| - |x+y| \quad \text{for } x,y \in X,$$

which is ridiculous. The two authors proved that $X_{\mathbb{R}} := \{|x|-|y| : x,y \in X\}$ becomes a real linear lattice. If all conditions above are satisfied and $X = X_{\mathbb{R}} + \mathrm{i}X_{\mathbb{R}}$, then $x \mapsto |x|$ is called an **absolute value**.

A **complex Banach lattice** is a complex Banach space together with an absolute value such that $\|x\| = \big\||x|\big\|$.

4.11.6 A constructive approach is due to Lotz [1968, pp. 17–18].

Let $X_{\mathbb{R}}$ be a real linear lattice. Then an absolute value is obtained on the complexification $X_{\mathbb{C}} = X_{\mathbb{R}} + \mathrm{i}X_{\mathbb{R}}$ by letting

$$|x+\mathrm{i}y| := \sup_{0 \le s < 2\pi} |x\cos s + y\sin s|. \tag{4.11.5.a}$$

The right-hand supremum exists under a mild completeness condition, which is satisfied for every real Banach lattice; see [SCHAE, p. 134]. Moreover, $X_{\mathbb{C}}$ becomes a complex Banach lattice with $\|x+\mathrm{i}y\| := \big\||x+\mathrm{i}y|\big\|$ as its norm.

Comparing (4.11.2.a) and (4.11.5.a) suggests that there could be a counterpart of (4.11.2.b). But this definition yields nothing new. Hence in contrast to the complexification of norms, the complex extension of the absolute value is unique. This implies that every real Banach lattice has its well-determined complexification. Conversely, every complex Banach lattice can be obtained in this way.

4.11.7 The complexification of "real rings" was already carried out by Gelfand [1941, p. 15]. Following a proposal of Kreĭn, he used the norm (4.11.2.a). The same procedure can be found in [BONS$^+$, pp. 68–71]. I stress, however, that Gelfand's approach is closely adapted to the setting of $C^\star$-algebras. In the case of group algebras one should switch to the definition (4.11.2.b).

5

Basic Results from the Post-Banach Period

Since this is a book about history, it was my intention to present the subjects in chronological order. On the other hand, intrinsic connections are more important. Thus the interval from 1932 to 1958 that, by definition, is called the *post-Banach period* (see p. xix) should not be taken as a dogma. This remark is supposed to justify the fact that the present chapter contains many results that were obtained much later. A typical example is Rosenthal's l_1-theorem; though proved only in 1974, its roots go back to Banach's monograph. In connection with bases, I have also presented the theory of wavelets, which was founded in the 1980s.

5.1 Analysis in Banach spaces

5.1.1 Convergence of series

5.1.1.1 The concept of **unconditional convergence** of series in Banach spaces is due to Orlicz [1929, Part II, p. 242]:

Es sei V ein B-Raum. Die Reihe $\sum_{n=1}^{\infty} x_n$ *nennen wir* **unbedingt konvergent***, wenn sie bei jeder Anordnung der Glieder konvergent ist.*

In [1933, Part I, p. 33], he proved *den folgenden elementaren Satz:*

Hinreichend und notwendig dafür, daß die Reihe $\sum_{n=1}^{\infty} x_n$ *in V unbedingt konvergiere, ist die Konvergenz der Teilreihe* $\sum_{i=1}^{\infty} x_{n_i}$ *für jede Indexfolge* $\{n_i\}$, $n_{i+1} > n_i$.

The latter property is now referred to as **subseries convergence**.

5.1.1.2 Orlicz–Pettis theorem:

If all subseries $\sum_{i=1}^{\infty} x_{n_i}$ are weakly convergent, then $\sum_{n=1}^{\infty} x_n$ converges unconditionally in norm.

In [BAN, p. 240] this result is attributed to Orlicz. Indeed, it follows by combining *Satz* 2 from Orlicz [1929, Part II, p. 244] with the criterion stated above. However, the underlying Banach space must be weakly sequentially complete. The general case was independently treated by Pettis [1938, pp. 281–282] and Dunford [1938, pp. 321–322]. Hence the name Orlicz–Pettis theorem, which is used in [DIE$_2^+$, p. 22] and [DIE$_2$, pp. 24, 45, 85], should be changed into Orlicz–Pettis–Dunford or just into Orlicz.

5.1.1.3 The most elegant characterization of unconditional convergence can be given in terms of directed systems; see 3.2.2.3.

Let $\mathscr{F}(\mathbb{I})$ denote the collection of all finite subsets of any index set $\mathbb{I}$. Obviously, $\mathscr{F}(\mathbb{I})$ is directed with respect to set-theoretic inclusion. A family $(x_i)_{i\in\mathbb{I}}$ in a Banach space is said to be **unconditionally summable**, or just **summable**, if the partial sums

$$s_{\mathbb{F}} := \sum_{i\in\mathbb{F}} x_i \quad \text{for } \mathbb{F}\in\mathscr{F}(\mathbb{I})$$

tend to a limit $s := \sum_{i\in\mathbb{I}} x_i$.

For a countable index set, the coincidence of this concept with the previous ones was established by Hildebrandt [1940].

5.1.1.4 A family $(x_i)_{i\in\mathbb{I}}$ is **absolutely summable** if

$$\sum_{i\in\mathbb{I}} \|x_i\| < \infty.$$

In the case of a sequence, one uses the name **absolute convergence**.

5.1.1.5 A classical theorem due to Dirichlet [1829] asserts that all rearrangements of a scalar series $\sum_{k=1}^{\infty} \xi_k$ converge (even to the same sum) if and only if $\sum_{k=1}^{\infty} |\xi_k| < \infty$. In other words, unconditional and absolute convergence of series coincide. This result extends to finite-dimensional Banach spaces. However, looking at orthogonal series in l_2 shows that there are unconditional convergent series that fail to converge absolutely. Mazur and Orlicz asked in *The Scottish Book* [MAUL•, Problem 122] whether this phenomenon occurs in every infinite-dimensional Banach space. An affirmative answer to this question was given by Dvoretzky/Rogers [1950, p. 192]:

The unconditionally convergent series coincide with the absolutely convergent series if and only if the space B is of finite dimension.

5.1.1.6 The **Orlicz theorem** [1933, Part I, p. 36; Part II, pp. 43–44] says that

$$\sum_{k=1}^{\infty} \|f_k|L_p\|^2 < \infty \quad \text{if } 1\le p\le 2$$

and

$$\sum_{k=1}^{\infty} \|f_k|L_p\|^p < \infty \quad \text{if } 2\le p<\infty$$

for every unconditionally convergent series $\sum_{k=1}^{\infty} f_k$ in L_p.

5.1.1.7 A convergent series is said to be **conditionally convergent** if it does not converge unconditionally. For such series Riemann [1854, p. 235] made a striking observation:

Offenbar kann nun die Reihe durch geeignete Anordnung der Glieder einen beliebig gegeben Werth erhalten.

This result, which was proved for real scalars, becomes false in the complex case.

Hence the following definition is justified:

We associate with every conditionally convergent series $\sum_{k=1}^{\infty} x_k$ in a Banach space its **domain of sums** $\mathrm{DS}(x_k)$; this is the collection of all elements $s_\pi = \sum_{k=1}^{\infty} x_{\pi(k)}$ that can be obtained by rearranging $\sum_{k=1}^{\infty} x_k$ into a convergent series.

For finite-dimensional Banach spaces, the **Steinitz theorem** [1913/16, Teil I, p. 163] says that $\mathrm{DS}(x_k)$ is a closed linear manifold. More precisely, we have

$$\mathrm{DS}(x_k) = s + M^{\perp}, \quad \text{where} \quad s = \sum_{k=1}^{\infty} x_k \quad \text{and} \quad M := \Big\{ x^* \in X^* : \sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty \Big\}.$$

Problem 106 of *The Scottish Book* [MAUL•] raised the question, *What happens in infinite-dimensional Banach spaces?* A fairly complete answer can be found in a readable monograph of (father and son) Kadets, which also contains a historical account; [KAD+, p. 24].

Here is a short summary of results:

Kornilov [1988, стр. 122] and Ostrovskiĭ [1986, стр. 84] constructed counterexamples that show that even in a Hilbert space, the domain of sums may be neither convex nor closed; [KAD+, pp. 24–25].

However, in special Banach spaces and subject to additional conditions, the Steinitz theorem remains true.

In a first step, Kadets [1954, стр. 109] treated L_r spaces with $1 < r < \infty$. Putting $p := \min\{r, 2\}$, he showed that $DS(x_k)$ is a closed linear manifold whenever

$$\sum_{k=1}^{\infty} \|x_k\|^p < \infty;$$

see also [KAD+, p. 30].

Later on, it turned out that the same conclusion holds in spaces of Rademacher type p; see [KAD+, p. 105]. Another generalization in terms of the modulus of smoothness will be presented in 5.5.4.8.

5.1.2 Integration of vector-valued functions

5.1.2.1 Without specifying the Banach function spaces X and Y, we consider an integral operator from X into Y defined by

$$K_{\mathrm{op}} : f(t) \mapsto g(s) = \int_M K(s,t) f(t) d\mu(t).$$

In most cases, the underlying kernel K can be viewed as a Y-valued function $\boldsymbol{k}$ that assigns to $t \in M$ the scalar function $\boldsymbol{k}(t) : s \mapsto K(s,t)$. In this way, the concrete integral above becomes an abstract integral

$$\int_M \boldsymbol{k}(t) f(t) d\mu(t).$$

This observation helps to explain why an integration theory for vector-valued functions is desirable.

Another reason for integrating functions with values in a Banach space stems from Spectral theory: resolvents are analytic operator-valued functions, and every analyst wants that Cauchy's integral formula should be available also in this setting.

One word about terminology. Functions $\boldsymbol{f} : M \to X$ are referred to as X**-valued**, **abstractly valued**, or **vector-valued**. Some authors use the term *Banach-space-valued*, which should be reserved for functions such as $p \mapsto l_p$.

5.1.2.2 For X-valued functions on a subset M of the Euclidean space, the following definition is due to Bochner [1933a, p. 263]:

Eine Funktion $\boldsymbol{g}$ auf einer messbaren Punktmenge M heisse endlichwertig, falls M in endlich viele messbare Teilmengen zerfällt, auf denen $\boldsymbol{g}$ konstant ist; und die Funktion $\boldsymbol{f}$ heisse **messbar** *auf M, falls es endlichwertige Funktionen $\boldsymbol{g}_n$ gibt, derart dass*

$$\boldsymbol{f}(t) = \lim_{n\to\infty} \boldsymbol{g}_n(t) \quad \textit{für fast alle } t \textit{ aus } M.$$

In what follows, I will use the name **Bochner measurable** X-valued function.

Originally, Bochner developed his theory only for the Lebesgue measure. Of course, the same approach can be used for arbitrary measure spaces $(M, \mathscr{M}, \mu)$, where μ is countably additive on the σ-algebra $\mathscr{M}$.

Gelfand [1938, p. 238] and Pettis [1938, p. 278] introduced **weak measurability** by assuming that whenever $x^* \in X^*$, all scalar-valued functions $\langle \boldsymbol{f}, x^* \rangle$ are measurable in the usual sense. As observed by Gelfand, both concepts coincide if X is separable. Pettis made a more precise statement:

A necessary and sufficient condition that $\boldsymbol{f}$ be [Bochner] *measurable is that it be weakly measurable and* [almost] *separably valued.*

The latter property means that there exist a separable subspace X_0 of X and a μ-null set N such that $\boldsymbol{f}(t) \in X_0$ for $t \notin N$.

A function $\boldsymbol{f}$ taking values in a dual space X^* is said to be **weakly*** **measurable** if all scalar-valued functions $\langle x, \boldsymbol{f} \rangle$ with $x \in X$ are measurable; see Gelfand [1938, p. 239].

5.1.2.3 The following example, which is due to Birkhoff [1935, p. 376], says "take care when dealing with the notion *almost everywhere*."

Consider the function $\boldsymbol{e} : [0,1] \to l_2\big([0,1]\big)$ that assigns to every point $t \in [0,1]$ the t^{th} unit family e_t in the non-separable Hilbert space $l_2\big([0,1]\big)$. Since $x = (\xi_t) \in l_2\big([0,1]\big)$ has at most countably many non-zero coordinates, it follows from $\big(\boldsymbol{e}(t)|x\big) = \xi_t$ that $(\boldsymbol{e}|x)$ vanishes almost everywhere with respect to the Lebesgue measure. On the other hand, $\|\boldsymbol{e}(t)|l_2\| = 1$ for all $t \in [0,1]$.

Conclusion: We need to distinguish between **almost everywhere** in the ordinary sense, **weakly almost everywhere**, and **weakly*** **almost everywhere**.

5.1.2.4 Roughly speaking, the **Bochner integral** for X-valued functions is a straightforward extension of the Lebesgue integral: replace the absolute value by the norm. However, Bochner [1933a, p. 273] had earlier observed that there is no full analogy with the scalar-valued case. It has become a main feature of the modern theory to characterize such Banach spaces in which a specific classical theorem remains true. Typical examples are the existence of Radon–Nikodym derivatives and the validity of Bessel's inequality; see 5.1.4.7 and 6.1.8.1.

For every Bochner measurable X-valued function $\boldsymbol{f}$, the real-valued function $t \mapsto \|\boldsymbol{f}(t)\|$ is measurable. Bochner [1933a, p. 265] referred to $\boldsymbol{f}$ as **summable** if $\|\boldsymbol{f}\|$ is summable. In this text, I use the term **Bochner integrable**.

Obviously, an $\mathscr{M}$**-simple** X-valued function

$$\boldsymbol{g} = \sum_{k=1}^{n} x_k \chi_{A_k} \quad \text{with } x_1, \dots, x_n \in X \text{ and } A_1, \dots, A_n \in \mathscr{M}$$

is integrable if all A_k's have finite measure. Then

$$\int_M \boldsymbol{g}(t)\, d\mu(t) := \sum_{k=1}^{n} x_k \mu(A_k).$$

In the general case, we let

$$\int_M \boldsymbol{f}(t)\, d\mu(t) := \lim_{n\to\infty} \int_M \boldsymbol{g}_n(t)\, d\mu(t),$$

where $(\boldsymbol{g}_n)$ is a suitable sequence of $\mathscr{M}$-simple functions. Bochner [1933a, p. 266] assumed that $\boldsymbol{g}_n(t) \to \boldsymbol{f}(t)$ and $\|\boldsymbol{g}_n(t)\| \le G(t)$ almost everywhere, with an integrable real-valued majorant G. Equivalently, we may require that $\int_M \|\boldsymbol{g}_n(t) - \boldsymbol{f}(t)\|\, d\mu(t) \to 0$; see [DIE$_2^+$, p. 44].

5.1.2.5 For every measure space $(M, \mathscr{M}, \mu)$ and $1 \le p < \infty$, the collection of all (equivalence classes of) Bochner measurable X-valued functions $\boldsymbol{f}$ such that $\|\boldsymbol{f}\| \in L_p(M, \mathscr{M}, \mu)$ is a Banach space $[L_p(M, \mathscr{M}, \mu), X]$ under the norm

$$\|\boldsymbol{f}|L_p\| := \left(\int_M \|\boldsymbol{f}(t)\|^p\, d\mu(t) \right)^{1/p}.$$

5.1.2.6 When Fréchet [1915, pp. 257–258] generalized Lebesgue's integral to functions on abstract sets, he used "infinite Riemann series" instead of "finite Riemann sums." Garrett Birkhoff [1935, p. 367], [1937, pp. 50–52] extended this approach to X-valued functions.

Let $(M, \mathscr{M}, \mu)$ be a finite measure space. We say that $\boldsymbol{f} : M \to X$ is **Birkhoff integrable** if given $\varepsilon > 0$, there exists a partition of M into countably many pairwise disjoint subsets $A_k \in \mathscr{M}$ such that

$$\left\| \sum_{k=1}^{\infty} \boldsymbol{f}(s_k)\mu(A_k) - \sum_{k=1}^{\infty} \boldsymbol{f}(t_k)\mu(A_k) \right\| \le \varepsilon \quad \text{for any choice of } s_k, t_k \in A_k.$$

The left-hand series are supposed to converge unconditionally. In this case, taking $\varepsilon_n \to 0$, we find a sequence of partitions for which the associated sums tend to a limit in X, the **Birkhoff integral**:

$$\left\| \sum_{k=1}^{\infty} \boldsymbol{f}\big(t_k^{(n)}\big)\mu\big(A_k^{(n)}\big) - \int_M \boldsymbol{f}(t)\,d\mu(t) \right\| \le \varepsilon_n \quad \text{for any choice of } t_k^{(n)} \in A_k^{(n)}.$$

If a function is Birkhoff integrable on M, then it is Birkhoff integrable on any subset $A \in \mathscr{M}$. Birkhoff [1935, p. 368] observed that

$$\|\boldsymbol{f}|L_1^{\text{birk}}\| := \sup_{A \in \mathscr{M}} \left\| \int_A \boldsymbol{f}(t)\,d\mu(t) \right\|$$

defines a norm on the linear space of all (equivalence classes of) Birkhoff integrable X-valued functions on M.

5.1.2.7 Gelfand [1938, pp. 253–257] used the following approach:

Die Funktion x_t, deren Werte E angehören, wird summierbar genannt, wenn für ein beliebiges lineares Funktional f die reelle Funktion fx_t summierbar ist.

Wenn x_t summierbar ist, so gibt es ein solches M, dass für ein beliebiges f die Ungleichung

$$\int_a^b |fx_t|\,dt \le M\,|f| \tag{5.1.2.7.a}$$

besteht.

Das Lebesguesche Integral einer abstrakten Funktion x_t wird ein solches Element ξ des Raumes $\overline{\overline{E}}$ [bidual] *genannt, dass für ein beliebiges Funktional die Gleichung*

$$\xi f = \int_a^b fx_t\,dt$$

besteht.

Jede summierbare Funktion besitzt ein Integral.

The same approach was independently proposed by Dunford [1938, pp. 307–308, 334, 338]. Therefore I will use the term **Gelfand–Dunford integral**.

Taking the least constant M in (5.1.2.7.a) yields a norm on the linear space of all (equivalence classes of) Gelfand–Dunford integrable X-valued functions:

$$\|\boldsymbol{f}|L_1^{\text{weak}}\| := \sup\left\{ \int_a^b |fx_t|\,dt \,:\, |f| \le 1 \right\}.$$

Of course, the definitions above can easily be extended to abstract measure spaces.

5.1.2.8 Pettis [1938, p. 280] defined a special kind of the Gelfand–Dunford integral by assuming that for *every* subset $A \in \mathscr{M}$, there exists an element $x_A \in X$ with

$$\langle x_A, x^* \rangle = \int_A \langle \boldsymbol{f}(t), x^* \rangle \, d\mu \quad \text{whenever } x^* \in X^*.$$

In this case, $\boldsymbol{f}$ is called **Pettis integrable**.

The X-valued set function $A \mapsto x_A$ turns out to be countably additive and absolutely continuous with respect to μ; see Pettis [1938, p. 283].

5.1.2.9 Comparing the various integrals yields that

$$\boxed{\begin{matrix}\text{Bochner}\\ \text{integrable}\end{matrix}} \overset{1}{\Longrightarrow} \boxed{\begin{matrix}\text{Birkhoff}\\ \text{integrable}\end{matrix}} \overset{2}{\Longrightarrow} \boxed{\begin{matrix}\text{Pettis}\\ \text{integrable}\end{matrix}} \overset{3}{\Longrightarrow} \boxed{\begin{matrix}\text{Gelfand–Dunford}\\ \text{integrable}\end{matrix}}$$

The implications above cannot be reversed.

Since absolute convergence of series is stronger than unconditional convergence, we do not have $\overset{1}{\Longleftarrow}$.

The function $\boldsymbol{e} : \mathbb{R}_+ \to c$ (space of convergent sequences), defined by $\boldsymbol{e}(t) := e_n$ for $t \in [n-1, n)$, is Gelfand–Dunford integrable, but its integral over the union of all intervals $[2n-1, 2n)$ does not belong to c; see Pettis [1938, pp. 301–302]. Thus $\overset{3}{\Longleftarrow}$ fails.

At the end of his paper, Pettis [1938, p. 303] asked, *Is the* $(\mathfrak{X})$ *integral* [read: the Pettis integral] *equivalent to that of Birkhoff?* A negative answer was given by Talagrand [TAL, p. 51], who constructed a highly complicated counterexample. On the other hand, Birkhoff and Pettis integrability coincide for Bochner measurable functions; see Pettis [1938, p. 292].

5.1.2.10 Viewing infinite sums $\sum_{k=1}^{\infty} x_k$ as integrals, we get the following relationships:

Bochner integrability $\Longleftrightarrow$ absolute summability

$$\sum_{k=1}^{\infty} \|x_k\| < \infty.$$

Birkhoff = Pettis integrability $\Longleftrightarrow$ unconditional summability

$$\sum_{k=1}^{\infty} x_{\pi(k)} \quad \text{converges for every permutation } \pi.$$

Gelfand–Dunford integrability $\Longleftrightarrow$ weak summability

$$\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty \quad \text{for all } x^* \in X^*.$$

In the latter case, which was considered for the first time by Gelfand [1938, p. 241], the weak limit $\sum_{k=1}^{\infty} x_k$ need not exist in X.

5.1.2.11 As observed by Pettis [1938, pp. 286–287], the norms $\|\cdot|L_1^{\text{birk}}\|$ and $\|\cdot|L_1^{\text{weak}}\|$ are equivalent on the linear space of Birkhoff integrable functions. It follows from results of Birkhoff [1935, p. 369] and Pettis [1938, p. 291] that the set of Birkhoff integrable functions is a closed subspace of the normed linear space of Pettis integrable functions. More precisely, it turns out to be the closed linear span of those measurable functions *assuming only a finite number of distinct values*.

Finally, we easily see that the Pettis integrable functions form a closed subspace of the Gelfand–Dunford integrable functions.

5.1.2.12 The following example is due to Birkhoff [1935, p. 376]:

Define $\boldsymbol{f}_n := \sum_{k=1}^{2^n} e_{nk}\chi_{nk}$, where (e_{nk}) is any orthonormal double sequence in a Hilbert space H and where χ_{nk} denotes the characteristic function of the dyadic interval $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$. Then $\|\boldsymbol{f}_n|L_1^{\text{birk}}\| = 2^{-n/2}$, but no limit $\boldsymbol{f} = \sum_{n=1}^{\infty}\boldsymbol{f}_n$ can exist. Indeed, since

$$\big(\boldsymbol{f}(t)|e_{mh}\big) = \big(\boldsymbol{f}_m(t)|e_{mh}\big) = \chi_{mh}(t) \quad \text{almost everywhere on } [0,1),$$

$\boldsymbol{f}(t)$ would have infinitely many Fourier coefficients equal to 1.

Conclusion: The linear spaces of H-valued functions that are integrable in the sense of Birkhoff, Pettis, or Gelfand–Dunford may fail to be complete. This defect reduces the flexibility of these integrals considerably, and in spite of the following quotation from [DIE$_2^+$, p. 57], the Bochner integral remains the favorite:

Presently the Pettis integral has very few applications. But our prediction is that when (and if) the general Pettis integral is understood it will pay off in deep applications.

A later remark of Talagrand [TAL, Introduction] has the same tendency:

The theory is now in search of new directions, as well as of applications of its many powerful results.

5.1.3 Representation of operators from L_1 into X

5.1.3.1 The generalized Steinhaus–Nikodym theorem (4.3.6 and 4.3.10) tells us that for a localizable measure space, every functional ℓ on $L_1(M,\mathscr{M},\mu)$ can be represented in the form

$$\ell(f) = \int_M f(t)g(t)\,d\mu(t),$$

where $g \in L_\infty(M,\mathscr{M},\mu)$ is uniquely determined up to a null function. Thus one may ask whether a similar result holds for operators T acting from $L_1(M,\mathscr{M},\mu)$ into an arbitrary Banach space.

The first representation theorem along these lines was obtained by Gelfand [1938, pp. 276–277], who treated the case $T : L_1[a,b] \to X^*$, where X is supposed to be *separable*. Subsequently, Dunford/Pettis [1940, pp. 345–346] replaced $[a,b]$ by a complete σ-finite measure space.

Under the assumptions above, the **Gelfand–Dunford–Pettis theorem** claims that there exists a weakly* measurable function $\boldsymbol{g}^* : M \to X^*$ with bounded range $\boldsymbol{g}^*(M)$ such that

$$\langle x, Tf \rangle = \int_M f(t) \langle x, \boldsymbol{g}^*(t) \rangle \, d\mu(t) \quad \text{for } f \in L_1 \text{ and } x \in X.$$

Conversely, every such $\boldsymbol{g}^*$ generates an operator $T : L_1(M, \mathscr{M}, \mu) \to X^*$.

The representing function is uniquely determined almost everywhere, and the ess-sup in the formula

$$\|T\| = \underset{t \in M}{\text{ess-sup}} \, \|\boldsymbol{g}^*(t)\|$$

must be taken with respect to this equivalence.

For a detailed presentation of this old-fashioned version the reader is referred to [BOU$_{6c}$, p. 46] and [DUN$_1^+$, p. 503]. Removing the superfluous separability condition turned out to be a hard task. Surprisingly, this was achieved as a by-product of the theory of measure theoretic liftings, which will be outlined in paragraphs 5.1.3.5 to 5.1.3.7.

5.1.3.2 In order to illustrate the Gelfand–Dunford–Pettis theorem, we consider the **integration operator** from $L_1[0,1]$ into $L_\infty[0,1] = L_1[0,1]^*$,

$$S : f(t) \mapsto \int_0^s f(t) \, dt.$$

The representing function is given by $\boldsymbol{g}^*(t) = \chi_{[t,1]}$, where $\chi_{[t,1]}$ denotes the characteristic function of the interval $[t,1]$. I stress that $\boldsymbol{g}^*$ fails to be Bochner measurable, since its range cannot be pushed into a separable subspace; see also 5.1.4.2.

5.1.3.3 In order to treat operators $T : L_1(M, \mathscr{M}, \mu) \to X$, we identify X with a subspace of X^{**} via the canonical map $K_X : X \to X^{**}$. Then the representing function $\boldsymbol{g}^{**}$ becomes X^{**}-valued and

$$\boldsymbol{g}^{**}(M)^{\circ\circ} = \big(K_X T(B_{L_1})\big)^{\circ\circ}.$$

Hence, for weakly compact operators, it follows that $\boldsymbol{g}^{**}(M) \subset X$. In this way, we get the **Dunford–Pettis–Phillips theorem**; [DUN$_1^+$, p. 507]:

If the underlying measure space is σ-finite, then every weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$ can be represented by a Bochner integral

$$Tf = \int_M f(t) \boldsymbol{g}(t) \, d\mu(t) \quad \text{for } f \in L_1,$$

where $\boldsymbol{g} : M \to X$ is a Bochner measurable function having a relatively weakly compact range $\boldsymbol{g}(M)$. Conversely, every such $\boldsymbol{g}$ generates a weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$.

In the case that M is an *Euclidean interval, finite or infinite,* this result was obtained by Dunford–Pettis [1940, p. 369]. Next, Phillips [1940, p. 534] dealt with σ-finite measure spaces, but under the assumption that T has a separable range.

On the other hand, Phillips [1943, p. 132] observed that for any uncountable index set $\mathbb{I}$, a representing function of the identity map from $l_1(\mathbb{I})$ into $l_2(\mathbb{I})$ could never be almost separably valued. This fact shows that σ-finiteness is not only a technical assumption. A way out of this dilemma will be described in 5.1.3.10.

5.1.3.4 The Dunford–Pettis–Phillips theorem can be used to prove that abstract L-spaces have the Dunford–Pettis property defined in Subsection 4.8.5:

Every weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$ takes weakly null sequences into norm null sequences.

Obviously, it suffices to check this statement for separable spaces, since we are concerned only with sequences. Hence no additional assumption on the underlying measure space is required; see Phillips [1940, p. 535] and [DUN_1^+, p. 508].

Next, assuming that the underlying measure has finite total mass, we consider a weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$. Then $L_2 \subseteq L_1$, and the closed unit ball of L_2 can be viewed as a weakly sequentially compact subset of L_1. Now it follows from the Dunford–Pettis property that $T(B_{L_2})$ is compact in X. Thus $T(L_2)$ must be separable. Finally, since L_2 is dense in L_1, we get the separability of $T(L_1)$. The same result holds for σ-finite measure spaces. Consequently, in this case, the separability assumption in the Dunford–Pettis–Phillips theorem is superfluous.

Nowadays, we are surprised that Phillips did not draw this conclusion in his paper from 1940. He needed another three years to reach the final goal; [1943, pp. 130–132].

5.1.3.5 A typical feature of measure and integration theory is the identification of functions f and g that coincide *almost everywhere*. In the next two paragraphs (and only there!), this equivalence is indicated by writing $f \equiv g$, while the symbol $f = g$ means that f and g are equal *everywhere*; see [ION, p. 12].

5.1.3.6 Around 1930, Haar asked von Neumann the following question:

Ist es möglich, jeder Äquivalenzklasse $\mathfrak{K}$ einen Repräsentanten $f_{\mathfrak{K}}$ derart zuzuordnen, daß für jedes allgemeine Polynom $p\big(u_1, \ldots, u_n\big)$ und irgendwelche Klassen $\mathfrak{K}_1, \ldots, \mathfrak{K}_n$ aus $p\big(\mathfrak{K}_1, \ldots, \mathfrak{K}_n\big) \equiv 0$ identisch $p\big(f_{\mathfrak{K}_1}, \ldots, f_{\mathfrak{K}_n}\big) = 0$ folgt ?

For the set of all bounded Lebesgue measurable complex-valued functions on the real line, von Neumann [1931] gave an affirmative answer. He also observed in footnote [1]), *daß diese Aufgabe ohne die Zusatzbedingung der Beschränktheit unlösbar ist.*

5.1.3.7 A linear **lifting** is a positive linear mapping ρ on the linear space of all bounded measurable functions on a measure space that satisfies the conditions

$$\rho(f) \equiv f, \qquad f \equiv g \text{ implies } \rho(f) = \rho(g), \qquad \text{and} \qquad \rho(1) = 1.$$

If we also have $\rho(fg) = \rho(f)\rho(g)$, then ρ is said to be **multiplicative**.

Maharam has reported that around 1942, von Neumann told her how to prove that every complete σ-finite measure space has a multiplicative lifting. But his hints got lost. Therefore she worked out another proof [1958], which is based on her representation theorem for measure algebras. A much simpler approach was given in a paper of A. and C. Ionescu Tulcea [1961, p. 543].

In 1960, the Bourbakists [BOU$_{6c}$, p. 42] still did not have a general lifting theorem at their disposal:

Tout sous-espace G de type dénombrable de l'espace de Banach $L^\infty(T,\mu)$ possède la propriété de relèvement.

5.1.3.8 Now I present the modern Gelfand–Dunford–Pettis theorem, which is due to Dieudonné [1951, p. 79]. Because in 1951, the existence of a linear lifting was known only for separable complete measure spaces, Dieudonné used it as a hypothesis. The point is that the following result holds for an arbitrary Banach space X.

Gelfand–Dunford–Pettis theorem; [ION, p. 89]:

If the underlying measure space is localizable and admits a linear lifting, then for every operator $T : L_1(M,\mathscr{M},\mu) \to X^*$ there exists a weakly* measurable function $\boldsymbol{g}^* : M \to X^*$ with bounded range $\boldsymbol{g}^*(M)$ such that

$$\langle x, Tf\rangle = \int_M f(t)\langle x, \boldsymbol{g}^*(t)\rangle d\mu(t) \quad \text{for } f \in L_1 \text{ and } x \in X.$$

Conversely, every such $\boldsymbol{g}^*$ generates an operator $T : L_1(M,\mathscr{M},\mu) \to X^*$. The representing function is uniquely determined *weakly** almost everywhere, and the ess-sup in the formula

$$\|T\| = \operatorname*{ess\text{-}sup}_{t\in M} \|\boldsymbol{g}^*(t)\|$$

must be taken with respect to this equivalence.

In order to prove this deep result we first observe that for every $x \in X$, the rule $f \mapsto \langle x, Tf\rangle$ defines a functional on L_1 that can be represented by a function $g_x^* \in L_\infty$:

$$\langle x, Tf\rangle = \int_M f(t) g_x^*(t) d\mu(t) \quad \text{whenever } f \in L_1.$$

The existence of g_x^* follows from the localizability of $(M,\mathscr{M},\mu)$; see 4.3.10.

Let ρ be any linear lifting. Passing from g_x^* to $\rho(g_x^*)$ yields a well-determined value $\rho\big(g_x^*\big)(t)$ for every $t \in M$ such that

$$|\rho(g_x^*)(t)| \le \|g_x^*|L_\infty\| \le \|T\|\|x\|.$$

Moreover, $g^*_{\alpha_1 x_1+\alpha_2 x_2} \equiv \alpha_1 g^*_{x_1} + \alpha_2 g^*_{x_2}$ implies

$$\rho\big(g^*_{\alpha_1 x_1+\alpha_2 x_2}\big)(t) = \alpha_1 \rho\big(g^*_{x_1}\big)(t) + \alpha_2 \rho\big(g^*_{x_2}\big)(t).$$

Forcing linearity is the crucial trick. Hence $x \mapsto \rho\left(g_x^*\right)(t)$ defines a functional on X, which will be denoted by $\boldsymbol{g}^*(t)$:

$$\langle x, \boldsymbol{g}^*(t)\rangle = \rho(g_x^*)(t) \quad \text{for } \textit{all } t \in M \text{ and } x \in X.$$

This proves that

$$\langle x, Tf\rangle = \int_M f(t) g_x^*(t) d\mu(t) = \int_M f(t)\langle x, \boldsymbol{g}^*(t)\rangle d\mu(t).$$

5.1.3.9 A. and C. Ionescu Tulcea, [1962, pp. 789–790] and [ION, pp. 89–90], made a surprising discovery: if $(M, \mathscr{M}, \mu)$ is localizable, then the validity of the Gelfand–Dunford–Pettis theorem for all operators $T : L_1(M, \mathscr{M}, \mu) \to X^*$ and arbitrary Banach spaces X implies that $(M, \mathscr{M}, \mu)$ admits a linear lifting.

5.1.3.10 Next, I discuss an up-to-date version of the representation theorem for weakly compact operators from $L_1(M, \mathscr{M}, \mu)$ into any Banach space, where the measure space need not be σ-finite. To this end, a modified concept of measurability is required:

A function $\boldsymbol{f} : M \to X$ is called **locally measurable** if its restrictions to all subsets with finite measure are measurable.

Some authors, for example [BOU$_{6a}$, Chap. IV, pp. 180–183] and [ION, p. 70], use the definition above as their starting point, which means that the *principe de localisation* is realized by decree; see 4.3.9.

Now we are in a position to state the modern **Dunford–Pettis–Phillips theorem;** [ION, p. 89]:

If the underlying measure space is localizable and admits a linear lifting, then every weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$ can be represented by a Bochner integral

$$Tf = \int_M f(t)\boldsymbol{g}(t)\, d\mu(t) \quad \text{for } f \in L_1,$$

where $\boldsymbol{g} : M \to X$ is a locally measurable function having a relatively weakly compact range $\boldsymbol{g}(M)$. Conversely, every such $\boldsymbol{g}$ generates a weakly compact operator $T : L_1(M, \mathscr{M}, \mu) \to X$.

The representing function is uniquely determined *locally* almost everywhere, and the ess-sup in the formula

$$\|T\| = \operatorname*{ess\text{-}sup}_{t \in M} \|\boldsymbol{g}(t)\|$$

must be taken with respect to this equivalence.

5.1.3.11 So far, we know that every complete σ-finite measure space admits a multiplicative lifting. However, sometimes σ-finiteness turns out to be an unnatural restriction.

A measure space $(M,\mathscr{M},\mu)$ is said to be **decomposable** if it can be represented as the union $M=\bigcup_{i\in\mathbb{I}} M_i$ of a family of pairwise disjoint subsets with finite measure such that $A\cap M_i\in\mathscr{M}$ for all $i\in\mathbb{I}$ implies $A\in\mathscr{M}$ and

$$\mu(A)=\sum_{i\in\mathbb{I}}\mu(A\cap M_i).$$

The right-hand expression may be defined as the supremum over all finite partial sums.

Putting together the "local" liftings on the components M_i yields a "global" lifting on M. I stress that according to our present knowledge, we need to assume completeness of the underlying measure space in order to ensure the existence of the local liftings; see Fremlin [1989, pp. 934–935].

At first glance, the concept above looks like a cheap trick; one just assumes what is obviously needed for a straightforward extension of the proof. However, this is not so. Subject to the "mild" condition that the measure space is locally determined, the existence of a lifting implies its decomposability; see [ION, p. 48] and [FREM$_2$, Vol. III].

5.1.3.12 Obviously, every σ-finite measure space is decomposable. A much deeper proposition can be found in [BOU$_{6b}$, p. 6]:

Soient T un espace localement compact, μ une [Radon] *mesure positive sur T. Il existe un ensemble localement dénombrable $\mathfrak{K}$ de parties compactes non vides de T, deux à deux disjointes et telles que $T-\bigcup_{K\in\mathfrak{K}} K$ soit localement μ-négligeable.*

According to Schwartz [SCHW$_2$, p. 46] and Grothendieck [GRO$_1$, Introduction, p. 25], this result goes back to Godement.

With $\mathfrak{K}$ referred to as a *concassage*, a generalization was proved in [BOU$_{6d}$, p. 18]. It seems that this concept became the historical starting point of all further considerations on decomposability. A survey on the state of the art in the mid 1960s was given by Kölzow [1966] and [KÖL].

5.1.3.13 Decomposable measure spaces are localizable. Fremlin [1978, p. 163] showed that the converse is not true. These observations justify that decomposable measure spaces are also called **strictly localizable**. To the best of my knowledge, the latter term appeared for the first time in the book of the Ionescu Tulceas, [ION, p. 17].

Summary:

The Steinhaus–Nikodym formula $L_1(M,\mathscr{M},\mu)^*=L_\infty(M,\mathscr{M},\mu)$ holds if and only if $(M,\mathscr{M},\mu)$ is localizable. An analogous representation theorem for operators from $L_1(M,\mathscr{M},\mu)$ into X^* requires stronger assumptions: completeness and strict localizability. It is an open problem whether one really needs completeness. On the other hand, strict localizability is "almost" necessary.

5.1.3.14 Finally, I remind the reader that Maharam algebras are generated by localizable measure spaces $(M,\mathscr{M},\mu)$; see 4.8.2.2. As stated in 4.8.2.8, the quality of $(M,\mathscr{M},\mu)$ can be improved. Of course, it is easily possible to get completeness. In a further step, we may achieve that μ becomes a positive Radon measure on a locally compact space M. This ensures, in particular, strict localizability. A modern presentation was given by Fremlin [1989, pp. 894, 915].

5.1.4 The Radon–Nikodym property: analytic aspects

5.1.4.1 We consider a function $\boldsymbol{f}$ from (a,b) into a Banach space X. Fix $t_0 \in (a,b)$, and suppose that the limit

$$\boldsymbol{f}'(t_0) := \lim_{t\to t_0} \frac{\boldsymbol{f}(t)-\boldsymbol{f}(t_0)}{t-t_0}$$

exists with respect to the norm topology. Then $\boldsymbol{f}'(t_0)$ is said to be the (strong) **derivative** of $\boldsymbol{f}$ at the point t_0.

5.1.4.2 One of the basic questions of *analysis in Banach spaces* was whether the **fundamental theorem of calculus** remains true for X-valued functions.

The first negative example in $l_\infty\big([0,1]\big)$ is due to Bochner [1933b]. Next, Clarkson [1936, p. 405] observed that *this phenomenon may still occur in separable spaces,* namely in $L_1[0,1]$.

To each point t, $0 \le t \le 1$, let correspond the element $\varphi_t \in L$ defined as follows:

$$\varphi_t(s) = \begin{cases} 1 & (s \le t), \\ 0 & (s > t). \end{cases}$$

Gelfand [1938, p. 265] used the same construction, and he added *eine Anmerkung, die nicht ohne Interesse sein dürfte.*

Der Raum $L^{(1)}$ und jeder separable Raum, der einen $L^{(1)}$ isomorphen Teil enthält, ist keinem konjugierten Raum isomorph.

This follows from his observation [1938, p. 264] that the fundamental theorem of calculus holds for separable duals.

5.1.4.3 Here is a list of further positive results.

Clarkson [1936, p. 411]:

A function $\varphi(t)$ from an interval $a \le t \le b$ into B, a uniformly convex space, which is of bounded variation, is (strongly) differentiable almost everywhere. The derivative, $\varphi'(t)$, is integrable (Bochner). If in addition $\varphi(t)$ is absolutely continuous, then

$$\varphi(t) = \varphi(a) + \int_a^t \varphi'(t)\,dt.$$

Gelfand [1938, p. 266]:

Damit die Funktion x_t das unbestimmte Integral einer Funktion sei $\big($*E ist dabei regulär* [read: reflexive]$\big)$, *ist es notwendig und hinreichend, dass die folgenden Bedingungen erfüllt sind:*

1°) *Bei beliebiger Zerlegung des Intervalls* (a,b) *in Teilintervalle gilt die Ungleichung*

$$\sum_{i=0}^{n-1} |x_{t_{i+1}} - x_{t_i}| \leq M \qquad (a \leq t_1 \leq \cdots \leq t_n \leq b),$$

wo M eine passend gewählte Konstante ist.

2°) *Ist* $\lim \sum |t_k - s_k| = 0$, *so ist* $\lim \sum |x_{t_k} - x_{s_k}| = 0$.

The theorem that states that uniformly convex spaces are reflexive was proved only in 1938/39; see 5.5.2.4. Therefore, at the time of their discovery the results of Gelfand and Clarkson seemed to be unrelated.

Dunford/Morse [1936, p. 415] obtained a Radon–Nikodym theorem *for all Banach spaces X with a basis* $\{\varphi_i\}$ *which satisfies the following postulate*:

If $a_1, a_2, \ldots$ *is any sequence of real numbers such that* $\sup_n \big\| \sum_{i=1}^{n} a_i \varphi_i \big\| < \infty$, *then the series* $\sum_{i=1}^{\infty} a_i \varphi_i$ *converges.*

According to [DAY, 1st edition, p. 69], such a basis is now called **boundedly complete**.

5.1.4.4 *A Banach space X will be said to have the* **property (D)** *if whenever* $\boldsymbol{f} : (0, 2\pi) \to X$ *is of bounded variation then* $\boldsymbol{f}$ *has a derivative a.e.*

This definition from Bochner/Taylor [1938, p. 915] seems to be the first case in which a classical theorem gave rise to the specification of a subclass of Banach spaces. Many others were to follow.

5.1.4.5 So far I have discussed only X-valued functions of a real variable. However, one may take a more general view. Clarkson [1936, p. 406] dealt with vector-valued *functions of elementary figures* of the n-dimensional Euclidean space. At the same time, Gowurin [1936, p. 259] considered Stieltjes integrals with respect to abstract functions of bounded variation. The final setting is that of a countably additive **X-valued measure**, or **vector measure**, defined on a σ-algebra $\mathscr{M}$:

$$\boldsymbol{m}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \boldsymbol{m}(A_k) \quad \text{whenever } A_h \cap A_k = \emptyset \text{ for } h \neq k.$$

Note that the right-hand series converges unconditionally.

Dinculeanu wrote the first monograph on *Vector Measures* [DINC], which was published in 1966.

5.1.4.6 Putting

$$|\boldsymbol{m}|(A) := \sup\left\{ \sum_{k=1}^{n} \|\boldsymbol{m}(A_k)\| : A_1, \ldots A_n \subseteq A,\ A_h \cap A_k = \emptyset \text{ if } h \neq k \right\}$$

yields a countably additive measure $|\boldsymbol{m}|$ on $\mathscr{M}$, the **variation** of $\boldsymbol{m}$. If $|\boldsymbol{m}|(M) < \infty$, then $\boldsymbol{m}$ is said to be of **bounded variation**; see Dunford [1936, p. 483].

Let μ be a finite measure on $\mathscr{M}$. We say that $\boldsymbol{m}$ is **μ-continuous** or **absolutely continuous** with respect to μ if $\mu(A)=0$ implies $\boldsymbol{m}(A)=\mathrm{o}$. This is the case precisely when the variation $|\boldsymbol{m}|$ is μ-continuous.

5.1.4.7 A Banach space X has the **Radon–Nikodym property** with respect to a finite measure space $(M,\mathscr{M},\mu)$ if for every μ-continuous X-valued measure $\boldsymbol{m}$ on $\mathscr{M}$ with bounded variation there exists a Bochner integrable X-valued function $\boldsymbol{f}$ such that

$$\boldsymbol{m}(A) = \int_A \boldsymbol{f}(t)\, d\mu(t) \quad \text{for } A \in \mathscr{M}.$$

The definition above is due to Chatterji [1968, p. 22].

5.1.4.8 The Radon–Nikodym property can be characterized as follows:

Every operator T from $L_1(M,\mathscr{M},\mu)$ into X admits a representation

$$Tf = \int_M f(t)\boldsymbol{g}(t)\, d\mu \quad \text{for } f \in L_1,$$

where $\boldsymbol{g}$ is a bounded Bochner measurable X-valued function.

The necessity of this condition was discovered by Dunford [1936, p. 485], and for a proof of its sufficiency I refer to [DIE$_2^+$, pp. 63–64].

The Dunford–Pettis–Philips theorem 5.1.3.3 tells us that the required representation is possible for every reflexive X; see Phillips [1943, pp. 134–135].

5.1.4.9 A Banach space is said to have the **Radon–Nikodym property** if the preceding conditions are satisfied for *all* finite measure spaces. However, as observed by Chatterji [1968, p. 26], we need only check the case of the Lebesgue measure on $[0,1]$. This means that the Radon–Nikodym property is equivalent to the Bochner–Taylor property (D) defined in 5.1.4.4.

5.1.4.10 Summarizing, I recall that reflexive spaces and separable duals have the Radon–Nikodym property. These criteria apply to all L_p's with $1<p<\infty$ and to l_1. On the other hand, c_0, l_∞, $C[a,b]$, $L_\infty[a,b]$, and $L_1[a,b]$ are counterexamples.

5.1.4.11 The Radon–Nikodym property carries over to closed subspaces. There is even a remarkable converse: if all *separable* closed subspaces of a Banach space have the Radon–Nikodym property, then so does the whole space; see Huff [1974, p. 114].

The spaces c_0, $l_1=c_0^*$, and $l_\infty=l_1^*$ show that the Radon–Nikodym property behaves badly under passage to duals or preduals.

5.1.4.12 Asplund [1968, pp. 31, 43] defined **strong differentiability spaces** X by the property that every continuous convex real-valued function defined on a convex open subset of X is Fréchet differentiable in a dense G_δ subset of its domain; see 5.1.8.2. He also showed that the requirement "G_δ" can be dropped. Moreover, it suffices if the condition above is fulfilled only for functions on X. Nowadays, following a proposal of Namioka/Phelps [1975, p. 735], we refer to such spaces as **Asplund spaces**.

A result of Uhl [1972, p. 114] and Stegall [1975, p. 218] says that X^* has the Radon–Nikodym property if and only if the dual of every separable subspace of X is separable.

Moreover, Namioka/Phelps [1975, p. 741] proved that the dual of an Asplund space has the Radon–Nikodym property. Finally, the reverse implication was added by Stegall [1978, p. 408], [1981, p. 515].

In summary: all properties just described characterize one and the same class of Banach spaces.

5.1.4.13 The Riesz formula $L_p[a,b]^* = L_{p^*}[a,b]$, see 2.2.7, extends to the vector-valued case if and only if X is an Asplund space:

$$\big[L_p(M,\mathscr{M},\mu),X\big]^* = \big[L_{p^*}(M,\mathscr{M},\mu),X^*\big]$$

for $1 \le p < \infty$ and any measure space with finite total mass. A complete proof can be found in [DIE$_2^+$, pp. 98–100]. Subject to the assumption that X^* has property (D), the "if" part had earlier been established by Bochner/Taylor [1938, p. 920].

5.1.4.14 In this subsection, I have discussed only the analytic part of the Radon–Nikodym story. The striking relations to the geometry of Banach spaces, to the theory of operator ideals and to the theory of martingales will be presented in Subsections 5.4.4, 6.3.18, and 6.8.11, respectively.

The excellent monograph [DIE$_2^+$] and the survey of Diestel/Uhl [1976] contain many historical remarks.

5.1.5 Representation of operators from L_p into X

Throughout this subsection, we let $1 < p < \infty$. The measure μ is supposed to be countably additive on a σ-algebra $\mathscr{M}$, and the collection of all subsets $A \in \mathscr{M}$ with finite measure will be denoted by $\mathscr{M}_\bullet(\mu)$.

5.1.5.1 Following Phillips [1940, p. 517], we consider finitely additive set functions $\boldsymbol{m} : \mathscr{M}_\bullet(\mu) \to X$ such that

$$\sum_{k=1}^{\infty} \frac{|\langle \boldsymbol{m}(A_k), x^* \rangle|^p}{\mu(A_k)^{p-1}} < \infty \tag{5.1.5.1.a}$$

for $x^* \in X^*$ and for any sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathscr{M}$ with $0 < \mu(A_k) < \infty$. Moreover, $\boldsymbol{m}(A) = o$ whenever $\mu(A) = 0$.

It can be shown that the restriction of $\boldsymbol{m}$ to any member of $\mathscr{M}_\bullet(\mu)$ is countably additive and μ-continuous; see Phillips [1940, p. 519].

The collection of these set functions, which is denoted by $[V_p(M,\mathscr{M},\mu),X]$, becomes a Banach space under the norm

$$\|\boldsymbol{m}|V_p\| := \sup\left\{\left(\sum_{k=1}^{\infty}\frac{|\langle \boldsymbol{m}(A_k),x^*\rangle|^p}{\mu(A_k)^{p-1}}\right)^{1/p} : \|x^*\|\leq 1, \quad A_1,A_2,\dots\right\},$$

where the A_k's are chosen as described above.

5.1.5.2 The main result of Phillips [1940, p. 528] says that

the general form of a linear transformation $U(\varphi)$ on L^p to X is

$$U(\varphi) = \int \varphi \, d\boldsymbol{m},$$

where $\boldsymbol{m} \in [V_p(M,\mathscr{M},\mu),X]$ and $\|U\| = \|\boldsymbol{m}|V_p\|$.

The set function $\boldsymbol{m}$ is obtained from U by putting $\boldsymbol{m}(A) := U(\chi_A)$ for $A \in \mathscr{M}_\bullet(\mu)$. The right-hand integral can be defined canonically: in a first step, one considers $\mathscr{M}_\bullet(\mu)$-simple functions, and the rest follows by continuous extension.

I stress that (5.1.5.1.a) generalizes Radon's condition (4.4.2.a) from the scalar-valued case. However, in the present situation, there is no Radon–Nikodym theorem at our disposal and we must be content with a vector measure.

5.1.5.3 In the case of the counting measure on an arbitrary set $\mathbb{I}$, the preceding result simplifies considerably.

The general form of an operator T from $l_p(\mathbb{I})$ into X is

$$T(\xi_i) = \sum_{i\in\mathbb{I}} \xi_i x_i,$$

where the family $(x_i)_{i\in\mathbb{I}}$ is **weakly p-summable** (see also 6.3.6.1):

$$\|(x_i)|w_p\| := \sup_{\|x^*\|\leq 1}\left(\sum_{i\in\mathbb{I}}|\langle x_i,x^*\rangle|^p\right)^{1/p} < \infty.$$

5.1.6 Representation of operators from $C(K)$ into X

5.1.6.1 The major step in the proof of the Riesz representation theorem consists in extending $\ell \in C[a,b]^*$ to a larger class of functions such that $\ell(\chi_{[a,t]})$ makes sense. When we deal with operators $T : C[a,b] \to X$, it may happen that the only possible value of $T(\chi_{[a,t]})$ turns out to be a member of $X^{**}\backslash X$. A fundamental discovery of Grothendieck [1953, p. 167] says that this phenomenon does not occur for weakly compact operators:

Soit K un espace compact, E un espace localement convexe séparé complet. Alors les fonctions vectorielles d'ensembles boréliens, à valeurs dans E, complètement additives (faiblement ou fortement) correspondent biunivoquement aux applications linéaires faiblement compactes de $C(K)$ de E.

For understanding this proposition, one must take into account that Grothendieck's Borel sets are just the Baire sets; see footnote [8] on p. 166:

Par abréger, nous appellerons, contrairement à l'usage courant, ensemble borélien toute partie de K dont la fonction caractéristique est une fonction de Baire.

5.1.6.2 A detailed approach to the theory of vector measures was given by Bartle/ Dunford/Schwartz. Among other results, they showed in [1953, p. 299] that every countably additive X-valued set function $\boldsymbol{m}$ on a σ-algebra $\mathscr{M}$ has a relatively weakly compact range:

$$\boldsymbol{m}(\mathscr{M}) := \{\, \boldsymbol{m}(A) : A \in \mathscr{M} \,\};$$

compare with Lyapunov's theorem 5.4.6.1.

In contrast to Grothendieck's point of view, the three authors [1953, pp. 299–300] used regular Borel measures. Regularity was defined via that of the scalar-valued measures $\boldsymbol{m}_{x^*} : A \mapsto \langle \boldsymbol{m}(A), x^* \rangle$ for all $x^* \in X^*$.

The upshot is a vector-valued version of Kakutani's theorem 4.6.11; see also [DUN_1^+, p. 493]:

Let K be a compact Hausdorff space. Then the formula

$$Tf = \int_K f(t)\, d\boldsymbol{m}(t) \quad \text{for } f \in C(K)$$

yields a one-to-one correspondence between the weakly compact operators from $C(K)$ into X and the regular countably additive X-valued Borel measures on K.

5.1.7 Vector-valued analytic functions on the complex plane

5.1.7.1 A function $\boldsymbol{f}$ defined on an open subset G of $\mathbb{C}$ and having its values in a *complex* Banach space X is called **analytic** or **holomorphic** if the derivative

$$\boldsymbol{f}'(\lambda_0) := \lim_{\lambda \to \lambda_0} \frac{\boldsymbol{f}(\lambda) - \boldsymbol{f}(\lambda_0)}{\lambda - \lambda_0}$$

exists at every point $\lambda_0 \in G$. The limit is taken with respect to the norm. At first glance, replacing the norm topology by the weak topology yields a weaker property. However, Dunford [1938, p. 354] proved that both concepts coincide.

5.1.7.2 Motivated by applications in spectral theory, Taylor considered analytic operator-valued functions, and proved a remarkable theorem [1938d, p. 576]. The following quotation is from his paper [1971•, p. 334]:

For each λ in an open set D of the complex plane let $A(\lambda)$ be a bounded linear operator from a complex Banach space X to a complex Banach space Y. Suppose that for each vector x in X, $A(\lambda)x$ is analytic as a function from D to Y. Then $A(\lambda)$ is analytic on D as an operator-valued function, using the uniform topology of operators. . . .

Taylor was so surprised by this result that when he announced it at a meeting of the American Mathematical Society in New York on February 26, 1938, he had not thought to consider what might follow from the weaker assumption that $y^*A(\lambda)x$ is analytic on D for each x in X and each continuous linear functional y^* on Y. This weaker assumption does indeed imply the analyticity of $A(\lambda)x$ for each x and hence also the further conclusion which Taylor had obtained.

5.1.7.3 Wiener [1923a] was the first to observe that the fundamental theorems of complex analysis carry over to the vector-valued setting. This fact was made more precise by Nagumo [1936, pp. 73–75], who considered *reguläre Funktionen* with values in $\mathfrak{L}(X)$. One can define integrals over rectifiable curves, the Cauchy formula

$$f(\lambda_0) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathscr{C}} \frac{f(\lambda)}{\lambda - \lambda_0} d\lambda$$

holds, and we have Taylor series. Replacing absolute values by norms, the classical proofs and estimates remain valid.

5.1.7.4 As stated in 2.6.5.2, Riesz had already discovered that analytic operator-valued functions of a complex variable play a decisive role in spectral theory. Nagumo [1936], Gelfand [1941], and Lorch [1943] extended these considerations to functions taking their values in Banach algebras. Detailed information along these lines will be given in Subsection 5.2.1. However, there are also important applications to the theory of semi-groups as well as to complex interpolation; see Subsections 5.3.3 and 6.6.3.

5.1.8 Gâteaux and Fréchet derivatives

5.1.8.1 So far, I have dealt only with X-valued functions of a "classical" variable. On the other hand, the name *"analyse fonctionnelle"* stems from the fact that one considers functions of functions, which means that the argument may be an element of a Banach space as well. This situation is reflected by the fact that Chapter IV of [HIL] has the title *"functions on vectors to vectors"*; see also [HIL$^+$, Chap. XXVI].

In the early period, the fundamental concepts such as derivatives were defined for concrete spaces. However, when Banach spaces entered the picture it was a relatively easy task to translate the former results into this new language.

5.1.8.2 Fréchet's *notion de différentielle* was first presented in the Comptes Rendus from spring 1911. Concerning this early work the reader is referred to Taylor's excellent accounts; [1974$^\bullet$], [1982/87$^\bullet$, Part III, pp. 41–53]. I only stress that at the beginning of his studies Fréchet did not know the book of [STOLZ, pp. 130–133], in which the "little oh" approach was created. However, in the subsequent paper [1912, p. 388] he recognized the priority of Stolz, Pierpont, and Young.

Here is an abstract version; Fréchet [1925b, p. 308]:

Nous dirons qu'une transformation ponctuelle $M = F(m)$ d'un point m d'un espace $(\mathscr{D})$ vectoriel en un point M d'un espace $(\mathscr{D})$ vectoriel, distinct ou non du précédent, est **différentiable** *au point m_0, s'il existe une transformation vectorielle linéaire $\Psi(\Delta m_0)$ de l'accroissement Δm_0 de la variable, qui ne diffère de l'accroissement correspondant $\Delta F(m_0)$ de la fonction que par un vecteur $o(\Delta m_0)$ infiniment petit par rapport à Δm_0.*

On a ainsi

$$\Delta F(m_0) = \Psi(\Delta m_0) + o(\Delta m_0),$$

la notation $o(\Delta m_0)$ signifiant que le rapport ε de la longueur du vecteur $o(\Delta m_0)$ à la longueur du vecteur Δm_0 tend vers zéro avec cette dernière longueur.

The quotation above justifies that the resulting concept is referred to today as **Fréchet differentiability**.

Since Fréchet's formulations are sometimes extremely vague, it seems to be unclear what he meant by an *espace $(\mathscr{D})$ vectoriel*; see Taylor [1982/87•, Part III, p. 31]. After having read his paper on affine spaces [1925a, p. 38] and [FRÉ, pp. 125–127, 140–141] very carefully, I came to the conclusion that this is just a normed linear space: *l'espace vectoriel au sens de M. Wiener*.

5.1.8.3 Passing to modern terminology, we now consider a function $\boldsymbol{f}$ from an open subset G of a Banach space X into a Banach space Y. Fix $x_0 \in G$, and suppose that the limit

$$\delta \boldsymbol{f}(x_0)h := \lim_{t\to 0} \frac{\boldsymbol{f}(x_0+th) - \boldsymbol{f}(x_0)}{t}$$

exists (with respect to the norm topology) for every $h \in X$. The parameter t is real. In general, the mapping $h \mapsto \delta \boldsymbol{f}(x_0)h$ need not be continuous or linear. In the case that $\delta \boldsymbol{f}(x_0)$ is a bounded linear operator, it will be called the **Gâteaux derivative** of $\boldsymbol{f}$ at the point x_0. This naming is a tribute to Gâteaux, who was killed during the first days of World War I. A short note of Gâteaux [1913] appeared in Comptes Rendus. The rest of his papers, written before spring 1914 and edited by Lévy, was posthumously published in 1919 and 1922.

Of course, Gâteaux [1922, p. 11] worked in a concrete Banach space:

Prenons par exemple la fonctionnelle continue réelle $U(z)$ de la fonction continue réelle $z(\alpha)$ $(0 \le \alpha \le 1)$.

Considérons $U(z+\lambda t_1)$ (t_1 fonction analogue à z). Supposons que

$$\left[\frac{d}{d\lambda} U(z+\lambda t_1)\right]_{\lambda=0}$$

existe quel que soit t_1. On l'appelle la **variation première** *de U au point z : $\delta U(z,t_1)$. C'est une fonctionnelle de z et de t_1, qu'on suppose habituellement linéaire, en chaque point z, par rapport à t_1.*

The same definition an be found in [1919, p. 83]. But in this paper Gâteaux used Fréchet's space formed by all scalar sequences.

5.1.8.4 Fréchet differentiability at a point x_0 implies continuity. We know from calculus that this conclusion may fail in the case of Gâteaux differentiability:

$$f(x,y) := \frac{x^2y}{x^4+y^2} \quad \text{if } x^2+y^2>0 \quad \text{and} \quad f(0,0) := 0.$$

5.1.8.5 As stated in [MICHA, Vol. I, p. 56], the following theorem was proved by Michal in 1940:

Une condition nécessaire et suffisante pour que, dans des espaces de Banach, $\lim_{\lambda\to 0}\frac{f(x_0+\lambda y)-f(x_0)}{\lambda}$ *soit une F-différentielle de $f(x)$ au point $x{=}x_0$, d'accroissement y, est que cette limite soit une fonction linéaire de y* [which includes continuity] *et qu'elle soit approchée uniformément en y par tout y dans un voisinage donné $\|y\| < a$.*

5.1.8.6 Fréchet [1925b, pp. 308–309] and Hildebrandt/Graves [1927, p. 136] referred to $\boldsymbol{\delta f}(x,\delta x)$, also $\boldsymbol{\delta f}(x,h)$ or $\boldsymbol{df}(x,dx)$, as a *differential*, while Govurin [1939b, p. 550] and Zorn [1946, p. 134] stressed the fact that $\boldsymbol{f}'(x) : h \mapsto \boldsymbol{\delta f}(x,h)$ is an operator from X into Y that deserves its own name: *derivative of $\boldsymbol{f}$ at the point x*. Worried by this confusion, Taylor made the following comments, [1974•, p. 381]:

In modern usage, it seems entirely natural to use $\boldsymbol{f}'$ to denote the mapping $x \mapsto \boldsymbol{f}'(x)$, and it seems to me that the name **derivative** *of $\boldsymbol{f}$ should be used for this function.*

...

The **differential** *of $\boldsymbol{f}$ at x is the value of the derivative of $\boldsymbol{f}$ at x. On the other hand, when we speak of the value of the differential, we are referring to $\boldsymbol{f}'(x)dx$, and this is not the same as the value of the derivative.*

In fact, given any Y-valued function $\boldsymbol{f}$ that is differentiable in all points of an open subset G of X, we have three levels:

- the function $\boldsymbol{f}'$ from G into $\mathfrak{L}(X,Y)$,
- the value of $\boldsymbol{f}'$ at $x \in X$, which is an operator,
- the value of the operator $\boldsymbol{f}'(x)$ at the increment $h \in X$, which is a member of Y.

5.1.8.7 Let G be an open subset of the direct product $X \times Y$ equipped with any suitable norm. Suppose that $\boldsymbol{F} : G \to Z$ is Fréchet differentiable at $(x_0,y_0) \in G$. Then

$$\boldsymbol{\delta F}(x_0,y_0)(h,k) = \delta_x\boldsymbol{F}(x_0,y_0)h + \delta_y\boldsymbol{F}(x_0,y_0)k,$$

where

$$\delta_x\boldsymbol{F}(x_0,y_0)h := \lim_{t\to 0}\frac{\boldsymbol{F}(x_0+th,y_0)-\boldsymbol{F}(x_0,y_0)}{t}$$

and

$$\delta_y\boldsymbol{F}(x_0,y_0)k := \lim_{t\to 0}\frac{\boldsymbol{F}(x_0,y_0+tk)-\boldsymbol{F}(x_0,y_0)}{t}$$

are the **partial derivatives** at the point (x_0,y_0). This straightforward generalization of the classical concept is due to Fréchet [1925b, p. 319]. In the work of Hildebrandt/Graves [1927, pp. 138–139], partial derivatives were used for a purpose that will be discussed in the next paragraph.

5.1.8.8 At the University of Chicago, the **implicit function theorem** provided the main impetus for dealing with Banach spaces. For reasons of priority, Hildebrandt [1953, footnote [1] on p. 115] insisted on the term "linear normed complete or LNC space." After the preliminary work of Lamson, which I have already discussed in 1.6.11, the final version was established by Hildebrandt/Graves [1927, p. 150]:

Let $\boldsymbol{F}$ be a Fréchet differentiable Y-valued function on an open subset G of the direct product $X \times Y$ whose derivative $\delta\boldsymbol{F}(x,y)$ depends continuously on (x,y). Suppose that $\boldsymbol{F}(x_0,y_0) = \mathsf{o}$ and that $\delta_y\boldsymbol{F}(x_0,y_0)$ is invertible. Then there is a uniquely determined Y-valued function $\boldsymbol{f}$ on some neighborhood $U_\delta(x_0) := \{x \in X : \|x - x_0\| < \delta\}$ such that

$$\boldsymbol{f}(x_0) = y_0, \quad \big(x, \boldsymbol{f}(x)\big) \in G \quad \text{and} \quad \boldsymbol{F}\big(x, \boldsymbol{f}(x)\big) = \mathsf{o}.$$

Moreover, $\boldsymbol{f}$ is Fréchet differentiable on $U_\delta(x_0)$ and

$$\delta\boldsymbol{f}(x) = \delta_y\boldsymbol{F}\big(x, \boldsymbol{f}(x)\big)^{-1}\delta_x\boldsymbol{F}\big(x, \boldsymbol{f}(x)\big).$$

The previous formulas hold for all $x \in U_\delta(x_0)$. A modern presentation is given in [DIEU, Section 10.2].

5.1.9 Polynomials and derivatives of higher order

5.1.9.1 In concrete spaces, the concept of a **polynomial** was introduced by Fréchet [1909]. Later on, he defined *polynomes abstraits* on the sophisticated class of *espaces algébrophiles*; [1929, pp. 75–76]. The case of Banach spaces was included.

Let

$$\Delta_h\boldsymbol{f}(x) := \boldsymbol{f}(x+h) - \boldsymbol{f}(x) \quad \text{for } \boldsymbol{f} : X \to Y \text{ and } x, h \in X.$$

A function $\boldsymbol{P} : X \to Y$ is called a **polynomial** of degree n if

$$\Delta_{h_1} \cdots \Delta_{h_{n+1}}\boldsymbol{P}(x)$$

vanishes identically for any choice of increments $h_1, \dots, h_{n+1} \in X$, and if there exist $h_1^\circ, \dots, h_n^\circ, x_0 \in X$ such that $\Delta_{h_1^\circ} \cdots \Delta_{h_n^\circ}\boldsymbol{P}(x_0) \neq \mathsf{o}$.

A function $\boldsymbol{f} : X \to Y$ is said to be **homogeneous** of degree n if

$$\boldsymbol{f}(\lambda x) = \lambda^n\boldsymbol{f}(x) \quad \text{for } \lambda \in \mathbb{K} \text{ and } x \in X.$$

The main result of Fréchet [1929, Résumé, p. 92] says that in the real case, every continuous polynomial of degree n admits a unique representation

$$\boldsymbol{P}(x) = \sum_{k=0}^{n} \boldsymbol{P}_k(x),$$

where the $\boldsymbol{P}_k$'s are continuous homogeneous polynomials of degree k.

Fréchet's definition of a polynomial is based only on the additive structure of the underlying spaces. Hence, in the complex case,

$$\boldsymbol{P}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k + \sum_{k=0}^{n} b_k \bar{\lambda}^k$$

would be a polynomial. Thus the second term must be excluded by an additional condition. According to Highberg [1937, p. 309], this can be achieved by assuming that $\boldsymbol{P}$ is Gâteaux differentiable.

5.1.9.2 The next quotation is taken from Gâteaux [1919, p. 75]. His polynomials are defined on the linear space of all complex sequences equipped with Fréchet's metric; see 3.4.1.1. However, the same formulation makes sense for arbitrary Banach spaces.

Une fonction $P(z) = P(z_1, \dots, z_p, \dots)$ sera dite un **polynome** *de degré n, si:*

1° *Elle est continue;*

2° *$P(\lambda z + \mu t)$ est un polynome de degré n par rapport aux variables complexes λ, μ, quels que soient les points z, t.*

Si $P(\lambda z + \mu t)$ est homogène en λ, μ, nous dirons que le polynome $P(z)$ est **homogène**.

Gâteaux observed that every polynomial can be written as the sum of homogeneous polynomials.

5.1.9.3 Based on the unpublished thesis of Martin [1932], a simpler definition was proposed by Michal/Martin [1934, p. 71]:

By a **polynomial** *$p(x)$ of degree n on a vector space $V_1(A)$ to a vector space $V_2(A)$ we shall mean a continuous function $p(x)$ such that $p(x + \lambda y)$ is a polynomial of degree n in λ of the number system A, with coefficients in $V_2(A)$.*

5.1.9.4 Finally, Michal presented a definition via multilinear mappings, which is now commonly accepted; [MICHA, Vol. I, pp. 30–35]:

Soient N_1 et N_2 deux espaces linéaires normés [real or complex].

...

S'il existe une fonction multilinéaire $h(x_1, \dots, x_n)$ définie sur N_1^n à valeurs dans N_2, complètement symétrique en $x_1, \dots, x_n$, par définition $h(x, \dots, x) = P_n(x)$ est un ≪ **polynome homogène** *de degré n≫ défini sur N_1 à valeurs dans N_2 et $h(x_1, \dots, x_n)$ est appelée la ≪* **form polaire** *≫ du polynome* [see 5.1.9.5].

...

Une fonction $P(x)$ définie sur N_1 à valeurs dans N_2 est appelée un **polynome** *de degré n si $P(x) = \sum_{i=0}^{n} P_i(x)$, où $P_i(x)$ est un polynome homogène de degré i, $P_n(x) \not\equiv o$, pour tout $x \in N_1$, $P_0(x)$ est un élément constant de N_2 indépendant de x.*

...

Les définitions furent donnés pour la première fois par l'auteur en 1932–1933 dans ses leçons au ≪California Institute of Technology≫.

The same approach was independently given in a fundamental paper of Mazur/Orlicz [1935, Part I, p. 50], who acknowledged:

Die hier an die Spitze gestellte allgemeine Definition eines Polynoms stammt von Herrn S. Banach; sie wurde von ihm bei noch nicht veröffentlichten Untersuchungen über die analytischen Operationen eingeführt.

Indeed, Banach had the intention to write a second volume of his famous book devoted to the non-linear theory; see [BAN, p. 231]. In the back of [SAKS], which appeared in 1937, we find the announcement that a MONOGRAFIE MATEMATYCZNE

S. BANACH, *Théorie générale des operations*

was in course of preparation. Unfortunately, due to the German invasion in World War II, the Polish school of functional analysis was largely demolished, and Banach died in 1945 without accomplishing his project.

5.1.9.5 The **polar form** of a homogeneous polynomial $\boldsymbol{P}$ of degree n is given by

$$\boldsymbol{L}(x_1,\dots,x_n) := \frac{1}{n!}\Delta_{x_1}\cdots\Delta_{x_n}\boldsymbol{P}(x) \quad \text{for } x_1,\dots,x_n \in X.$$

Because of $\Delta_h\Delta_{x_1}\cdots\Delta_{x_n}\boldsymbol{P}(x) = \boldsymbol{O}$, the right-hand expression does not depend on x. This definition is due to Martin [1932, p. 36] and Mazur/Orlicz [1935, Part I, pp. 54, 62]; the latter used the term *erzeugende Operation*. Obviously,

$$\boldsymbol{P}(x) = \boldsymbol{L}(\overbrace{x,\dots,x}^{n \text{ times}}).$$

The authors mentioned above also proved the **polarization formula**

$$\boldsymbol{L}(x_1,\dots,x_n) = \frac{1}{2^n n!}\sum_{\varepsilon_k=\pm 1}\varepsilon_1\cdots\varepsilon_n\boldsymbol{P}\Big(\sum_{k=1}^{n}\varepsilon_k x_k\Big); \tag{5.1.9.5.a}$$

see [DINE_1, pp. 4–5].

The polar form is a symmetric multilinear mapping $\boldsymbol{L}$ from the n-fold Cartesian product $X\times\cdots\times X$ into Y. **Symmetry** means that

$$\boldsymbol{L}(x_{\pi(1)},\dots,x_{\pi(n)}) = \boldsymbol{L}(x_1,\dots,x_n)$$

for every permutation π of $\{1,\dots,n\}$, and concerning n-**linearity**, the reader is referred to 5.7.1.6. These concepts were introduced by Mazur/Orlicz [1935, Part I, pp. 50–51] and Govurin [1939a, p. 543].

5.1.9.6 The various definitions of polynomials are summarized in the following table. We assume that $\boldsymbol{P}$ is continuous.

$\Delta_{h_1}\cdots\Delta_{h_{n+1}}\boldsymbol{P}=\boldsymbol{O}$ Fréchet 1909/1929	$+$	Gâteaux differentiable (in the complex case)

$\boldsymbol{P}(x+\lambda h)$ polynomial in λ Michal/Martin 1934	$\xrightarrow{\text{Hartogs [1906]}}$ $\xleftarrow[\text{obvious}]{}$	$\boldsymbol{P}(\lambda_1 x_1+\cdots+\lambda_n x_n)$ polynomial in $\lambda_1,\ldots,\lambda_n$ $n=2$, Gâteaux 1914

$\boldsymbol{P}(x)=\boldsymbol{L}(x,\ldots,x)$ Banach–Michal $\sim$ 1932

The equivalence of these approaches was verified by Mazur/Orlicz [1935, Part I, p. 64] (real case) and Highberg [1937, pp. 312–314] (complex case). Taylor has described the state of the art achieved by Michal's school at the California Institute of Technology in [1938a, pp. 303–308]. An elegant presentation of the theory of complex polynomials can be found in [HIL, pp. 65–90] or [HIL$^+$, pp. 760–779].

5.1.9.7 From [HIL, p. 75] we know that *a homogeneous polynomial* [in the sense of Michal/Martin] *is always Gâteaux differentiable; it is Fréchet differentiable if and only if it is continuous.*

5.1.9.8 A homogeneous polynomial $\boldsymbol{P}$ of degree n is **continuous** precisely when it has a finite **norm**:

$$\|\boldsymbol{P}\| := \sup\big\{\, \|\boldsymbol{P}(x)\| : \|x\| \le 1 \,\big\}.$$

A corresponding result holds for n-linear mappings $\boldsymbol{L}$ from $X_1\times\cdots\times X_n$ into Y. In this case, one defines

$$\|\boldsymbol{L}\| := \sup\big\{\, \|\boldsymbol{L}(x_1,\ldots,x_n)\| : \|x_1\| \le 1,\ldots,\|x_n\| \le 1 \,\big\}.$$

If $\boldsymbol{L}$ is the polar form of $\boldsymbol{P}$, then it easily follows from (5.1.9.5.a) that

$$\|\boldsymbol{P}\| \le \|\boldsymbol{L}\| \le \frac{n^n}{n!}\|\boldsymbol{P}\|; \tag{5.1.9.8.a}$$

see Taylor [1938a, p. 307]. The scalar-valued polynomial

$$\boldsymbol{P}(x) := \prod_{k=1}^{n} \xi_k \quad \text{for } x=(\xi_1,\ldots,\xi_n)\in l_1^n$$

shows that the upper estimate of $\|\boldsymbol{L}\|$ is sharp; see 5.1.10.11. On the other hand, for polynomials on a Hilbert space we have $\|\boldsymbol{P}\|=\|\boldsymbol{L}\|$; see Banach [1938, p. 42] and Taylor [1938a, p. 317]. The special case $n=2$ was already known in the early period of Hilbert space theory; [RIE, p. 127] and Riesz [1930, p. 29]:

$$\|A\| = \sup\{\, |(Ax|x)| : \|x\| \le 1 \,\} \quad \text{for every self-adjoint operator } A.$$

Let $c(n,X)$ denote the smallest $c \geq 1$ such that $\|\boldsymbol{L}\| \leq c\|\boldsymbol{P}\|$ for all continuous homogeneous polynomials of degree n on a fixed Banach space X. As indicated by the preceding examples, the sequence of these **polarization constants** is closely related to the geometry of X. Further information can be found in [DINE$_2$, pp. 41–66].

5.1.9.9 Polynomials and symmetric n-linear mappings naturally occur in the definition of **derivatives** of higher order.

Differentiation of $t_2 \mapsto \delta \boldsymbol{f}(x+t_2h_2)h_1$ at $t_2=0$ yields $\delta^2 \boldsymbol{f}(x)(h_1,h_2)$. Proceeding by induction, we get

$$\delta^n \boldsymbol{f}(x)(h_1,\cdots,h_n) := \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \boldsymbol{f}\Big(x+t_1h_1+\cdots+t_nh_n\Big)_{t_1=\ldots=t_n=0};$$

see [HIL, p. 73]. If for any choice of increments $h_1,\ldots,h_n \in X$, this limit exists and is a continuous n-linear mapping, then we refer to $\delta^n \boldsymbol{f}(x)$ as the n^{th} **Gâteaux derivative** of $\boldsymbol{f}$ at the point x.

Similarly, the process of **Fréchet differentiation** can be iterated. Starting with $\delta \boldsymbol{f}(x) \in \mathfrak{L}(X,Y)$, we obtain

$$\delta^2 \boldsymbol{f}(x) \in \mathfrak{L}(X,\mathfrak{L}(X,Y)), \quad \delta^3 \boldsymbol{f}(x) \in \mathfrak{L}(X,\mathfrak{L}(X,\mathfrak{L}(X,Y))), \quad \ldots .$$

The conceptual difference between the derivatives of both authors can be bridged by a (nowadays trivial) identification; [DIEU, 5.7.8]:

Let $\boldsymbol{L}: X \times Y \to Z$ be any bounded bilinear mapping. Then, for fixed $x \in X$, the rule $L_x : y \mapsto \boldsymbol{L}(x,y)$ defines an operator from Y into Z. Hence $L: x \mapsto L_x$ is a member of $\mathfrak{L}(X,\mathfrak{L}(Y,Z))$ such that $\|\boldsymbol{L}\| = \|L\|$. Moreover, every $L \in \mathfrak{L}(X,\mathfrak{L}(Y,Z))$ is obtained in this way.

Unfortunately, I was not able to locate the historical source of this observation. To the best of my knowledge, Govurin [1939b, p. 550] was the first to mention the fact that *the values of* $\boldsymbol{f}''$ *lie in* $\{X \to \{X \to Y\}\}$, *i.e.,* $\{X,X \to Y\}$. The latter symbol stands for the collection of all bilinear mappings from $X \times X$ into Y, which we denote by $\mathfrak{L}(X \times X, Y)$; see 5.7.1.2. Actually, the formula $\mathfrak{L}(X \times X,Y) = \mathfrak{L}(X,\mathfrak{L}(X,Y))$ seems to be the generalization of a folklore result: the correspondence between bilinear forms and operators; see 5.7.1.5. In a purely algebraic setting, this isomorphy is stated in the very first proposition of Bourbaki's *Algèbre multilinéaire*; [BOU$_{2b}$, p. 2].

5.1.9.10 *Différentielles d'ordre supérieur* were introduced by Gâteaux in his theory of analytic functions; [1919, pp. 83, 89], [1922, pp. 19–20]; see also Fréchet [1925b, p. 321]. Here is a quotation from Gâteaux's first paper:

$$\delta^n F(z) = \left[\frac{d^n}{d\lambda^n} F(z+\lambda\,\delta z)\right]_{\lambda=0}$$

est une fonction holomorphe de z et de δz. *De plus, c'est une forme de degré n en* δz.

La série de Maclaurin se généralise par la série

$$F(z) = F(\mathrm{o}) + \delta F(\mathrm{o}, z) + \cdots + \tfrac{1}{n!}\delta^n F(\mathrm{o}, z) + \cdots,$$

where $\delta^n F(\mathrm{o}, z)$ denotes the value of $\delta^n F(\mathrm{o})$ at the increment z.

5.1.9.11 Graves [1927, p. 173] extended Taylor's formula to functions $\boldsymbol{f}$ from an open subset of a Banach space X into a Banach space Y. Since in this case there cannot hold any *mean value theorem*, the remainder must be given by an integral. This was the reason why Graves developed a theory of Riemann integration. Here is his final result:

$$\boldsymbol{f}(x+h) = \boldsymbol{f}(x) + \sum_{k=1}^{n} \frac{1}{k!}\delta^k \boldsymbol{f}(x,h) + \int_0^1 \frac{(1-t)^n}{n!}\delta^{n+1}\boldsymbol{f}(x+th,h)\,dt.$$

For simplicity, one may assume that $\boldsymbol{f}$ has Gâteaux derivatives up to order $n+1$ such that $\delta^{n+1}\boldsymbol{f}(x+th,h)$ depends continuously on $0 \le t \le 1$. It would be enough to know the values of $\boldsymbol{f}$ on the segment $\{x+th : 0 \le t \le 1\}$. Hence the theorem is actually concerned with Y-valued functions on the interval $[0,1]$. A modern presentation can be found in [DIEU, Section 8.14].

5.1.9.12 Readable accounts of the history of abstract polynomials were given by Taylor [1970•, pp. 327–332], [1982/87•, Part III, pp. 36–40], and by [DINE$_1$, p. 49], [DINE$_2$, pp. 73–81].

5.1.10 Analytic functions on Banach spaces

Let $\mathscr{H}(G)$ denote the collection of all analytic (holomorphic) complex-valued functions on an open set G of $\mathbb{C}$. This linear space carries a most natural topology: the topology of uniform convergence on the compact subsets of G. In this way, $\mathscr{H}(G)$ becomes a locally convex linear space, but not a Banach space. Due to this fact, the theory of analytic functions with abstract arguments has likewise been developed in the locally convex setting. However, since this text is concerned with Banach spaces, the following presentation will be restricted to this case.

5.1.10.1 A function $\boldsymbol{f}$ from an open subset G of a complex Banach space X into a complex Banach space Y is said to be **Fréchet analytic** or, simply, **analytic** if it has a Fréchet derivative at every point of G.

The concept of **Gâteaux analyticity** is obtained analogously by assuming that the limit

$$\delta\boldsymbol{f}(x)h := \lim_{\lambda\to 0}\frac{\boldsymbol{f}(x+\lambda h) - \boldsymbol{f}(x)}{\lambda}$$

exists for all $x, h \in X$. The parameter λ is complex. In contrast to the definition of the (real) Gâteaux derivative, one does not require that $\delta\boldsymbol{f}(x)h$ be continuous and linear with respect to the variable h. In fact, Taylor [1937, p. 283] discovered that linearity follows automatically; see also Zorn [1945a, p. 587] and [HIL, p. 73].

Trivial examples show that continuity may fail:

Let ℓ be any non-continuous linear form on X. Then

$$\delta^n \mathrm{e}^{\ell(x)}(h_1,\dots,h_n) = \mathrm{e}^{\ell(x)}\ell(h_1)\cdots\ell(h_n)$$

or more simply, $\delta\ell(o)h = \ell(h)$.

According to Zorn [1945a, p. 585], such functions *are to be considered as freaks, counter examples rather than examples.*

5.1.10.2 Without any proof or reference, Graves [1935, pp. 651–653] sketched a theory of analytic functions. In particular, he stated the following equivalence, which was independently found and proved by Taylor [1937, p. 285]:

Fréchet analyticity $\Leftrightarrow$ *Gâteaux analyticity plus continuity*.

On the right-hand side, the requirement of continuity can be replaced by the weaker condition of **local boundedness**: the function is bounded in some neighborhood of every point. Zorn [1945b, p. 579] claimed that this result is taken from Hille's Colloquium Lectures *Topics in the Theory of Semigroups* (1944). Since such a publication of Hille never appeared, it seems likely that Zorn meant a preliminary version of [HIL, p. 81], where the property in question was used in the definition of analyticity.

The equivalence above can be viewed as a generalization of Hartogs's theorem [1906].

5.1.10.3 A **power series** is an expression

$$\boldsymbol{P}_0 + \boldsymbol{P}_1(x) + \cdots + \boldsymbol{P}_n(x) + \cdots \tag{5.1.10.3.a}$$

built from homogeneous polynomials $\boldsymbol{P}_n : X \to Y$ of degree n. More precisely, one uses the term **Fréchet** power series or **Gâteaux** power series according as all polynomials $\boldsymbol{P}_n$ are continuous or not.

The convergence is commonly taken pointwise and with respect to the norm topology of Y. However, it can be shown that whenever (5.1.10.3.a) converges weakly for all x in an open subset, then it even converges in norm; see [HIL, p. 79].

5.1.10.4 The following result shows that the classical Weierstrass approach to the theory of analytic functions also works in the infinite-dimensional setting.

Taylor [1937, p. 284]:

If $\boldsymbol{f}(x)$ is analytic in the region defined by $\|x-x_0\|<\rho$, it may be expanded in the form

$$\boldsymbol{f}(x) = \boldsymbol{f}(x_0) + \delta\boldsymbol{f}(x_0, x-x_0) + \cdots + \tfrac{1}{n!}\delta^n\boldsymbol{f}(x_0, x-x_0) + \cdots \ .$$

This series converges uniformly in every compact subset K extracted from the sphere $\|x-x_0\| \le \theta\rho$, where θ, $0<\theta<1$, is arbitrary. Moreover, the series

$$\|\boldsymbol{f}(x_0)\| + \|\delta\boldsymbol{f}(x_0, x-x_0)\| + \cdots$$

converges uniformly in K.

5.1.10.5 Taylor [1937, p. 287] also defined the **radius of convergence** of a power series. But he added at the end of his paper that this concept *is not as important here as in the classical theory, for the region of convergence of an abstract power series is not necessarily that defined by an inequality* $\|x\| < \rho$.

5.1.10.6 The **radius of uniform convergence** of a power series

$$\boldsymbol{P}_0 + \boldsymbol{P}_1(x - x_0) + \cdots + \boldsymbol{P}_n(x - x_0) + \cdots \tag{5.1.10.6.a}$$

about a point x_0 is the supremum $r(x_0)$ of all $r \geq 0$ such that (5.1.10.6.a) converges uniformly in the ball $\{x \in X : \|x - x_0\| < r\}$.

The definition above is usually attributed to Nachbin [NACH, p. 11], who also generalized Hadamard's formula:

$$r(x_0) = \frac{1}{\limsup\limits_{n\to\infty} \|\boldsymbol{P}_n\|^{1/n}}.$$

Note, however, that the right-hand expression had already been used by Martin [1932, p. 58]. He referred to this quantity as "radius of analyticity."

5.1.10.7 Let $\boldsymbol{f}$ be an analytic function on an open subset G of a Banach space X. According to Nachbin [NACH, p. 26], the **radius of boundedness** of $\boldsymbol{f}$ at a point $x_0 \in G$ is the supremum $r_b(x_0)$ of all $r > 0$ such that $\|\boldsymbol{f}(x)\|$ is bounded for $\|x - x_0\| < r$. Moreover, he proved that $r_b(x_0)$ is the minimum of the radius of uniform convergence of the Taylor series of $\boldsymbol{f}$ at x_0 and the distance of x_0 to the boundary of G.

5.1.10.8 The complex-valued function

$$\boldsymbol{f}(x) := \sum_{n=1}^{\infty} \langle x, x_n^* \rangle^n \quad \text{for all } x \in X$$

is analytic on X if and only if (x_n^*) is a weak* zero sequence in X^*. Taylor [1938b, pp. 474–475] had earlier considered a special example of this kind:

$$\boldsymbol{f}_0(x) := \sum_{n=1}^{\infty} (\xi_n)^n \quad \text{for } x = (\xi_n) \in l_p \text{ and } 1 \leq p < \infty. \tag{5.1.10.8.a}$$

He observed that B_{l_p} is the largest ball centered at o in which the series (5.1.10.8.a) converges uniformly, whereas we have pointwise convergence in the whole space X. In other words, the radius of uniform convergence may be smaller than the radius of convergence.

5.1.10.9 The **Josefson–Nissenzweig theorem** says that in every infinite-dimensional Banach space, we can find a weak* zero sequence such that $\|x_n^*\| = 1$; see Josefson [1975], Nissenzweig [1975], and [DIE$_2$, Chap. XII].

Consequently, Taylor's phenomenon described above occurs in every infinite-dimensional Banach space; see [DINE$_1$, p. 169].

5.1.10.10 A subset B of a Banach space X is **bounding** if every analytic function $f : X \to \mathbb{C}$ is bounded on B.

Since every $x^* \in X^*$ is an analytic function on X, the principle of uniform boundedness implies that bounding sets are bounded.

Of course, every relatively compact subset is bounding. Dineen [1972, p. 463] showed that the converse holds in every Banach space X for which the closed unit ball of X^* is weakly* sequentially compact.

The special case of a Hilbert space had already been treated by Alexander [1968], who also introduced the concept of a bounding subset. Schottenloher [1976, p. 210] gave a simple proof for separable Banach spaces.

An Asplund space is defined by the property that every continuous convex function on X is Fréchet differentiable in a dense G_δ set; see 5.1.4.12. Replacing Fréchet differentiability by Gâteaux differentiability yields the larger class of **weak Asplund spaces**; see Asplund [1968, p. 31] and [FAB]. A theorem, commonly attributed to Mazur [1933, pp. 76–77], says that all separable spaces satisfy the condition above. Note that the term "weak" does not refer to any weak topology; it means only that the property in question is weakened: one uses *weak* differentiability instead of *strong* differentiability.

Weak Asplund spaces are useful in the present context, since the closed unit balls of their duals are weakly* sequentially compact; see Stegall [1981, p. 517].

The preceding results suggest l_∞ as a favorite candidate of a Banach space that may contain a bounding subset that fails to be relatively compact. Indeed, according to Dineen [1971, p. 64], the set of all unit sequences has the desired properties. By the way, this example implies the well-known fact that c_0 cannot be complemented in l_∞.

A characterization of Banach spaces in which all bounding subsets are relatively compact is still unknown.

The complicated nature of the concept under consideration is indicated by the fact that Josefson [2000] was able to construct a bounding set whose convex hull fails to be bounding.

5.1.10.11 The theory of analytic functions on abstract spaces is a descendant of classical complex analysis in the sense of Cauchy and Weierstrass. If we regard functional analysis as infinite-dimensional analysis, then such a generalization of analyticity becomes a must.

Hilbert [1909, pp. 66–67]:

Um allgemeinere transzendente Probleme mittels der Methode der unendlichvielen Variablen in Angriff zu nehmen, ist es zuvor notwendig, die Theorie der Funktionen unendlichvieler Variablen weiter zu entwickeln und dies geschieht vor Allem durch Einführung des Begriffes der **analytischen Funktion** *von unendlichvielen Variablen.*
...

Wir verstehen unter einer Potenzreihe der unendlichvielen Variablen einen Ausdruck von der Gestalt

$$\mathfrak{P}(x_1,x_2,\ldots) = c + \sum_{(p)} c_p x_p + \sum_{(p,q)} c_{pq} x_p x_q + \sum_{(p,q,r)} c_{pqr} x_p x_q x_r + \cdots$$

wo $c, c_p, c_{pq}, c_{pqr}, \ldots$ *irgendwelche gegebene reelle oder komplexe Grössen und* $x_1, x_2, x_3, \ldots$ *die Variablen bedeuten.*

The outcome is impressive. However, in contrast to the case of operator-valued functions of a complex variable, the theory was mainly growing in its own right. To the best of my knowledge, the feedback to the rest of mathematics has been moderate. Here are some examples:

An n-dimensional Banach space X is isometric to l_1^n if and only if there exists on X a homogeneous polynomial $\boldsymbol{P}$ of degree n for which the right-hand inequality in (5.1.9.8.a) becomes an equality: $\|\boldsymbol{L}\| = \frac{n^n}{n!}\|\boldsymbol{P}\|$; see Sarantopoulos [1987, p. 345] and [DINE$_2$, p. 46].

The Josefson–Nissenzweig theorem 5.1.10.9 was partly initiated by questions from infinite-dimensional analyticity; see Josefson [1975, p. 79].

The function $T \mapsto \det(I+T)$ is analytic on the trace class $\mathfrak{S}_1(H)$. But I do not know any consequence of this fact; what one really uses is the analyticity of the function $\zeta \mapsto \det(I-\zeta T)$; see Subsection 6.5.2.

5.2 Spectral theory

5.2.1 Operational calculus

The following considerations could be carried out in an arbitrary Banach algebra with unit; see [BONS$^+$, pp. 27–38]. However, we concentrate our attention on the most interesting case: $\mathfrak{L}(X)$, where X is a complex Banach space.

5.2.1.1 The **resolvent set** $\rho(T)$ of $T \in \mathfrak{L}(X)$ consists of all complex scalars λ for which the **resolvent** $R(\lambda,T) := (\lambda I - T)^{-1} \in \mathfrak{L}(X)$ exists.

The **spectrum** $\sigma(T)$ is the complement of $\rho(T)$.

For operators on l_2, Riesz had already observed that the resolvent $R(\lambda,T)$ is an analytic function (in the sense of Weierstrass) on the open set $\rho(T)$. This fact was reformulated in [STONE, p. 141] by passing to the complex functions $\big(R(\lambda,T)f|g\big)$ with arbitrary elements f ad g from the underlying Hilbert space H. Applying Liouville's theorem, Stone proved an impressive result, [STONE, p. 149]:

The spectrum of a bounded linear transformation T with domain $\mathfrak{H}$ *contains at least one point.*

Finally, Taylor discovered how Stone's reasoning can be extended to complex Banach spaces. Among others he obtained the expansion

$$R(\lambda,T)=\sum_{n=0}^{\infty}R(\lambda_0,T)^{n+1}(\lambda-\lambda_0)^n \quad \text{for } |\lambda_0-\lambda|\,\|R(\lambda_0,T)\|<1$$

and the estimate

$$\|R(\lambda,T)\|\le\frac{1}{|\lambda|-\|T\|} \quad \text{for } |\lambda|>\|T\|;$$

see Taylor [1938c, pp. 73–74] or [TAY$^+$, pp. 272–278].

Taylor was not aware, in 1937, of the work of Riesz; see Taylor [1971$^\bullet$, p. 333]. Otherwise, he would have seen *le développement uniformément convergent*

$$A_\zeta=A_{\zeta_0}+(\zeta-\zeta_0)A_{\zeta_0}^2+(\zeta-\zeta_0)^2A_{\zeta_0}^3+\cdots\ ,$$

which was established in [RIE, p. 117] for $A_\zeta:=A(I-\zeta A)^{-1}$; see 2.6.5.2.

5.2.1.2 At the Stanford meeting of the American Mathematical Society on April 24, 1943, Dunford and Taylor presented independently an **operational** or **functional calculus**, which was also foreshadowed in [RIE, p. 117]:

On peut aller plus loin en appliquant à l'étude de A_ζ les différentes méthodes de la Théorie des fonctions; en particulier, on pourra y appliquer le **calcul des résidus**.

Let $\mathscr{F}(T)$ denote the collection of all functions f that are analytic on some neighborhood of $\sigma(T)$. This neighborhood need not be connected and can depend on f. Put

$$f(T):=\frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}}f(\lambda)R(\lambda,T)d\lambda \tag{5.2.1.2.a}$$

or, more suggestively,

$$f(T):=\frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}}\frac{f(\lambda)}{\lambda-T}d\lambda.$$

Here $\mathscr{C}$ consists of a finite number of rectifiable Jordan curves contained in the domain of f and surrounding the spectrum $\sigma(T)$ in the positive sense; see Dunford [1943b, p. 193], Taylor [1943, p. 661], [DUN$_1^+$, p. 568], and [TAY$^+$, p. 312].

According to Dunford [1943b, p. 194] and Taylor [1943, p. 662], the mapping $f\mapsto f(T)$ is a ring homomorphism from $\mathscr{F}(T)$ into $\mathfrak{L}(X)$.

Dunford's spectral mapping theorem [1943b, p. 195] says that

$$f\big(\sigma(T)\big)=\sigma\big(f(T)\big).$$

5.2.1.3 Dunford [1943b, p. 190] knew that Gelfand [1941, pp. 20–23] had already developed an operational calculus. However, he claimed that formula (5.2.1.2.a) *was used only in an incidental way*. This statement is wrong. In fact, Gelfand took a more general point of view that made his approach less transparent.

Let $\mathscr{A}$ be any Banach algebra. Note that $\mathscr{G} := \{x \in \mathscr{A} : \sigma(x) \subset G\}$ is an open subset of $\mathscr{A}$ for every open subset G of $\mathbb{C}$. With every scalar-valued analytic function $f : \lambda \mapsto f(\lambda)$ on G one may associate an analytic $\mathscr{A}$-valued function on $\mathscr{G}$,

$$\boldsymbol{f}(x) := \frac{1}{2\pi \mathrm{i}} \oint_{\mathscr{C}} f(\lambda)(\lambda - x)^{-1} d\lambda,$$

where the curve $\mathscr{C}$ is chosen as above. In this way, Gelfand obtained *eine isomorphe Abbildung* $f \mapsto \boldsymbol{f}$.

In [1941, p. 20, footnote [6]], he said:

Die Anwendung des Integrals von Cauchy in ähnlichen Fragen (*Untersuchung von Operatoren*) *findet sich schon bei F. Riesz ....*

Gelfand's spectral mapping theorem [1941, p. 23] reads as follows:

Wenn $f(\lambda)$ *in dem Bereich der Werte der Funktion* $x(M)$ *regulär ist, so existiert im Ring ein solches Element y, dass* $y(M) = f\big(x(M)\big)$ *für alle maximalen Ideale M ist.*
See 4.10.2.5 concerning the meaning of $x(M)$ and $y(M)$.

5.2.1.4 Contour integrals were also used by Lorch [1942, p. 238] when he dealt with the question of reducibility: *find all projections P which commute with T*.

The fundamental [now: **spectral**] **projection** *is the integral*

$$P = \frac{1}{2\pi \mathrm{i}} \oint_{\mathscr{C}} \frac{d\lambda}{\lambda I - T}$$

evaluated over a simple closed contour lying entirely in the resolvent set of T.

The special case in which $\mathscr{C}$ surrounds an isolated point of $\sigma(T)$ was already considered by Nagumo [1936, p. 80].

Lorch [1942, p. 239] also introduced the following concept:

A **spectral set** *is any set obtained from the interiors of curves* $\mathscr{C}$ *by the operations of complementation, addition, and intersection performed a finite number of times.*

In the hands of Dunford [1943b, p. 196] the definition above took the following form:

By a **spectral set** *of T will be meant any subset of* $\sigma(T)$ *which is both open and closed in* $\sigma(T)$.

The map that assigns to every spectral set the corresponding projection is a finitely additive operator-valued measure on the Boolean algebra $\mathscr{C}_{\mathrm{lop}}\big(\sigma(T)\big)$. Unfortunately, *the attempt to extend this homomorphism so as to admit of denumerable addition and intersection has to be abandoned;* see Lorch [1942, p. 239].

Dunford's reaction to this "SOS" will be described in Subsection 5.2.5.

5.2.1.5 A particular kind of contour integral for operator-valued functions was used by Hille [1939, p. 3] in his theory of analytic semi-groups.

5.2.1.6 The basics of spectral theory and operational calculus are presented in several classical monographs: [HIL], [RIE$^+$, Chap. XI], [DUN$_1^+$, Chap. VII], [TAY], and [LOR, Chap. IV].

5.2.2 Fredholm operators

5.2.2.1 With every operator $T \in \mathfrak{L}(X,Y)$ we associate its **null space** and its **range**:

$$N(T) := \{x : Tx = \mathrm{o}\} \quad \text{and} \quad M(T) := \{Tx : x \in X\}.$$

Letting $X_0 := X/N(T)$ and $Y_0 := \overline{M(T)}$ yields the canonical factorization

$$T : X \xrightarrow{Q} X_0 \xrightarrow{T_0} Y_0 \xrightarrow{J} Y\,,$$

where T_0 is a one-to-one operator with dense range. We refer to T as a **homomorphism** if T_0 is an isomorphism from X_0 onto Y_0. By the bounded inverse theorem 2.5.2, this happens if and only if $M(T)$ is closed.

5.2.2.2 According to Hausdorff [1932, p. 307], the equation $Tx = y$ is said to be **normally solvable** if the following criterion holds:

For $y \in Y$ there exists a solution $x \in X$ of $Tx = y$ if and only if $\langle y, y^*\rangle = 0$ whenever $T^*y^* = \mathrm{o}$. In shorthand: $M(T) = N(T^*)^\perp$.

The bipolar theorem tells us that normal solvability is equivalent to the closedness of $M(T)$; see 3.3.3.5. This observation shows why the theory of operator equations was the historical source of duality theory.

Looking at the dual equation $T^*y^* = x^*$, we have two different concepts of normal solvability. Indeed, it may be required that either $\langle x^*, x\rangle = 0$ for $x \in N(T)$ or $\langle x^*, x^{**}\rangle = 0$ for $x^{**} \in N(T^{**})$. Luckily enough, both concepts coincide. But this is one of Banach's deepest results, which will be discussed next.

5.2.2.3 The desired conclusion is contained in the following supplement to the bounded inverse theorem and its relatives; see 2.5.2.

Closed range theorem: For $T \in \mathfrak{L}(X,Y)$, the following are equivalent.

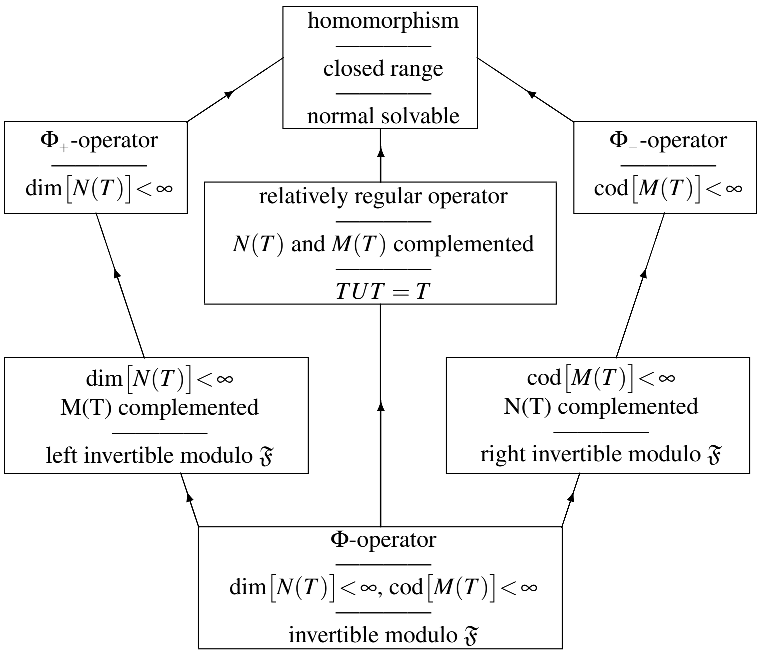
(1) $M(T)$ is strongly closed in Y.
(2) $M(T)$ is weakly closed in Y.
(1*) $M(T^*)$ is strongly closed in X^*.
(2*) $M(T^*)$ is weakly closed in X^*.
(3*) $M(T^*)$ is weakly* closed in X^*.

Note that *strongly closed* means closed with respect to the norm topology.

The crucial step is the implication (1^*)⇒(3^*). In general, there are strongly closed subspaces of X^* that fail to be weakly* closed. Take, for example, the subspace c_0 of $l_\infty = l_1^*$. But for ranges this cannot happen. Indeed, consider a bounded net x_α^* in $M(T^*)$ weakly* converging to $x^* \in X^*$. Since T^* is a homomorphism, there exists a bounded net in Y^* with $x_\alpha^* = T^* y_\alpha^*$. By the weak* compactness theorem, (y_α^*) has a weak* cluster point $y^* \in Y^*$. Hence $x^* = T^* y^* \in M(T^*)$, which proves that $M(T^*)$ is closed in the bounded weak* topology or $\mathscr{K}$-closed; see 3.3.4.3. Thus $M(T^*)$ is weakly* closed, by 3.6.7.

This is a modern version of Banach's original proof, [1929, Part II, pp. 236–237], [BAN, p. 146], which was still used in [DUN_1^+, p. 488]. Nowadays, one can do better: [RUD_1, p. 96] and [TAY^+, pp. 235–236]. To the best of my knowledge, this significant simplification is due to Taylor/Halberg [1957, p. 97].

5.2.2.4 In the rest of this subsection, we deal with special classes of homomorphisms:



5.2.2.5 For elements of a Banach algebra, the concept of *relative regularity* was introduced by Rickart [1946, p. 536]. Following this idea, Atkinson [1953, p. 39] referred to $T \in \mathfrak{L}(X,Y)$ as **relatively regular** if there exists an operator $U \in \mathfrak{L}(Y,X)$ for which $TUT = T$. He observed that such operators are homomorphisms. Only later, was a "geometric" characterization found; Pietsch [1960, p. 349]:

A homomorphism $T \in \mathfrak{L}(X,Y)$ is relatively regular if and only if its null space and its range are complemented. The required projections are given by $P := UT$ and $Q := TU$.

5.2.2.6 Let $\mathfrak{A}$ be any operator ideal. Then $T \in \mathfrak{L}(X,Y)$ is said to be **invertible modulo** $\mathfrak{A}$ if there exist $U,V \in \mathfrak{L}(Y,X)$ as well as $A \in \mathfrak{A}(X)$ and $B \in \mathfrak{A}(Y)$ such that

$$\overbrace{UT = I_X - A}^{\text{left invertible}} \quad \text{and} \quad \overbrace{TV = I_Y - B}^{\text{right invertible}}.$$

Invertibility modulo $\mathfrak{A}$ is not changed by passing to the closed hull $\overline{\mathfrak{A}}$. The Riesz theory implies that $\mathfrak{F}$ can even be replaced by $\mathfrak{K}$, which is larger than $\overline{\mathfrak{F}}$.

Calkin [1941, p. 849] was the first to consider quotients $\mathfrak{L}(H)/\mathfrak{A}(H)$, where $\mathfrak{A}(H)$ is an ideal in the ring $\mathfrak{L}(H)$. In particular, he dealt with $\mathfrak{L}(H)/\mathfrak{K}(H)$, which is now referred to as the **Calkin algebra**.

5.2.2.7 Homomorphisms $T : X \to Y$ for which either $\dim\big[N(T)\big]$ or $\operatorname{cod}\big[M(T)\big]$ is finite were already considered by Dieudonné [1943]. Nowadays, these operators are called **semi-Fredholm**. In this text, I mostly use the Russian terminology: **Φ_+- and Φ_--operators**, respectively.

A **Fredholm operator**, or just a **Φ-operator**, is one such that $\dim\big[N(T)\big]$ and $\operatorname{cod}\big[M(T)\big]$ are finite. In shorthand,

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

The quantity

$$\operatorname{ind}(T) := \overbrace{\dim\big[N(T)\big]}^{\text{nullity}} - \overbrace{\operatorname{cod}\big[M(T)\big]}^{\text{deficiency}}$$

is called the **index** of T. The term "index" was already used by Noether [1921, p. 55].

5.2.2.8 In the definition of semi-Fredholm operators, it is tacitly assumed that they have closed ranges. However, Kato [1958, p. 275] discovered that in the case of Φ_--operators, this property automatically follows from $\operatorname{cod}\big[M(T)\big] < \infty$.

5.2.2.9 In the setting of Hilbert spaces, operators that admit a left-inverse modulo $\mathfrak{K}$ were considered for the first time by Mikhlin [1948, стр. 45]:

Если A – произвольный линейный оператор и существует такой ограниченный линейный оператор M, что $MA = E + T$, где E – тождественный, а T – вполне непрерывный оператор, то мы будем говорить, что оператор A допускает регуляризацию, и будем называть оператор M регуляризирующим.

Engl. transl.: *Let A be an arbitrary linear operator. Suppose that there exists a bounded operator M such that $MA = E + T$, where E is the identity map and T is a completely continuous* [read: compact] *operator. Then we say that A admits a regularization, and M is called regularizing.*

He also proved that operators *admitting a regularization* are normally solvable; Mikhlin [1948, стр. 50].

In 1948, living in Nigeria, the English mathematician Atkinson submitted a paper [1951] to the Russian journal Математический Сборник in which he created a theory of generalized (his terminology) Fredholm operators. His basic criterion characterizes such operators by the fact that they are invertible modulo $\mathfrak{F}$. Another fundamental result is the product formula for the index; Atkinson [1951, стр. 8]:

If $T \in \Phi(X,Y)$ and $S \in \Phi(Y,Z)$, then $ST \in \Phi(X,Z)$ and

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

A little bit later and independently, Yood developed similar ideas. He proved [1951, p. 609] that $T \in \mathfrak{L}(X,Y)$ is a relatively regular $\Phi_{\pm}$-operator if and only if it admits a $\substack{\text{left}\\ \text{right}}$-inverse modulo $\mathfrak{F}$.

5.2.2.10 The following duality results are immediate consequences of the closed range theorem, $N(T) = M(T^*)^{\perp}$ and $N(T^*) = M(T)^{\perp}$, as well as the algebraic characterizations stated above:

$X \xrightarrow{T} Y$		$X^* \xleftarrow{T^*} Y^*$
homomorphism	$\Leftrightarrow$	homomorphism
relatively regular	$\substack{\Rightarrow\\ \not\Leftarrow}$	relatively regular
Φ_+-operator	$\Leftrightarrow$	Φ_--operator
Φ_--operator	$\Leftrightarrow$	Φ_+-operator
relatively regular Φ_+-operator	$\substack{\Rightarrow\\ \not\Leftarrow}$	relatively regular Φ_--operator
relatively regular Φ_--operator	$\substack{\Rightarrow\\ \not\Leftarrow}$	relatively regular Φ_+-operator
Φ-operator	$\Leftrightarrow$	Φ-operator

Counterexamples are provided by the canonical embedding from c_0 into l_∞ and any surjection from l_1 onto $L_1[0,1]$. These operators fail to be relatively regular, but their duals are.

5.2.2.11 Perturbation theory deals with the behavior of specific properties when the underlying objects are changed in a controlled way. The basic idea is to replace complicated objects by simpler ones such that the difference is "small."

In the setting of operators we know two typical kinds of *smallness*: an operator may have a small norm or it may belong to a small ideal.

Dieudonné [1943, pp. 76–78] pointed out that the zero operator of an infinite-dimensional space behaves very badly under perturbations, though it is a (relatively regular) homomorphism. This was the reason why he posed additional assumptions:

Soit u un homomorphisme de E dans F, tel que $u^{-1}(O)$ ait un nombre fini p de dimensions. Si w est une application linéaire continue de E dans F, de norme assez petite, $v = u + w$ est un homomorphisme de E dans F et $v^{-1}(O)$ a au plus p dimensions.

Soit u un homomorphisme de E dans F, tel que l'espace quotient $F/u(E)$ ait un nombre fini p de dimensions. Si w est une application linéaire continue de E dans F, de norme assez petite, $v = u + w$ est un homomorphisme de E dans F et $F/v(E)$ a au plus p dimensions.

The preceding theorems show that Dieudonné was the first to deal with $\Phi_{\pm}$-operators. Next, Yood [1951, p. 602] proved that these properties, which he called *property A* and *property B*, are preserved upon addition of a compact operator: if $T \in \Phi_{\pm}(X,Y)$ and $K \in \mathfrak{K}(X,Y)$, then $T+K \in \Phi_{\pm}(X,Y)$ and $\mathrm{ind}(T+K) = \mathrm{ind}(T)$. In the case of Φ-operators, this result was independently obtained by Atkinson [1951, стр. 9] and Gohberg [1951]. Using the concept of invertibility modulo $\mathfrak{K}$ reduces many proofs to simple algebraic manipulations.

Atkinson [1951, стр. 10] also observed that the index is stable under small perturbations:

For $T_0 \in \Phi(X,Y)$ there exists a $\rho > 0$ such that $T \in \Phi(X,Y)$ and $\mathrm{ind}(T) = \mathrm{ind}(T_0)$ if $T \in \mathfrak{L}(X,Y)$ and $\|T - T_0\| \le \rho$.

The first summary of the theory of Fredholm operators and related topics was given by Gohberg/Kreĭn [1957]. The standard monograph on perturbation theory is [KATO].

5.2.2.12 The following formula was independently discovered by many people when they computed the index of elliptic differential operators: Calderón [1967, p. 1194], Fedosov [1970, стр. 83], and Hörmander [1969, p. 137].

Let $T \in \mathfrak{L}(X,Y)$, $U \in \mathfrak{L}(Y,X)$, $A \in \mathfrak{F}(X)$, and $B \in \mathfrak{F}(Y)$ be such that

$$UT = I_X - A \quad \text{and} \quad TU = I_Y - B.$$

Then

$$\mathrm{ind}(T) = \mathrm{trace}(A) - \mathrm{trace}(B). \tag{5.2.2.12.a}$$

In order to illuminate the large flexibility of traces as a tool in Banach space theory, I sketch a proof of this fact.

First of all, we choose $V \in \mathfrak{L}(Y,X)$ as well as projections $P \in \mathfrak{F}(X)$ and $Q \in \mathfrak{F}(Y)$ such that

$$VT = I_X - P, \quad M(P) = N(T), \quad \text{and} \quad TV = I_Y - Q, \quad N(Q) = M(T).$$

In this case, the formula

$$\mathrm{ind}(T) = \dim\big[N(T)\big] - \mathrm{cod}\big[M(T)\big] = \mathrm{rank}(P) - \mathrm{rank}(Q)$$

is obvious. The rest follows by algebraic manipulations. Indeed, we may conclude from $U - PU = VTU = V - VB$ that $U - V = PU - VB$ has finite rank. Hence

$$\mathrm{trace}(P-A) = \mathrm{trace}\big[(U-V)T\big] = \mathrm{trace}\big[T(U-V)\big] = \mathrm{trace}(Q-B).$$

Applying (5.2.2.12.a) yields an elegant proof of $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$.

Calderón [1967, p. 1194], [HÖR, p. 188], and Pietsch [1982a] discovered that the index-trace formula can be generalized as follows:

Let $\mathfrak{A}$ be any operator ideal whose n^{th} power has a trace; see 6.5.1.1. If

$$UT = I_X - A \quad \text{and} \quad TU = I_Y - B$$

with $A \in \mathfrak{A}(X)$ and $B \in \mathfrak{A}(Y)$, then

$$\operatorname{ind}(T) = \operatorname{trace}(A^n) - \operatorname{trace}(B^n).$$

5.2.2.13 The concept of a Fredholm operator goes back to Noether [1921], who considered *singuläre Integralgleichungen* of the form

$$a(s)f(s) - \frac{b(s)}{2\pi}\int_0^{2\pi} \cot\Big(\frac{s-t}{2}\Big) f(t)\,dt + \frac{1}{2\pi}\int_0^{2\pi} K(s,t)f(t)\,dt = g(s).$$

The functions a and b are supposed to be 2π-periodic and continuous, K is a Hilbert–Schmidt kernel.

The singularity is isolated in the **periodic Hilbert transform**

$$H_{\text{ilb}}^{2\pi} : f(t) \mapsto \frac{1}{2\pi}\int_0^{2\pi} \cot\Big(\frac{s-t}{2}\Big) f(t)\,dt,$$

the integral being taken in the sense of Cauchy's principal value. We have

$$(H_{\text{ilb}}^{2\pi})^2 = -I + P_0, \qquad (5.2.2.13.\text{a})$$

where

$$P_0 : f(t) \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt \quad \text{(constant function)}$$

is a 1-dimensional projection.

With the help of the multiplication operators

$$A : f(s) \mapsto a(s)f(s), \qquad B : f(s) \mapsto b(s)f(s)$$

and the compact perturbation

$$K : f(t) \mapsto \frac{1}{2\pi}\int_0^{2\pi} K(s,t)f(t)\,dt,$$

the equation above can be written in shorthand:

$$Af - BH_{\text{ilb}}^{2\pi} f + Kf = g.$$

Using the **Riesz projection** $P := \frac{1}{2}(\mathrm{i}H_{\text{ilb}}^{2\pi} + I + P_0)$ yields another (quite handy) version,

$$A - BH_{\text{ilb}}^{2\pi} + \mathrm{i}BP_0 = (A + \mathrm{i}B)P + (A - \mathrm{i}B)(I - P),$$

which is due to Gohberg [1964, стр. 121–122].

In the following, $\equiv$ denotes equality modulo compact operators. This means that we work in the Calkin algebra $\mathfrak{L}(X)/\mathfrak{K}(X)$. Then (5.2.2.13.a) passes into

$$(H_{\text{ilb}}^{2\pi})^2 \equiv -I. \qquad (5.2.2.13.\text{b})$$

We also have

$$AH_{\mathrm{ilb}}^{2\pi} \equiv H_{\mathrm{ilb}}^{2\pi}A \quad \text{and} \quad BH_{\mathrm{ilb}}^{2\pi} \equiv H_{\mathrm{ilb}}^{2\pi}B. \tag{5.2.2.13.c}$$

Hence

$$(A - BH_{\mathrm{ilb}}^{2\pi})(A + BH_{\mathrm{ilb}}^{2\pi}) \equiv A^2 + B^2. \tag{5.2.2.13.d}$$

This formula was implicitly proved in the classical paper of Noether [1921, pp. 44–46], who also discovered (5.2.2.13.c); see Mikhlin [1948, стр. 39, 44] for explicit statements.

Finally, it follows from (5.2.2.13.d) that $A - BH_{\mathrm{ilb}}^{2\pi} + K$ is a Fredholm operator whenever

$$a(s)^2 + b(s)^2 = \big(a(s) + \mathrm{i}b(s)\big)\big(a(s) - \mathrm{i}b(s)\big) \neq 0 \quad \text{for } 0 \leq s < 2\pi.$$

According to Noether [1921, p. 55], the index of $A - BH_{\mathrm{ilb}}^{2\pi} + K$ coincides with the index (winding number) of the function

$$\frac{a(s) - \mathrm{i}b(s)}{a(s) + \mathrm{i}b(s)}.$$

Actually, his formula looks simpler, since he dealt only with real functions a and b.

Though Noether did not specify the underlying space, it is clear that he meant $L_2(\mathbb{T})$ defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. His followers extended the consideration to $L_p(\mathbb{T})$ with $1 < p < \infty$ and to spaces of 2π-periodic Lipschitz–Hölder continuous functions. More information about the Hilbert transform will be given in 6.1.10.3.

I recommend the following well-known books on singular integral operators, which contain valuable historical comments: [GOH$_4^+$], [MICH$^+$], [PRÖ]. The reader may also consult Dieudonné's short note [1985$^\bullet$].

5.2.3 Riesz operators

5.2.3.1 In my opinion, the Riesz–Schauder spectral theory of compact operators is the absolute highlight of functional analysis in non-Hilbert spaces. Therefore it was quite natural to ask whether the same results could be obtained for larger classes of operators. In a first step, Nikolskiĭ [1936] discovered that one may use operators $T \in \mathfrak{L}(X)$ with some compact power.

5.2.3.2 **Riesz operators** $T \in \mathfrak{L}(X)$ are defined by the following equivalent properties, which summarize the main features of the famous Riesz theory of compact operators.

(1) Every $I - \zeta T$ with $\zeta \in \mathbb{C}$ is a Fredholm operator.

(2) Every $I - \zeta T$ with $\zeta \in \mathbb{C}$ is a Fredholm operator that has finite ascent and finite descent.

(3) The canonical image of T in the Calkin algebra is quasi-nilpotent.

(4) For every $\varepsilon > 0$ there exists a natural n such that $T^n(B_X)$ can be covered by a finite number of ε^n-balls.

(5) The resolvent $(I-\zeta T)^{-1}$ is a meromorphic function on the complex plane for which the singular part of the Laurent expansion at every pole ζ_0 has finite rank coefficients $A_1^\circ,\dots,A_d^\circ$:

$$(I-\zeta T)^{-1}=\overbrace{\frac{A_d^\circ}{(\zeta-\zeta_0)^d}+\dots+\frac{A_1^\circ}{\zeta-\zeta_0}}^{\text{singular part}}+\overbrace{\sum_{k=0}^{\infty}B_k^\circ(\zeta-\zeta_0)^k}^{\text{regular part}}. \tag{5.2.3.2.a}$$

The concept of a Riesz operator was introduced by Ruston [1954, p. 319], who also discovered characterization **(3)**:

$$\lim_{n\to\infty}|||T^n|||^{1/n}=0, \quad \text{where } |||T^n|||=\inf\big\{\|T^n-K\|:K\in\mathfrak{K}(X)\big\}.$$

He referred to such operators as **asymptotically quasi-compact**; [1954, p. 322]. Yosida [1939, p. 297] had already defined **quasi-compact** (quasi-completely-continuous) operators by the property that $\|T^m-K\|<1$ for some exponent m and some compact operator K. Then $I-\zeta T$ is a Fredholm operator whenever $|\zeta|\le 1$.

The equivalence of **(1)** and **(2)** was proved by Pietsch [1960, p. 361] and, once more, by Caradus [1966, p. 64]. Condition **(4)** appears in [BARN$^+$, p. 11] and is there attributed to Smyth; see also Pietsch [1961, p. 17]. A lengthy discussion of condition **(5)** follows in the next paragraph.

5.2.3.3 Stated in modern terminology, Fredholm's determinant theory [1903] says that the Fredholm resolvent of an integral operator T induced by a continuous kernel can be represented as the quotient of entire functions:

$$T(I-\zeta T)^{-1}=\frac{D(\zeta,T)}{d(\zeta,T)};$$

see also 6.5.2.1. The **Fredholm denominator** or **Fredholm divisor**

$$d(\zeta,T)=\sum_{n=0}^{\infty}d_n\zeta^n$$

is complex-valued, whereas the values of the **Fredholm numerator** or **Fredholm minor** (of first order)

$$D(\zeta,T)=\sum_{n=0}^{\infty}D_n\zeta^n$$

are operators generated by continuous kernels depending on the parameter ζ. This reflects the fact that the Fredholm resolvent $T(I-\zeta T)^{-1}$ is a meromorphic operator-valued function.

Using the Riesz decomposition, Schauder [1930c, footnote [18])] proved that this result remains true in the determinant-free setting:

Betrachtet man die Funktionaloperation $y=x+\zeta f(x)$ mit vollstetigen $f(x)$, so ist die Lösung $x=\varphi(y,\zeta)$ eine meromorphe Funktion von ζ.

The same conclusion was obtained by Nagumo [1936, pp. 79–80] with the help of complex integration:

Die Resolvente R_ζ *einer vollstetigen Transformation A* (*im linearen, normierten vollständigen Raum* $\mathfrak{B}$) *ist eine meromorphe Funktion von* ζ.

Dunford [1943b, p. 198] and Taylor [1943, p. 660] treated the case of an arbitrary operator whose resolvent has a pole of order n at the point ζ_0. In this case, ascent and descent of $I-\zeta_0 T$ are finite, and the common value of both quantities is n. If, moreover, the coefficients of the singular part in (5.2.3.2.a) are of finite rank, then $I-\zeta_0 T$ is a Fredholm operator.

5.2.3.4 The theory of Riesz operators is represented in the following monographs: [BARN$^+$, pp. 8–12], [CAR$^+$, Chap. 3], [DOW, Chap. 3], [PIE$_4$, pp. 138–149].

5.2.3.5 Let X be any infinite-dimensional Banach space. Then the operators

$$S:(x,y)\mapsto(y,o) \quad \text{and} \quad T:(x,y)\mapsto(o,x)$$

are nilpotent on $X\oplus X$ and therefore Riesz. It follows from

$$S+T:(x,y)\mapsto(y,x) \quad \text{and} \quad ST:(x,y)\mapsto(x,o)$$

that $(S+T)^2$ is the identity and that ST is a projection with infinite-dimensional range. Hence $S+T$ and ST fail to be Riesz. This counterexample is due to West [1966, p. 136], who also showed that the uniform limit of a sequence of Riesz operators need not be Riesz.

5.2.3.6 We have just seen that the Riesz operators do not form an ideal. Hence one may look for "ideal" subclasses. Indeed, there exists a largest ideal $\mathfrak{R}_{\text{ad}}$ for which the components $\mathfrak{R}_{\text{ad}}(X)$ contain only Riesz operators. The symbol $\mathfrak{R}_{\text{ad}}$ is justified by the fact that Yood [1954, pp. 617–618] defined $\mathfrak{R}_{\text{ad}}(X)$ as the preimage of the radical in the Calkin algebra $\mathfrak{L}(X)/\mathfrak{K}(X)$; see 4.10.2.6. Another characterization was given by Gohberg/Markus/Feldman [1960, стр. 56]:

An operator $T\in\mathfrak{L}(X)$ is a member of $\mathfrak{R}_{\text{ad}}(X)$ if and only if it is an *admissible* Fredholm perturbation: $S\in\Phi(X)$ implies $S+T\in\Phi(X)$.

The full ideal $\mathfrak{R}_{\text{ad}}$ was introduced in [PIE$_2$, p. 57] and [PIE$_3$, p. 66]:

The component $\mathfrak{R}_{\text{ad}}(X,Y)$ consists of all operators $T\in\mathfrak{L}(X,Y)$ such that I_X+AT, or equivalently I_Y+TA, is a Fredholm operator for every $A\in\mathfrak{L}(Y,X)$.

As shown by Pietsch [1978], it suffices that the operators I_X+AT have finite-dimensional null spaces. I stress that the Gohberg–Markus–Feldman criterion via admissible Fredholm perturbations may fail in the case $\Phi(X,Y)=\emptyset$.

Of course, $\mathfrak{R}_{\text{ad}}$ is closed. Following Kleinecke [1963], one refers to operators in $\mathfrak{R}_{\text{ad}}$ as **inessential**.

5.2.3.7 An operator ideal $\mathfrak{A}$ is **proper** if it does not contain the identity map of any infinite-dimensional Banach space. A long-standing conjecture says that $\mathfrak{R}_{\text{ad}}$ is the largest proper ideal.

5.2.3.8 The concept of a strictly singular operator was introduced by Kato [1958, p. 284] as a tool in perturbation theory, and Pełczyński [1965] found the dual counterpart.

An operator $T \in \mathfrak{L}(X,Y)$ is said to be **strictly singular** if TJ_M^X cannot be an injection for any infinite-dimensional closed subspace M of X:

$$\begin{tikzcd} X \arrow[r, "T"] & Y \\ M \arrow[u, "J_M^X"] \arrow[ur, "TJ_M^X"'] & \end{tikzcd} .$$

These operators form a closed ideal, which is denoted by $\mathfrak{S}_{\text{in}}$.

An operator $T \in \mathfrak{L}(X,Y)$ is said to be **strictly cosingular** if $Q_N^Y T$ cannot be a surjection for any infinite-codimensional closed subspace N of Y:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "Q_N^Y T"'] & Y \arrow[d, "Q_N^Y"] \\ & Y/N \end{tikzcd} .$$

These operators form a closed ideal, which is denoted by $\mathfrak{C}_{\text{os}}$.

Kato [1958, p. 287] had already observed that $\mathfrak{S}_{\text{in}}$ is a closed ideal, whereas Pełczyński did not succeed in showing that $\mathfrak{C}_{\text{os}}$ is stable under addition. This was achieved only by Vladimirskiĭ [1967], who based his proof on the right-hand criterion below. Its left-hand counterpart appeared for the first time in [GOL$_1$, p. 84].

An operator $T \in \mathfrak{L}(X,Y)$ is strictly singular if and only if every infinite-dimensional closed subspace M of X contains an infinite-dimensional closed subspace M_0 such that $TJ_{M_0}^X$ is compact.

An operator $T \in \mathfrak{L}(X,Y)$ is strictly cosingular if and only if every infinite-codimensional closed subspace N of Y is contained in an infinite-codimensional closed subspace N_0 such that $Q_{N_0}^Y T$ is compact.

The theory of strictly singular (semicompact) and strictly cosingular (co-semicompact) operators was presented in full length in [PRZ$^+$, pp. 252–274]. In Kato's book [KATO, p. 238], strictly singular operators were mentioned only in a footnote, which indicates that he underestimated his own creation.

5.2.3.9 The relations between the ideals considered above are presented in the following diagram:

$$\begin{tikzcd} & \mathfrak{R}_{\text{ad}} & \\ \mathfrak{S}_{\text{in}} \arrow[ur] & & \mathfrak{C}_{\text{os}} \arrow[ul] \\ & \mathfrak{K} \arrow[ul] \arrow[ur] & \\ & \overline{\mathfrak{F}} \arrow[u] & \end{tikzcd}$$

Kato [1958, p. 285] proved that $\mathfrak{K} \subseteq \mathfrak{S}_{\text{in}}$, and he asked whether even equality holds. This conjecture was disproved by Goldberg/Thorp [1963], who showed that $\mathfrak{S}_{\text{in}}(l_1, l_2) = \mathfrak{L}(l_1, l_2)$. Nowadays, we know that uncountably many different closed ideals lie between $\mathfrak{K}$ and $\mathfrak{S}_{\text{in}}$ as well as between $\mathfrak{K}$ and $\mathfrak{C}_{\text{os}}$; see Pietsch [2002].

The inclusions $\mathfrak{S}_{\text{in}} \subseteq \mathfrak{R}_{\text{ad}}$ and $\mathfrak{C}_{\text{os}} \subseteq \mathfrak{R}_{\text{ad}}$ are implied by results of Kato [1958, p. 286] and Vladimirskiĭ [1967]; see also [PRZ$^+$, pp. 315–317].

5.2.3.10 There is a "one-sided" duality between $\mathfrak{S}_{\text{in}}$ and $\mathfrak{C}_{\text{os}}$:

$$T^* \in \mathfrak{S}_{\text{in}} \ \Rightarrow \ T \in \mathfrak{C}_{\text{os}} \quad \text{and} \quad T^* \in \mathfrak{C}_{\text{os}} \ \Rightarrow \ T \in \mathfrak{S}_{\text{in}}.$$

Note, however, that the converse implications fail in general.

Pełczyński [1965, Part II, p. 40] discovered an illuminating example:
The canonical embedding $K_{c_0} : c_0 \to l_\infty$ is strictly cosingular. On the other hand, the dual operator $K_{c_0}^*$ does not even belong to $\mathfrak{R}_{\text{ad}} \supset \mathfrak{S}_{\text{in}}$.

Further examples were given by Whitley [1964].

5.2.4 Invariant subspaces

Throughout this subsection, subspaces are supposed to be *closed*.

5.2.4.1 To begin with, I present some quotations.

Michaels [1977, p. 56]:
Perhaps the best-known unsolved problem in functional analysis is the invariant subspace problem: Does every bounded operator on (separable, infinite-dimensional, complex) Hilbert space have a nontrivial invariant subspace ?

Radjavi/Rosenthal [1982$^\bullet$, p. 33]:
The invariant subspace problem is the most famous unsolved problem in the theory of bounded linear operators.

Beauzamy [BEAU$_3$, Introduction]:
The most fundamental question in operator theory is the Invariant Subspace Problem, which is still unsolved in Hilbert spaces.

5.2.4.2 The first partial answer was found by Aronszajn/Smith, who attributed the following credit, [1954, p. 345]:

Some years ago, J. von Neumann informed the first author of this paper that in the early 1930s he proved the existence of proper invariant subspaces for completely continuous operators in a Hilbert space; the proof was never published.

The authors mentioned above were able to extend this result to Banach spaces.

Aronszajn–Smith theorem; [DUN$_2^+$, p. 1120]:
Every compact operator T on a complex Banach space X (of dimension greater than one) has a non-trivial **invariant subspace** M. That is,

$$x \in M \ \text{ implies } \ Tx \in M \quad \text{and} \quad \{o\} \neq M \neq X.$$

5.2.4.3 An operator $T \in \mathfrak{L}(X)$ is **polynomially compact** if there exists a non-zero polynomial P for which $P(T)$ becomes compact.

Using methods from non-standard analysis, Bernstein/Robinson [1966] showed that polynomially compact operators on Hilbert spaces have invariant subspaces. Instantly thereafter, Halmos [1966] obtained the same result by classical methods.

5.2.4.4 In [1973, стр. 55], the experts were shocked by a sensation.

Lomonosov's theorem:
Пусть A – ненулевой линейный вполне непрерывный оператор, действующий в бесконечномерном комплексном банаховом пространстве. Тогда найдется нетривиальное подпространство, инвариантное для каждого линейного ограниченного оператора, который коммутирует с A.

Engl. transl., Lomonosov [1973, p. 213]:
Let A be a nonzero, linear completely continuous [read: compact] *operator acting in an infinite-dimensional complex Banach space. Then there exists a nontrivial subspace which is invariant under each bounded linear operator which commutes with A.*

Lomonosov's elegant and short proof relied on the Schauder fixed point theorem. Subsequently, Hilden accomplished an even simpler proof, which, for whatever reasons, was published not by himself, but by Michaels [1977].

5.2.4.5 Lomonosov's theorem is the most exciting positive result. Now I turn to the negative part.

In the Maurey–Schwartz seminar [PAR$^{\Sigma}_{75}$, exposés 14 and 15, February 24 and March 2, 1976,] Enflo sketched the basic idea of how to produce a counterexample:

It is clear that every operator with a cyclic vector on a Banach space can be represented as multiplication by x on the set of polynomials under some norm. So what we will do is to construct a norm on the space of polynomials and prove that the shift operator under this norm has only trivial invariant subspaces.

The completion of Enflo's work took nine years. Finally, the revised form of his manuscript was received by Acta Mathematica on March 21, 1985; see Enflo [1987]. There is also an elaboration by Beauzamy [1985], which appeared a little bit earlier.

In the meantime, Read [1984] had published his own construction, and in a next step [1985], he gave a counterexample on l_1. Comparing both approaches, one may say that either the operator is simple and the space is complicated, or the space is simple and the operator is complicated.

Let me conclude this story with a piece of yellow press.
Read [1984, p. 337]:
This is the only counterexample which the author knows to be valid. P. Enflo has produced two preprints purporting to contain examples of operators without an invariant subspace, obtained by very different methods to our own. However, these preprints have been in existence since 1976 and 1981, and neither has yet been published.

Read [1985, p. 317]:

I would like to apologise for the wording of the comments in my last paper concerning the preprints of P. Enflo, which caused offence to some people. I truly did not wish to offend him. We believe that Enflo's solution has been found to be correct.

Beauzamy [BEAU_3, p. 345]:

C. Read also produced a counter-example. His paper benefitted of quick refereeing and "jumped over" the waiting line for publication in the Bulletin of the London Mathematical Society, so it appeared in july 84. Similar facilities were also offered, by the editors of this Journal, to the present author, who declined.

To give such precision is uncommon, and would be of no interest, if C. Read had not, several times, unelegantly and unsuccessfully, tried to claim priority towards the solution of the problem. This behavior might be condemned with stronger words, but we remember we are presently writing for posterity.

5.2.4.6 Next, I present a beautiful supplement to the Riesz–Schauder theory, which is due to Ringrose [1962]; see also [DOW, pp. 57–66].

Fix any complex Banach space X. By a **nest** we mean a family of subspaces, say $\mathscr{N}$, that is linearly ordered with respect to set-theoretic inclusion.

The collection of all nests can be partially ordered, again by set-theoretic inclusion. Moreover, this partial order is inductive, so that Zorn's lemma applies. Hence maximal nests $\mathscr{M}$ exist, and these are characterized as follows:

(1) $\{o\} \in \mathscr{M}$ and $X \in \mathscr{M}$.

(2) If $\mathscr{N}$ is any subfamily of $\mathscr{M}$, then the subspaces

$$\bigcap\{N : N \in \mathscr{N}\} \quad \text{and} \quad \overline{\bigcup\{N : N \in \mathscr{N}\}}$$

are in $\mathscr{M}$.

(3) Let $M_- := \overline{\bigcup\{M_0 : M_0 \in \mathscr{M}, M_0 \subset M\}}$ for $M \in \mathscr{M}$. Then M/M_- is at most 1-dimensional.

It follows from the Aronszajn–Smith theorem 5.2.4.2 that every compact operator $T \in \mathfrak{K}(X)$ has a maximal nest $\mathscr{M}$ consisting of invariant subspaces.

With every member $M \neq \{o\}$ of $\mathscr{M}$ we associate its **diagonal coefficient** $\alpha_M \in \mathbb{C}$: Let $\alpha_M := 0$ if $M = M_-$. Otherwise, fix any $x \in M \setminus M_-$. Then $Tx \in M$ can be written in the form $Tx = \alpha_M x + x_0$, where $x_0 \in M_-$. Note that α_M does not depend on the particular choice of x.

The main result of Ringrose [1962, p. 376] says that a non-zero scalar λ is a diagonal coefficient of T if and only if it is an eigenvalue of T. Moreover, the diagonal multiplicity of λ is equal to its algebraic multiplicity as an eigenvalue. Thus we have an extension of Schur's classical theorem; see [1909, p. 490]:

For every (n,n)-matrix T there exists a unitary (n,n)-matrix U such that $U^\star TU$ has superdiagonal (upper triangular) form, and the coefficients on the diagonal are just the eigenvalues of T.

5.2.4.7 The Aronszajn–Smith theorem is trivial for operators with an eigenvector. Then there exists a 1-dimensional invariant subspace. Hence the invariant subspace problem is of interest only for **quasi-nilpotent operators**: $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$. The term **Volterra operator** is also common.

In contrast to the finite-dimensional setting, it may happen that a non-zero compact operator $T \in \mathfrak{K}(X)$ does not have any eigenvector. The most important example is the integration operator on $L_2[0,1]$,

$$S : f(t) \mapsto \int_0^s f(t)\,dt.$$

Brodskiĭ [1957] and Donoghue [1957] proved independently from each other that all invariant subspaces of this operator have the form

$$M_c := \{f \in L_2[0,1] : f(t) = 0 \quad \text{for } 0 \le t \le c\} \quad \text{with } 0 \le c \le 1.$$

Plainly, these subspaces form a maximal nest and their diagonal coefficients are zero.

5.2.4.8 Coming back to the quotations from the first paragraph of this subsection, I make a heretical comment:

Of course, the invariant subspace problem is a striking challenge. Suppose that it has an affirmative answer for Hilbert spaces. To what extent would this fact change the further development of operator theory? I leave the answer to the experts.

Concerning Banach space theory, the situation is clearer. Using the Gowers–Maurey space, Read [1999] constructed a strictly singular operator without invariant subspaces. Thus the YES–NO borderline runs between $\mathfrak{K}$ and $\mathfrak{S}_{\text{in}}$. With regard to the collection of all operator ideals this gap is relatively small. On the other hand, as observed in 5.2.3.9, there exist uncountably many closed operator ideals $\mathfrak{A}$ such that $\mathfrak{K} \subset \mathfrak{A} \subset \mathfrak{S}_{\text{in}}$. However, proving the Aronszajn–Smith theorem for all members of such an intermediate ideal would not a constitute significant progress. Of course, one could try to extend the Ringrose theory of superdiagonalization to operators in $\mathfrak{A}$. What else?

5.2.4.9 For further information the reader is referred to the monographs [BEAU$_3$] and [RAD$^+$] as well as to the survey articles of Chalendar/Esterle [2000$^\bullet$] and Enflo/Lomonosov [2001].

5.2.5 Spectral operators

5.2.5.1 The starting point in the theory of spectral operators on Banach spaces was von Neumann's classical theorem: Every normal operator on a Hilbert space admits a spectral resolution whose projections are orthogonal; see [DUN_2^+, Chap. X]. Hence the basic intention of Banach space people can be described as follows: save as much as possible of the famous results obtained in the Hilbert space setting. The theory was mainly developed by Dunford [1943a, 1943b, 1952]. He gave a readable survey in [1958]. Finally, I refer to the third part of Dunford/Schwartz, which is entirely devoted to this subject. The monograph [DOW] contains some additional information.

Related concepts will be discussed in Subsection 6.9.8.

5.2.5.2 By a countably additive **spectral measure** in a complex Banach space X we mean a map $\boldsymbol{E}$ that assigns to every Borel set of the complex plane a projection in $\mathfrak{L}(X)$ such that

(1) $\boldsymbol{E}(\mathbb{C}) = I$ and $\boldsymbol{E}(\emptyset) = O$.

(2) $\boldsymbol{E}(A \cap B) = \boldsymbol{E}(A)\boldsymbol{E}(B)$ whenever $A, B \in \mathscr{B}_{\text{orel}}(\mathbb{C})$.

(3) $\boldsymbol{E}\Big(\bigcup\limits_{k=1}^{\infty} A_k\Big) = \sum\limits_{k=1}^{\infty} \boldsymbol{E}(A_k)$ whenever $A_h \cap A_k = \emptyset$ for $h \neq k$; the right-hand series is supposed to converge in the strong operator topology.

Every countably additive spectral measure is bounded:

$$\|\boldsymbol{E}(A)\| \leq c \quad \text{for some } c \geq 1 \text{ and all } A \in \mathscr{B}_{\text{orel}}(\mathbb{C}).$$

5.2.5.3 A countably additive spectral measure $\boldsymbol{E}$ is said to be a **resolution of the identity** for the operator $T \in \mathfrak{L}(X)$ if the following conditions are satisfied. First of all, it is assumed that

$$T\boldsymbol{E}(A) = \boldsymbol{E}(A)T \quad \text{whenever } A \in \mathscr{B}_{\text{orel}}(\mathbb{C}).$$

Then the range of every projection $\boldsymbol{E}(A)$ is invariant under T; and it makes sense to require that the spectrum of the induced operator $T(A)$ be contained in the closure of A.

Of course, one may speculate about the question, *What is the spectrum of the zero operator O in the zero space $\{o\}$?* Nevertheless, it seems to me that all authors tacitly consider the condition $\sigma(T(A)) \subseteq \overline{A}$ as fulfilled whenever $\boldsymbol{E}(A) = O$.

5.2.5.4 An operator T is called **spectral** if it has a resolution of the identity. In this case, the associated spectral measure is uniquely determined, and $\boldsymbol{E}\big(\sigma(T)\big) = I$. That is, $\boldsymbol{E}$ has compact support.

According to Dunford [1958, p. 218],

The main problem, of course, is that of discovering conditions on an operator which, on the one hand, are sufficient to insure the existence of a countably additive resolution of the identity and, on the other hand, are stated in a form that may be applied to the more concrete problems of mathematical study.

Such conditions were given in [DUN_3^+, pp. 2134–2169]; see also Dunford [1958].

5.2.5.5 Quasi-nilpotent operators are spectral. For any fixed Banach space, they have one and the same resolution of the identity that looks like a Dirac measure:

$$\boldsymbol{E}(A) := \begin{cases} I & \text{if } 0 \in A, \\ O & \text{if } 0 \notin A. \end{cases}$$

More generally, every operator with a finite spectrum is spectral, since the problem of countable additivity disappears; see 5.2.1.4.

5.2.5.6 A spectral operator S is said to be of **scalar type**, or simply a **scalar operator**, if

$$S = \int_{\mathbb{C}} \lambda\, \boldsymbol{E}(\lambda), \tag{5.2.5.6.a}$$

where $\boldsymbol{E}$ is the resolution of the identity for S. One may also start from any countably additive spectral measure $\boldsymbol{E}$ that vanishes outside a compact subset of the complex plane.

5.2.5.7 An operator T is spectral if and only if it can be written in the form $T = S+N$, where S and N are commuting operators such that S has scalar type and N is quasi-nilpotent. This decomposition is unique. The **scalar part** S is given by (5.2.5.6.a), where $\boldsymbol{E}$ is the resolution of the identity for T.

Let $\mathfrak{A}$ be any closed operator ideal. Then Foguel [1958, p. 52] proved that $T \in \mathfrak{A}(X)$ implies $S, N \in \mathfrak{A}(X)$ and $E(A) \in \mathfrak{A}(X)$ whenever A is a Borel set with $0 \notin \overline{A}$.

5.2.5.8 Let N be the **quasi-nilpotent part** of the spectral operator T. Then, for every function f that is analytic on some neighborhood of $\sigma(T)$, the operator $f(T)$ defined by formula (5.2.1.2.a) can be expressed in the form

$$f(T) = \int_{\mathbb{C}} f(\lambda) \boldsymbol{E}(d\lambda) + \tfrac{1}{1!} \int_{\mathbb{C}} f'(\lambda) \boldsymbol{E}(d\lambda) N + \cdots + \tfrac{1}{n!} \int_{\mathbb{C}} f^{(n)}(\lambda) \boldsymbol{E}(d\lambda) N^n + \cdots .$$

5.2.5.9 Of course, all Hermitian operators on a Hilbert space turn out to be spectral; the values of their spectral measures, which live on the real line, are orthogonal projections. More general are the normal operators: $TT^\star = T^\star T$. These can be written in the form $T = A + \mathrm{i}B$, where the Hermitian parts A and B commute. In this case, a resolution of the identity of T is obtained by forming the product of the spectral measures $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ associated with A and B , respectively:

$$\boldsymbol{E}(U + \mathrm{i}V) = \boldsymbol{E}\big((U + \mathrm{i}\mathbb{R}) \cap (\mathbb{R} + \mathrm{i}V)\big) = \boldsymbol{E}(U + \mathrm{i}\mathbb{R}) \boldsymbol{E}(\mathbb{R} + \mathrm{i}V),$$
$$\boldsymbol{E}(U + \mathrm{i}\mathbb{R}) = \boldsymbol{E}_A(U), \quad \text{and} \quad \boldsymbol{E}(\mathbb{R} + \mathrm{i}V) = \boldsymbol{E}_B(V)$$

for $U, V \in \mathscr{B}_{\mathrm{orel}}(\mathbb{R})$. Consequently, normal operators have scalar type.

A conclusion in the converse direction follows from a result of Wermer [1954, pp. 355–356]; see also [DUN$_3^+$, pp. 1945–1950]:

Every scalar type operator T on a Hilbert space H is **similar** to a normal operator S, which means that there exists an isomorphism $A \in \mathfrak{L}(H)$ such that $T = ASA^{-1}$.

For an example, the reader is referred to 5.6.5.6.

McCarthy/Tzafriri [1968] obtained a surprising result:

Every scalar type operator on the space $L_\infty[0,1]$ is similar to a multiplication operator $f(t) \mapsto d(t)f(t)$, where d has only finitely many values.

The latter observation tells us that $L_\infty[0,1]$ carries very few scalar type operators; see also 5.6.7.4. In particular, there is no compact scalar type operator on $L_\infty[0,1]$ other than O.

5.2.5.10 A compact operator has scalar type if and only if it admits a representation

$$T = \sum_{k=1}^{\infty} \lambda_k E_k, \tag{5.2.5.10.a}$$

where $(\lambda_k) \in c_0$ and (E_k) is a sequence of finite rank projections such that

$$\sup\left\{\Big\|\sum_{k\in\mathbb{F}} E_k\Big\| : \mathbb{F} \in \mathscr{F}(\mathbb{N})\right\} < \infty \quad \text{and} \quad E_h E_k = O \quad \text{for } h \neq k.$$

This criterion shows that such operators form an extremely small subset of $\mathfrak{K}(X)$; in my opinion it is too small. For example, an operator of the form (5.2.5.10.a) with

$$\lim_{k\to\infty} \|E_k\| = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\lambda_k|\|E_k\| < \infty$$

fails to be of scalar type.

Recall that countably additive spectral measures are bounded. This property is natural for Hilbert spaces, since one commonly works with orthogonal projections. However, in Banach spaces we must deal with arbitrary projections, and assuming any upper bound is an artificial requirement.

Next, I describe a fact that is even more frustrating. As every analyst knows, spectral theory grew out of the problem of the vibrating string (fixed at the end points, say a and b). The solution at a time t can be written as a trigonometric series that converges unconditionally in $L_2[a,b]$. Unfortunately, the unconditionality gets lost when we pass to any space $L_p[a,b]$ with $p \neq 2$. On the other hand, the expansions induced by compact spectral operators must be unconditional. Consequently, in non-Hilbert spaces and without additional assumptions, Dunford's theory does not apply to integral operators generated by symmetric continuous kernels.

Dunford was aware of the defect just described, [1958, p. 223]:

In fact by demanding a countably additive resolution of the identity we probably rule out most of the differential operators in L_p with $p \neq 2$.

The dreams of those working in spectral theory at the beginning of the forties were expressed by Murray [1942] in an address delivered to the Washington meeting of the American Mathematical Society on May 2, 1941. Most of them have *"gone with the wind."*

5.3 Semi-groups of operators

5.3.1 Deterministic and stochastic processes

5.3.1.1 A family of operators $U_t \in \mathfrak{L}(X)$ with $t > 0$ is a one-parameter **semi-group** if

$$U_{t_1+t_2} = U_{t_1} U_{t_2} \quad \text{for } t_1, t_2 > 0.$$

In this subsection, I show how such semi-groups emerge.

5.3.1.2 Let $\mathscr{M}$ be a σ-algebra on a set M. Measures μ are assumed to be countably additive.

A map φ from M into itself is said to be **measure-preserving** if $B \in \mathscr{M}$ implies $\varphi^{-1}(B) \in \mathscr{M}$ and $\mu\big(\varphi^{-1}(B)\big) = \mu(B)$. Many authors assume, in addition, that φ is invertible and that $\varphi(B) \in \mathscr{M}$ whenever $B \in \mathscr{M}$. In this case, φ^{-1} is measure-preserving as well.

The rule

$$U_\varphi : f(\eta) \mapsto f\big(\varphi(\xi)\big) \quad \text{for all } \xi \in M$$

defines an isometric and positive operator from $L_p(M, \mathscr{M}, \mu)$ into itself, $1 \le p \le \infty$. This "trivial" fact has remarkable consequences. With any measure-preserving transformation φ, which is in general non-linear, we associate a linear object, namely U_φ. Thanks to this "linearization," spectral theory can be used as an important tool.

5.3.1.3 The preceding observations go back to Koopman [1931], who treated *autonomous mechanical systems*:

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \quad \text{and} \quad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \quad \text{for } k = 1, \dots, n. \tag{5.3.1.3.a}$$

The Hamiltonian function $H(p_1, \dots, p_n; q_1, \dots, q_n)$ is supposed to be independent of t. If the "phase space" M is the whole $\mathbb{R}^{2n}$, then suitable conditions guarantee that every point lies on one and only one solution $t \mapsto \big(p_1(t), \dots, p_n(t); q_1(t), \dots, q_n(t)\big)$ of the equations (5.3.1.3.a). Thus

$$\varphi_t : \big(p_1(0), \dots, p_n(0); q_1(0), \dots, q_n(0)\big) \mapsto \big(p_1(t), \dots, p_n(t); q_1(t), \dots, q_n(t)\big)$$

defines a family of one-to-one maps from $\mathbb{R}^{2n}$ onto itself such that

$$\varphi_{t_1+t_2} : \mathbb{R}^{2n} \xrightarrow{\varphi_{t_2}} \mathbb{R}^{2n} \xrightarrow{\varphi_{t_1}} \mathbb{R}^{2n} \quad \text{for} \quad t_1, t_2 \in \mathbb{R}.$$

The classical Liouville theorem says that the φ_t's preserve the Lebesgue measure. The considerations can be restricted to an appropriate subvariety M of $\mathbb{R}^{2n}$ that, for example, consists of all points with $H(p_1, \dots, p_n; q_1, \dots, q_n) = \text{const}$. Also in this case there exists an invariant measure.

Passing from φ_t to U_{φ_t}, Koopman arrived at a group of unitary operators on a Hilbert space: $U_{\varphi_{t_1+t_2}} = U_{\varphi_{t_1}} U_{\varphi_{t_2}}$.

5.3.1.4 By a **stochastic kernel** we mean a mapping $\mathsf{P}: \mathscr{M} \times M \to [0,1]$ such that $A \mapsto \mathsf{P}(A,\xi)$ is a probability measure for every fixed $\xi \in M$ and such that $\xi \mapsto \mathsf{P}(A,\xi)$ is $\mathscr{M}$-measurable for every fixed $A \in \mathscr{M}$.

Recall from 4.2.5.4 that $ca(M,\mathscr{M})$ denotes the collection of all bounded and countably additive set functions on $\mathscr{M}$. Put

$$\nu(A) := \int_M \mathsf{P}(A,\xi)\mu(d\xi) \quad \text{for } \mu \in ca(M,\mathscr{M}) \text{ and } A \in \mathscr{M}.$$

Then $U_{\mathrm{P}} : \mu \to \nu$ defines a positive operator from $ca(M,\mathscr{M})$ into itself with $\|U_{\mathrm{P}}\| = 1$.

A stochastic kernel may be interpreted as a random transformation that sends the point ξ into the set A with probability $\mathsf{P}(A,\xi)$. The operator U_{P} tells us what happens with an initial distribution μ.

Probabilists often write $\int_M f(\xi)\mu(d\xi)$ instead of $\int_M f(\xi)d\mu(\xi)$. The advantage of this modification becomes clear when the underlying measure depends on a parameter: $\int_M f(\xi)\mathsf{P}(d\xi,\eta)$. A similar notation was already used by Kolmogoroff [1931a, p. 420], who wrote $\int_{M_\xi} f(\xi)\mu(dM)$.

5.3.1.5 We now consider a family of stochastic kernels P_t such that

$$\mathsf{P}_{t_1+t_2}(A,\eta) = \int_M \mathsf{P}_{t_1}(A,\xi)\mathsf{P}_{t_2}(d\xi,\eta) \quad \text{for } A \in \mathscr{M},\ \eta \in M \text{ and } t_1, t_2 > 0.$$

This formula is referred to as the **Chapman–Kolmogoroff equation**. Subject to the assumption that the P_t's have densities, it was proved by Chapman [1928, p. 39] in his theory of Brownian displacements. In the general case, Kolmogoroff [1931a, p. 420] used it as an axiom. The Chapman–Kolmogroff equation implies that the operators U_{P_t} form a semi-group: $U_{\mathrm{P}_{t_1+t_2}} = U_{\mathrm{P}_{t_1}} U_{\mathrm{P}_{t_2}}$.

5.3.1.6 An infinite matrix (π_{hk}) is called **stochastic** if

$$\pi_{hk} \geq 0 \quad \text{and} \quad \sum_{h=1}^{\infty} \pi_{hk} = 1 \quad \text{for} \quad k = 1, 2, \ldots \,.$$

Letting

$$\mathsf{P}(A,k) := \sum_{h \in A} \pi_{hk}$$

yields a stochastic kernel P on $\mathscr{P}(\mathbb{N}) \times \mathbb{N}$ such that

$$U_{\mathrm{P}} : (\xi_k) \mapsto \Big(\sum_{k=1}^{\infty} \pi_{hk}\xi_k \Big)$$

defines a positive operator from l_1 into itself with $\|U_{\mathrm{P}}\| = 1$.

The case of (n,n)-matrices was considered by Markov (senior) in 1906. This definition gave rise to the concept of a *Markov chain*, which was later extended to that of a *Markov process*; see [SCHN$^\cup$, pp. 443–445] and Pier [1991$^\bullet$, p. 68].

5.3.2 The Hille–Yosida theorem

Until the end of Subsection 5.3.4, the underlying Banach space X is assumed to be complex.

5.3.2.1 We will deal only with **semi-groups of class** (C_0), which means that

$$\lim_{t\searrow 0} U_t x = x \quad \text{for all } x \in X.$$

Equivalently, letting $U_0 := I$, we may require that the function $t \mapsto U_t$ be continuous at 0 with respect to the strong operator topology.

5.3.2.2 Every operator $A \in \mathfrak{L}(X)$ generates the semi-group $U_t := \mathrm{e}^{At}$, where the *exponential function* is defined by

$$\mathrm{e}^{At} := I + \tfrac{1}{1!}At + \tfrac{1}{2!}A^2t^2 + \cdots \quad \text{or} \quad \mathrm{e}^{At} := \lim_{n\to\infty} \Big(I + \tfrac{1}{n}At\Big)^n. \tag{5.3.2.2.a}$$

As independently shown by Nathan [1935, p. 518], Nagumo [1936, p. 72], and Yosida [1936, pp. 24–26], this construction yields only all *uniformly continuous* semi-groups:

$$\lim_{t\searrow 0} \|U_t - I\| = 0.$$

Since e^{At} makes sense for every $t \in \mathbb{R}$, we even get a one-parameter group. Moreover, everything works in abstract Banach algebras.

5.3.2.3 The starting point of the theory of semi-groups was a paper of Stone [1932] in which he studied one-parameter groups of *unitary* operators on a Hilbert space. His main result says that such groups can be represented in the form $U_t := \mathrm{e}^{\mathrm{i}At}$, where A is *self-adjoint*. The factor i in the exponent was introduced for aesthetic reasons; it disappeared in the Banach space setting.

5.3.2.4 The main goal of the theory is to investigate the interplay between a semi-group U_t and its **infinitesimal generator**

$$Ax := \lim_{h\searrow 0} \frac{U_h x - x}{h}$$

whose domain $D(A)$ consists of all elements $x \in X$ for which the right-hand limit exists. We have

$$\int_0^h U_t x \, dt \in D(A) \quad \text{and} \quad \lim_{h\searrow 0} \frac{1}{h} \int_0^h U_t x \, dt = x \quad \text{for all } x \in X.$$

Hence $D(A)$ is dense in X. Moreover, A turns out to be closed.

5.3.2.5 Around 1948, the theory of semi-groups was created by Hille and Yosida. They considered contractions: $\|U_t\| \le 1$. In this case, all positive numbers belong to the resolvent set of A, and $R(\lambda, A) := (\lambda I - A)^{-1}$ satisfies the inequality

$$\|\lambda R(\lambda, A)\| \le 1 \quad \text{whenever } \lambda > 0. \tag{5.3.2.5.a}$$

Conversely, this property guarantees that A is the infinitesimal generator of such a semi-group. The operators U_t can be obtained from A as follows.

Hille's way, [HIL, p. 238]: If A is bounded, then a clever transformation of the right-hand side of (5.3.2.2.a) yields

$$e^{At} = \lim_{n\to\infty}\left(I - \tfrac{t}{n}A\right)^{-n} = \lim_{n\to\infty}\left(\frac{n}{t}\right)^n R\left(\tfrac{n}{t},A\right)^n.$$

This observation suggests the definition

$$U_t x := \lim_{n\to\infty}\left(\frac{n}{t}\right)^n R\left(\tfrac{n}{t},A\right)^n \quad \text{for all } x\in X.$$

Yosida's way, [1948, pp. 18–19]: The operators $A(\lambda) := \lambda^2 R(\lambda,A) - \lambda I$ are bounded whenever $\lambda > 0$, and we have

$$\lim_{\lambda\to\infty} A(\lambda)x = Ax \quad \text{for all } x\in X.$$

Hence it makes sense to put

$$U_t x := \lim_{\lambda\to\infty} e^{A(\lambda)t}x \quad \text{for all } x\in X.$$

5.3.2.6 Let U_t be a semi-group of class (C_0). The uniform boundedness principle implies the existence of a constant $M \geq 1$ such that $\|U_t\| \leq M$ for $0 < t \leq 1$. Moreover, we find a real constant ω such that

$$\|U_t\| \leq Me^{\omega t} \quad \text{for all } t>0. \tag{5.3.2.6.a}$$

This estimate of growth was stated for the first time by Hille [1939, p. 13]; see also [HIL, p. 230], Miyadera [1952, p. 110], and Phillips [1953, p. 201]. The infimum of all possible ω's is given by

$$\omega_0 = \lim_{t\to\infty}\frac{\log\|U_t\|}{t} = \inf_{t>0}\frac{\log\|U_t\|}{t}.$$

I stress the analogy with formula (4.10.2.4.a), which yields the spectral radius.

5.3.2.7 Miyadera [1952, pp. 113–114] considered the operators $U_t : f(x) \mapsto f(x+t)$ on the Banach space of all measurable functions on $(0,\infty)$ for which

$$\|f\| := \int_0^\infty |f(x)|e^{-\sqrt{x}}dx$$

is finite. The infinitesimal generator of this semi-group is the smallest closed extension of the map $f(x) \mapsto f'(x)$, where f ranges over all continuously differentiable functions with compact support.

It follows from $\|U_t\| = e^{\sqrt{t}}$ that $\|U_t\| \leq Me^{\omega t}$ fails for $M=1$ and any $\omega \in \mathbb{R}$. This example shows that the original Hille–Yosida theorem did not cover all semi-groups of class (C_0); see 5.3.2.5.

5.3.2.8 Suppose that U_t satisfies (5.3.2.6.a). Passing to $V_t := U_t e^{-\omega t}$ yields a bounded semi-group: $\|V_t\| \le M$. Moreover, if A denotes the infinitesimal generator of U_t, then $B := A - \omega I$ is the infinitesimal generator of V_t, and we have $R(\lambda, B) = R(\lambda + \omega, A)$.

5.3.2.9 Consider a bounded semi-group: $\|U_t\| \le M$. Gelfand [1939, p. 713] observed that

$$\|x\|_1 := \sup\{\|U_t x\| : t > 0\}$$

defines an equivalent norm on X under which the operators U_t become contractions and $\|x\| \le \|x\|_1 \le M\|x\|$.

Similarly, assuming that

$$\|\lambda^n R(\lambda, A)^n\| \le M \quad \text{whenever } \lambda > 0 \text{ and } n = 1, 2, \dots,$$

Feller [1953, pp. 168, 174] obtained a renorming

$$\|x\|_2 := \sup\{\|\lambda^n R(\lambda, A)^n x\| : \lambda > 0,\ n = 1, 2, \dots\}.$$

Then $\|\lambda R(\lambda, A)\| \le 1$ (with respect to the new norm) and $\|x\| \le \|x\|_2 \le M\|x\|$.

5.3.2.10 The final version of the **Hille–Yosida theorem** is due to Feller [1953, pp. 173–174], Miyadera [1952, p. 110], and Phillips [1953, pp. 201–202]:

A densely defined and closed operator A on a Banach space X is the infinitesimal generator of a (C_0) semi-group satisfying $\|U_t\| \le M e^{\omega t}$ if and only if

$$\|(\lambda - \omega)^n R(\lambda, A)^n\| \le M \quad \text{whenever } \lambda > \omega \text{ and } n = 1, 2, \dots, \tag{5.3.2.10.a}$$

or

$$\|(\mathrm{Re}(\lambda - \omega)^n R(\lambda, A)^n\| \le M \quad \text{whenever } \mathrm{Re}(\lambda) > \omega \text{ and } n = 1, 2, \dots. \tag{5.3.2.10.b}$$

In particular, the resolvent set of A contains the half-plane defined by $\mathrm{Re}(\lambda) > \omega$.

Note that (5.3.2.5.a) is equivalent to (5.3.2.10.a) for $n = 1$, $M = 1$ and $\omega = 0$.

5.3.2.11 The formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} U_t x \, dt \quad \text{for all } x \in X$$

means that $R(\lambda, A)$ is the **Laplace transform** of U_t. By formal inversion one obtains

$$U_t x = \lim_{\beta \to \infty} \frac{1}{2\pi} \int_{-\beta}^{+\beta} e^{(\omega + i\eta)t} R(\omega + i\eta, A) x \, d\eta \quad \text{for all } x \in D(A) \tag{5.3.2.11.a}$$

and any choice of $\omega > \omega_0$; see [HIL, p. 232] or [AREN$^+$, p. 201]. Driouich/El-Mennaoui [1999, p. 57] showed that in *UMD* spaces (6.1.10.1), formula (5.3.2.11.a) even holds for *all* $x \in X$; see also [AREN$^+$, p. 202]. This fact illustrates the fascinating interplay between the validity of analytic statements and the geometry of the underlying Banach spaces.

In the case that A is bounded, the operational calculus described in Subsection 5.2.1 yields

$$U_t = e^{At} = \frac{1}{2\pi i} \oint_{\mathscr{C}} e^{\lambda t} R(\lambda, A) \, d\lambda,$$

the path $\mathscr{C}$ surrounding the spectrum of A in the positive sense; [HIL, p. 227].

5.3.2.12 The classical monographs on semi-groups are [HIL] as well as the revised edition [HIL$^+$] written in collaboration with Phillips. The reader is also referred to [AREN$^+$], [BUT$_1^+$], [DAV], [DUN$_1^+$, Chap. VIII], [GOL$_2$], and [PAZY]; each of these books contains historical comments.

5.3.3 Analytic semi-groups

5.3.3.1 Operators U_ζ constituting an **analytic** or **holomorphic semi-group** depend on a complex parameter ζ that ranges over a sector

$$\Sigma_\alpha := \{\, \zeta = re^{i\varphi} \in \mathbb{C} : r > 0,\ |\varphi| < \alpha \,\}, \quad \text{where } 0 < \alpha \le \pi/2.$$

As indicated by the naming, $\zeta \to U_\zeta$ is required to be analytic on Σ_α. One further assumes the semi-group property

$$U_{\zeta_1+\zeta_2} = U_{\zeta_1}U_{\zeta_2} \quad \text{for all } \zeta_1, \zeta_2 \in \Sigma_\alpha$$

as well as continuity at $\zeta = 0$,

$$\lim_{\zeta \to 0} \|U_\zeta x - x\| = 0 \quad \text{for all } x \in X.$$

Since different authors, for example [HIL$^+$, p. 325], use various (small!) modifications of the definition above, some care is advisable.

Analytic semi-groups were introduced by Hille [1939, p. 14]. To this end, he developed the background of a theory of operator-valued analytic functions; see Subsection 5.1.7.

5.3.3.2 A densely defined and closed operator A on a Banach space X is the infinitesimal generator of an analytic semi-group satisfying $\|U_\zeta\| \le M_0 e^{\omega_0|\zeta|}$ for $\zeta \in \Sigma_{\alpha_0}$ if and only if there exist α, ω, and M such that

$$\|(\lambda - \omega)R(\lambda, A)\| \le M \quad \text{whenever } \lambda - \omega \in \Sigma_{\alpha+\pi/2}. \tag{5.3.3.2.a}$$

I do not discuss the tricky relations between the constants α_0, ω_0, M_0, and α, ω, M. According to [PAZY, p. 63], *These relations can be discovered by checking carefully the details of the proof.*

Comparing (5.3.2.10.b) and (5.3.3.2.a) shows that $|\mathrm{Re}(\lambda) - \omega|$ is replaced by $|\lambda - \omega|$, which makes the condition stronger. On the other hand, we need the estimate $\|(\lambda - \omega)^n R(\lambda, A)^n\| \le M$ only for $n = 1$.

5.3.3.3 Independently of each other, Beurling [1970, p. 388] and Kato [1970, p. 496] proved that subject to the condition

$$\limsup_{t \searrow 0} \|U_t - I\| < 2,$$

a (C_0) semi-group extends analytically to some sector Σ_α. In the case that the U_t's are contractions on a uniformly convex Banach space, the assumption above is even necessary; see [PAZY, p. 68]. For historical comments, I refer to Neuberger [1993•].

A quantitative version of this result was used by Pisier [1982, p. 378] in his proof of *B-convexity* $=$ *K-convexity*; see 6.1.7.12 and [DIE_1^+, pp. 268–272]:

Given $0<r<2$, there exist constants $0<\alpha\le\pi/2$ and $M\ge 1$ such that every real semi-group of contractions satisfying $\|U_t - I\| \le r$ for all $t>0$ admits an analytic extension to the sector Σ_α with $\|U_\zeta\| \le M$ for all $\zeta \in \Sigma_\alpha$.

5.3.4 The abstract Cauchy problem

5.3.4.1 Let $A : D(A) \to X$ be a linear mapping whose domain of definition is dense in X, and fix any element $x \in D(A)$. We look for continuously differentiable functions $\boldsymbol{u} : [0,\infty) \to D(A)$ that satisfy the *ordinary differential equation*

$$\frac{d\boldsymbol{u}(t)}{dt} = A\boldsymbol{u}(t) \quad \text{for all } t \ge 0 \tag{5.3.4.1.a}$$

as well as the *initial condition*

$$\boldsymbol{u}(0) = x. \tag{5.3.4.1.b}$$

Every such $\boldsymbol{u}$ is called a solution of the **abstract Cauchy problem** for A with initial value x.

5.3.4.2 We are interested in the case that the abstract Cauchy problem has a unique solution for all $x \in D(A)$. This is certainly true for infinitesimal generators of (C_0) semi-groups. Then $D(A)$ is invariant under all U_t's, and $\boldsymbol{u}(t) := U_t x$ yields the required solution.

Conversely, we define $U_t : x \mapsto \boldsymbol{u}(t)$, where $\boldsymbol{u}$ is the solution with $\boldsymbol{u}(0) = x \in D(A)$. The operators U_t so obtained extend continuously to all of X.

Abstract Cauchy problems were first considered by Hille [1952, pp. 32–33], who additionally assumed the solutions to satisfy a growth condition $\|\boldsymbol{u}(t)\| \le M e^{\omega t}$ for $t > 0$. This superfluous restriction was subsequently removed by Phillips [1954, pp. 244–245]. In the same paper he showed that uniqueness of the solutions implies that the problem is *well-posed*:

$\boldsymbol{u}(t)$ depends continuously on the initial value x for every fixed $t > 0$.

5.3.4.3 In the case of an analytic semi-group, we have $U_t x \in D(A)$ for $t > 0$ and for *all* $x \in X$. Then $\boldsymbol{u}(t) := U_t x$ solves (5.3.4.1.a), but only for $t > 0$, and instead of (5.3.4.1.b) we have $\lim_{t \searrow 0} \|\boldsymbol{u}(t) - x\| = 0$.

5.3.4.4 For an arbitrary (C_0) semi-group, the function $\boldsymbol{u}(t) := U_t x$ makes sense whenever $x \in X$, though it may be nowhere differentiable. One refers to $\boldsymbol{u}(t) := U_t x$ as a *mild, weak*, or *generalized solution*; see [PAZY].

5.3.4.5 We now treat the *inhomogeneous equation*

$$\frac{d\boldsymbol{u}(t)}{dt} = A\boldsymbol{u}(t) + \boldsymbol{f}(t) \quad \text{for all } t \geq 0$$

under the *initial condition*

$$\boldsymbol{u}(0) = x.$$

By "variation of the constants," we get

$$\boldsymbol{u}(t) = U_t x + \int_0^t U_{t-\tau}\boldsymbol{f}(\tau)d\tau. \tag{5.3.4.5.a}$$

However, this expression does not always yield a solution, even when $\boldsymbol{f}$ is continuous. Indeed, if $x = o$ and $\boldsymbol{f}(\tau) = U_\tau x_0$, then $\boldsymbol{u}(t) = tU_t x_0$ may fail to be differentiable. On the other hand, Phillips [1953, pp. 215–216] proved that (5.3.4.5.a) is a solution for any continuously differentiable function $\boldsymbol{f}$.

5.3.4.6 The standard references about abstract Cauchy problems are [KREIN] and [PAZY]. The former contains valuable historical remarks.

5.3.5 Ergodic theory

In this subsection, the underlying Banach spaces may be real or complex.

5.3.5.1 Ergodic theory deals with the long-term average behavior of dynamical systems. Its historical source was the following problem.

Consider an autonomous mechanical system as described in 5.3.1.3. The value $f(\xi) \in \mathbb{R}$ of a measurable function f at the state ξ in the phase space M can be viewed as a measurement of a physical quantity.

If μ is finite, then every integrable function f has a

$$\text{space-average} = \frac{1}{\mu(M)} \int_M f(\xi)d\mu(\xi).$$

On the other hand, we may (if possible!) form the

$$\text{time-average} = \lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(\varphi_s(\xi)\big)\, ds$$

along a trajectory (orbit) starting in a point ξ at time 0. The **ergodic hypothesis** claims that under suitable conditions,

$$\textit{space-average} \;=\; \textit{time-average}$$

for almost all $\xi \in M$.

5.3.5.2 The functional analytic essence of the preceding considerations is the question whether and in what sense

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t U_s\, ds$$

exists for a given semi-group of operators $U_t \in \mathfrak{L}(X)$.

This problem has a discrete counterpart that shows all essential phenomena. Thus from now on, I will concentrate on the asymptotic behavior of the arithmetic means

$$\frac{I+T+\cdots+T^{n-1}}{n},$$

where $T\in\mathfrak{L}(X)$. According to the underlying convergence as $n\to\infty$, one gets different kinds of **ergodic theorems**.

5.3.5.3 The **individual**, or **pointwise**, **ergodic theorem** is concerned with operators in function spaces, and it says that

$$\frac{f(\xi)+Tf(\xi)+\cdots+T^{n-1}f(\xi)}{n}$$

converges almost everywhere. The first result along these lines was proved by George Birkhoff [1931]. Translated in modern terminology, he treated operators U_φ generated by measure-preserving transformations φ of compact manifolds M. In [DUN$_1^+$, p. 675] we find a pointwise ergodic theorem for operators on the space $L_p(M,\mathscr{M},\mu)$ such that $\|T:L_1\to L_1\|\le 1$ and $\|T:L_\infty\to L_\infty\|\le 1$.

5.3.5.4 The **mean ergodic theorem** holds in general Banach spaces. However, its name, which was coined by Riesz [1938], refers to convergence in L_2: convergence in **mean**.

The historical source is a result of von Neumann [1932b, p. 74]:

The arithmetic means

$$\frac{x+Tx+\cdots+T^{n-1}x}{n}$$

converge in norm for every unitary operator T on a Hilbert space H and every $x\in H$.

In a next step, Visser [1938] obtained weak convergence for contractions. Subsequently, Riesz [1938, p. 275] established norm convergence. He also observed that the same method works in the Banach spaces L_p with $1<p<\infty$; see [1938, p. 276].

The final version of the **mean ergodic theorem** is due to the joint efforts of Kakutani [1938b, p. 296], Yosida [1938b, p. 157], and Yosida/Kakutani [1941, p. 194]:

Let $T\in\mathfrak{W}(X)$ be a weakly compact operator such that the norms $\|T^n\|$ are bounded. Then

$$\lim_{n\to\infty}\left\|\frac{x+Tx+\cdots+T^{n-1}x}{n}-Px\right\|=0 \quad \text{for all } x\in X,$$

where P is the projection from X onto $N(I-T)$ along $\overline{M(I-T)}$.

5.3.5.5 The assumption of weak compactness is obviously satisfied for all operators on reflexive Banach spaces. On the other hand, it fails for operators on L_1 generated by measure-preserving transformations. Hence there arose some need for a modified ergodic theorem in L_1 spaces. This goal was achieved by Garrett Birkhoff [1938,

p. 157] and Kakutani [1941a, p. 534]. They based their approach on the concept of an abstract L-space, which I have discussed in 4.8.3.1.

This is the final result:

Let T be a positive operator on an abstract L-space X such that the norms $\|T^n\|$ are bounded and such that the arithmetic means

$$\frac{x+Tx+\cdots+T^{n-1}x}{n}$$

are bounded above, with respect to the underlying order, for every $x \geq \mathsf{o}$. Then the sequence of these means converges in norm for all $x \in X$.

The proof uses the fact that order intervals in abstract L-spaces are weakly sequentially compact; see Kakutani [1941a, p. 535]. Thus, also in this setting, weak compactness plays a decisive role.

In retrospect, one may say that ergodic theory had a considerable impact on the development of weak topology.

5.3.5.6 So far I have treated the case in which

$$\frac{I+T+\cdots+T^{n-1}}{n} \quad \to \quad P$$

converges with respect to the strong operator topology. A parallel theory provides conditions that even guarantee uniform convergence. Then one speaks of **uniform ergodic theorems**. Here is an example:

Let $T \in \mathfrak{L}(X)$ be an operator such that the norms $\|T^n\|$ are bounded and such that $\|T^m - K\| < 1$ for some exponent m and some $K \in \mathfrak{K}(X)$. Then X is the direct sum of the closed subspaces $N(I-T)$ and $M(I-T)$, the projection P from X onto $N(I-T)$ along $M(I-T)$ has finite rank, and

$$\left\| \frac{I+T+\cdots+T^{n-1}}{n} - P \right\| = O\Big(\frac{1}{n}\Big).$$

Moreover, $\lambda_0 = 1$ is a simple pole of the resolvent $(\lambda I - T)^{-1}$.

The first uniform ergodic theorem as well as the concept of quasi-compactness are due to Kryloff/Bogolioùboff [1937, p. 1387]. Modern presentations can be found in [DUN_1^+, p. 711] and [KREN, pp. 86–94]. It is remarkable that quasi-compact operators also play a decisive role in the theory of Riesz operators; see 5.2.3.2. Their spectral theory was developed by Yosida [1938a], [1939].

5.3.5.7 I now describe a concrete situation in which the preceding condition is satisfied. The following property of a stochastic kernel (see 5.3.1.4) was introduced by Doeblin [1937]; see also [DOOB, p. 192]:

(D) There exist a finite measure μ as well as constants $0<q<1$ and $\varepsilon>0$ such that $\mu(A) \le \varepsilon$ implies $\mathsf{P}(A,\xi) \le q$ for all $\xi \in M$.

In the case of a stochastic matrix (see 5.3.1.6), we must find some n such that

$$\sum_{h=n+1}^{\infty} \pi_{hk} \le q < 1 \quad \text{for} \quad k = 1, 2, \dots \ .$$

Yosida/Kakutani [1941, pp. 222–225] showed that (D) ensures the existence of an operator $K \in \mathfrak{K}\big(ca(M,\mathscr{M})\big)$ with $\|U_{\mathrm{P}}^2 - K\| < 1$; see 5.3.1.4 for the definition of U_{P}. Thus we have quasi-compactness, and the uniform ergodic theorem holds.

Every deterministic transformation $\varphi : M \to M$ defines a stochastic kernel P on $\mathscr{P}(M) \times M$, where the probability $\mathsf{P}(A,\xi)$ is 1 for $\varphi(\xi) \in A$ and 0 otherwise. If (D) is true, then it follows from $\mathsf{P}\big(\{\varphi(\xi)\},\xi\big) = 1$ that $\mu\big(\{\varphi(\xi)\}\big) > \varepsilon$ for all $\xi \in M$. Hence $\varphi(M)$ must be finite. Consequently, in the infinite case, Doeblin's property makes sense only for "genuine" random transformations.

5.3.5.8 The *Ergebnisbericht* [HOPF] had a strong impact on the early development of ergodic theory. I also refer to the surveys of (George) Birkhoff [1942] and Kakutani [1950] as well as to [JAC]. A *"question of priority"* between George Birkhoff and von Neumann is discussed in a paper of Zund [2002•]. Modern textbooks are [KREN] and [PET].

5.4 Convexity, extreme points, and related topics

In this section, all linear spaces are supposed to be *real*. Nevertheless, complex scalars are not excluded, since linear spaces over $\mathbb{C}$ can be regarded as linear spaces over $\mathbb{R}$.

5.4.1 The Kreĭn–Milman theorem

5.4.1.1 Let C be a convex subset in a linear space. We refer to $x_0 \in C$ as an **extreme point** if $x_0 \pm x \in C$ implies $x = o$.

In the words of Minkowski [≤1909, p. 157]:

Wir bezeichnen einen Punkt $\mathfrak{p}$ *in* $\mathfrak{K}$ [read: C] *als einen* **extremen Punkt** *von* $\mathfrak{K}$*, wenn* $\mathfrak{p}$ *auf keine Weise als inwendiger Punkt einer ganz in* $\mathfrak{K}$ *enthaltenen Strecke erscheint.*

All extreme points of C form the **extreme boundary**, denoted by $\partial_e C$.

5.4.1.2 The **Kreĭn–Milman theorem** says that every compact and convex subset K of a locally convex linear space is the closed convex hull of $\partial_e K$. In particular, every non-empty compact and convex subset has at least one extreme point.

Minkowski's original version [≤1909, p. 160] was stated in $\mathbb{R}^3$:

Überblicken wir diese Tatsachen, so erkennen wir, daß in einem konvexen Bezirk $\mathfrak{K}$ ein beliebiger Punkt stets entweder selbst ein extremer Punkt von $\mathfrak{K}$ ist oder einer Strecke oder einem Dreieck oder einem Tetraeder angehört, deren zwei, drei, vier Eckpunkte lauter extreme Punkte von $\mathfrak{K}$ sind.

Ein konvexer Bezirk $\mathfrak{K}$ ist auf diese Weise durch die Gesamtheit seiner extremen Punkte völlig bestimmt als der kleinste konvexe Bezirk, der alle diese Punkte aufnimmt.

The generalization from $\mathbb{R}^3$ to $\mathbb{R}^n$ is due to Carathéodory [1911, p. 200] and Steinitz [1913/16, Part III, p. 16].

Minkowski became interested in convex bodies in connection with his famous *Geometric Number Theory*, Carathéodory [1911, p. 195] *bediente sich der Sprache der Geometrie im n-dimensionalen Raume* when he investigated Fourier coefficients of positive harmonic functions, and Steinitz used these tools in his studies of conditionally convergent series. In summary, the latter [1913/16, Part I, p. 129] stated:

Die Theorie der konvexen Punktmengen hat sich in neuerer Zeit überhaupt mehr und mehr als ein wichtiges Hilfsmittel für Untersuchungen in den verschiedensten Gebieten der Mathematik bewährt.

5.4.1.3 The first result in the infinite-dimensional setting is due to Price [1937, p. 63], who considered norm compact and convex subsets of a strictly convex Banach space. The real breakthrough was achieved by Kreĭn/Milman [1940, p. 134] when they treated bounded and regularly convex subsets of a dual Banach space. A little bit later, Yosida/Fukamiya [1941, p. 49, footnote 3] discovered the same result.

As shown in 3.3.3.6, the regularly convex subsets of a dual Banach space are just the weakly* closed convex subsets. Hence, by Alaoglu's theorem, bounded and regularly convex subsets are compact in the weak* topology; see Shmulyan [1939c, p. 332]. Thus it came as no surprise when Kelley [1951] established the Kreĭn–Milman theorem for compact convex subsets in all topological linear spaces having the following separation property:

For any compact convex set K and any point x not in K there is a continuous linear functional whose value at x is greater than its value at any point of K.

The crucial idea of Kelley's paper was not his "slight generalization" of the statement, but the proof that uses Zorn's lemma instead of transfinite arguments. In fact, this discovery had already been made by Godement [1948, pp. 36–38], whose approach inspired the presentation in [BOU_{5a}, Chap. II, pp. 81–84]. Curiously, almost at the same time the algebraist E. Artin [1950] wrote a letter to Zorn in which he communicated a proof based on the "unmentionable lemma."

5.4.1.4 An important supplement to the Kreĭn–Milman theorem goes back to Milman [1947, стр. 119]; its modern version is due to [BOU$_{5a}$, Chap. II, p. 84].

Let K be a compact subset of a locally convex space whose closed convex hull is compact as well. Then all extreme points of $\overline{\text{conv}}(K)$ are contained in K.

5.4.1.5 In what follows, when speaking of the extreme points of a Banach space X we mean the extreme points of its closed unit ball $B_X := \{x \in X : \|x\| \leq 1\}$. Looking at these objects makes sense only if they are somehow exceptional. In this respect, the spaces L_p with $1 < p < \infty$ turn out to be trivial. More precisely, if the closed unit ball is "round", then *all* points on the sphere $S_X := \{x \in X : \|x\| = 1\}$ are extreme; see 5.5.4.2.

5.4.1.6 Interesting examples of extreme points were given by Kreĭn/Milman [1940, pp. 136–137]; see also 5.4.5.4.

For every topological space M, they observed that f is an extreme point of $C_b(M)$ if and only if $|f(t)| = 1$ for all $t \in M$. In the special case of $C[0,1]$ there are only two extreme points: the constant functions $+1$ and -1; see Price [1937, p. 66].

The space $L_1(M, \mathscr{M}, \mu)$ has no extreme points whenever $(M, \mathscr{M}, \mu)$ is atomless.

As a consequence of these facts, Kreĭn/Milman inferred that $C[0,1]$ and $L_1[0,1]$ are, in the isometric sense, *not adjoint to any Banach space*; compare with 5.1.4.2.

Another corollary (folklore) says that there exist equivalent norms on l_1 under which l_1 fails to be a dual space: Just take a hyperplane, say $\sum_{k=1}^{\infty} \xi_k = 0$, whose unit ball has no extreme points.

By the way, using the Kadets–Klee property 5.5.3.9, Davis/Johnson [1973b, p. 487] proved that every non-reflexive Banach space can be renormed such that it is not isometric to a dual space.

5.4.1.7 The extreme points of a $C^\star$-algebra were identified by Kadison [1951, p. 326]. In particular, for the full operator algebra $\mathfrak{L}(H)$ he identified all partial isometries U for which at least one of the products $U^\star U$ or $UU^\star$ is the identity.

From [SAKAI, pp. 10–14] we know that the extreme points of the collection of all positive elements with norm less than or equal to 1 are just the projections.

5.4.1.8 Arens/Kelley [1947, p. 502] showed that for any compact Hausdorff space K, the extreme points of $C(K)^*$ are given by the Dirac measures $+\delta_t$ as well as their antipodes $-\delta_t$.

5.4.1.9 Isometries of Banach spaces carry extreme points into extreme points. This observation was used by Arens/Kelley [1947, p. 503] for a simple proof of the Banach–Stone theorem 4.5.5; see also [DUN$_1^+$, pp. 442–443]. The same technique applies to the study of isometries in $C^\star$-algebras; see Kadison [1951, p. 330]. The commutative case, namely $C(K)$, had already been treated by Milman [1948, стр. 1242–1243].

5.4.1.10 Next, I present an example that goes back to Bogoliouboff/Kryloff [1937, pp. 111–112].

Let $\mathscr{M}$ be a σ-algebra of subsets of any set M, fix a family Φ of $\mathscr{M}$-measurable maps φ from M into itself, and denote by K the collection of all Φ-invariant normalized measures μ on $\mathscr{M}$:

$$\mu(\varphi^{-1}(A)) = \mu(A) \quad \text{for } \varphi \in \Phi \text{ and } A \in \mathscr{M}.$$

Then the extreme boundary $\partial_e K$ consists of the **ergodic** members $\mu \in K$:

$$\mu\big(A \triangle \varphi^{-1}(A)\big) = 0 \quad \text{for all } \varphi \in \Phi \quad \text{implies} \quad \text{either } \mu(A) = 0 \text{ or } \mu(\complement A) = 0.$$

For a proof of this fact the reader is referred to [PHE$_1$, 2nd edition, p. 75].

5.4.1.11 By a **convex polyhedron** we mean a set that is the convex hull of a finite number of points.

Klee [1960a, p. 266] showed that the closed unit ball of any finite-dimensional subspace of c_0 is always a polyhedron. Hence c_0 does not contain *isometric* copies of l_2^n for $n \ge 2$. Of course, by Dvoretzky's theorem 6.1.2.2, there exist "almost" isometric copies.

5.4.2 Integral representations

5.4.2.1 The theory of integral representations was predicted by Kreĭn/Milman [1940, p. 136]:

Our theorem permits us to say that every point of a regularly convex space is, in a certain sense, the center of gravity of masses, distributed on the extreme points of this set.

In fact, this physical formulation goes back to Carathéodory [1911, p. 200]:

Die Punkte des kleinsten konvexen Bereichs $\mathfrak{K}$, der eine abgeschlossene Punktmenge $\mathfrak{M}$ enthält, können sämtlich als Schwerpunkte von Massenbelegungen auf der Menge $\mathfrak{M}$ gedeutet werden, wobei alle in Betracht kommenden Massen positiv sind und die Gesamtmasse 1 *besitzen.*

5.4.2.2 Let μ be a normalized Baire measure on a compact and convex set K of a locally convex linear space. Here *normalized* means that $\mu(K) = 1$. Then there exists a unique element $x_0 \in K$ such that

$$\langle x_0, x^* \rangle = \int_K \langle x, x^* \rangle \, d\mu(x) \quad \text{for all } x^* \in X^*.$$

One refers to x_0 as the **barycenter**, or resultant, of μ and writes

$$x_0 = \int_K x \, d\mu(x).$$

5.4.2.3 Given normalized Baire measures μ and ν, we obtain a partial ordering by letting $\mu \preceq \nu$ whenever

$$\int_K f(x)d\mu(x) \le \int_K f(x)d\nu(x) \tag{5.4.2.3.a}$$

for all convex continuous real-valued functions f on K. Since the f's attain their maxima on $\partial_e K$, the relation $\mu \preceq \nu$ intuitively means that the mass of ν is more concentrated in the neighborhoods of $\partial_e K$ than that of μ; see [CHO, pp. 119–120].

Substituting the functions $f(x) = \pm\langle x, x^*\rangle$ into (5.4.2.3.a) yields

$$\int_K \langle x, x^*\rangle d\mu(x) = \int_K \langle x, x^*\rangle d\nu(x) \quad \text{for all } x^* \in X^*.$$

Hence μ and ν have the same barycenters.

The key lemma says that every normalized Baire measure μ is majorized by a maximal normalized Baire measure μ_0:

$$\mu \preceq \mu_0 \quad \text{and} \quad \mu_0 \preceq \nu \;\Rightarrow\; \mu_0 = \nu.$$

Applying this result to the Dirac measure δ_x produces a maximal measure μ_x with the barycenter $x \in K$.

Though the definition of $\preceq$ was invented by Choquet [1960, p. 335], the use of partial orderings for proving the existence of representing measures goes back to Bishop/de Leeuw [1959, p. 307].

The final result is the **Choquet–Bishop–de Leeuw theorem** or, for short,
Choquet's theorem:
Every point x_0 in a compact and convex subset K of a locally convex linear space is the barycenter of a maximal normalized Baire measure μ on K.

5.4.2.4 In this paragraph, I discuss the question in which sense a maximal normalized Baire measure μ is "almost" located on the extreme boundary. An observation of Milman [1947, p. 119] suggested that $\mu(\complement\overline{\partial_e(K)}) = 0$; see [CHO, p. 133]. However, this statement is pointless, since $\overline{\partial_e(K)}$ can be very big. One may even have $\overline{\partial_e(K)} = K$. Take, for example, the closed unit ball of l_2 equipped with the weak topology. Next, Choquet [1956, p. 7] discovered that the situation becomes pleasant by assuming K to be metrizable. Then $\partial_e(K)$ is a G_δ-set, and without passing to its closed hull, $\mu(\complement\partial_e(K)) = 0$ holds. The general case was treated by Bishop/de Leeuw [1959]. In the concluding section of their seminal paper, these authors constructed various counter-examples. They showed that $\partial_e(K)$ can be "arbitrarily bad"; in particular, it need not be a Borel set. Hence the required property $\mu(\complement\partial_e(K)) = 0$ may have no meaning. Nevertheless, a weaker conclusion is obtained: $\mu(B) = 0$ for every Baire subset of K with $B \cap \partial_e(K) = \emptyset$.

5.4.2.5 Of course, one was interested to find necessary and sufficient conditions under which the representing measures in Choquet's theorem are unique.

Let K be a compact and convex subset of a locally convex linear space X, and note that

$$C := \{\, (\lambda x, \lambda) : x \in K,\; \lambda > 0 \,\}$$

is a convex cone in the direct sum $X \oplus \mathbb{R}$. If the subspace $C - C$ becomes a linear lattice with respect to the induced partial ordering, then K is called a **(Choquet) simplex**. Choquet/Meyer [1963, pp. 145–146] showed that this simple, but non-illustrative, property characterizes the required uniqueness.

Equivalent conditions can be formulated in terms of the space $A(K)$ that is formed by all *affine* continuous real-valued functions f on K:

$$f\big((1-\lambda)x_0+\lambda x_1\big) = (1-\lambda)f(x_0)+\lambda f(x_1) \quad \text{for } x_0, x_1 \in K \text{ and } 0<\lambda<1.$$

We equip $A(K)$ with its natural partial ordering and the sup-norm.

- $A(K)$ has the Riesz decomposition property:
  Given non-negative functions $f, g_1, \dots, g_n$ such that $f \le \sum_{k=1}^n g_k$, there are non-negative functions$f_1, \dots, f_n$ such that $f = \sum_{k=1}^n f_k$ and $f_k \le g_k$.
- The dual $A(K)^*$ is a linear lattice.
- The dual $A(K)^*$ is an abstract L-space .

These criteria are due to Lindenstrauss [LIND, pp. 62–66] and Semadeni [1965, p. 144]; see also [LAC, pp. 183–186].

Bauer [1961, p. 120] proved that $A(K)$ is a Banach lattice if and only if the simplex K has a closed extreme boundary. Then every continuous function $f : \partial_e(K) \to \mathbb{R}$ admits a unique affine continuous extension to K. In this way, $A(K)$ and $C\big(\partial_e(K)\big)$ can be identified.

Poulsen [1961] constructed a simplex in which the extreme points are dense.

5.4.2.6 Let K be any compact Hausdorff space. Then all positive and normalized functionals in $C(K)^*$ form a simplex whose extreme points are the Dirac measures δ_t with $t \in K$; see 5.4.1.8. Every normalized Baire measure is represented by itself:

$$\mu = \int_K \delta_t \, d\mu(t).$$

5.4.2.7 Next, I discuss a standard example that illustrates Choquet's integral representation.

A complex-valued function f on $\mathbb{R}$ is called **positive definite** if

$$\sum_{h=1}^n \sum_{k=1}^n \zeta_h f(s_h - s_k) \overline{\zeta}_k \ge 0$$

for any choice of $s_1, \dots, s_n \in \mathbb{R}$, $\zeta_1, \dots, \zeta_n \in \mathbb{C}$, and $n = 1, 2, \dots$.

The **Bochner theorem** says that every positive definite continuous function f can be written in the form

$$f(s) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} \, d\mu(t), \tag{5.4.2.7.a}$$

where μ is a finite Baire measure: $\mu(\mathbb{R}) = f(0)$.

In order to identify (5.4.2.7.a) as a Choquet representation, one has to check that the convex set of all positive definite continuous functions f with $f(0) \leq 1$ is weakly* compact in $L_\infty(\mathbb{R}) = L_1(\mathbb{R})^*$. Its extreme points are the functions $e_t(s) := e^{ist}$ with $t \in \mathbb{R}$ as well as the zero function.

The considerations above can be extended to functions defined on locally compact abelian groups; see Raĭkov [1940] and [WEIL$_2$, pp. 56–57, 122]. The classical Bochner theorem appears in [BOCH$_1$, pp. 74–77, 224]. An account of its long history is given in [HEW$_2^+$, p. 325].

5.4.2.8 Let $\mathscr{H}_1^{\text{harm}}(\mathbb{D})$ denote the collection of all harmonic functions u on the open unit disk $\mathbb{D} := \{(r,\theta) : 0 \leq r < 1,\ 0 \leq \theta < 2\pi\}$ for which

$$\|u|\mathscr{H}_1^{\text{harm}}\| := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(r,\theta)|\,d\theta$$

is finite. This class was introduced by Plessner [1923, p. 13]. However, as in the case of the Hardy class $\mathscr{H}_1(\mathbb{D})$, the Banach space structure of $\mathscr{H}_1^{\text{harm}}(\mathbb{D})$ became relevant much later; see 6.7.12.3.

Let $\mathscr{H}_+^{\text{harm}}(\mathbb{D})$ stand for the collection of all non-negative harmonic functions u on $\mathbb{D}$. Using the mean value property, Plessner [1923, p. 16] observed that

$$\frac{1}{2\pi r}\int_0^{2\pi} |u(r,\theta)|\,d\theta = \frac{1}{2\pi r}\int_0^{2\pi} u(r,\theta)\,d\theta = u(0,0)$$

whenever $u \in \mathscr{H}_+^{\text{harm}}(\mathbb{D})$. Hence $\mathscr{H}_+^{\text{harm}}(\mathbb{D}) \subset \mathscr{H}_1^{\text{harm}}(\mathbb{D})$. More precisely, $\mathscr{H}_+^{\text{harm}}(\mathbb{D})$ is a generating convex cone of $\mathscr{H}_1^{\text{harm}}(\mathbb{D})$.

The **Poisson kernel** is defined by

$$P(r,\theta) := \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

A classical theorem of Herglotz [1911, p. 511] says that the **Poisson formula**

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta - t)\,d\mu(t) \tag{5.4.2.8.a}$$

yields a one-to-one correspondence $\mu \mapsto u$ between the (non-negative) Baire measures on $\mathbb{T}$ and the non-negative harmonic functions on $\mathbb{D}$:

$$\|u|\mathscr{H}_1^{\text{harm}}\| = u(0,0) = \mu(\mathbb{T}) = \|\mu\|.$$

In this way, the Dirac measures δ_t pass into the rotated Poisson kernels $P(r,\theta - t)$. Next, $\mathscr{H}_1^{\text{harm}}(\mathbb{D}) = \mathscr{H}_+^{\text{harm}}(\mathbb{D}) - \mathscr{H}_+^{\text{harm}}(\mathbb{D})$ implies that the map $\mu \mapsto u$ extends to an isometry between $C(\mathbb{T})^*$ and $\mathscr{H}_1^{\text{harm}}(\mathbb{D})$. The functional $\ell \in C(\mathbb{T})^*$ that yields $u \in \mathscr{H}_1^{\text{harm}}(\mathbb{D})$ is obtained as follows:

Define

$$\ell_r(f) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta) f(\theta)\,d\theta \quad \text{for } f \in C(\mathbb{T}) \text{ and } 0 \leq r < 1.$$

Then, by Alaoglu's theorem and $\|\ell_r\| \le \|u|\mathscr{H}_1^{\mathrm{harm}}\|$, we obtain the desired preimage ℓ as the weak* limit of a subsequence (ℓ_{r_n}), where $r_n \nearrow 1$. This proof is due to Plessner [1923, p. 14], who used Helly's *Auswahlprinzip*; see 3.4.2.4.

Finally, it turns out that the non-negative harmonic functions with $u(0,0)=1$ form a compact and convex set with respect to the topology of uniform convergence on the compact subsets of $\mathbb{D}$. Its extreme points are the rotated Poisson kernels $P(r,\theta-t)$ indexed by $t\in\mathbb{T}$; see Holland [1973] for an elementary proof. In this sense, (5.4.2.8.a) is the ancestor of Choquet's integral representation.

5.4.2.9 I conclude with a list of monographs devoted to Choquet's theory: [ALF], [BECK], [CHO], and [PHE$_1$].

5.4.3 Gelfand–Naĭmark–Segal representations

[HEW$_1^+$, p. 311]: *A central technique in contemporary analysis is the study of topologico-algebraic objects of a given class by means of their continuous homomorphisms into the most elementary objects of the same class.*

5.4.3.1 A **representation** of a $C^\star$-algebra $\mathscr{A}$ by operators on a Hilbert space H is a $\star$-homomorphism $\pi : \mathscr{A} \to \mathfrak{L}(H)$:

$$\pi(\xi_1 a_1 + \xi_2 a_2) = \xi_1 \pi(a_1) + \xi_2 \pi(a_2) \quad \text{for } a_1, a_2 \in \mathscr{A} \text{ and } \xi_1, \xi_2 \in \mathbb{C},$$

$$\pi(ab) = \pi(a)\pi(b) \quad \text{and} \quad \pi(a^\star) = \pi(a)^\star \quad \text{for } a,b \in \mathscr{A}.$$

As observed by Gelfand/Naĭmark [1948, § 2, Theorem 2], we automatically have continuity: $\|\pi(a)\| \le \|a\|$.

5.4.3.2 Following Segal [1947, p. 75], by a **state** on a $C^\star$-algebra $\mathscr{A}$ we mean a positive linear functional $\omega : \mathscr{A} \to \mathbb{C}$ of norm one:

$$\omega(x^\star x) \ge 0 \quad \text{for } x \in \mathscr{A} \qquad \text{and} \quad \|\omega\| = 1.$$

Positivity is equivalent to $\|\omega\| = \omega(\mathbf{1})$; see Bohnenblust/Karlin [1955, p. 226] and [SAKAI, p. 9].

Here are two basic formulas:

$$\omega(x^\star y) = \overline{\omega(y^\star x)} \quad \text{and} \quad |\omega(y^\star x)|^2 \le \omega(x^\star x)\omega(y^\star y) \quad \text{for } x,y \in \mathscr{A}.$$

With every state ω we associate its **Gelfand–Naĭmark–Segal representation:** Form the left ideal $\mathscr{N}_\omega := \{x \in \mathscr{A} : \omega(x^\star x) = 0\}$, and denote the members of the quotient $\mathscr{A}/\mathscr{N}_\omega$ by $\chi(x) = x + \mathscr{N}_\omega$. Letting $\big(\chi(x)|\chi(y)\big) := \omega(y^\star x)$ yields an inner product, and the required Hilbert space H_ω is obtained by completion. Note that $\pi_\omega(a) : \chi(x) \mapsto \chi(ax)$ defines a linear map on $\mathscr{A}/\mathscr{N}_\omega$ that admits a continuous extension. In this way, we assign to every element $a \in \mathscr{A}$ an operator $\pi_\omega(a)$ acting on H_ω. Obviously, π_ω is a representation.

5.4.3.3 A representation π is called **irreducible** if there does not exist any subspace M, different from $\{o\}$ and H that is invariant under all operators $\pi(a)$ with $a \in \mathscr{A}$.

Segal [1947, pp. 79–80] proved that the representation π_ω associated with ω is irreducible if and only if the underlying state is an extreme point of the collection of all states. Since this collection turns out to be a weakly* compact and convex subset in the dual of $\mathscr{A}$, the Kreĭn–Milman theorem yields the existence of sufficiently many irreducible representations.

Extreme states ω are also referred to as **pure**. They can be characterized as follows: Every linear functional φ such that $0 \le \varphi(x^\star x) \le \omega(x^\star x)$ for $x \in \mathscr{A}$ is of the form $\varphi = c\,\omega$ with $0 \le c \le 1$.

In the case of a commutative $C^\star$-algebra $C(K)$ one just gets the Dirac measures or, from the algebraic point of view, the multiplicative linear functionals; see 4.10.2.5, 5.4.1.8, and Kreĭn/Kreĭn [1940, p. 429].

5.4.4 The Radon–Nikodym property: geometric aspects

5.4.4.1 A point x_0 in a subset A of X is called **exposed** if there exists a functional $x_0^* \in X^*$ such that $\langle x, x_0^* \rangle < \langle x_0, x_0^* \rangle$ for all $x \in A \setminus \{x_0\}$. In the finite-dimensional case, this concept was introduced by Straszewicz [1935, p. 139]. Then

$$x_n \in A \text{ and } \langle x_n, x_0^* \rangle \to \langle x_0, x_0^* \rangle \quad \text{imply} \quad \|x_n - x_0\| \to 0. \tag{5.4.4.1.a}$$

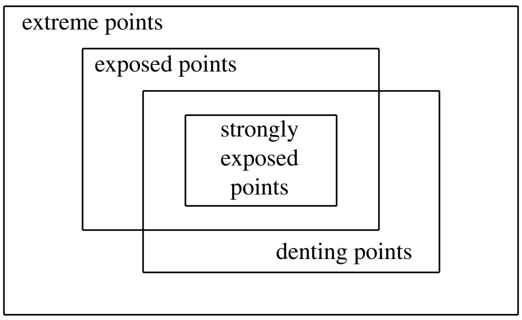
In general Banach spaces, the latter conclusion does not hold automatically. Therefore Lindenstrauss [1963b, p. 140] says that a point $x_0 \in A$ is **strongly exposed** if we also have (5.4.4.1.a).

Rieffel [1966, p. 75] refers to $x_0 \in A$ as a **denting point** if

$$x_0 \notin \overline{\text{conv}}\{A \setminus B_\varepsilon(x_0)\} \quad \text{for all } \varepsilon > 0,$$

where $B_\varepsilon(x_0) := \{x \in X : \|x - x_0\| \le \varepsilon\}$.

The following diagram illustrates the relationship between the different kinds of points.



Some elementary examples can be found in [PHE$_2$, pp. 76–77].

5.4.4.2 The central result of the geometric theory says that the Radon–Nikodym property of a Banach space can be characterized by the following equivalent conditions:

Every non-empty bounded, closed, and convex subset

$$\begin{tikzcd}
\boxed{\text{A}}\ \begin{array}{c}\text{is contained}\\ \text{in the closed convex hull}\\ \text{of its strongly exposed points}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{A}_\text{o}}\ \begin{array}{c}\text{has at least one}\\ \text{strongly exposed point}\end{array} \arrow[d, Rightarrow] \\
\boxed{\text{B}}\ \begin{array}{c}\text{is contained}\\ \text{in the closed convex hull}\\ \text{of its denting points}\end{array} \arrow[r, Rightarrow] & \boxed{\text{B}_\text{o}}\ \begin{array}{c}\text{has at least one}\\ \text{denting point}\end{array} \arrow[ul, Rightarrow]
\end{tikzcd}$$

All implications indicated by $\Rightarrow$ and $\Downarrow$ are trivial. The historical starting point for getting the central arrow was a fundamental result of Rieffel [1966, p. 71], which yields that (B_o) ensures the Radon–Nikodym property. Phelps [1974, p. 85] closed this circle by showing that every space with the Radon–Nikodym property satisfies condition (A).

There were several intermediate steps:

First of all, Rieffel [1966, p. 71] introduced the concept of a **dentable set** A in which, by definition, for every $\varepsilon > 0$ we can find $x_\varepsilon \in A$ such that

$$x_\varepsilon \notin \overline{\text{conv}}\{A \setminus B_\varepsilon(x_\varepsilon)\}. \tag{5.4.4.2.a}$$

This means that in contrast to the existence of a denting point x_0, the x_ε's may depend on $\varepsilon > 0$. He proved that a Banach space has the Radon–Nikodym property whenever all non-empty bounded (or only all bounded, closed, and convex) subsets are dentable.

Another decisive contribution is due to Maynard, who modified the concept of dentability: in (5.4.4.2.a) the closed convex hull is replaced by the σ-convex hull,

$$\sigma\text{-conv}(B) := \Big\{ \sum_{k=1}^{\infty} \lambda_k x_k \ \text{(convergent)} : \ \sum_{k=1}^{\infty} \lambda_k = 1, \ \lambda_k \geq 0, \ x_k \in B \Big\}.$$

In this way, he arrived at the following criterion, [1973, p. 497]:

A Banach space B has the Radon–Nikodym property iff B is a σ-dentable space. The latter property means that every bounded set of B is σ-dentable.

Davis/Phelps [1974] and Huff [1974] were able, independently of each other, to remove the σ- in the final result, but not in its proof.

5.4.4.3 Let C denote a bounded, closed, and convex subset of X. Given $x^* \in X^*$ and $\varepsilon > 0$, we refer to

$$S(x^*, \varepsilon, C) := \Big\{ x \in C : \langle x, x^* \rangle \geq \sup_{u \in C} \langle u, x^* \rangle - \varepsilon \Big\}$$

as a **slice**. This concept appeared for the first time in a paper of Phelps [1974, p. 79], who made the obvious remark that C is dentable if and only if it possesses slices with arbitrarily small diameters.

5.4.4.4 One may consider spaces with the **Kreĭn–Milman property** in which, by definition, every non-empty bounded, closed, and convex subset

C: is the closed convex hull of its extreme points	$\Longleftrightarrow$	$\mathrm{C_o}$: has at least one extreme point

The implication $\Rightarrow$ is trivial and $\Leftarrow$ was shown by Lindenstrauss [1966a, p. 60].

[DIE_2^+, p. 209]:

Scanning the long list of spaces with the Kreĭn–Mil'man property and noting the resemblance to the known possessors of the Radon–Nikodým property, Diestel, in 1972, was unable to resist the temptation to ask: Are the Kreĭn–Mil'man and the Radon–Nikodým property equivalent?

Huff/Morris [1975] and Bourgain/Talagrand [1981] verified this conjecture for dual spaces and for Banach lattices, respectively. In the general case, an answer is still missing. Of course, since $(\mathrm{B_o})\Rightarrow(\mathrm{C_o})$, the Radon–Nikodym property is stronger than the Kreĭn–Milman property.

5.4.4.5 Another kind of point was introduced by Namioka [1967], who motivated his definition as follows:

Let C be a compact subset of a Banach space E. Then, of course, the norm topology and the weak topology agree on C. Now suppose that C is only weakly compact. Then the identity map: $(C, weak) \to (C, norm)$ is no longer continuous in general. Nevertheless one may still ask how the set of points of continuity of this map is distributed in C.

Let C be a bounded, closed, and convex subset of a Banach space. Then $x_0 \in C$ is called a **point of** (weak-to-norm) **continuity** if the identity map from C (equipped with the weak topology) onto C (equipped with the norm topology) is continuous at x_0.

Bourgain [1980a, pp. 138–139] said that a Banach space has property $(*)$ if every non-empty bounded, closed, and convex subset contains a point of continuity. Nowadays, one speaks of the **point of continuity property**.

Every denting point of a bounded, closed, and convex subset is an extreme point as well as a point of continuity. Schachermayer [1987, p. 673] was able to prove a striking converse:

If there are "many" extreme points and "many" points of weak-to-norm continuity, then there are in fact "many" denting points.

Using the abbreviations

PCP : point of continuity property,
KMP : Kreĭn-Milman property,
RNP : Radon–Nikodym property,

we have the shorthand formula

$$\text{PCP} + \text{KMP} = \text{RNP}.$$

Surprisingly, the three properties are equivalent on Banach lattices; see [GHO$^+$, p. 75].

5.4.4.6 Localization in the context of Radon–Nikodym theory means that one assumes a certain property for a *fixed* subset and *not for all* subsets. This "local" standpoint appeared in the late 1970s, for example, in a paper of Bourgain [1977]. In contrast to the "global" view of [DIE$_2^+$], it was stressed in the lecture notes [BOUR$_1$] and [BOUR$_3$].

Recall that a Banach space X has the Radon–Nikodym property if given any finite measure space $(M, \mathscr{M}, \mu)$, for every μ-continuous X-valued measure $\boldsymbol{m}$ on $\mathscr{M}$ with bounded variation there exists a Bochner integrable X-valued function $\boldsymbol{f}$ such that

$$\boldsymbol{m}(A) = \int_A \boldsymbol{f}(t)\, d\mu(t) \quad \text{for } A \in \mathscr{M}.$$

In order to define the **Radon–Nikodym property** for a bounded, closed, and convex subset C, we take into account only those X-valued measures $\boldsymbol{m}$ for which

$$\boldsymbol{m}(A)/\mu(A) \in C \quad \text{whenever } A \in \mathscr{M} \text{ and } \mu(A) > 0.$$

By the way, the latter condition ensures that $\boldsymbol{m}$ is μ-continuous.

In this new setting, the classical theorem that asserts that every reflexive Banach space has the Radon–Nikodym property reads as follows:

Every weakly compact and convex set has the Radon–Nikodym property.

Next, I state the modified version of Rieffel's criterion; see Namioka/Asplund [1967, p. 443], [BENY$_1^+$, p. 104], and [BOUR$_3$, p. 31]:

A bounded, closed, and convex set has the Radon–Nikodym property if and only if each of its closed and convex subsets is dentable.

It also turned out that every bounded, closed, and convex subset with the Radon–Nikodym property is the closed convex hull of its strongly exposed points; see [BENY$_1^+$, p. 110], [BOUR$_3$, p. 55], and 5.5.4.4.

5.4.4.7 Further information on the geometric Radon–Nikodym theory can be found in [DIE$_2^+$], [BENY$_1^+$, Chap. 5], [BOUR$_1$], [BOUR$_3$], and [PHE$_2$, Section 5], as well as in surveys of James [1980] and Fonf/Lindenstrauss/Phelps [2001].

Complex analogues of extreme points, exposed points, denting points, etc. are discussed in Subsection 6.9.3.

5.4.5 Convex and concave functions

5.4.5.1 A real-valued function f on a convex subset C is called **convex** if

$$f\big((1-\lambda)x_0+\lambda x_1\big) \le (1-\lambda)f(x_0)+\lambda f(x_1) \quad \text{for } x_0,x_1 \in C \text{ and } 0<\lambda<1.$$

In the opposite case, when

$$f\big((1-\lambda)x_0+\lambda x_1\big) \ge (1-\lambda)f(x_0)+\lambda f(x_1) \quad \text{for } x_0,x_1 \in C \text{ and } 0<\lambda<1,$$

the function f is said to be **concave**.

For many purposes, it is suitable to consider convex functions that are *lower semi-continuous* and may take the value $+\infty$.

5.4.5.2 With every convex function $f : C \to \mathbb{R}$ we associate its **conjugate**

$$f^*(x^*) := \sup\big\{\langle x,x^*\rangle - f(x) \,:\, x\in C\big\}.$$

Note that f^* is a convex function on the convex set $C^* := \{x^* \in X^* : f^*(x^*) < \infty\}$.

Iterating this process yields the **biconjugate** f^{**}. As shown by Brøndsted [1964, pp. 17–18], the relation $f(x) = f^{**}(x)$ holds for all points $x\in C$ at which f is lower semi-continuous.

In the finite-dimensional setting, the concept of a conjugate function was invented by Fenchel [1949]. Termed "komplementär," conjugate functions had already occurred in the theory of Orlicz spaces; see 6.7.14.3.

5.4.5.3 Suppose that the convex set C is open. Then the **subgradient** of a convex function $f : C \to \mathbb{R}$ at the point $x_0\in C$ is defined by

$$\partial f(x_0) := \big\{x^* \in X^* \,:\, f(x) \ge f(x_0) + \langle x - x_0, x^*\rangle \quad \text{if } x\in C\big\};$$

see Moreau [1963] (in Hilbert spaces) and Brøndsted/Rockafellar [1965]. The multi-valued function $\partial f : x \mapsto \partial f(x)$ is called the **subdifferential** of f. For continuous convex functions, $\partial f(x_0)$ is a singleton precisely when f has a Gâteaux derivative at x_0; see 5.1.8.3.

It easily turns out that f attains its minimum at x_0 if and only if $\mathsf{o}\in\partial f(x_0)$.

5.4.5.4 **Bauer's maximum principle** [1958, p. 392] asserts that every convex upper semi-continuous real-valued function on a compact and convex subset of a locally convex linear space attains its maximum at some extreme point. For a simplified proof the reader is referred to [CHO, p. 102].

The preceding result is by no means trivial, since the extreme boundary need not be closed. Indeed, Straszewicz [1935, p. 143] observed that this pathology already occurs in $\mathbb{R}^3$. Take, for example, the convex hull of the set A formed by the circle $\{(x,y,0) : (x-1)^2+y^2=1\}$ and the points $(0,0,\pm1)$. Then all members of A except $(0,0,0)$ are extreme points.

5.4.5.5 Suppose that A and B are compact convex subsets of locally convex linear spaces. Let $f : A \times B \to \mathbb{R}$ be a continuous function such that $f(x,y)$ is convex in x and concave in y. Then

$$\min_{x\in A}\max_{y\in B} f(x,y) = \max_{y\in B}\min_{x\in A} f(x,y).$$

This **minimax theorem** goes back to von Neumann [1929]. More general versions are due to (Ky) Fan [1952, 1953], Kneser [1952], Sion [1958], Heinz König [1968], and Terkelsen [1972]. (Ky) Fan based his proof on a fixed point theorem, while König used the Hahn–Banach theorem.

Minimax theorems belong to game theory. However, there are applications to other fields. For example, (Ky) Fan's theorem has been successfully used in the theory of p-summing operators; see [PIE_3, pp. 40, 232, 237].

5.4.6 Lyapunov's theorem and the bang-bang principle

5.4.6.1 Let $\boldsymbol{m}$ be a countably additive X-valued measure on a σ-algebra $\mathscr{M}$. A subset $A \in \mathscr{M}$ with $\boldsymbol{m}(A) \neq 0$ is called an **atom** if $A_0 \in \mathscr{M}$ and $A_0 \subseteq A$ imply either $\boldsymbol{m}(A_0) = o$ or $\boldsymbol{m}(A_0) = \boldsymbol{m}(A)$. By definition, **non-atomic** vector measures have no atoms.

A famous theorem of Lyapunov [1940] says that every non-atomic countably additive vector measure taking its values in a finite-dimensional Banach space has a convex compact range $\boldsymbol{m}(\mathscr{M}) := \{\boldsymbol{m}(A) : A \in \mathscr{M}\}$.

An elegant proof was given by Lindenstrauss [1966b].

Lyapunov [1946] constructed a non-atomic countably additive l_2-valued measure (of bounded variation) for which the preceding assertion fails. One can even show that Lyapunov's theorem holds only in finite-dimensional spaces; see [DIE_2^+, p. 265].

5.4.6.2 In the infinite-dimensional case, we know from 5.1.6.2 that $\boldsymbol{m}(\mathscr{M})$ is relatively weakly compact. Therefore it makes sense to look for exposed points. The following result is due to Anantharaman [1972, pp. 19–20]:

The range $\boldsymbol{m}(\mathscr{M})$ and its closed convex hull have the same exposed points, and these are even strongly exposed.

5.4.6.3 According to Rybakov [1970, стр. 250], for every countably additive X-valued measure $\boldsymbol{m}$ there exists $x_0^* \in X^*$ such that $\boldsymbol{m}$ is absolutely continuous with respect to the scalar measure $\langle \boldsymbol{m}, x_0^* \rangle$.

Anantharaman [1972, pp. 16–17] characterized the required functional by the property that it exposes some $\boldsymbol{m}(A_0)$:

$$\langle \boldsymbol{m}(A), x_0^* \rangle < \langle \boldsymbol{m}(A_0), x_0^* \rangle \quad \text{for all } A \in \mathscr{M} \text{ with } A \neq A_0.$$

5.4.6.4 Lyapunov's theorem is intimately related to the **bang-bang principle** of control theory whose preliminary version goes back to LaSalle [1960, p. 7]:

Any state of a system that can be reached by a control taking values in a convex compact set can also be reached by a control whose values are extreme points of this set.

5.5 Geometry of the unit ball

This section deals with some aspects of the *metric* theory of Banach spaces, which means that the properties under consideration depend on the given norms.

The process of passing to an equivalent norm is referred to as **renorming**. We adopt a notational convention proposed by Klee [1953, p. 36]: a Banach space will be called **$\mathbb{P}$-able** if there exists an equivalent norm that has property $\mathbb{P}$. Examples: convexifiable, smoothable.

Following an advice of Diestel [DIE$_1$, Preface], we assume that all scalars are real: *Many of the proofs hold with minor modifications for the complex case as well. However, several proofs require rather drastic surgery to be adapted to the complex case; rather than take a chance with "a successful operation in which the patient died", I have presented only the proofs for real scalars feeling that here is where the intuition best serves valid understanding of the geometric phenomena discussed.*

Diestel's rule has (of course!) exceptions. In Subsection 5.5.5, when considering vector-valued analytic functions, we will need complex scalars.

Here is a list of related monographs (chronological order):
[DAY, Chap. VII], [DIE$_1$], [SCHÄ], [IST], [DEV$^+$], [HAB$^+$, Chap. 11–12], [MEG, Chap. 5], [FAB$^+$, Chap. 8–10].
The reader is also referred to the surveys of Milman [1971a] and Godefroy [2001].

5.5.1 Strict convexity and smoothness

5.5.1.1 A Banach space X or, more precisely, its norm is called **strictly convex** if

$$\|x_0+x_1\| = \|x_0\|+\|x_1\| \;\text{ and }\; x_0 \neq o, x_1 \neq o \quad \text{imply} \quad x_1 = cx_0 \quad \text{with some } c > 0.$$

In [DAY, 1st edition, p. 111], the name *strictly convex* was replaced by **rotund**. Since this proposal has been accepted only partly, the modern terminology is non-uniform.

The concept above is usually attributed to Clarkson [1936, p. 404, footnote]. Termed as **strictly normed** (строго нормируемо), spaces with such a norm were also considered by Kreĭn [1938, pp. 183–184] and Shmulyan [1939b, стр. 77]. The first author observed that X is strictly normed if and only if every non-zero functional can attain its norm in at most one point. He also stated an equivalent property that is now commonly used as a definition:

$$\left\|\frac{x_0+x_1}{2}\right\| < 1 \quad \text{whenever } \|x_0\| = \|x_1\| = 1 \text{ and } x_0 \neq x_1.$$

Kreĭn's paper was published in an "exotic" book [AKH$^+$], whose English translation appeared only in 1962. Therefore it is no surprise that both characterizations were rediscovered by Ruston [1949, p. 157].

The following quotation from Fréchet [1925a, pp. 39–40] shows that in contrast to the story told so far, the concept of strict convexity is, in fact, 10 years older:

Nous pourrons appeler **espace métrique** [strange!], *un espace* $(\mathscr{D})$ *vectoriel où on a*

$$\|\overline{AB}\| < \|\overline{AC}\| + \|\overline{CB}\|$$

toutes les fois que les points A, B, C *ne sont pas alignés.*

Comme exemple d'un tel espace, on peut indiquer non seulement les espaces usuels à 1, 2, 3 *et même n dimension, mais aussi l'espace de Hilbert.*

5.5.1.2 Let $x_0^* \neq \mathsf{o}$ and $c \in \mathbb{R}$. We refer to $M := \{u \in X : \langle u, x_0^* \rangle = c\}$ as a **supporting hyperplane** (*Stützebene*) of a set A if there exists a point $x_0 \in A$ such that

$$\langle x, x_0^* \rangle \leq c = \langle x_0, x_0^* \rangle \quad \text{for all } x \in A.$$

Generalizing Minkowski's classical theorem [≤1909, p. 139], Mazur [1933, p. 74] proved that every boundary point of a convex body lies on at least one supporting hyperplane.

Ascoli [1932, pp. 53–55] observed that for every norm and all $x, h \in X$, the one-sided limit

$$G_+(x,h) := \lim_{t \searrow 0} \frac{\|x+th\| - \|x\|}{t}$$

always exists. In a next step, Mazur [1933, p. 78] discovered that the two-sided limit

$$G(x,h) := \lim_{t \to 0} \frac{\|x+th\| - \|x\|}{t}$$

exists if and only if rB_X has a *unique* supporting hyperplane through the point x, where $r := \|x\| > 0$. In this case, $G(x,h)$ is automatically continuous and linear with respect to h. In other words, the norm is Gâteaux differentiable at x; see 5.1.8.3.

5.5.1.3 A Banach space X or, more precisely, its norm is said to be **(Gâteaux) smooth** if $\|\cdot\|$ is Gâteaux differentiable at every $x \neq \mathsf{o}$. An equivalent condition requires that there exist a tangent hyperplane (*Tangentialhyperebene*) through every point of the unit sphere. The term "smooth" was coined by Klee [1953, p. 36].

5.5.1.4 Obviously, the natural norms of the spaces L_p with $1 < p < \infty$ are strictly convex and smooth. This is not so in the limiting cases $p = 1$ and $p = \infty$. However, the situation can be improved by clever renormings.

First of all, Clarkson [1936, p. 413] discovered a strictly convex norm on $C[0,1]$:

$$|||f||| := \left(\sup_{0 \leq t \leq 1} |f(t)|^2 + \sum_{n=1}^{\infty} \tfrac{1}{2^{2n}} |f(t_n)|^2 \right)^{1/2},$$

where (t_n) is a dense sequence in $[0,1]$. He obtained the following corollary:

Any separable Banach space may be given a new norm, equivalent to the original norm, with respect to which the space is strictly convex.

Clarkson's result was improved by Day [1955, p. 519], who showed that every separable space admits a renorming that is at the same time strictly convex and smooth. Finally, Klee [1959, p. 56] added a third property: the dual norm can be made strictly convex too.

A theorem of Day [1957] says that all abstract L-spaces are strictly convexifiable. In contrast to this result, there exist abstract M-spaces that are neither strictly convexifiable nor smoothable; take, for example, $l_\infty(\mathbb{I})$ with an uncountable index set $\mathbb{I}$.

Day [1955, pp. 523–525] discovered a useful norm on $c_0(\mathbb{I})$:
For every $x = (\xi_i)$ in $c_0(\mathbb{I})$, he defined *a sort of decreasing rearrangement*,

$$a_n(x) := \inf_{|\mathbb{F}|<n} \sup_{i \notin \mathbb{F}} |\xi_i|.$$

Then

$$|||x||| := \Bigg(\sum_{n=1}^{\infty} \frac{1}{2^{2n}} a_n(x)^2 \Bigg)^{1/2}$$

is strictly convex for all $\mathbb{I}$'s; see also 5.5.3.5.

5.5.1.5 The following table, in which $\mathbb{I}$ denotes an uncountable index set, is due to Day [1955, p. 517]:

		strictly convex renorming	
		yes	no
smooth renorming	yes	l_1	no example seems to be known
	no	l_∞, $l_1(\mathbb{I})$	$l_\infty(\mathbb{I})$

5.5.1.6 Shmulyan [1939b, стр. 82] obtained the following theorems:

Assume that the closed unit ball of a Banach space X is weakly sequentially compact. Then the norm of X^* is Gâteaux differentiable if and only if the norm of X is strictly convex.

Assume that every functional $x^* \in X^*$ attains its norm. Then the norm of X^* is strictly convex if and only if the norm of X is Gâteaux differentiable.

Nowadays, these statements may be summarized by saying that in the reflexive case, there is a full duality between strict convexity and smoothness. However, in 1939 it was still unknown that the two assumptions above characterize reflexivity.

According to Alaoglu/Birkhoff [1940, p. 301, footnote [13]], von Neumann remarked to the authors that the following conditions are equivalent (take care!):

The unit sphere has two tangent planes at some point.

The unit sphere in the conjugate space contains a straight line segment.

Without knowing Shmulyan's work, Klee [1953, p. 37] proved the following results for general Banach spaces:

If X^* is smooth, then X is strictly convex. If X^* is strictly convex, then X is smooth.

This is the best that can be obtained, since the converse implication fails in both cases. Indeed, we know from 5.5.1.5 that l_1 is strictly convexifiable, whereas l_∞ has no smooth renorming. In the second case, an implicit counterexample was constructed by Klee [1959, p. 62]:

Let N be a non-reflexive closed subspace of a separable Banach space X such that $\operatorname{cod}(N) \geq 2$. Then X admits a smooth norm for which X/N is not smooth.

Troyanski [1970] turned l_1 into a smooth Banach space whose dual lacks strict convexity.

5.5.2 Uniform convexity and uniform smoothness

5.5.2.1 Recall from 4.4.8 that a Banach space X or, more precisely, its norm is called **uniformly convex** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\|\frac{x_0+x_1}{2}\right\| \leq 1-\delta \quad \text{whenever } \|x_0\|=\|x_1\|=1 \text{ and } \|x_0-x_1\| \geq \varepsilon.$$

This property, which is due to Clarkson [1936, pp. 396–397], was characterized by Day [1944, p. 375] in terms of the **modulus of convexity**,

$$\delta_X(\varepsilon) := \inf\left\{1-\left\|\frac{x_0+x_1}{2}\right\| : \|x_0\|=\|x_1\|=1 \text{ and } \|x_0-x_1\|=\varepsilon\right\} \text{ for } 0<\varepsilon\leq 2.$$

The function $\varepsilon \mapsto \delta_X(\varepsilon)$ is non-decreasing, but may fail to be convex; Chi/Gurariĭ [1969, p. 74], Figiel [1976a, p. 124], [LIND$_2^+$, p. 67], and Liokumovich [1973, стр. 44].

Uniform convexity means that $\delta_X(\varepsilon)$ is always *strictly* positive.

5.5.2.2 Shmulyan [1940a, p. 645] refers to a norm as *fortement uniformément dérivable* (**uniformly Fréchet differentiable**) if the limit

$$G(x,h) := \lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$$

exists and is uniform for $\|x\|=1$ and $\|h\|=1$. He showed [1940a, p. 647] that this happens if and only if the dual space is uniformly convex.

An equivalent property, termed as **uniform flatteness**, was considered by Day [1944, p. 375]. He also introduced a **modulus of flattening**. However, his definitions did not survive in their original form, since they were *awkward to handle*; see [DAY, 3rd edition, p. 147]. Much better suited is the **modulus of smoothness**,

$$\rho_X(t) := \sup\left\{\frac{\|x+th\|+\|x-th\|}{2}-1 : \|x\|=\|h\|=1\right\} \quad \text{for } t>0.$$

This concept, due to Lindenstrauss [1963a, p. 241], was suggested by *Exercise* 15 in [BOU$_{5b}$, Chap. V, pp. 144–145]; see also [KÖT$_1$, pp. 366–368]. The function $t \mapsto \rho_X(t)$ is non-decreasing and convex; see [LIND$_2^+$, p. 64]. One has $\rho_X(t) \le t$.

A Banach space X or, more precisely, its norm is called **uniformly smooth** if $\rho_X(t) = o(t)$ as $t \to 0$.

5.5.2.3 Day [1944, p. 384] proved that X is uniformly convex/smooth if and only if X^* is uniformly smooth/convex. Nowadays this duality can be obtained from

$$\rho_{X^*}(t) = \sup_{0<\varepsilon\le 2} \{\varepsilon t/2 - \delta_X(\varepsilon)\} \quad \text{and} \quad \rho_X(t) = \rho_{X^{**}}(t); \tag{5.5.2.3.a}$$

see Lindenstrauss [1963a, pp. 242–243].

5.5.2.4 Uniformly convex spaces are reflexive; see Milman [1938], Kakutani [1939, p. 172], and Pettis [1939]. By duality, the same conclusion holds for uniformly smooth spaces. The latter case was first treated by Shmulyan [1940a, p. 648].

Day [1941] disproved the conjecture that every reflexive space is uniformly convexifiable or uniformly smoothable. Take, for example, $[l_2, l_\infty^n]$. This defect has led to the brilliant theory of superreflexivity to be discussed in Subsection 6.1.9.

5.5.2.5 The concepts of uniform convexity and uniform smoothness are not only dual to each other; they are even equivalent modulo renorming:

$$\begin{array}{c} X \quad \text{is uniformly convexifiable} \\ \Updownarrow \\ X \quad \text{is superreflexive} \\ \Updownarrow \\ X^* \quad \text{is superreflexive} \\ \Updownarrow \\ X^* \quad \text{is uniformly convexifiable} \\ \Updownarrow \\ X \quad \text{is uniformly smoothable}\ . \end{array}$$

It would be worthy to replace this tricky chain of reasoning by a direct construction.

5.5.2.6 So far we have stated that the existence of a uniformly convex norm implies the existence of an equivalent uniformly smooth norm. Even more is true: an averaging process due to Asplund [1967] yields an equivalent norm that has both properties simultaneously.

5.5.2.7 For the classical spaces, the asymptotic behavior of the moduli is given by the following formulas; see Hanner [1956, p. 244], Kadets [1956, стр. 187–190], and Lindenstrauss [1963a, p. 243]:

$$\begin{array}{lllll} \delta_{L_p}(\varepsilon) \asymp \varepsilon^2 & \text{and} & \rho_{L_p}(t) \asymp t^p & \text{if} & 1 < p \le 2, \\ \delta_{L_q}(\varepsilon) \asymp \varepsilon^q & \text{and} & \rho_{L_q}(t) \asymp t^2 & \text{if} & 2 \le q < \infty, \end{array}$$

as $\varepsilon \to 0$ and $t \to 0$, respectively. Thus we have a special case of Asplund's theorem: all spaces L_r with $1 < r < \infty$ are simultaneously uniformly convex and uniformly smooth.

5.5.2.8 Nordlander [1960, p. 15] proved that

$$\delta_X(\varepsilon) \le 1 - \sqrt{1 - \tfrac{1}{4}\varepsilon^2} \le \tfrac{1}{4}\varepsilon^2$$

whenever $\dim(X) \ge 2$. Moreover, by (5.5.2.3.a),

$$\rho_X(t) \ge \tfrac{1}{4}t^2 \quad \text{if} \quad 0 \le t \le 2.$$

These results show that Hilbert spaces are the "most uniformly convex" and "most uniformly smooth" among all Banach spaces.

5.5.2.9 For many years, convexity and smoothness were mainly viewed as qualitative properties. However, the preceding inequalities suggest that the asymptotic behavior of the moduli δ_X and ρ_X may be used for quantitative considerations. We owe this decisive observation to Pisier, who proposed the following definitions, [1975a, p. 336]:

Let $1 < p \le 2$. A Banach space X, or more precisely its norm, is said to be p-**smooth** if its modulus of smoothness is of *power type* p:

$$\rho_X(t) \le c_0 t^p \quad \text{if} \quad 0 < t < \infty,$$

where $c_0 > 0$ is some constant.

Let $2 \le q < \infty$. A Banach space X, or more precisely its norm, is said to be q-**convex** if its modulus of convexity is of *power type* q:

$$\delta_X(\varepsilon) \ge c_0 \varepsilon^q \quad \text{if} \quad 0 < \varepsilon \le 2,$$

where $c_0 > 0$ is some constant.

The subsequent criteria are implicitly contained in Pisier's work; see [1975a, p. 337]. An explicit presentation can be found in [BEAU$_2$, pp. 310–312].

A Banach space X is p-smooth if and only if there exists a constant $c \ge 1$ such that

$$\left(\frac{\|x+h\|^p + \|x-h\|^p}{2} - \|x\|^p\right)^{1/p} \le c\|h\|$$

for $x, h \in X$.

A Banach space X is q-convex if and only if there exists a constant $c \ge 1$ such that

$$\left\|\frac{x_0 - x_1}{2}\right\| \le$$
$$\le c\left(\frac{\|x_0\|^q + \|x_1\|^q}{2} - \left\|\frac{x_0 + x_1}{2}\right\|^q\right)^{1/q}$$

for $x_0, x_1 \in X$.

The concepts above are dual to each other:

A Banach space X is p-smooth if and only if X^* is p^*-convex.

A Banach space X is q-convex if and only if X^* is q^*-smooth.

5.5.2.10 A striking result of Pisier [1975a, p. 340] says that every uniformly convexifiable/smoothable Banach space is q-convexifiable for some $2 \le q < \infty$ and p-smoothable for some $1 < p \le 2$.

We do not know whether there exists an equivalent norm that realizes the best values of p and q simultaneously (if they are attained at all). In any case, Asplund's averaging technique 5.5.2.6 yields a q_0-convex and p_0-smooth renorming, where either $p_0 < p$ or $q < q_0$. That is, we get an arbitrarily small deterioration.

5.5.2.11 In the limiting case $p = q = 2$, Figiel/Pisier [1974] obtained a beautiful result:

Every 2-convexifiable and 2-smoothable Banach space X admits an equivalent norm that is induced by an inner product. In other words, X is "Hilbertizable."

5.5.2.12 Many of the previous results were proved by techniques from the local theory of Banach spaces; vector-valued Walsh–Paley martingales played a decisive role. For details the reader is referred to 6.1.9.9.

5.5.3 Further concepts related to convexity and smoothness

There is a host of further concepts describing the shape of the unit ball of a Banach space or the differentiability of its norm. A long list of such properties was given by Fan/Glicksberg [1958, pp. 554–555].

5.5.3.1 Shmulyan [1939d, p. 648] refers to a norm as *strongly differentiable* (**Fréchet differentiable**; see 5.1.8.2) if for every fixed $x \neq o$, the limit

$$G(x,h) := \lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exists uniformly with respect to all h in the closed unit sphere; compare with 5.5.2.2. In [MEG, p. 504] a Banach space with such a norm is said to be **Fréchet smooth**.

5.5.3.2 Suppose that the norm of the Banach space X is (Gâteaux) smooth. Then there exists a **support mapping** that assigns to every member $x \in S_X$ the functional $x^* \in S_{X^*}$ that is uniquely defined by $\langle x, x^* \rangle = 1$. A result of Shmulyan [1939b, стр. 82], [1939d, p. 650] implies that the support mapping is continuous from S_X (equipped with the norm topology) into S_{X^*} (equipped with the weak* topology).

Another criterion of Shmulyan [1940a, p. 645] yields that the norm of X is Fréchet smooth if and only if $x \mapsto x^*$ is continuous from S_X (equipped with the norm topology) into S_{X^*} (equipped with the norm topology); see also Phelps [1960a].

Next, Diestel/Faires [1974, p. 265] defined a **very smooth norm** by the property that $x \mapsto x^*$ is continuous from S_X (equipped with the norm topology) into S_{X^*} (equipped with the weak topology).

The first systematic studies of the support mapping were carried out by Giles [1971].

5.5.3.3 Of course, one was looking for a convexity property that, roughly speaking, is dual to Fréchet smoothness. Next, I describe the outcome of these efforts.

According to Lovaglia [1955, p. 226], a Banach space X or, more precisely, its norm is called **locally uniformly convex** if for every $\varepsilon > 0$ and every $x_0 \in X$ with $\|x_0\| = 1$ there exists a $\delta > 0$ such that

$$\left\|\frac{x+x_0}{2}\right\| \leq 1-\delta \quad \text{whenever } \|x\|=1 \text{ and } \|x-x_0\| \geq \varepsilon.$$

In contrast to the concept of uniform convexity, the point is that δ not only depends on ε, but may also depend on x_0.

5.5.3.4 Lovaglia [1955, p. 232]:
If X^* is locally uniformly convex, then X is Fréchet smooth.

It was clear from the very beginning that the converse implication does not hold. A 'strong' counterexample is due to Yost [1981, p. 302]: every reflexive infinite-dimensional space, in particular l_2, admits a Fréchet differentiable norm whose dual norm fails to be locally uniformly convex. Thus one had to find weaker concepts. The most satisfactory results along these lines were obtained by Cudia [1961], [1964].

5.5.3.5 The first locally uniformly convex renorming was produced by Kadets [1959, стр. 57], who treated $C[0,1]$ and hence all separable spaces. Next, Rainwater [1969] observed that Day's norm on $c_0(\mathbb{I})$, defined in 5.5.1.4, is locally uniformly convex. Much deeper is **Troyanski's theorem** [1971, p. 177]:
Every weakly compactly generated Banach space is locally uniformly convexifiable.

This beautiful result has many consequences. Combined with Asplund's averaging technique it yields that every reflexive space admits an equivalent norm such that both the space and its dual are simultaneously locally uniformly convex and Fréchet smooth; see [DIE$_1$, p. 167].

5.5.3.6 For every separable Banach space X, the following are equivalent:
(1) X^* is separable.
(2) X admits an equivalent Fréchet differentiable norm.
(3) X admits an equivalent norm such that the associated dual norm is locally uniformly convex on X^*.

The equivalence of **(1)** and **(2)** was independently proved by Restrepo [1964, p. 413] and Kadets [1965, pp. 183, 186], whereas Asplund [1968, pp. 32, 41] added **(3)**.

This result implies that every Fréchet smoothable Banach space is Asplund, or equivalently, that its dual possesses the Radon–Nikodym property; see 5.1.4.12.

5.5.3.7 Each of the following conditions implies reflexivity of a Banach space X:
(1) X is uniformly convex; Milman [1938], Kakutani [1939], and Pettis [1939].
(2) X^* is Fréchet smooth; see the next paragraph.
(3) X^{**} is locally uniformly convex; see 5.5.3.4.
(4) X^{***} is smooth; Giles [1974, p. 118].
(5) X^{****} is strictly convex; Dixmier [1948, p. 1070].

The preceding results are sharp in the following sense:

By Smith [1976], there exist non-reflexive spaces, for example a renormed James space, whose biduals are smooth. Moreover, the non-reflexive space c_0 admits a renorming such that its dual becomes locally uniformly convex; see [DEV$^+$, p. 48].

The philosophical essence of the considerations above is that convexity and smoothness properties are getting worse when we pass from the original norm to the norm in the bidual space, etc.

5.5.3.8 Next, I give a typical example that shows how mathematicians "mollify" the history of their subject.

Several authors attribute Statement **(2)** from the previous paragraph to Shmulyan; see [DAY, 3rd edition, p. 148], [DEV$^+$, p. 88], and [DIE$_1$, p. 43]. However, searching in his impressive work, I found merely the following results.

[1940a, p. 645]:
Si dans les espaces $\overline{E}$, $\overline{\overline{E}}$ les normes sont partout fortement dérivables, alors E est régulier.

[1941, p. 550]:
Si dans l'espace $\overline{E}$ la norme $\|f\|$ est partout fortement dérivable et chaque fonctionnelle linéaire $F(f)$ atteint sa plus grande valeur sur la sphère unitaire [note that $f\in\overline{E}$ and $F\in\overline{\overline{E}}$]*, alors l'espace E est régulier.*

[1941, p. 550]:
Pour que $\|f\|$ soit partout fortement dérivable il est nécessaire que chaque fonctionnelle linéaire $f(x)$ atteint sa plus grande valeur sur la sphère unitaire [note that $x\in E$ and $f\in\overline{E}$].

Thus a correct chronicler should say that the theorem under discussion is due to Shmulyan modulo James's characterization of reflexivity, which was proved much later; see 3.4.3.8.

5.5.3.9 A Banach space X or, more precisely, its norm is said to have the **Radon–Riesz property** = **Kadets–Klee property** if

$$x_k \xrightarrow{\text{w}} x \quad \text{and} \quad \|x_k\| \to \|x\| \quad \text{imply} \quad \|x_k - x\| \to 0.$$

These names are motivated by the fact that Radon [1913, p. 1363] and Riesz [1929, p. 60] proved the above implication for all L_p's with $1 < p < \infty$, whereas Kadets and Klee used renormings with this property for constructing homeomorphisms between certain separable Banach spaces; see 5.5.4.1.

Shmulyan [1939d, p. 651] had earlier observed that uniformly convex Banach spaces have the property above. Next, Výborný [1956, p. 352] obtained the same conclusion under a weaker assumption: local uniform convexity.

Note that l_1 is neither strictly convex nor smooth. Nevertheless, the Schur property implies that l_1 has the "four men" property.

5.5.4 Applications of convexity and smoothness

In view of the fact that many Banach spaces admit strictly convex or smooth renormings, these properties do not have far-reaching consequences in structure theory. Nevertheless, improving the quality of a norm can make life easier. Many more conclusions can be drawn from uniform convexity or uniform smoothness.

5.5.4.1 The famous **Kadets theorem**, which says that all separable infinite-dimensional Banach spaces are homeomorphic, was proved in several steps. First of all, Kadets [1955, стр. 141] solved the uniformly convex case. Next, he treated separable reflexive spaces with the help of a clever renorming [1958b, стр. 15]; see 5.5.3.9. This method was extended by him (paper in Ukrainian, 1959) as well as independently by Klee [1960b, p. 28] to separable duals. The final aim was achieved by the use of another tool: Schauder bases; see 5.6.1.5.

5.5.4.2 Price [1937, p. 57] observed that in a strictly convex space all boundary points of the closed unit ball are extreme. Even more, they are exposed. Next, Lindenstrauss [1963b, p. 143] considered locally uniformly convex spaces. In this case the boundary points are strongly exposed.

5.5.4.3 For any non-empty bounded subset B of a Banach space X, the expression

$$p_B(T) := \sup\{\,\|Tx\| : x \in B\,\}$$

defines a semi-norm on $\mathfrak{L}(X,Y)$; see 3.3.4.1. An operator **attains** p_B if there exists $x_0 \in B$ such that $\|Tx\| \le \|Tx_0\|$ for all $x \in B$.

Following Bourgain [1977, p. 266], we say that a Banach space has the **Bishop–Phelps property** if for every non-empty bounded, closed, and absolutely convex subset B of X and for any choice of Y, the collection of all p_B-attaining operators is dense in $\mathfrak{L}(X,Y)$ with respect to the norm topology; see 3.4.3.9.

The main result of Bourgain's paper [1977, p. 269] says that the Bishop–Phelps property and the Radon–Nikodym property are equivalent.

The following theorems hold in spaces with the Bishop–Phelps property; see Lindenstrauss [1963b, p. 142]:

If X is *strictly convexifiable*, then every bounded, closed, and absolutely convex subset is the closed convex hull of its *exposed* points.

If X is *locally uniformly convexifiable*, then every bounded, closed, and absolutely convex subset is the closed convex hull of its *strongly exposed* points.

5.5.4.4 Troyanski's renorming theorem 5.5.3.5 has a remarkable consequence; see Troyanski [1971, p. 178]:

Every weakly compact convex set is the closed convex hull of its strongly exposed points.

The separable version was earlier obtained by Lindenstrauss [1963b, p. 144].

Nowadays, the approach via renorming has become superfluous, since we have a more elegant proof in the framework of the Radon–Nikodym theory; see Namioka [1967, p. 149], Bourgain [1976, p. 200], and 5.4.4.6.

5.5.4.5 Strict convexity has a trivial but far-reaching application in approximation theory; see Fortet [1941, p. 23]:

The best approximation of every element in a strictly convex Banach space X by elements of a closed subspace M is unique (if it exists).

Unfortunately, the natural norms of the most interesting spaces, $C[0,1]$ and $L_1[0,1]$, are not strictly convex.

5.5.4.6 The concept of uniform convexity was introduced by Clarkson for a special purpose: he looked for spaces with the Radon–Nikodym property; see 5.1.4.3.

5.5.4.7 In view of 5.5.2.7, the following result of Kadets [1956, стр. 186] implies the Orlicz theorem 5.1.1.6, but only for $1<p<\infty$.

Let δ_X denote the modulus of convexity. Then

$$\sum_{k=1}^{\infty} \delta_X\big(\|x_k\|\big) < \infty$$

for every unconditionally convergent series $\sum_{k=1}^{\infty} x_k$.

5.5.4.8 Another result of Kadets, stated in 5.1.1.7, can be generalized as follows; see Fonf [1972, стр. 210]:

Let ρ_X denote the modulus of smoothness. Then the Steinitz theorem holds for every convergent series $\sum_{k=1}^{\infty} x_k$ such that

$$\sum_{k=1}^{\infty} \rho_X\big(\|x_k\|\big) < \infty.$$

5.5.4.9 Brodskiĭ/Milman [1948, стр. 837] introduced the following concept:

A Banach space has **normal structure** if for every bounded convex subset C that contains more than one element, there exists an element $x_0 \in C$ such that

$$\sup\{\, \|x_0-y\| \,:\, y\in C \,\} < \overbrace{\sup\{\, \|x-y\| \,:\, x,y\in C \,\}}^{\text{diameter of } C}.$$

A quick look at

$$A_0 := \big\{\varphi\in C[0,1] \,:\, 0\le\varphi(t)\le 1 \text{ for } 0\le t\le 1, \quad \varphi(1)=1\big\}$$

shows that $C[0,1]$ does not have normal structure.

Clearly, l_2 has normal structure with respect to its natural norm. Bynum [1972, p. 234] discovered that this property fails for the equivalent norm

$$\|x\|_\infty := \max\left\{\|x_+|l_2\|, \|x_-|l_2\|\right\},$$

where x_+ and x_- denote the positive and negative parts of x, respectively. Therefore reflexivity does not imply normal structure. Bynum's norm also shows that normal structure is not preserved under passing to the dual space; the dual norm of $\|\cdot\|_\infty$ is given by

$$\|x\|_1 := \|x_+|l_2\| + \|x_-|l_2\|.$$

Finally, I mention a result of Zizler [ZIZ, pp. 20, 27]:

Every separable space has normal structure under a suitable norm.

5.5.4.10 In modern terminology, the main result of Brodskiĭ/Milman [1948] reads as follows:

If K is a non-empty weakly compact convex subset of a Banach space with normal structure, then all isometries from K onto itself have a common fixed point, the **center** of K.

5.5.4.11 Let A be any subset of a Banach space. A mapping $\boldsymbol{f} : A \to A$ is called **non-expansive** if

$$\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| \le \|x - y\| \quad \text{for all } x, y \in A.$$

Kirk [1965, p. 1006] observed that in contrast to Banach's theorem 2.1.12, non-expansive mappings need not possess a fixed point: consider $\boldsymbol{f} : \varphi(t) \to t\varphi(t)$ on the subset A_0 defined in 5.5.4.9.

Consequently, additional assumptions are required. Indeed, Browder [1965, p. 1041] and Göhde [1965, p. 252] showed independently that for every non-empty bounded, closed, and convex subset K of a uniformly convex Banach space, every non-expansive map $\boldsymbol{f} : K \to K$ has a fixed point. In a next step, Baillon [1978, exposé 7, p. 10] obtained a corresponding fixed point theorem for uniformly smooth spaces.

The decisive approach is due to Kirk [1965, p. 1004], who proved the existence of a fixed point subject to the condition that K is a non-empty weakly compact convex subset of a Banach space with normal structure. In fact, he stated his theorem only for bounded, closed, and convex subsets of a reflexive Banach space with normal structure, but the proof works also in the setting described above; see Belluce/Kirk [1966].

It is not hard to check that all uniformly convex spaces have normal structure; see Belluce/Kirk [1967, p. 477] or [ZIZ, p. 27]. Finally, Turett [1982, p. 283] showed that the duals of uniformly convex spaces have normal structure as well. Thus Kirk's result is more general than those of Browder–Göhde and Baillon.

For further information the reader is referred to [BENY$_1^+$, pp. 65–82], [GOE$^+$, pp. 27–71], and [IST, pp. 188–205].

5.5.5 Complex convexity

5.5.5.1 A *real* Banach space X is strictly convex (see 5.5.1.1) if

$$\|x_0\| = 1 \text{ and } \|x_0 + \lambda x\| \leq 1 \text{ for all } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 1 \quad \text{imply} \quad x = o.$$

By analogy with this definition, one refers to a *complex* Banach space X as **complex strictly convex** if

$$\|x_0\| = 1 \text{ and } \|x_0 + \lambda x\| \leq 1 \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| \leq 1 \quad \text{imply} \quad x = o.$$

Every real strictly convex space is also complex strictly convex. However, $L_1[0,1]$ is complex strictly convex without being real strictly convex.

5.5.5.2 An interesting application of the above concept is due to Thorp/Whitley [1967, p. 641], who proved that a Banach space X is complex strictly convex if and only if it satisfies the **strong maximum modulus principle**:

Every analytic X-valued function $\boldsymbol{f}$ defined on a connected open subset of $\mathbb{C}$ is constant whenever $\|\boldsymbol{f}(\lambda)\|$ attains its maximum.

Observe that $\lambda \mapsto \|\boldsymbol{f}(\lambda)\|$ is subharmonic for every analytic function taking its values in an arbitrary Banach space. If $\|\boldsymbol{f}(\lambda)\|$ attains its maximum, then $\|\boldsymbol{f}(\lambda)\|$ is a constant scalar. However, the term "strong" refers to a stronger conclusion, namely $\boldsymbol{f}(\lambda) = \text{const}$. This does not always hold: Take the function $\boldsymbol{f} : \lambda \mapsto (1, \lambda)$ on the open unit disk $\mathbb{D}$ with values in $\mathbb{C}^2$ under the max-norm. Then $\|\boldsymbol{f}(\lambda)\| = \max(1, |\lambda|) = 1$ for all $\lambda \in \mathbb{D}$.

The reader may also consult [DINE_1, p. 164] and [IST, p. 252].

5.5.5.3 A notion of *complex uniform convexity* or *uniform PL-convexity* was studied by Globevnik [1975], Davis/Garling/Tomczak-Jaegermann [1984], and Dilworth [1986]. Note that *PL* is an abbreviation for *pluri-subharmonic*; see 6.9.3.4.

5.6 Bases

The theory of bases is a world of its own that was presented in Singer's monumental treatise [SIN_1, SIN_2]. Unfortunately, the third volume (*Applications and Bases in Concrete Spaces*) has never appeared. There is also a textbook at an elementary level: [MAR]. Surveys were written by Milman [1970], McArthur [1972], and James [1982].

5.6.1 Schauder bases and basic sequences

5.6.1.1 The concept of a basis was defined by Schauder [1927, p. 48]:

Es gibt in E eine feste endliche oder unendliche Folge von normierten „Basiselementen" $\{\mathfrak{e}_i\}$, $\|\mathfrak{e}_i\| = 1$ $(i = 1, 2, \ldots, \infty)$, so daß jedes $\mathfrak{e}$ aus E eine eindeutige Darstellung

$$\mathfrak{e} = \sum_{i=1}^{\infty} c_i \mathfrak{e}_i \quad \textit{d.h.} \quad \lim_{n\to\infty} \Big\| \mathfrak{e} - \sum_{i=1}^{n} c_i \mathfrak{e}_i \Big\| = 0$$

zuläßt.

Für ein festes n ist der Koeffizient $c_n(\mathfrak{e})$ eine „beschränkte Funktionaloperation“ in E, d.h. es gibt eine Zahl M_n, so daß

$$|c_n(\mathfrak{e})| \leq M_n \|\mathfrak{e}\|, \quad \mathfrak{e} \in E.$$

5.6.1.2 Nowadays, by a **basis**, or a **Schauder basis**, of a Banach space X we mean a sequence (x_k) such that every element $x \in X$ admits a *unique* representation

$$x = \sum_{k=1}^{\infty} \xi_k x_k \quad \text{with } \xi_k \in \mathbb{K}.$$

As already observed in [BAN, p. 111], the second part of Schauder's definition is redundant. Indeed, Banach showed that X becomes complete under the norm

$$|||x||| := \sup_n \Big\| \sum_{k=1}^{n} \xi_k x_k \Big\|.$$

Since $\|x\| \leq |||x|||$, the inverse mapping theorem implies that both norms are equivalent: $|||x||| \leq c\|x\|$. Hence the projections

$$P_n : x \mapsto \sum_{k=1}^{n} \xi_k x_k$$

as well as the **coordinate functionals** $x_k^* : x \to \xi_k$ are automatically continuous.

We refer to (x_k) as **monotone** if all projections P_n have norm 1. The preceding considerations show that this can always be arranged by a suitable renorming of the underlying space.

A basis (x_k) with $\|x_1\| = \|x_2\| = \cdots = 1$ is called **normalized**.

5.6.1.3 Of course, every Banach space with a basis is separable. The famous **basis problem** raised the question whether the converse holds,

[BAN, p. 111]:

On ne sait pas si tout espace du type (B) séparable admet une base.

A negative answer was given by Enflo when he constructed a Banach space without the approximation property. Details will be discussed in Subsection 5.7.4.

5.6.1.4 Every basis (x_k) of a Banach space X defines a one-to-one correspondence

$$\sum_{k=1}^{\infty} \xi_k x_k \longleftrightarrow (\xi_k)$$

between X and the associated sequence space

$$S(X, x_k) := \Big\{ (\xi_k) : \sum_{k=1}^{\infty} \xi_k x_k \ \text{converges in } X \Big\}.$$

However, passing from abstract elements to sequences turns out to be of little help. This is due to the fact that in most cases, we do not have any effective description of those (ξ_k)'s contained in $S(X, x_k)$. Exceptions are orthonormal bases of the separable Hilbert space and certain bases of Besov spaces.

5.6.1.5 Knowing that a Banach space admits a basis may be very advantageous. Most important is the fact that such spaces have the bounded approximation property; see 5.7.4.2.

Another convincing application is the **Kadets theorem** [1967, стр. 61]:
All separable infinite-dimensional Banach spaces are homeomorphic to each other.

The original proof of this fundamental result was obtained by reducing the general case to the case in which there exists a basis. The reasoning of Kadets as well as comprehensive historical remarks can be found in [BES$^+$, pp. 215–232]; see also 5.5.4.1.

Finally, I mention that in a first attempt, Schauder [1927, p. 52] proved his famous fixed point theorem for Banach spaces with a basis. It even seems that he introduced the concept of a basis just for this purpose. Only later, was it discovered that Schauder's result holds in arbitrary Banach spaces; see [1930b, p. 173].

5.6.1.6 By a **basic sequence** we mean a sequence (x_k) that is a basis of its closed linear span. Basic sequences are characterized by the property that for a suitable constant $c \geq 1$,

$$\Big\| \sum_{k=1}^{m} \xi_k x_k \Big\| \leq c \Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| \quad \text{if } \xi_1, \xi_2, \ldots \in \mathbb{K} \text{ and } 1 \leq m \leq n. \tag{5.6.1.6.a}$$

Here we assume tacitly that all elements x_k are different from o.

The preceding inequality was stated for the first time in a seminal paper by Bessaga/Pełczyński [1958a, p. 152]. Its necessity follows from Banach's observation sketched in 5.6.1.2. To the best of my knowledge, the elementary proof of its sufficiency, which can be found in [HAB$^+$, pp. 190–191] or [MEG, p. 359], has never been recorded in any research paper.

5.6.1.7 Banach [BAN, p. 238] pointed out that in contrast to the situation for bases, basic sequences exist in abundance:

Remarquons toutefois que tout espace du type (B) à une infinité de dimensions renferme un ensemble linéaire fermé à une infinité de dimensions qui admet une base.

This remark, which had been communicated without proof, goes back to Mazur. His original reasoning was published by Pełczyński [1962a, p. 371]. In the meantime, different proofs were given by Bessaga/Pełczyński [1958a, p. 157] and Gelbaum [1958, pp. 29–30]. An approach based on Borsuk's antipodal theorem is due to Day [1962, p. 655].

5.6.2 Biorthogonal systems

5.6.2.1 A **biorthogonal system** is a pair of families $(x_i)_{i\in\mathbb{I}}$ in X and $(x_i^*)_{i\in\mathbb{I}}$ in X^* such that $\langle x_i, x_i^*\rangle = 1$ and $\langle x_i, x_j^*\rangle = 0$ if $i \neq j$. In the sequel, we deal only with the case that the index set $\mathbb{I}$ is countable.

5.6.2.2 A biorthogonal system may have the following properties.

total : $\langle x, x_i^*\rangle = 0$ for all $i\in\mathbb{I}$ implies $x = o$; see [BAN, pp. 42, 58],

fundamental : the finite linear combinations of the x_i's are dense in X, or $\langle x_i, x^*\rangle = 0$ for all $i\in\mathbb{I}$ implies $x^* = o$; see [BAN, p. 58],

bounded : there exists a constant $c \geq 1$ such that $\|x_i\|\|x_i^*\| \leq c$ for all $i\in\mathbb{I}$.

According to Banach [BAN, p. 237], a fundamental and total biorthogonal system is said to be **complete**. He claimed on p. 238 that *il existe des systèmes biorthogonaux complets dans tout espace du type* (B) *séparable.* Since a proof was given only by Markoushevitch [1943], a complete biorthogonal sequence is also referred to as a **Markoushevitch basis**.

The question whether there is a fundamental, total, and bounded biorthogonal sequence in every separable Banach space remained open for a long period. A positive answer was found step by step.

fundamental	total	bounded	authors
+		$1+\varepsilon$	Davis/Johnson [1973a]
	+	$1+\varepsilon$	Davis/Johnson [1973a]
+	+	some finite bound	Ovsepian/Pełczyński [1975]
+	+	$1+\varepsilon$	Pełczyński [1976]

5.6.2.3 Every Schauder basis (x_k) together with the associated coordinate functionals x_k^* is a fundamental, total, and bounded biorthogonal sequence.

5.6.2.4 In every n-dimensional Banach space there are biorthogonal systems such that $\|x_1\| = \cdots = \|x_n\| = 1$ and $\|x_1^*\| = \cdots = \|x_n^*\| = 1$.

According to [BAN, p. 238], this result goes back to the Polish mathematician Auerbach. As a credit to him, a finite basis with the properties above is commonly called an **Auerbach basis**. Various proofs of existence were given only after the end of World War II. First of all, Day [1947] and Taylor [1947] presented geometric arguments. An analytic approach, due to Kadets, is sketched in a footnote of [TIM, стр. 407 (Russ.) or p. 393 (Engl.)]. The same idea was rediscovered by Ruston [1962].

Every Auerbach basis yields a representation of the identity map:

$$I = \sum_{k=1}^{n} x_k^* \otimes x_k.$$

Using this fact, we can show that n-dimensional subspaces of a Banach space are always complemented by a projection of norm at most n; see 4.9.1.9 and 6.1.1.7. Far-reaching applications to the theory of tensor products and nuclear operators will be discussed in 5.7.2.8 and 5.7.3.8, respectively.

5.6.2.5 The trigonometric functions $e_k(t) := \mathrm{e}^{\mathrm{i}kt}$ and the functionals

$$e_k^* : f(t) \mapsto \frac{1}{2\pi} \int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt} dt$$

form a fundamental, total, and bounded biorthogonal system in $C(\mathbb{T})$ and $C(\mathbb{T})^*$. However, $(\mathrm{e}^{\mathrm{i}kt})_{k\in\mathbb{Z}}$ fails to be a basis of $C(\mathbb{T})$; see 5.6.4.4.

5.6.3 Bases and structure theory

5.6.3.1 Let me begin with a quotation from [LIND$_1^+$, p. 15]:

The existence of a Schauder basis in a Banach space does not give very much information on the structure of the space. If one wants to study in more detail the structure of a Banach space by using bases one is led to consider bases with various special properties.

5.6.3.2 Lorch [1939b, p. 564] stated that the concept of a Schauder basis *is open to two objections: It involves the notion of order of the elements of the basis; and for that reason it does not permit of immediate extension to nonseparable spaces.*

As a consequence he came up with the following definition; see [1939b, p. 565]:

A set of elements $\{\varphi_\alpha ; \alpha \in M\}$ with $\varphi_\alpha \in \mathfrak{B}$ will be said to be a basis of $\mathfrak{B}$ if

(*i*) $K^{-1} \le \|\varphi_\alpha\| \le K$, *for some number K, for all $\alpha \in M$;*

(*ii*) *To every $f \in \mathfrak{B}$, there is associated a unique set of numbers $\{a_\alpha ; \alpha \in M\}$. The association is such that $a_\alpha = 0$ except for at most a denumerable set of $\alpha \in M$ and $f = \sum_{\alpha\in M} a_\alpha \varphi_\alpha$, the convergence to the element f being independent of the order of summation.*

5.6.3.3 In his spectral-theoretic studies, Murray [1942, p. 82] proposed the following definition:

Given any subsequence (f_{i_α}) of a basis (f_i) such that $\sum \eta_{i_\alpha} f_{i_\alpha}$ converges for every f [read: $f = \sum_{i=1}^{\infty} \eta_i f_i$] *one would consider the corresponding projection $Ef = \sum \eta_{i_\alpha} f_{i_\alpha}$. However, it is not known whether every infinite subsequence of a basis series is convergent*; [see 5.1.1.1]. *When every such subsequence is convergent, we will call the basis an* **absolute basis**.

The term "absolute" was also used by Karlin [1948], Gelbaum [1950, 1951], and Bessaga/Pełczyński [1958a, 1958b]. Nowadays, following James [1950, p. 518], the more appropriate name **unconditional basis** is commonly accepted, whereas non-absolute bases are referred to as **conditional**.

The unit sequences form an unconditional basis of l_p with $1 \le p < \infty$ and of c_0. On the other hand,

$$(1,0,0,0,0,\dots),\ (1,-1,0,0,0,\dots),\ (0,1,-1,0,0,\dots),\ (0,0,1,-1,0,\dots),\ \dots$$

and

$$(1,0,0,0,0,\dots),\ (1,1,0,0,0,\dots),\ (1,1,1,0,0,\dots),\ (1,1,1,1,0,\dots),\ \dots$$

are conditional bases of l_1 and c_0, respectively; see Singer [1962a, p. 364] and [SIN_1, pp. 423–424]. The first example of a conditional basis of c_0 was given by Gelbaum [1950, p. 188]:

$$(1,0,0,0,0,\dots), (-1,1,0,0,0,\dots), (1,-1,1,0,0,\dots), (-1,1,-1,1,0,\dots), \dots .$$

5.6.3.4 The state of the art around 1940 is documented by the fact that Murray [1942, p. 82] asked the question, *Is every basis an absolute basis?* A "very" negative answer was given by Karlin [1948, p. 983]:

There exists a separable Banach space [for example, $C[0,1]$] *in which no basis is an absolute basis.*

5.6.3.5 If X has an unconditional basis (x_k), then the associated sequence space

$$S(X,x_k) := \Big\{ (\xi_k) : \sum_{k=1}^{\infty} \xi_k x_k \ \text{converges in } X \Big\}$$

is a linear lattice. One can even find an equivalent norm under which $S(X,x_k)$ becomes a Banach lattice.

5.6.3.6 In the following, I will discuss a number of important results concerned with isomorphic embeddings of l_p or c_0 into a given Banach space. The theorems to be presented have a common feature: they show that assuming the existence of an unconditional basis may lead to significant improvements of the conclusions.

As observed in the previous paragraph, every unconditional basis induces a natural ordering. This fact helps to explain why some of the subsequent results even hold in Banach lattices.

5.6.3.7 First of all, I deal with the space c_0. This case is relatively simple.

Bessaga–Pełczyński c_0-theorem; [1958a, p. 160]:

A Banach space does not contain a copy of c_0 if and only if every series $\sum_{k=1}^{\infty} x_k$ such that $\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle| < \infty$ for all $x^* \in X^*$ is unconditionally convergent.

The condition above is certainly fulfilled for weakly sequentially complete spaces. The James space, which will be discussed in 7.4.1.4, shows that in general, weak sequential completeness is not necessary. However, if there exists an unconditional basis, then the situation becomes pleasant; Bessaga/Pełczyński [1958b, p. 169]. The final result is due to Lozanovskiĭ [1967] (without proof) and Meyer-Nieberg [1973, p. 309]; see also [LIND$_2^+$, p. 34]:

A Banach lattice does not contain a copy of c_0 if and only if it is weakly sequentially complete.

5.6.3.8 Next, I present one of the deepest results in Banach space theory, which in the real case was proved by Rosenthal [1974a]; see also 7.3.2.4. The adaptation to the complex setting is due to Dor [1975].

Rosenthal's l_1-theorem:

A Banach space does not contain a copy of l_1 if and only if every bounded sequence has a weak Cauchy subsequence.

Several related criteria were proved by Odell/Rosenthal [1975, p. 376].

5.6.3.9 The next result is classical, [BAN, p. 189]:

Si l'espace conjugué $\overline{E}$ est séparable, l'espace E l'est également.

The converse implication fails: there are separable spaces with a non-separable dual. The most prominent example is l_1. More generally, James [1950, p. 521] proved that X^* is non-separable if X contains a copy of l_1. Maybe, this observation was already known to Banach when he posed the following problem, [BAN, p. 243]:

Étant donné un espace E du type (B) [missing: séparable] *tel que l'espace conjugué $\overline{E}$ n'est pas séparable, existe-t-il dans E une suite bornée d'éléments ne contenant aucune suite partielle faiblement convergente*?

For a correct interpretation, one has to take into account [BAN, p. 9, footnote [1])]:

Les suites convergentes dans notre sens sont appelées d'habitude ≪suites satisfaisant à la condition de Cauchy≫.

In view of Rosenthal's l_1-theorem, Banach's problem reads as follows:

Does a separable Banach space contain a copy of l_1 if and only if its dual is non-separable?

It took more than 40 years until James [1974a, p. 738] and Davis/Figiel/Johnson/ Pełczyński [1974, p. 325] were able to produce counterexamples.

For spaces with an unconditional basis, James [1950, pp. 520 and 522] showed that Banach's question has a positive answer. In a next step, Lotz (unpublished) proved the same for Banach lattices; see [MEY-N, p. 367].

5.6.3.10 The following theorem of James [1950, p. 521] can be considered as the origin of all results concerned with isomorphic embeddings:

A Banach space with an unconditional basis does not contain copies of l_1 or c_0 if and only if it is reflexive.

Lozanovskiĭ [1967] observed (without proof) that the assertion above remains true for Banach lattices; see also [LIND$_2^+$, p. 35].

This criterion fails for general Banach spaces. As in 5.6.3.7, a counterexample is provided by the famous James space.

5.6.3.11 Tsirelson [1974] disproved a long-standing conjecture when he discovered a *reflexive Banach space with an unconditional basis* that does not contain a copy of any l_p with $1 < p < \infty$. Subsequently, Figiel/Johnson [1974] even constructed a uniformly convex space with the same property. Further details can be found in Subsection 7.4.3 and [CASA$^+$].

This counterexample had destroyed the dream that every infinite-dimensional Banach space contains an elementary piece, namely some l_p or c_0; see Lindenstrauss [1970a, p. 165].

5.6.3.12 A basis (x_k) is called **boundedly complete** if

$$\sup_n \Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| < \infty$$

implies that $\sum_{k=1}^{\infty} \xi_k x_k$ converges in X, and **shrinking** means that

$$\|x^*\|_n := \sup\Big\{ |\langle x, x^*\rangle| \,:\, \|x\| = 1,\ x \in \overline{\operatorname{span}}(x_n, x_{n+1}, \ldots)\Big\} \to 0 \quad \text{for all } x^* \in X^*.$$

The first property was used by Dunford/Morse [1936, p. 415] in order to ensure the validity of a Radon–Nikodym theorem for vector-valued functions; see 5.1.4.3. Unaware of this early appearance, James [1950, p. 519] introduced both properties as a tool in the structure theory of Banach spaces. The naming is due to [DAY, 1st edition, p. 69].

As shown by James [1950, p. 522], a basis (x_k) is shrinking if and only if the sequence of its coordinate functionals is a basis of X^*. In this case, the dual basis (x_k^*) turns out to be boundedly complete.

Moreover, every Banach space with a boundedly complete basis is isomorphic to the dual of a Banach space with a shrinking basis. This result goes back to Alaoglu [1940, p. 257] and Karlin [1948, p. 978].

5.6.3.13 The following examples are obvious.

The unit sequences form a boundedly complete basis of l_1 and a shrinking basis of c_0. In l_p with $1<p<\infty$ this basis is boundedly complete as well as shrinking.

5.6.3.14 As already mentioned in 5.6.1.4, the sequence space $S(X, x_k)$ associated with a given basis can be identified only in a small number of cases. The situation is even worse, since different normalized bases (x_k) and (y_k) of one and the same space X may yield different $S(X, x_k)$ and $S(X, y_k)$. Thus we are faced with the provocative question to decide which bases of a given Banach space, say $C[0,1]$, are "good" and which are "bad."

5.6.3.15 The trouble just sketched gave rise to the following concept, which was suggested by Theorem 7 in [BAN, p. 112]:

Bases (x_k) and (y_k) in Banach spaces X and Y, respectively, are called **equivalent** if $S(X,x_k) = S(Y,y_k)$. By the closed graph theorem, this is the case if and only if the correspondence

$$x = \sum_{k=1}^{\infty} \lambda_k x_k \longleftrightarrow \sum_{k=1}^{\infty} \lambda_k y_k = y$$

defines an isomorphism between X and Y. Then there exists a constant $c \geq 1$ such that

$$\frac{1}{c}\left\| \sum_{k=1}^{n} \lambda_k x_k \right\| \leq \left\| \sum_{k=1}^{n} \lambda_k y_k \right\| \leq c \left\| \sum_{k=1}^{n} \lambda_k x_k \right\|$$

for $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $n = 1, 2, \ldots$. If we want to indicate the value of c, then (x_k) and (y_k) are said to be c-**equivalent**. The preceding definitions also make sense for basic sequences.

The term *equivalent* was independently coined by Gurevich [1956, стр. 47] and Bessaga/Pełczyński [1958a, p. 152], while Arsove [1958, p. 283] referred to such bases as *similar*.

We know from Pełczyński/Singer [1964, p. 18] that *in every infinite-dimensional Banach space with a basis there exists a continuum of mutually non-equivalent normalized bases.* This result generalizes an observation of Babenko [1948, стр. 160], who proved that $(|t|^{\alpha} \mathrm{e}^{ikt})_{k \in \mathbb{Z}}$ (in its natural order) is a conditional basis of $L_2(-\pi, +\pi)$ whenever $0 < |\alpha| < \frac{1}{2}$; see also Gelbaum [1951]. It follows, mainly from the Hausdorff–Young theorem, that Babenkov's bases are non-equivalent for different α's.

5.6.3.16 Restricting the considerations to unconditional bases yields a slight improvement of the situation:

There exist three and only three infinite-dimensional Banach spaces in which *all* unconditional normalized bases are equivalent: l_1, l_2, and c_0.

This theorem has been established step by step:

- The case of the Hilbert space l_2 will be discussed in 5.6.5.5.
- Pełczyński [1960, p. 224] discovered that l_p is isomorphic to $[l_p, l_2^n]$ whenever $1 < p < \infty$. The canonical bases of both spaces are unconditional, but fail to be equivalent for $p \neq 2$.
- Using deep results from the theory of absolutely summing operators, Lindenstrauss/Pełczyński [1968, p. 295] proved that every unconditional normalized basis of l_1 and c_0 is equivalent to the standard basis.
- Finally, Lindenstrauss/Zippin [1969, p. 115] showed that there are no other spaces than l_1, l_2, and c_0 in which all unconditional normalized bases are equivalent.

5.6.3.17 Going a little bit further, one may consider normalized bases that become equivalent by a suitable permutation of their elements. The first result along these lines was obtained by Edelstein/Wojtaszczyk [1976, p. 275], who proved that besides l_1, l_2, and c_0, the following spaces have a unique unconditional basis, *up to permutation*:

$$l_1 \oplus l_2, \quad l_2 \oplus c_0, \quad c_0 \oplus l_1, \quad \text{and} \quad l_1 \oplus l_2 \oplus c_0.$$

For more information the reader is referred to [BOUR$^+$].

I stress that such problems were considered much earlier. Köthe [1982$^\bullet$, p. 581] and Toeplitz guessed that for isomorphic perfect sequence spaces λ and μ, *there exists a permutation of coordinates and a diagonal transformation the product of which transforms* λ *into* [read: onto] μ. This conjecture was disproved by Pełczyński's counterexample mentioned in the previous paragraph.

5.6.3.18 Starting from a normalized basic sequence (x_k), one can construct **block basic sequences**:

Take an arbitrary partition of $\mathbb{N} = \{1, 2, \dots\}$ into a sequence of consecutive intervals and choose coefficients $\lambda_k \in \mathbb{K}$ such that the elements

$$u_i := \sum_{k_i \le k < k_{i+1}} \lambda_k x_k$$

are normalized; see Bessaga/Pełczyński [1958a, p. 152].

This technique was used for the first time by Banach [BAN, pp. 194–197] when he showed that every infinite-dimensional closed subspace M of c_0 contains a subspace M_0 isomorphic to c_0. Bessaga/Pełczyński [1958a, p. 157] observed that M_0 can be chosen as a complemented subspace of c_0. An analogous theorem holds for the spaces l_p with $1 \le p < \infty$; see Pełczyński [1960, p. 214].

5.6.3.19 The main tool in proving the previous results is the **Bessaga–Pełczyński selection principle** [1958a, p. 154]:

Every weak null sequence (w_n) with $\inf_n \|w_n\| > 0$ contains a basic subsequence (w_i°). If the underlying Banach space X has a basis (x_k), then for any $\varepsilon > 0$ it can be arranged that (w_i°) is $(1+\varepsilon)$-equivalent to a block basic sequence of (x_k).

Streamlined proofs are given in [DUL$_1$, p. 105] and [WOJ$_1$, pp. 41–42].

Pełczyński [1964b, p. 544] observed that a basic sequence cannot weakly converge to some element $x \ne \mathsf{o}$.

5.6.3.20 The famous **Calkin theorem** [1941, pp. 841, 847] says that $\mathfrak{K}(l_2)$ is the sole non-trivial closed ideal in $\mathfrak{L}(l_2)$; see also 6.3.1.2. Basic sequences were used as a crucial tool when Gohberg/Markus/Feldman [1960, стр. 63] extended this result to the spaces l_p with $1 \le p < \infty$ and c_0. It still seems to be an open problem whether the same situation may occur in other Banach spaces.

5.6.3.21 The property of being a Schauder basis (x_k) is stable under small perturbations: a sequence (y_k) is a basis if it satisfies one of the following conditions. In both cases, the bases (x_k) and (y_k) are equivalent.

Paley/Wiener [PAL$^+$, p. 100], Boas [1940, p. 139]:

$$\Big\| \sum_{k=1}^{n} \xi_k (x_k - y_k) \Big\| \leq q \Big\| \sum_{k=1}^{n} \xi_k x_k \Big\|$$

for any choice of $\xi_1, \ldots, \xi_n \in \mathbb{K}$, $n = 1, 2, \ldots$, and some $0 < q < 1$.

Kreĭn/Milman/Rutman [1940]:

$$\sum_{k=1}^{\infty} \|x_k - y_k\| \|x_k^*\| < 1.$$

5.6.3.22 Of course, the concept of unconditionality can also be defined for basic sequences. Then we get a supplement to (5.6.1.6.a) that goes back to James [1950, p. 518], Gurevich [1953, p. 154], and Bessaga/Pełczyński [1958a, p. 152]:

Unconditional basic sequences are characterized by the property that

$$\Big\| \sum_{k \in A} \xi_k x_k \Big\| \leq c \Big\| \sum_{k \in B} \xi_k x_k \Big\| \quad \text{if } A \subset B \text{ and } \xi_k \in \mathbb{K}, \tag{5.6.3.22.a}$$

or (with a possibly different constant $c \geq 1$)

$$\Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| \leq c \Big\| \sum_{k=1}^{n} \eta_k x_k \Big\| \quad \text{if } |\xi_k| \leq |\eta_k|. \tag{5.6.3.22.b}$$

Here A and B are supposed to be finite subsets of $\mathbb{N}$ and $n = 1, 2, \ldots$.

5.6.3.23 Bessaga/Pełczyński [1958b, p. 173] posed a crucial problem, which according to Pełczyński [1983, p. 243], goes back to Mazur [about 1955, unpublished]:

Does every infinite-dimensional Banach space contain an infinite-dimensional subspace with an absolute basis?

The negative answer will be discussed in 7.4.5.1.

5.6.3.24 The concept of a **symmetric basis** was independently invented by Singer [1961, p. 159] and Kadets/Pełczyński [1962, p. 172]. The latter used the attribute *permutatively homogeneous*. This property means that for any permutation π of $\mathbb{N}$, the rearranged basis $(x_{\pi(k)})$ is equivalent to the original basis (x_k). Obviously, every symmetric basis is unconditional.

Following [SIN$_1$, p. 563], we call a basis (x_k) **subsymmetric** if it is unconditional and all basic sequences (x_{k_n}) such that $1 \leq k_1 < k_2 < \cdots$ are equivalent to each other.

Singer [1962b, p. 185] and Kadets/Pełczyński [1962, p. 172] showed that symmetry implies subsymmetry. Originally, Singer [1962b, p. 185] had claimed that both concepts coincide. However, this was disproved by a counterexample of Garling [1968, pp. 179–181]; see also [LIND$_1^+$, p. 115] and [SIN$_1$, p. 583].

5.6.4 Bases in concrete Banach spaces

5.6.4.1 To begin with, I summarize the most important results that will be discussed in what follows.

	$C[0,1]$	$L_1[0,1]$	$L_2[0,1]$	$L_p[0,1]$ $1<p<\infty$ $p\neq 2$
Fourier system	no basis Bois-Reymond [1873]	no basis Lebesgue [1909]	orth. basis F. Riesz [1907a] Fischer [1907]	basis M. Riesz [1927] ——— no unc. basis Karlin [1948]
Walsh system	senseless	no basis Fine [1949]	orth. basis Walsh [1923]	basis Paley [1932] ——— no unc. basis Karlin [1948]
Haar system	senseless	basis Schauder [1928]	orth. basis Haar [1910]	basis Schauder [1928] ——— unc. basis Marcinkiewicz [1937]
Faber–Schauder system	basis Faber [1909, 1910] Schauder [1927]	continuous coordinate functionals would be finite linear combinations of Dirac measures – a contradiction –		
Franklin system	basis Franklin [1928]	basis Ciesielski [1963/66]	orth. basis Franklin [1928]	basis Ciesielski [1963/66] ——— unc. basis Bochkarev [1974]
remarks	no unc. bases Karlin [1948]	no unc. bases Pełczyński [1960, 1961]	many non-equivalent cond. bases Babenko [1948]	

The reader may also consult the survey of Figiel/Wojtaszczyk [2001].

5.6.4.2 The classical theory of Fourier series goes back to Fourier, Dirichlet, Riemann, and many others. It is concerned with the **trigonometric systems**

$$\overbrace{1, \sin t, \cos t, \sin 2t, \cos 2t, \ \ldots}^{\text{real version}} \quad \text{and} \quad \overbrace{\ldots\ , \mathrm{e}^{-2\mathrm{i}t}, \mathrm{e}^{-\mathrm{i}t}, 1, \mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{2\mathrm{i}t}, \ \ldots}^{\text{complex version}},$$

where t is a real variable modulo 2π. More precisely, the domain of definition is $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. I stress that t and $z=\mathrm{e}^{\mathrm{i}t}$ are often identified. In this sense, the symbol $\mathbb{T}$ will also be used to denote the unit circle of the complex plane, $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$.

5.6.4.3 Throughout, the spaces $L_p(\mathbb{T})$ are equipped with the norm

$$\|f|L_p\| := \Big(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p dt\Big)^{1/p}.$$

5.6.4.4 The n^{th} partial Fourier sum of a function $f\in L_1(\mathbb{T})$ is defined by

$$f_n(s) := \gamma_0 + \sum_{k=1}^{n}(\alpha_k\cos ks+\beta_k\sin ks) = \sum_{|k|\le n}\gamma_k e^{iks},$$

where

$$\alpha_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, \quad \beta_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt,$$

and

$$\gamma_k := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt.$$

It follows that

$$f_n(s) = \frac{1}{2\pi}\int_0^{2\pi} D_n(s-t)f(t)\,dt.$$

The function

$$D_n(t) := \frac{\sin\frac{(2n+1)t}{2}}{\sin\frac{t}{2}}$$

is called the **Dirichlet kernel**.

Viewing $S_n : f \mapsto f_n$ as an operator on $C(\mathbb{T})$ or $L_1(\mathbb{T})$ yields

$$\|S_n|C\to C\| = \|S_n|L_1\to L_1\| = L_n := \frac{1}{2\pi}\int_0^{2\pi}|D_n(t)|\,dt.$$

The fact that the **Lebesgue constants** L_n tend to ∞ was proved in [LEB$_2$, pp. 86–87]. Later on, Fejér [1910, p. 30] showed that

$$\lim_{n\to\infty}\frac{L_n}{\log n} = \frac{4}{\pi^2},$$

and Szegő [1921, p. 165] discovered the formula

$$L_n = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k(2n+1)-1}}{2k^2-1},$$

which yields $1 = L_0 < L_1 < L_2 < \cdots$.

As already stated in 2.4.2, the principle of uniform boundedness implies that the trigonometric system cannot be a basis of $C(\mathbb{T})$ or $L_1(\mathbb{T})$.

5.6.4.5 In view of the classical Fischer–Riesz theorem, $(e^{ikt})_{k\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{T})$. For $1 < p < \infty$, Marcel Riesz [1927, p. 230] showed that

$$\sup_n \|S_n|L_p\to L_p\| < \infty;$$

see also 5.2.2.13 and 6.1.10.3. Hence $(e^{ikt})_{k\in\mathbb{Z}}$ is also a basis of $L_p(\mathbb{T})$. On the other hand, using Orlicz's results [1933] on unconditionally convergent series, Karlin [1948, p. 980] observed that for $p\neq 2$, this basis is only conditional.

5.6.4.6 The k^{th} **Rademacher function** takes alternately the values $+1$ and -1 on the dyadic intervals of length $\frac{1}{2^k}$. That is,

$$r_k(t) := (-1)^{i-1} \quad \text{whenever } t \in \Big[\tfrac{i-1}{2^k}, \tfrac{i}{2^k}\Big) \text{ and } i = 1, \dots, 2^k.$$

The definition above is due to Rademacher [1922, p. 130] and Khintchine [1923, p. 110]. Those who dislike asymmetric intervals $[a,b)$ may put $r_k(t) := 0$ at the points $t = 0, \dots, \frac{i}{2^k}, \dots, 1$. In this case, one has the concise formula $r_k(t) = \operatorname{sgn}(\sin 2^k \pi t)$.

5.6.4.7 For every finite subset $\mathbb{A}$ of $\mathbb{N} = \{1, 2, \dots\}$, the corresponding **Walsh function** $w_{\mathbb{A}} : [0,1) \to \{-1, +1\}$ is defined by

$$w_{\mathbb{A}} := \prod_{k \in \mathbb{A}} r_k,$$

where $r_1, r_2, \dots$ are the Rademacher functions. These functions constitute an orthonormal basis of $L_2[0,1)$.

There are several ways to enumerate the Walsh system. Following Paley [1932, p. 241], most authors assign to $w_{\mathbb{A}}$ the index

$$n(\mathbb{A}) := \sum_{k \in \mathbb{A}} 2^{k-1} \in \mathbb{N}_0 = \{0, 1, \dots\}$$

with the understanding that $n(\varnothing) = 0$. Then $w_0(t) = 1$ and $w_{2^{k-1}} = r_k$.

The original definition of Walsh [1923, p. 8] was by induction. Take $\varphi_0 = 1$ and suppose that φ_n has already been defined. Then we let

$$\varphi_{2n}(t) := \begin{cases} \varphi_n(2t) & \text{if } 0 \le t < 1/2, \\ +\sigma_n \varphi_n(2t-1) & \text{if } 1/2 \le t < 1, \end{cases} \quad \text{and} \quad \varphi_{2n+1}(t) := \begin{cases} \varphi_n(2t) & \text{if } 0 \le t < 1/2, \\ -\sigma_n \varphi_n(2t-1) & \text{if } 1/2 \le t < 1. \end{cases}$$

The sign $\sigma_n = \pm 1$ is chosen such that φ_{2n} becomes continuous at $t = \frac{1}{2}$. This process yields a natural enumeration of the Walsh functions: φ_n has just n jumps.

Walsh observed a far-reaching analogy with the trigonometric system; see also Vilenkin [1947] and Fine [1949, p. 376]. This is mainly due to the fact that e^{ikt} and w_k are characters on the compact abelian groups $\mathbb{T}$ and $\mathbb{E}^{\mathbb{N}}$ (Cantor group), respectively.

However, there are also significant differences. Most interesting, Young [1976, p. 307] discovered that for $p \neq 2$, the Fourier system and the Walsh system do not yield equivalent bases in $L_p(\mathbb{T})$ and $L_p[0,1)$, respectively. One may ask whether this result remains true if one allows permutations; see 5.6.3.17.

5.6.4.8 We now consider the projection S_n that assigns to every $f \in L_1[0,1)$ the n^{th} partial Walsh sum

$$f_n(s) := \sum_{k=0}^{n} c_k w_k(s), \quad \text{where } c_k := \int_0^1 f(t) w_k(t)\, dt.$$

Paley [1932, p. 247] proved that $\|S_{2^k} : L_1 \to L_1\| = 1$. Hence (w_n) is fundamental in $L_1[0,1)$. However, according to Fine [1949, p. 389], it is not a basis:

$$\|S_{n_k} : L_1 \to L_1\| \ge \tfrac{2k+3}{3} \quad \text{if } n_k = \tfrac{4^{k+1}-1}{3} = 1 + 4 + \dots + 4^k.$$

For $1 < p < \infty$, Paley [1932, p. 255] proved an analogue of the famous Marcel Riesz theorem 5.6.4.5 that implies that the Walsh functions form a basis of $L_p[0,1)$:

$$\sup_n \|S_n : L_p \to L_p\| < \infty.$$

Adapting Karlin's proof for the trigonometric system [1948, p. 980], one can show that the Walsh basis is only conditional, $p \neq 2$; see also 5.1.1.6.

5.6.4.9 The **Haar functions** are defined by

$$\chi_k^{(i)}(t) := \begin{cases} +2^{k/2} & \text{if } \frac{2i-2}{2^{k+1}} \le t < \frac{2i-1}{2^{k+1}}, \\ -2^{k/2} & \text{if } \frac{2i-1}{2^{k+1}} \le t < \frac{2i}{2^{k+1}}, \\ 0 & \text{otherwise,} \end{cases}$$

$k = 0, 1, \dots$ and $i = 1, \dots, 2^k$. An enumeration is obtained by letting $\chi_{2^k+i-1} := \chi_k^{(i)}$. This system, enlarged by $\chi_0(t) = 1$, was introduced by Haar. In his thesis [1910, p. 361], written under the supervision of Hilbert, he looked for an orthonormal system in $L_2[0,1)$ such that, in contrast to the trigonometric system, the expansions of all continuous functions are uniformly convergent.

Schauder [1928, p. 317] showed that the Haar system is a basis in $L_p[0,1)$ with $1 \le p < \infty$. This result was improved by Marcinkiewicz [1937, p. 86]. Using Paley's fundamental work [1932], he discovered that the Haar system is an unconditional basis in $L_p[0,1)$ whenever $1 < p < \infty$. A modern approach by means of martingales is due to Burkholder [1985], who obtained the inequality

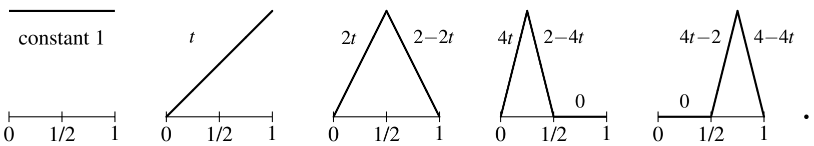
$$\Big\| \sum_{n=0}^{N} \varepsilon_n \xi_n \chi_n \Big\| \le c_p \Big\| \sum_{n=0}^{N} \xi_n \chi_n \Big\| \quad \text{for } \xi_n \in \mathbb{R} \text{ and } \varepsilon_n = \pm 1;$$

see (5.6.3.22.a) and (5.6.3.22.b). The constant $c_p = \max(p, p^*) - 1$ is best possible.

5.6.4.10 The following principle was stated by Faber [1909, p. 82]:

Es ist nun zu erwarten, daß man einen um so besseren Einblick in die Struktur der stetigen Funktionen gewinnen wird, je mehr es gelingt, dieselben durch möglichst übersichtliche und möglichst voneinander unabhängige, also jedenfalls abzählbare Daten zu charakterisieren.

In order to approach this goal, he showed [1909, pp. 84–85] that the following sequence of piecewise linear functions is fundamental in $C[0,1]$:



The general member is a tent of height 1 at the middle of the dyadic interval $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$.

These functions (apart from the first one) can be obtained, up to a norming factor, as the indefinite integrals of the corresponding Haar functions; see Faber [1910, p. 112].

Faber's construction was rediscovered by Schauder [1927, pp. 48–49] when he looked for a basis of $C[0,1]$. Therefore the name **Faber–Schauder system** is justified.

Since the coordinate functionals are finite linear combinations of Dirac measures, the Faber–Schauder system cannot be a basis of any $L_p[0,1]$ with $1 \le p < \infty$.

5.6.4.11 Applying the Gram–Schmidt process to the Faber–Schauder system, Franklin [1928, p. 523] obtained an orthonormal basis of $L_2[0,1]$ that has many interesting properties. The members of the **Franklin system** are continuous piecewise linear functions. The n^{th} function has $n-1$ knots at the dyadic points $\frac{1}{2}; \frac{1}{4}, \frac{3}{4}; \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}; \frac{1}{16}, \dots$.

Ciesielski [1963/66, Part I, p. 145] discovered that the Franklin system is a basis of $L_p[0,1]$ for all $1 \le p < \infty$. Next, Bochkarev [1974, § 2] showed that this basis is unconditional in the reflexive case. His approach was simplified by Ciesielski/Simon/Sjölin [1977]. Moreover, these authors proved that the Franklin system is equivalent to the Haar system for $1 < p < \infty$.

5.6.4.12 For $\alpha, \beta > -1$, the **Jacobi polynomials** are orthogonal with respect to the weighted scalar product

$$(f|g)_{\alpha,\beta} := \int_{-1}^{+1} f(t)\overline{g(t)}(1-t)^{\alpha}(1+t)^{\beta}\,dt.$$

Thus one may ask whether they form a basis of the Banach space defined by the norm

$$\|f|L_p\|_{\alpha,\beta} := \Big(\int_{-1}^{+1} |f(t)|^p(1-t)^{\alpha}(1+t)^{\beta}\,dt\Big)^{1/p}.$$

Pollard [1947/49, Part III, p. 189] for $\alpha, \beta > -\frac{1}{2}$ and Muckenhoupt [1969, p. 306] for $\alpha, \beta > -1$ showed that the answer is affirmative if and only if

$$\left|\tfrac{1}{p} - \tfrac{1}{2}\right| < \min\left\{\tfrac{1}{2}, \tfrac{1}{4\alpha+4}, \tfrac{1}{4\beta+4}\right\}.$$

Laguerre and Hermite polynomials were treated by Askey/Wainger [1965]. They proved, for example, that the **Hermite functions** form a basis of $L_p(\mathbb{R})$ if and only if $\frac{4}{3} < p < 4$.

5.6.4.13 Finally, I list some monographs devoted to special systems of functions:

Fourier system	:	[ZYG],
Walsh system	:	[GOL$^+$], [SCHI$^+$],
Haar system	:	[NOV$^+$],
Faber–Schauder system	:	[KAS$^+$, Chap. 6], [SCHI$^+$, Chap. 5],
Franklin system	:	[KAS$^+$, Chap. 6], [SCHI$^+$, Chap. 5],
orthogonal polynomials	:	[SZE],
general systems	:	[KAC$^+$], [OLEV].

5.6.5 Unconditional basic sequences in Hilbert spaces

5.6.5.1 According to Bari [1951, стр. 71] and Pełczyński/Singer [1964, p. 23], a basic sequence (x_k) in a Banach space is said to be **Besselian** if

$$\text{the convergence of} \quad \sum_{k=1}^{\infty} \xi_k x_k \quad \text{implies} \quad \sum_{k=1}^{\infty} |\xi_k|^2 < \infty. \tag{5.6.5.1.a}$$

In the reverse case, when

$$\sum_{k=1}^{\infty} |\xi_k|^2 < \infty \quad \text{implies} \quad \text{the convergence of} \quad \sum_{k=1}^{\infty} \xi_k x_k, \tag{5.6.5.1.b}$$

one refers to (x_k) as **Hilbertian**.

The properties above can be characterized by inequalities that are supposed to hold for all $\xi_1, \dots, \xi_n \in \mathbb{K}$ and $n = 1, 2, \dots$:

$$A\Big(\sum_{k=1}^{n} |\xi_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^{n} \xi_k x_k\Big\|, \tag{5.6.5.1.a*}$$

$$\Big\|\sum_{k=1}^{n} \xi_k x_k\Big\| \le B\Big(\sum_{k=1}^{n} |\xi_k|^2\Big)^{1/2}. \tag{5.6.5.1.b*}$$

The existence of the positive constants A and B follows from the closed graph theorem.

In the Hilbert space setting, estimates of the form (5.6.5.1.a*) and (5.6.5.1.b*) were used for the first time by Lorch [1939b, p. 569] and Boas [1941, p. 361], while properties (5.6.5.1.a) and (5.6.5.1.b) appeared in papers of Gelbaum [1950, pp. 192–193] and Bari [1951, стр. 71].

5.6.5.2 By a **Riesz sequence** (x_k) we mean a basic sequence that is at the same time Besselian and Hilbertian:

$$A\Big(\sum_{k=1}^{n} |\xi_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^{n} \xi_k x_k\Big\| \le B\Big(\sum_{k=1}^{n} |\xi_k|^2\Big)^{1/2}. \tag{5.6.5.2.a}$$

Defining the operator J by $Je_k = x_k$, one can restate this property as follows; see Bari [1951, стр. 79]:

A Riesz sequence is the image of the canonical basis (e_k) of l_2 with respect to some injection $J : l_2 \to X$.

5.6.5.3 It follows from Khintchine's inequality (6.1.7.2.a) that the Rademacher functions form a Riesz sequence in all spaces $L_p[0,1)$ with $1 \le p < \infty$.

5.6.5.4 For the rest of this section, we restrict our attention to Hilbert spaces. In this case, Boas [1941, pp. 361–362] and Bari [1951, стр. 78–81] proved a useful criterion; see also [GOH$_3^+$, pp. 310–311]:

A sequence (x_k) in a Hilbert space is Riesz if and only if $G := \big((x_h|x_k)\big)$ defines an invertible operator on l_2.

In view of

$$A^2 \sum_{k=1}^{n} |\xi_k|^2 \leq \sum_{h=1}^{n} \sum_{k=1}^{n} \xi_h (x_h|x_k) \overline{\xi}_k \leq B^2 \sum_{k=1}^{n} |\xi_k|^2,$$

the Gram matrix $G := \big((x_h|x_k)\big)$ is positive definite. Thus $G^{-1/2} = \big(\gamma_{jk}\big)$ exists, and we may infer from $G^{-1/2}GG^{-1/2} = I$ that the elements

$$y_j = \sum_{k=1}^{\infty} \gamma_{jk} x_k \quad \text{with } j = 1, 2, \ldots$$

form an orthonormal basis of H. In the finite-dimensional case, this orthogonalization method was studied by Schweinler/Wigner [1970]. Compared with the usual Gram–Schmidt process, it has a significant advantage, since certain properties of (x_k) are transferred to (y_j). This fact plays a decisive role in wavelet theory; see 5.6.6.4.

5.6.5.5 The most important characterization of Riesz sequences in Hilbert spaces was established by Lorch [1939b, p. 568]; see also Bari [1951, стр. 80] and Gelfand [1951, стр. 224]:

A normalized basic sequence in a Hilbert space is Riesz if and only if it is unconditional.

The original proof relied on a fundamental result of Orlicz. In modern terminology, Lorch used the fact that a Hilbert space has Rademacher type 2 and Rademacher cotype 2; see 6.1.7.7.

As a consequence, it follows that all unconditional bases in l_2 are equivalent; see 5.6.3.16. In the context of perfect sequences spaces (3.3.2.5), a similar result was earlier proved by Köthe/Toeplitz [1934, p. 217].

5.6.5.6 Finally, I present a simple application of Riesz sequences in spectral theory.

Suppose that $T \in \mathfrak{L}(H)$ is **similar** to a compact and normal operator $S \in \mathfrak{L}(H)$, which means that there exists an isomorphism $A \in \mathfrak{L}(H)$ such that $T = ASA^{-1}$. Consider the classical spectral representation

$$Sx = \sum_{k=1}^{\infty} \lambda_k (x|x_k) x_k \quad \text{for } x \in H$$

and put

$$y = Ax, \quad y_k = Ax_k \quad \text{and } y_k^* = (A^{-1})^* x_k.$$

Then

$$Ty = \sum_{k=1}^{\infty} \lambda_k (y|y_k^*) y_k \quad \text{for } y \in H.$$

The point is that instead of the orthonormal sequence (x_k), we get the biorthogonal Riesz sequences (y_k) and (y_k^*) for which

$$Ty_k = \lambda_k y_k \quad \text{and} \quad T^* y_k^* = \overline{\lambda}_k y_k^*.$$

In particular, T is a scalar type operator; see 5.2.5.6 and 5.2.5.9.

5.6.6 Wavelets

In this subsection, I will use the old-fashioned notation of functions, namely $f(t)$ instead of f. Thanks to this convention, the translates can be denoted by $f(t-k)$ and the dyadic dilations by $f(2^n t)$.

5.6.6.1 In the early 1980s, a new kind of orthonormal basis entered the scene. The English term "wavelet" was coined by Grossmann/Morlet [1984], and its French translation is "ondelette." Over the short period of 10 years, wavelet theory became a self-reliant mathematical discipline. There is a theoretical background related to functional analysis, harmonic analysis and approximation theory. Most important are the far-reaching applications.

5.6.6.2 An (orthonormal) **wavelet** is a function ψ such that the functions

$$2^{n/2}\psi(2^n t-k) \quad \text{with } k,n\in\mathbb{Z}$$

form an orthonormal basis of the Hilbert space $L_2(\mathbb{R})$. In this case, the sequence $\big(2^{n/2}\psi(2^n t-k)\big)_{k,n\in\mathbb{Z}}$ is called a **wavelet basis**. The decisive feature is that all members are produced out of one (the mother wavelet) by translations and dilations.

Wavelets are commonly defined on $\mathbb{R}$ or, more generally, on $\mathbb{R}^n$. However, we also have a theory of periodic wavelets.

The classical example is the **Haar wavelet**,

$$\chi(t) := \begin{cases} +1 & \text{if } 0\le t<\frac{1}{2}, \\ -1 & \text{if } \frac{1}{2}\le t<1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6.6.2.a}$$

5.6.6.3 All important wavelet bases are constructed from a **multiresolution analysis**. This is, by definition, a family of closed subspaces V_n of $L_2(\mathbb{R})$ indexed by $\mathbb{Z}$ that has the following properties:

(1) $\cdots\subset V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset\cdots$.

(2) $\bigcap\limits_{n\in\mathbb{Z}} V_n=\{\mathrm{o}\}$ and $\overline{\bigcup\limits_{n\in\mathbb{Z}} V_n}=L_2(\mathbb{R})$.

(3) $f(t)\in V_n \;\Leftrightarrow\; f(t-\frac{k}{2^n})\in V_n$ for $k,n\in\mathbb{Z}$.

(4) $f(t)\in V_n \;\Leftrightarrow\; f(2t)\in V_{n+1}$ for $n\in\mathbb{Z}$.

(5) There exists a **scaling function** φ whose translates $\varphi(t-k)$ with $k\in\mathbb{Z}$ form a Riesz basis of V_0.

5.6.6.4 Defining the Fourier transform by

$$F_{\text{our}} : f(t) \mapsto \widehat{f}(s) := \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{-\mathrm{i}st}\, dt,$$

Mallat [1989, p. 71] proved that the translates $\varphi(t-k)$ of a square integrable function φ form a Riesz sequence in $L_2(\mathbb{R})$ if and only if

$$A^2 \le \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(s+2\pi k)|^2 \le B^2 \quad \text{for almost all } s\in\mathbb{R}, \tag{5.6.6.4.a}$$

the constants A and B being the same as in (5.6.5.2.a). In the particular case that $A=B=1$, we get an orthonormal system. Hence, subject to the assumption (5.6.6.4.a), the expression

$$\frac{\widehat{\varphi}(s)}{\Big(\sum\limits_{k\in\mathbb{Z}} |\widehat{\varphi}(s+2\pi k)|^2\Big)^{1/2}}$$

is the Fourier transform of a scaling function whose translates are even orthonormal. The same result is obtained by applying the orthonormalization method described in 5.6.5.4; see [MEY, pp. 26–28]. As a consequence, property **(5)** in 5.6.6.3 can be sharpened: the $\varphi(t-k)$'s are orthonormal.

5.6.6.5 In order to find a wavelet associated with the multiresolution analysis $(V_n)_{n\in\mathbb{Z}}$, we observe that $L_2(\mathbb{R})$ is the orthogonal sum of the subspaces

$$W_n := V_{n+1} \cap V_n^{\perp}.$$

Thus it suffices to construct a function ψ whose translates $\psi(t-k)$ with $k\in\mathbb{Z}$ yield an orthonormal basis of W_0. This was achieved by Mallat [1989, pp. 76, 81–82] with the help of Fourier techniques, while Daubechies [1988, p. 944] used a direct approach, which will be described next.

5.6.6.6 Let $\varphi(t) \in V_0 \subset V_1$ be a scaling function of the multiresolution analysis $(V_n)_{n\in\mathbb{Z}}$. Since $\big(\varphi(2t-k)\big)_{k\in\mathbb{Z}}$ is a Riesz basis of V_1, there exists a sequence $(c_k)\in l_2(\mathbb{Z})$ such that

$$\varphi(t) = \sum_{k\in\mathbb{Z}} c_k \varphi(2t-k).$$

This formula has various names: **scaling equation**, **dilation equation**, **refinement equation**, and **two-scale equation**.

In view of 5.6.6.4, we may assume that the translates $\varphi(t-k)$ form an orthonormal basis of V_0. Then the required wavelet is obtained by

$$\psi(t) := \sum_{k\in\mathbb{Z}} (-1)^{k-1} \overline{c}_k\, \varphi(2t+k-1).$$

Having in mind the terminology used in [MEY, p. 67], Strichartz [1994, p. 26] made a sarcastic comment:

In the French literature the scaling function φ is sometimes called "le père" and the wavelet ψ is called "la mère", but this shows a scandalous misunderstanding of the human reproduction; in fact, the generation of wavelets more closely resembles the reproductive life style of amoebas.

In addition, I stress that there are mothers without fathers: a case of Immaculate Conception. Examples, based on an idea of Meyer, were discussed by Lemarié [1989, p. 32] and Mallat [1989, p. 85]; see also [DAU, p. 136].

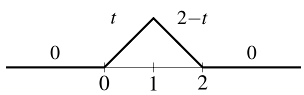
5.6.6.7 The most elementary multiresolution analysis is given by the subspaces $V_n^{(0)}$ of $L_2(\mathbb{R})$ consisting of all functions that are constant on the dyadic intervals $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ with $k \in \mathbb{Z}$. The characteristic function of $[0,1)$ can be used as a scaling function φ. Then $\varphi(t) = \varphi(2t) + \varphi(2t-1)$ is the scaling equation, and $\chi(t) = \varphi(2t) - \varphi(2t-1)$ yields the Haar wavelet (5.6.6.2.a).

5.6.6.8 Some important wavelets are built from **splines**. These are functions with a prescribed smoothness that may jump in prescribed nodes. The term **spline** was motivated by Schoenberg in [1946, p. 134]:

A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar of wood or some other elastic material. The spline is placed on the sheet of graph paper and held in place at various points by means of certain heavy objects (called "dogs" or "rats") such as to take the shape of the curve we wish to draw.

Concerning the history of splines the reader is referred to [SCHU, pp. 10–11].

5.6.6.9 As a first example, we consider the subspaces $V_n^{(1)}$ of $L_2(\mathbb{R})$ whose members are the continuous functions linear on the dyadic intervals $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ with $k \in \mathbb{Z}$. Then the properties **(1)** to **(4)** of a multiresolution analysis are obvious. Moreover, it can be seen that the translates of the tent function



yield a Riesz basis of $V_0^{(1)}$. Hence $(V_n^{(1)})_{n \in \mathbb{Z}}$ is a multiresolution analysis.

Extending the preceding considerations yields **spline wavelets** of higher order. The underlying subspaces $V_n^{(m)}$ consist of all square integrable functions f having continuous derivatives $f, \dots, f^{(m-1)}$ whose restrictions to the dyadic intervals $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ with $k \in \mathbb{Z}$ are polynomials of degree less than or equal to $m = 2, 3, \dots$.

Foreshadowing the general theory, the associated wavelets had been earlier discovered by Strömberg [1981]. His mother wavelet of order m is orthogonal to all splines with nodes in $\dots,-\frac{3}{2},-\frac{2}{2},-\frac{1}{2},0,+1,+2,+3,\dots$ and has an additional node in $\frac{1}{2}$. Thus it is unique up to a scalar factor. Spline wavelets of a different kind were independently found by Battle [1987] and Lemarié [1988].

The support of the Strömberg as well as of the Battle–Lemarié wavelets is all of $\mathbb{R}$. This was the reason why Daubechies [1988] constructed wavelets with compact (but quite large) supports. Later one, Chui/Wang [1992] discovered spline wavelets supported by $[0,2m-1]$. However, the requirement of compact support has its price: instead of orthogonal bases one gets only Riesz bases.

5.6.6.10 The quality of a wavelet ψ is measured by its smoothness and its decay as $t \to \pm\infty$. The number of vanishing moments $\int_{-\infty}^{+\infty} t^k \psi(t)\,dt = 0$ also plays a decisive role. However, there is an uncertainty principle, [DAU, p. 152]:

Orthogonal wavelets cannot have the best of both worlds: they cannot be C^∞ and have exponential decay.

5.6.6.11 The pioneering period of wavelet theory is summarized in the following table.

[1910] Haar defined his orthonormal system on $[0,1]$, which became the ancestor of all wavelets.

[1981] Strömberg constructed a modified Franklin system and higher-order spline systems.

[1984] Grossmann/Morlet coined the term “wavelet of constant shape.”

[1986] Meyer (summer 1985) discovered an “ondelette” that is a member of the Schwartz space $S(\mathbb{R})$. The generalization to $\mathbb{R}^n$ was done by Lemarié/Meyer [1986, pp. 2–3].

[1988] Daubechies developed a theory of compactly supported wavelets.

[1989] Mallat introduced (in collaboration with Meyer, autumn 1986) the concept of a multiresolution analysis.

Meyer [MEY, p. 125] states that he found smooth wavelets by chance. This situation changed in the light of his joint work with Mallat: *All that had appeared miraculous in our previous calculations now became very simple and quite natural*.

5.6.6.12 Wavelet theory has its roots in engineering as well as in pure mathematics. The geophysicist Morlet searched for petroleum, his collaborator Grossmann worked in quantum physics, Daubechies did her Ph.D. in physics, and Mallat was interested in signal processing.

On the other hand, it had been proved that the Franklin system is a basis in various function spaces. In order to cover Besov spaces, Ciesielski and his pupils constructed spline bases with a higher degree of smoothness; see 6.7.7.4. This was the reason why Meyer looked for smooth wavelets. Generalizing Marcinkiewicz's theorem for the Haar system, he observed in collaboration with Lemarié that his wavelet yields an unconditional basis of $L_p(\mathbb{R}^n)$ whenever $1 < p < \infty$; see Lemarié/Meyer [1986, p. 10]. This result was extended in two directions. Gripenberg [1993] showed that the same conclusion holds for any wavelet ψ on $\mathbb{R}$ such that

$$\operatorname*{ess\text{-}sup}_{t\in\mathbb{R}} |\psi(t)| < \infty \quad \text{and} \quad \int_0^\infty \tau \operatorname*{ess\text{-}sup}_{|t|>\tau} |\psi(t)|\, d\tau < \infty,$$

while Jaffard/Meyer [1989] constructed unconditional bases in $L_p(G)$, where G is an open subset of $\mathbb{R}^n$.

It came as a surprise when Maurey [1980] proved, by an implicit method, that the isomorphic Hardy spaces $H_1(\mathbb{T})$ and $H_1(\mathbb{R})$ have unconditional bases; see 6.7.12.18. Explicit unconditional bases of $H_1(\mathbb{T})$ were found by Carleson [1980, p. 415] and Wojtaszczyk [1980, pp. 294, 298–299]; see 6.7.12.18. Finally, Strömberg [1981, pp. 477–478] and Lemarié/Meyer [1986, p. 10] discovered unconditional wavelet bases of $H_1(\mathbb{R})$. Therefore it can be said without exaggeration that constructing bases of H_1-spaces was a decisive source of wavelet theory; see [WOJ$_2$, Chap. 8].

5.6.6.13 The pros and cons of wavelets are contrasted in the preface of [WALT]:

Wavelet series converge pointwise when others don't, wavelet series are more localized and pick up edge effects better, wavelets use fewer coefficients to represent certain signals and images.

Unfortunately, not all is rosy. Wavelet expansions change excessively under arbitrary translations – much worse than Fourier series. The same is true for other operators such as convolution and differentiation.

5.6.6.14 In the first half of the 1990s several textbooks on wavelets were published: [CHUI], [DAU], [HERN$^+$], and [WALT]. Wojtaszczyk gave a nice presentation from the point of view of Banach spaces; [WOJ$_2$]. In any case, the reader should take a look at Daubechies's article *Where do wavelets come from?*, [1996$^\bullet$]. Many details of the history of wavelets can be found in a remarkable book written by a female non-mathematician; [HUB$^\bullet$]. Another historical reference is [KAH$^{+\bullet}$]. In a readable account of the prehistory, Meyer [1993$^\bullet$] says:

Wavelet analysis originated from J. Morlet's deep intuition, from his numerical experiments on seismic signals and from an exceptional collaboration between Morlet and Grossmann.

5.6.7 Schauder decompositions

5.6.7.1 The concept of a Schauder basis gives rise to a straightforward generalization that is due to Grinblyum [1950] and Fage [1950]:

A sequence of closed subspaces $M_k \neq \{o\}$ is called a **Schauder decomposition** of a Banach space X if every element $x \in X$ admits a *unique* representation

$$x = \sum_{k=1}^{\infty} x_k \quad \text{with } x_k \in M_k \text{ for } k = 1, 2, \dots . \tag{5.6.7.1.a}$$

Then we refer to $X = \bigoplus_{k=1}^{\infty} M_k$ as an infinite direct sum. The coordinate mappings $C_k : x \mapsto x_k$ satisfy the relations $C_k^2 = C_k$ and $C_h C_k = O$ if $h \neq k$. The fact that these projections are continuous can be proved in the same way as for Schauder bases; Sanders [1965b]. The case of two direct summands, $X = M \oplus N$, was already considered by Murray [1937]; see 4.9.1.8.

5.6.7.2 Most properties of Schauder bases possess an analogue for Schauder decompositions. For example, a Schauder decomposition is said to be **unconditional** if all series (5.6.7.1.a) converge unconditionally.

The fundamentals of the theory of Schauder decompositions were developed by McArthur in collaboration with his Ph.D. students Ruckle [1964] and Sanders [1965a, 1965b].

5.6.7.3 A Schauder decomposition $X = \bigoplus_{k=1}^{\infty} M_k$ is **finite-dimensional** whenever $\dim(M_k) < \infty$ for $k = 1, 2, \dots$. If $\dim(M_k) = 1$, then we are just in the case of a Schauder basis. I stress that Szarek [1987, p. 89] produced a Banach space $X = [l_2, M_k]$ with an unconditional finite-dimensional Schauder decomposition, which, however, fails to have a Schauder basis. The basic step of his construction is a sophisticated choice of the finite-dimensional spaces M_k.

5.6.7.4 Every Banach space with a finite-dimensional Schauder decomposition is separable and has the bounded projection approximation property; see 5.7.4.3. Hence we know from Enflo that there are separable spaces without such a decomposition. The situation is even more pathological: Allexandrov/Kutzarova/Plichko [1999] observed that the Gowers–Maurey space 7.4.5.1 contains a subspace that does not admit any Schauder decomposition, finite-dimensional or not.

In the non-separable case, Dean [1967] discovered the following example; see also [MAR, p. 96]:

The space l_∞ does not have a Schauder decomposition.

This result implies that l_∞ as well as its isomorphic counterpart $L_\infty[0,1]$ carry very few scalar type operators; see 5.2.5.9.

5.7 Tensor products and approximation properties

5.7.1 Bilinear mappings

5.7.1.1 Our starting point is a purely algebraic definition quoted from [BOU$_{2b}$, p. 1]:

Soint A un anneau commutatif [for our purpose, $\mathbb{R}$ or $\mathbb{C}$] *ayant un élément unité, E,F,G trois A-modules unitaires* [for example, linear spaces]. *On dit qu'une application f de l'ensemble produit $E\times F$ dans G est une application* **bilinéaire** *si, pour tout $y\in F$, l'application partielle $x\mapsto f(x,y)$ est une application linéaire de E dans G, et si, pour tout $x\in E$, l'application partielle $y\mapsto f(x,y)$ est une application linéaire de F dans G.*

5.7.1.2 Given Banach spaces X, Y, and Z, we denote by $\mathfrak{L}(X\times Y,Z)$ the collection of all continuous bilinear mappings $\boldsymbol{T}$ from $X\times Y$ into Z. It turns out that $\mathfrak{L}(X\times Y,Z)$ becomes a Banach space under the norm

$$\|\boldsymbol{T}\| := \sup\{\,\|\boldsymbol{T}(x,y)\| : \|x\|\le 1,\ \|y\|\le 1\,\}.$$

The definition of $\|\boldsymbol{T}\|$ was given for the first time by Mazur/Orlicz [1935, Part II, p. 187], and Govurin [1939a, p. 543] observed the completeness of $\mathfrak{L}(X\times Y,Z)$.

Kerner [1931, pp. 159–160], [1933, p. 548] proved that *separate* continuity of a bilinear mapping in both variables x and y implies continuity; see also Mazur/Orlicz [1935, Part I, p. 65], Dieudonné/Schwartz [1950, p. 94], and [BOU$_{3b}$, p. 83].

5.7.1.3 Frobenius [1878, p. 2] developed a symbolical calculus of bilinear forms:

Es werden im Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben. Ich werde z.B. eine Form mit einer Constanten multipliciren und zwei Formen addiren. Ich werde aber nicht zwei Formen mit einander multipliciren [add: coordinatewise]. *Aus diesem Grund kann kein Missverständniss entstehen, wenn ich die aus A and B zusammengesetze Form P mit*

$$P=\Sigma_1^n \frac{\partial A}{\partial y_\kappa}\frac{\partial B}{\partial x_\kappa}$$

bezeichne und sie das **Produkt** *der Formen A und B nenne.*

On p. 4, Frobenius wrote:

Die Gleichung $P=AB$ ist eine symbolische Zusammenfassung der n^2 Gleichungen

$$p_{\alpha\beta}=\sum_\kappa a_{\alpha\kappa}b_{\kappa\beta}\qquad(\alpha,\beta=1,2,\ldots,n).$$

Hilbert [1906a, p. 179] extended the definition of Frobenius to bounded bilinear forms $A(x,y)$ and $B(x,y)$ depending on infinitely many variables:

... *folglich muß die unendliche Reihe*

$$\frac{\partial A(x,y)}{\partial y_1}\frac{\partial B(x,y)}{\partial x_1}+\frac{\partial A(x,y)}{\partial y_2}\frac{\partial B(x,y)}{\partial x_2}+\cdots$$

absolut konvergieren; dieselbe stellt dann notwendig wiederum eine Bilinearform der Variablen $x_1,x_2,\ldots,\ y_1,y_2,\ldots$ *dar, die wir mit*

$$A(x,.)B(.,y)$$

bezeichnen und die **Faltung** *der Bilinearformen* A,B *nennen.*

5.7.1.4 I stress that linear mappings also played a role in the theories of Frobenius and Hilbert: they were used as "linear substitutions" of the variables of bilinear forms.

Frobenius [1878, pp. 1–2] stated:

Wird zunächst nur die eine Reihe der Variablen einer bilinearen Form einer linearen Substitution unterworfen, so gehen in die Ausdrücke für die Coefficienten der transformirten Form die Coefficienten der ursprünglichen Form in der nämlichen Weise ein wie die Substitutionscoefficienten.

...

Diese Erwägungen leiten mich darauf, statt der Transformation der bilinearen Form die Zusammensetzung der linearen Substitutionen zu behandeln.

Ironically, Hilbert [1906a, p. 181] preferred the converse process. He wrote the transformed bilinear form as a convolution:

$$A(.\,,\bullet)O(.\,,x')O(\bullet,y').$$

Certainly, persisting in this viewpoint would have led into a dead end. Luckily enough, already in 1913 Riesz pointed out the right way; see 2.1.5.

5.7.1.5 At first glance, one could come to the conclusion that contrasting

linear mappings and bilinear forms

is only a sham fight. Indeed, the formula

$$\boldsymbol{t}(x,y)=\langle Tx,y\rangle \quad \text{for } x\in X \text{ and } y\in Y$$

yields a one-to-one correspondence between operators $T\in\mathfrak{L}(X,Y^*)$ and bilinear forms $\boldsymbol{t}\in\mathfrak{L}(X\times Y,\mathbb{K})$. In this way, $\mathfrak{L}(X,Y^{**})$ and $\mathfrak{L}(X\times Y^*,\mathbb{K})$ can be identified. If Y is reflexive, the same can be done for $\mathfrak{L}(X,Y)$ and $\mathfrak{L}(X\times Y^*,\mathbb{K})$. In particular, $\mathfrak{L}(l_2)=\mathfrak{L}(l_2\times l_2,\mathbb{K})$. However, in the non-reflexive case, dealing with $\mathfrak{L}(X,Y)$ requires some care. The trick consists in passing from $T:X\to Y$ to $K_YT:X\to Y\to Y^{**}$. This phenomenon occurred for the first time in the factorization theorem of integral operators; see 5.7.3.11.

Compared with the draw in the competitions between

nets and filters and *measures and integrals,*

I have the impression that *operators* won over *bilinear forms*. Notes in favor of bilinear forms were written by Gilbert/Leih, who made an *attempt to remedy the situation*; see [1980, p. 183] and 6.3.11.8.

5.7.1.6 In this subsection our considerations were restricted to bilinear mappings. This theory can easily be generalized to the n-linear case.

As shown in Subsection 5.1.9, multilinear mappings are an indispensable tool for dealing with derivatives of higher degree. In this context, bilinear forms have scored at least one (decisive!) point in the match against linear mappings.

5.7.2 Tensor products

5.7.2.1 Again we start with some purely algebraic considerations from [BOU$_{2b}$, pp. 3–5]:

Nous allons voir qu'on peut ramener la notion d'application bilinéaire à celle d'application linéaire, grâce à la notion de produit tensoriel.

Etant donnés deux A-modules unitaires E et F, nous allons montrer qu'il existe un A-module M, et une application bilinéaire φ de $E \times F$ dans M, telle que, si f est une application bilinéaire de $E \times F$ dans un A-module quelconque N, il existe une application linéaire g de M dans N satisfaisant à la relation $f = g \circ \varphi$.

Remarquons d'abord que, si M possède cette propriété, le sous-module de M, engendré par l'ensemble $\varphi(E \times F)$, la possède également; on peut donc se borner au cas où on impose en outre à M la condition d'être engendré par $\varphi(E \times F)$.

... *un tel module M est unique à une isomorphie près.*

On appelle **produit tensoriel** *d'une A-module E et d'une A-module F, et on note $E \otimes F$, le quotient du module des combinaisons linéaires formelles d'éléments de $E \times F$, par le sous-module où s'annulent toutes les fonctions bilinéaires.*

5.7.2.2 Bourbaki's elegant approach from 1948 grew out of a "handmade" construction that was performed by Whitney [1938, pp. 496–498] and, independently, by Schatten [1943, p. 196]. The latter acknowledged, *These problems were suggested to me by Professor F.J. Murray, who also pointed out the algebraic discussion.*

Given linear spaces X and Y, both authors considered the collection of all formal expressions

$$\sum_{k=1}^{n} x_k \otimes y_k \quad \text{with } x_1,\dots,x_n \in X \text{ and } y_1,\dots,y_n \in Y$$

subject to the agreement that

$$(x_1+x_2)\otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1+y_2) = x \otimes y_1 + x \otimes y_2,$$

and

$$\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y).$$

5.7.2.3 Murray/von Neumann [1936, pp. 127–129] had earlier invented **direct products** of Hilbert spaces as a tool in their fundamental studies of *rings of operators*. The name **tensor product** was coined by Whitney [1938] in the setting of abelian groups. On pp. 495–496 he made an observation that is extremely important from the viewpoint of this text:

If the linear spaces are topological, a topology may be introduced into the tensor product. … In the case of Hilbert spaces, there is a natural definition of the topology in the product. In the intermediate case of Banach spaces, probably the norm $|\alpha|$ may be defined as the lower bound of numbers $\sum |g_i|\,|h_i|$ for expressions $\sum g_i \cdot h_i$ of α.

(*footnote: This definition was suggested to me by H.E. Robbins.*)

Here is the opinion of an expert from the theory of operator algebras; [TAK, p. 229]:

Unlike the finite-dimensional case, the tensor product of infinite-dimensional Banach spaces behaves mysteriously. Crossnorms in the tensor product are highly nonunique.

This remark is misleading. Von Neumann [1937, p. 297] had earlier discovered a 1-parameter scale of different cross norms on $l_2^n \otimes l_2^n$; see 4.10.1.5. Next, in his famous *Résumé*, Grothendieck [1956b, p. 37] presented 14 "natural" tensor norms on finite-dimensional Banach spaces. Thus the non-uniqueness is not caused by "infinity." Of course, due to non-reflexivity and missing approximation properties, the extension of tensor norms from finite-dimensional to infinite-dimensional spaces is a tricky process.

Let us try to reveal the secret, step by step. The final discussion has to wait until Subsection 6.3.11.

5.7.2.4 According to Schatten [1943, p. 205], by a **cross norm** we mean a rule α that assigns to every element $u = \sum_{k=1}^n x_k \otimes y_k$ in the algebraic tensor product $X \otimes Y$ a non-negative number $\alpha(u)$ such that the following conditions are satisfied:

(CN$_0$) $\alpha(x \otimes y) = \|x\|\,\|y\|$ for $x \in X$ and $y \in Y$.

(CN$_1$) $\alpha(u+v) \le \alpha(u) + \alpha(v)$ for $u, v \in X \otimes Y$.

(CN$_2^\circ$) $\alpha(\lambda u) = |\lambda|\alpha(u)$ for $u \in X \otimes Y$ and $\lambda \in \mathbb{K}$.

The final objects of the theory of tensor products are the Banach spaces $X \mathbin{\widetilde{\otimes}_\alpha} Y$ obtained by completing $X \otimes Y$ with respect to a given cross norm α as well as their duals $(X \mathbin{\widetilde{\otimes}_\alpha} Y)^*$, whose members may be viewed as bilinear forms on $X \times Y$.

5.7.2.5 The pioneering work of Schatten [1943, p. 209] contains an important conceptual step. Besides cross norms defined on a *fixed* tensor product, he also considered **general** cross norms that are defined on *all* tensor products, simultaneously. However, he did not require any compatibility between the values of α on different domains $X \otimes Y$ and $X_0 \otimes Y_0$.

Nowadays, we assume that

$$(\mathbf{CN}_2) \qquad \begin{aligned} &\alpha\Big(\sum_{k=1}^{n} Ax_k \otimes By_k\Big) \le \|A\|\,\alpha\Big(\sum_{k=1}^{n} x_k \otimes y_k\Big)\,\|B\| \\ &\text{for } \sum_{k=1}^{n} x_k \otimes y_k \in X \otimes Y,\ A \in \mathfrak{L}(X, X_0) \text{ and } B \in \mathfrak{L}(Y, Y_0). \end{aligned}$$

In the case that $A \in \mathfrak{L}(X)$ and $B \in \mathfrak{L}(Y)$, this condition is due to Schatten/von Neumann [1948, p. 561]. They refer to such cross norms as **uniform**, Grothendieck [1956b, p. 8′] used the term $\otimes$**-norm**, and the modern name is **tensor norm**.

5.7.2.6 Next, we reduce **(CN$_2$)** to the special case in which A and B are functionals:

$$(\mathbf{CN}_2^*) \qquad \begin{aligned} &\Big|\sum_{k=1}^{n} \langle x^*, x_k\rangle\langle y_k, y^*\rangle\Big| \le \|x^*\|\,\alpha\Big(\sum_{k=1}^{n} x_k \otimes y_k\Big)\,\|y^*\| \\ &\text{for } \sum_{k=1}^{n} x_k \otimes y_k \in X \otimes Y,\ x^* \in X^*, \text{ and } y^* \in Y^*. \end{aligned}$$

In a slightly different form, this condition was introduced by Schatten [1943, p. 206]. Grothendieck [1956b, p. 9] refers to a cross norm satisfying **(CN$_2^*$)** as **reasonable**.

5.7.2.7 Schatten mainly dealt with two examples:

the *greatest* cross norm, now the **projective tensor norm**,

$$\pi(u) := \inf \sum_{k=1}^{n} \|x_k\|\,\|y_k\|,$$

where the infimum ranges over all representations

$$u = \sum_{k=1}^{n} x_k \otimes y_k,$$

and the *least reasonable* cross norm, now the **injective tensor norm**,

$$\varepsilon(u) := \sup\Big\{\Big|\sum_{k=1}^{n} \langle x^*, x_k\rangle\langle y_k, y^*\rangle\Big| : \|x^*\| \le 1,\ \|y^*\| \le 1\Big\}.$$

That is, for any reasonable cross norm α we have $\varepsilon \le \alpha \le \pi$.

The following notations were used:

greatest tensor norm	least tensor norm
N_1 Schatten [1943, p. 206]	N_0 Schatten [1943, p. 208]
γ [SCHA$_1$, p. 36]	λ [SCHA$_1$, p. 30]
$\|\cdot\|_\wedge$ Grothendieck [1956b, p. 8]	$\|\cdot\|_\vee$ Grothendieck [1956b, p. 8]
π [PAR$_{53}^{\Sigma}$, exposé 1, p. 2]	ε [PAR$_{53}^{\Sigma}$, exposé 7, p. 3]

5.7.2.8 If $\dim(X) = n$, then one can show with the help of an Auerbach basis that

$$\pi(u) \le n\varepsilon(u) \quad \text{for } u \in X \otimes Y$$

and any choice of Y. The factor n is sharp: take, for example, $Y = l_1^n$. Hence Grothendieck [1956b, p. 74], [GRO$_1$, Chap. II, p. 136] conjectured that the norms π and ε are never equivalent on the tensor product of two infinite-dimensional spaces. This problem remained open for 30 years. A counterexample was finally discovered by Pisier [1983, p. 197]. He constructed an infinite-dimensional Banach space X_{Pis} such that $X_{\text{Pis}} \tilde{\otimes}_\pi X_{\text{Pis}} = X_{\text{Pis}} \tilde{\otimes}_\varepsilon X_{\text{Pis}}$, both algebraically and topologically; see 7.4.2.4.

5.7.2.9 Grothendieck [GRO$_1$, Chap. I, p. 51] proved that given $\varepsilon > 0$, every member $u \in X \tilde{\otimes}_\pi Y$ can be represented in the form

$$u = \sum_{k=1}^{\infty} x_k \otimes y_k,$$

where $x_k \in X$, $y_k \in Y$, and

$$\sum_{k=1}^{\infty} \|x_k\| \|y_k\| \le \pi(u) + \varepsilon.$$

5.7.2.10 We know from [GRO$_1$, Chap. I, p. 59] that

$$L_1(M, \mathscr{M}, \mu) \tilde{\otimes}_\pi X = [L_1(M, \mathscr{M}, \mu), X]$$

for every measure space $(M, \mathscr{M}, \mu)$. Similarly, by [GRO$_1$, Chap. I , p. 90],

$$C(K) \tilde{\otimes}_\varepsilon X = [C(K), X],$$

where the right-hand Banach space $[C(K), X]$ consists of all X-valued continuous functions on a compact Hausdorff space K. In particular, we have

$$L_1(M, \mathscr{M}, \mu) \tilde{\otimes}_\pi L_1(N, \mathscr{N}, \nu) = L_1(M \times N, \mathscr{M} \times \mathscr{N}, \mu \times \nu)$$

and

$$C(K) \tilde{\otimes}_\varepsilon C(L) = C(K \times L),$$

respectively.

5.7.2.11 For every continuous bilinear mapping $\boldsymbol{T} : X \times Y \to Z$, the rule

$$\sum_{k=1}^{n} x_k \otimes y_k \mapsto \sum_{k=1}^{n} \boldsymbol{T}(x_k, y_k)$$

defines a linear mapping from $X \otimes Y$ into Z that is continuous with respect to the greatest cross norm and admits, therefore, a continuous extension on $X \tilde{\otimes}_\pi Y$. In this way, the Banach spaces

$$\mathfrak{L}(X \times Y, Z) \quad \text{and} \quad \mathfrak{L}(X \tilde{\otimes}_\pi Y, Z)$$

can be identified. In particular, we have $\mathfrak{L}(X \times Y, \mathbb{K}) = (X \tilde{\otimes}_\pi Y)^*$.

5.7.2.12 In a next step, Grothendieck [1956b, p. 12] considered bilinear forms $\boldsymbol{t}$ on $X \times Y$ for which the induced linear form on $X \otimes Y$ is continuous with respect to a given cross norm.

The most interesting case occurs when we take the least reasonable cross norm ε. Then the dual $(X \widetilde{\otimes}_\varepsilon Y)^*$ consists of all bilinear forms $\boldsymbol{t}$ that can be represented in the form

$$\boldsymbol{t}(x,y) = \int_{B_{X^*} \times B_{Y^*}} \langle x, x^* \rangle \langle y^*, y \rangle \, d\mu(x^*, y^*) \quad \text{for } x \in X \text{ and } y \in Y,$$

where μ is a Baire measure on the Cartesian product of the weakly* compact unit balls B_{X^*} and B_{Y^*}; see [GRO$_1$, Chap. I, pp. 95–96]. This formula is the reason why Grothendieck referred to those bilinear forms as **integral**.

5.7.2.13 The results obtained by Schatten/von Neumann in the period from 1943 to 1948 were presented in Schatten's monograph [SCHA$_1$], which suffers from a lack of concrete examples. For the purpose of an abstract Fredholm determinant theory, Ruston introduced *direct products* of Banach spaces. His contributions are summarized in [RUST]. Last but not least, inspired by the work of Dieudonné/Schwartz on locally convex linear spaces, Grothendieck developed a third approach. Unfortunately, his thesis [GRO$_1$] as well as his *Résumé* [1956b] are very hard to read. Written in the first half of the 1950s, they remained a mystery to the mathematical community for more than 10 years. Though Amemiya/Shiga [1957] made a feeble attempt to understand the master's ideas, the real breakthrough was achieved only in a seminal paper of Lindenstrauss/Pełczyński [1968].

5.7.2.14 In this subsection, our considerations were restricted to tensor products of two factors. All concepts and results can easily be generalized to $X_1 \otimes \cdots \otimes X_n$; see Ruston [1951b], [RUST], and Grothendieck [1956a].

5.7.3 Nuclear and integral operators

5.7.3.1 An operator $T \in \mathfrak{L}(X,Y)$ is called **nuclear** if it can be written in the form

$$T = \sum_{k=1}^{\infty} x_k^* \otimes y_k \quad \text{such that} \quad \sum_{k=1}^{\infty} \|x_k^*\| \, \|y_k\| < \infty, \tag{5.7.3.1.a}$$

$x_1^*, x_2^*, \ldots \in X^*$ and $y_1, y_2, \ldots \in Y$. It is readily seen that the infinite series of the terms $x_k^* \otimes y_k : x \mapsto \langle x, x_k^* \rangle y_k$ converges in $\mathfrak{L}(X,Y)$. Therefore nuclear operators can be approximated by finite rank operators.

The **nuclear norm** is given by

$$\nu(T) := \inf \sum_{k=1}^{\infty} \|x_k^*\| \, \|y_k\|,$$

where the infimum ranges over all **nuclear representations** (5.7.3.1.a).

5.7.3.2 Let $\mathfrak{N}(X,Y)$ denote the collection of all nuclear operators from X into Y. Then

$$\mathfrak{N} := \bigcup_{X,Y} \mathfrak{N}(X,Y)$$

is an ideal in the sense of 2.6.6.1. Moreover, every component $\mathfrak{N}(X,Y)$ becomes a Banach space under the norm ν, which satisfies the condition

$$\nu(BTA) \leq \|B\|\nu(T)\|A\| \quad \text{for } A \in \mathfrak{L}(X_0,X),\, T \in \mathfrak{N}(X,Y),\, B \in \mathfrak{L}(Y,Y_0);$$

see [GRO$_1$, Chap. I, p. 84].

5.7.3.3 Grothendieck [GRO$_1$, Chap. I, pp. 82–83] characterized nuclear operators by factorizations of the form

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & Y \\ l_\infty \arrow[r, "D_t"'] & l_1 \arrow[u, "B"'] \end{tikzcd} \tag{5.7.3.3.a}$$

where the **diagonal operator** $D_t : (\xi_k) \mapsto (\tau_k \xi_k)$ is induced by a sequence: $t = (\tau_k) \in l_1$. We have $\nu(D_t) = \|t|l_1\|$, and given $\varepsilon > 0$, it can be arranged that

$$\|B\|\,\|t|l_1\|\,\|A\| \leq \nu(T) + \varepsilon.$$

5.7.3.4 The following examples are due to Grothendieck [1956a, pp. 370–372].

Nuclear operators from l_1 into itself are generated by infinite matrices $T = (\tau_{hk})$ for which

$$\nu(T : l_1 \to l_1) = \sum_{h=1}^{\infty} \sup_k |\tau_{hk}|$$

is finite; see 2.6.3.3 (transposed matrix) and 6.5.2.8.

Every continuous kernel K on $[0,1] \times [0,1]$ defines a nuclear operator from $C[0,1]$ into itself such that

$$\nu(K : C \to C) = \int_0^1 \sup_s |K(s,t)|\,dt.$$

This means that the theory of nuclear operators applies to its historical source: integral operators in the classical sense; compare with 5.7.3.9.

5.7.3.5 The notion of nuclearity was introduced in the early 1950s for two reasons. Inspired by the theory of distributions, Grothendieck dealt with tensor products of locally convex linear spaces and discovered an abstract version of Schwartz's kernel theorem. This explains the origin of the term *nuclear*; see also 6.3.21.6.

Secondly, Ruston [1951a] and Grothendieck [1956a] extended Fredholm's determinant theory to operators in Banach spaces. To this end they needed a sufficiently large class of operators for which the concept of a trace is meaningful.

For a Hilbert space H, the Schatten–von Neumann class $\mathfrak{S}_1(H)$ turned out to be appropriate. Since $\mathfrak{S}_1(H)=\mathfrak{N}(H)$, it was thought that $\mathfrak{N}$ would be the right choice for general Banach spaces; indeed, the nuclear operators were referred to as *opérateurs à trace* or *trace class operators*; see Grothendieck [1951, p. 1557] and [RUST, p. 166]. Ironically, Enflo's negative solution of the approximation problem implies that there are trace class operators without traces!

5.7.3.6 Next, I discuss the original definition of trace class operators, which is based on the formula $H\,\widetilde{\otimes}_\pi H = \mathfrak{S}_1(H)$; see Schatten/von Neumann [1946, p. 625].

Ruston [1951a, p. 111]:

Thus to every $\mathfrak{A} \in \mathfrak{B} \otimes_\gamma \mathfrak{B}^*$ [read: $X^*\widetilde{\otimes}_\pi X$] *there corresponds a linear operator A, which is easily seen to be unique. The set of linear operators that can be obtained in this way we call the* **trace class** (tc) *of the Banach space* $\mathfrak{B}$.

In a footnote he added:

It should be observed that we have not proved in general that to each operator of the trace class there corresponds a unique element of $\mathfrak{B} \otimes_\gamma \mathfrak{B}^*$.

In [GRO_1, Chap. I, p. 36], this fundamental question was referred to as the **problème de biunivocité**. The negative answer is discussed in the next subsection.

Since Grothendieck worked in locally convex linear spaces, he got three different kinds of operators, which, however, coincide on Banach spaces; [GRO_1, Chap. I, p. 80]:

Si E et F sont des espaces de Banach, les applications à trace, les applications de Fredholm et les **applications nucléaires** *de E et F sont les mêmes, et l'ensemble de ces opérateurs s'identifie par definition à un espace quotient de* $E'\widehat{\otimes}F$*; on notera encore* $u \to \|u\|_1$ *la norm quotient, appelée aussi norme-trace de u.*

5.7.3.7 Recall that the nuclear norm of T is defined by

$$\nu(T) := \inf \sum_{k=1}^{\infty} \|x_k^*\|\,\|y_k\|,$$

where the infimum ranges over all *infinite* representations (5.7.3.1.a). If T has finite rank, then one may consider the infimum over all *finite* representations (2.6.2.1.b):

$$\nu^\circ(T) := \inf \sum_{k=1}^{n} \|x_k^*\|\,\|y_k\|.$$

Obviously, ν° is nothing but the greatest cross norm on $X^*\otimes Y$ and $\nu(T) \le \nu^\circ(T)$; see 5.7.2.7. The uniqueness problem can be rephrased by asking whether ν and ν° coincide; see $(\mathbf{AP}_2)$ and $(\mathbf{AP}_4)$ in 5.7.4.1.

5.7.3.8 The German term **Spur** was coined by Dedekind [1882, p. 5] in the framework of algebraic number theory:

Unter der Spur der Zahl θ verstehen wir die Summe aller mit ihr conjugirten Zahlen; wir bezeichnen diese offenbar rationale Zahl mit $S(\theta)$; dann ist

$$S(\theta) = e_{1,1} + e_{2,2} + \cdots + e_{n,n}.$$

Therefore we refer to

$$\operatorname{trace}(T) := \sum_{k=1}^{n} \tau_{kk}$$

as the trace of the (n,n)-matrix $T = (\tau_{hk})$. An abstract approach is due to [BOU$_{2b}$, pp. 48–50], who defined the **trace** of a linear mapping

$$T = \sum_{k=1}^{n} x_k^* \otimes x_k$$

on a finite-dimensional linear space by

$$\operatorname{trace}(T) := \sum_{k=1}^{n} \langle x_k^*, x_k \rangle.$$

This is a purely algebraic approach, since the x_k^*'s are just assumed to be linear forms. The same definition, now with $x_1^*, \ldots, x_n^* \in X^*$, can be used for finite rank operators on an arbitrary Banach space X; see Ruston [1951a, p. 112]. In this way, we obtain a linear functional $T \mapsto \operatorname{trace}(T)$ on $\mathfrak{F}(X)$.

Very important is the commutative law:

$$\operatorname{trace}(AT) = \operatorname{trace}(TA) \quad \text{for } T \in \mathfrak{F}(X,Y) \text{ and } A \in \mathfrak{L}(Y,X).$$

This formula relates traces defined on (possibly different) components $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$.

It is natural to think of extending the trace by continuity. Using Auerbach's lemma, we obtain the inequality

$$|\operatorname{trace}(T)| \le n\|T\| \quad \text{whenever } \operatorname{rank}(T) \le n.$$

Since the identity map of l_2^n shows that the factor n cannot be improved, the trace is discontinuous with respect to the operator norm in infinite dimensions.

On the other hand, in view of

$$\Big| \sum_{k=1}^{n} \langle x_k^*, x_k \rangle \Big| \le \pi\Big(\sum_{k=1}^{n} x_k^* \otimes x_k \Big),$$

every member of $X^* \widetilde{\otimes}_\pi X$ has a well-defined trace. However, this positive result does not carry over to nuclear operators, since $T \in \mathfrak{N}(X)$ may have different preimages in $X^* \widetilde{\otimes}_\pi X$; see 5.7.3.6.

5.7.3.9 An operator $T \in \mathfrak{L}(X,Y)$ is said to be **integral** if there exists a constant $c \geq 0$ such that

$$|\,\text{trace}(ST)| \leq c\|S\| \quad \text{for all } S \in \mathfrak{F}(Y,X). \tag{5.7.3.9.a}$$

The **integral norm** $\iota(T)$ is defined to be the smallest c for which (5.7.3.9.a) holds.

Note that an operator T is integral if and only if the associated bilinear form $\langle Tx, y^* \rangle$ is integral on $X \times Y^*$; see 5.7.2.12.

5.7.3.10 As in the case of nuclear operators, the class of all integral operators constitutes an ideal

$$\mathfrak{I} := \bigcup_{X,Y} \mathfrak{I}(X,Y).$$

Moreover, we have

$$\iota(BTA) \leq \|B\|\iota(T)\|A\| \quad \text{for} \in \mathfrak{L}(X_0,X),\ T \in \mathfrak{I}(X,Y),\ B \in \mathfrak{L}(Y,Y_0);$$

see [GRO$_1$, Chap. I, p. 128].

5.7.3.11 In analogy with (5.7.3.3.a), Grothendieck [GRO$_1$, Chap. I, p. 127] characterized integral operators by factorizations of the form

$$\begin{tikzcd} X \arrow[r, "K_Y T"] \arrow[d, "A"'] & Y^{**} \\ C(K) \arrow[r, "Id"'] & L_1(K,\mu) \arrow[u, "B"'] \end{tikzcd} .$$

Here μ is a normalized Borel measure on a compact Hausdorff space K. Instead of $Id : C(K) \to L_1(K,\mu)$ one may use $Id : L_\infty(M,\mathscr{M},\mu) \to L_1(M,\mathscr{M},\mu)$ with a finite measure space $(M,\mathscr{M},\mu)$. Both identity maps are prototypes of integral operators.

5.7.3.12 Grothendieck showed that the identity map from l_1 into c_0 is integral. Indeed,

$$\begin{tikzcd} l_1 \arrow[r, "Id"] \arrow[d, "R^*"'] & c_0 \\ L_\infty[0,1] \arrow[r, "Id"'] & L_1[0,1] \arrow[u, "R"'] \end{tikzcd} .$$

The operator $R : f \mapsto (f|r_k)$ assigns to every function the sequence of its Rademacher coefficients. The above factorization seems to be a part of the folklore. Grothendieck's original proof in [1956c, p. 93] was rather complicated.

5.7.3.13 According to [GRO$_1$, Chap. I, p. 127], nuclear operators are integral:

$$\mathfrak{N} \subset \mathfrak{I} \quad \text{and} \quad \iota(T) \leq \nu(T) \quad \text{for all } T \in \mathfrak{N}.$$

Since $Id : l_1 \to c_0$ is integral, but non-compact, the left-hand inclusion turns out to be proper. Conversely, we know from [GRO$_1$, Chap. I, p. 132] that $\mathfrak{W} \circ \mathfrak{I} \subseteq \mathfrak{N}$:

$$ST \in \mathfrak{N}(X,Z) \ \text{ and } \ \nu(ST) \leq \|S\|\iota(T) \ \text{ for } \ T \in \mathfrak{I}(X,Y) \ \text{ and } \ S \in \mathfrak{W}(Y,Z).$$

However, both norms ι and ν may be non-equivalent on specific components $\mathfrak{F}(X,Y)$.

5.7.3.14 An elementary observation in [GRO$_1$, Chap. I, p. 85] says that

$$T \in \mathfrak{N} \quad \text{implies} \quad T^* \in \mathfrak{N} \quad \text{and} \quad \nu(T^*) \leq \nu(T),$$

whereas the converse implication may fail; see Figiel/Johnson [1973, p. 199].

On the other hand,

$$T \in \mathfrak{I} \quad \text{if and only if} \quad T^* \in \mathfrak{I} \quad \text{and} \quad \iota(T^*) = \iota(T).$$

I was not able to find the latter result (verbatim) in any of Grothendieck's papers. However, it is contained in [PAR$^{\Sigma}_{53}$, exposé 16, p. 5].

5.7.4 Approximation properties

5.7.4.1 A Banach space X has the **approximation property** if the following condition is satisfied:

(AP) For every compact subset K and every $\varepsilon > 0$ there exists an operator $T \in \mathfrak{F}(X)$ such that $\|x - Tx\| \leq \varepsilon$ whenever $x \in K$.

This means that $\mathfrak{F}(X)$ is dense in $\mathfrak{L}(X)$ with respect to the $\mathscr{K}$-topology; see 3.3.4.3.

Grothendieck [GRO$_1$, Chap. I, p. 165] discovered many properties that are equivalent to **(AP)**; see also [DEF$^+$, p. 64] or [DIE$^+_2$, pp. 238–245]:

(AP$_1$) $\nu^\circ(T) \leq c\,\nu(T)$ for $T \in \mathfrak{F}(Y,X)$, every Banach space Y, and some constant $c \geq 1$.

(AP$^\circ_1$) $\nu^\circ(T) \leq c\,\nu(T)$ for $T \in \mathfrak{F}(X)$ and some constant $c \geq 1$.

(AP$_2$) $\nu^\circ(T) = \nu(T)$ for $T \in \mathfrak{F}(Y,X)$ and every Banach space Y.

(AP$^\circ_2$) $\nu^\circ(T) = \nu(T)$ for $T \in \mathfrak{F}(X)$.

(AP$_3$) The natural map from $Y \widetilde{\otimes}_\pi X$ into $Y \widetilde{\otimes}_\varepsilon X$ is one-to-one for every Banach space Y.

(AP$^\circ_3$) The natural map from $X^* \widetilde{\otimes}_\pi X$ into $X^* \widetilde{\otimes}_\varepsilon X$ is one-to-one.

(AP$_4$) The surjection from $Y^* \widetilde{\otimes}_\pi X$ onto $\mathfrak{N}(Y,X)$ is one-to-one for every Banach space Y.

(AP$^\circ_4$) The surjection from $X^* \widetilde{\otimes}_\pi X$ onto $\mathfrak{N}(X)$ is one-to-one.

(AP$_5$) The functional $T \mapsto \text{trace}(T)$, defined for finite rank operators, admits a continuous extension to $\mathfrak{N}(X)$.

(AP$_6$) $\overline{\mathfrak{F}}(Y,X) = \mathfrak{K}(Y,X)$ for every Banach space Y.

It seems to be an open problem whether this list can be enlarged by

(AP$^\circ_6$) $\overline{\mathfrak{F}}(X) = \mathfrak{K}(X)$.

Here is a short and direct proof of the implication **(AP$_1$)⇒(AP$_2$)**:

Given $\varepsilon > 0$, we choose a representation

$$T = \sum_{k=1}^{\infty} y_k^* \otimes x_k$$

and some n such that

$$\sum_{k=1}^{\infty} \|y_k^*\|\|x_k\| \leq \nu(T) + \varepsilon \quad \text{and} \quad c \sum_{k=n+1}^{\infty} \|y_k^*\|\|x_k\| \leq \varepsilon.$$

Then

$$\begin{aligned} \nu^\circ(T) &\leq \nu^\circ\Big(\sum_{k=1}^{n} y_k^* \otimes x_k\Big) + \nu^\circ\Big(\sum_{k=n+1}^{\infty} y_k^* \otimes x_k\Big) \leq \\ &\leq \sum_{k=1}^{n} \|y_k^*\|\|x_k\| + c\nu\Big(\sum_{k=n+1}^{\infty} y_k^* \otimes x_k\Big) \leq \nu(T) + 2\varepsilon. \end{aligned}$$

5.7.4.2 A Banach space X has the **bounded approximation property** if we can find a constant $c \geq 1$ such that the following condition is satisfied:

(BAP) For every compact subset K and every $\varepsilon > 0$ there exists an operator $T \in \mathfrak{F}(X)$ of norm not greater than c and such that $\|x - Tx\| \leq \varepsilon$ whenever $x \in K$.

Obviously, it suffices if **(BAP)** holds for all finite subsets K.

If the specific value of c is relevant, one speaks of the c**-bounded** or c**-metric approximation property**; Johnson [1971, pp. 481–482]. In the case $c = 1$, the term **metric approximation property** is used; [GRO$_1$, Chap. I, pp. 178–179].

The following characterization is due to Pełczyński [1971, p. 239] and Johnson/Rosenthal/Zippin [1971, p. 503]:

A separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a space with a basis.

5.7.4.3 Lindenstrauss [LIND, p. 25] introduced the **bounded projection approximation property** by requiring that (**BAP**) be satisfied not only with a finite rank operator, but even with a finite rank projection. Nowadays, following a proposal of Johnson/Rosenthal/Zippin [1971, p. 489], most authors prefer the name π**-property**; see also Johnson [1970].

5.7.4.4 Next, we define the **commuting bounded approximation property**:

There exists a net $(T_i)_{i \in \mathbb{I}}$ of finite rank operators on X such that

$$\lim_{i \in \mathbb{I}} T_i x = x \quad \text{for } x \in X, \quad \|T_i\| \leq c \quad \text{and} \quad T_i T_j = T_j T_i \quad \text{for } i, j \in \mathbb{I}.$$

Based on unpublished work of Rosenthal, the separable version of this concept was introduced by Johnson [1972, p. 309].

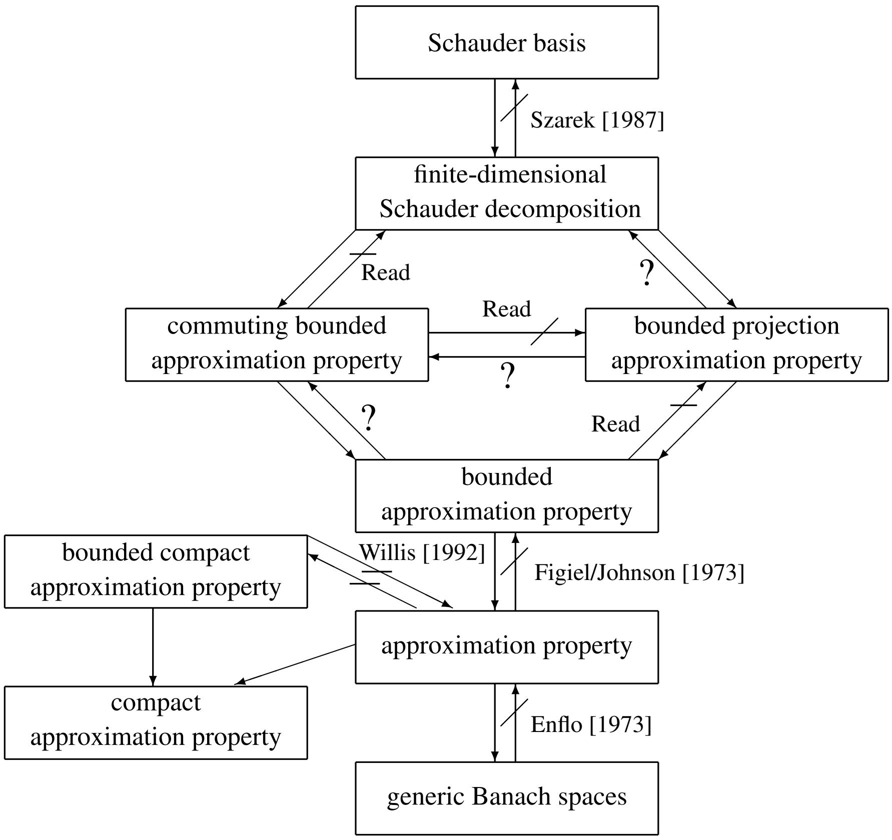
Using the commuting bounded approximation property and further purely functional analytic tools, Lusky [1992] was able to show that the disk algebra has a basis. The original proof is due to Bochkarev [1974].

5.7.4.5 Finally, an obvious weakening of **(AP)** leads to the **compact approximation property**, which for separable spaces already occurred in [BAN, p. 237]. Instead of having finite rank, the approximating operator is required only to be compact.

(CAP) For every compact subset K and every $\varepsilon > 0$ there exists an operator $T \in \mathfrak{K}(X)$ such that $\|x - Tx\| \leq \varepsilon$ whenever $x \in K$.

The **bounded compact approximation property** and the **metric compact approximation property** are obtained by obvious modifications.

5.7.4.6 The hierarchy of most (but not all) approximation properties is presented in the following table. Further variants can be found in a survey of Casazza [2001]. For better comparability, all spaces are supposed to be separable. I have indicated those authors who produced the corresponding counterexamples.



The arrows $\not\longrightarrow$ in the middle of the table are taken from a preprint of Read on *Different forms of the approximation property*, which was written around 1989 but has remained unpublished over the years; see also Casazza [2001, p. 301]. It is still open whether those implications labeled by a question mark are true or false. However, I stress that the non-separable space l_∞ has, of course, the bounded projection approximation property, while it fails the commuting bounded approximation property; see Casazza [2001, p. 311].

Casazza [1987, p. 117] proved that a separable Banach space admits a finite-dimensional Schauder decomposition if (and only if) it has, simultaneously, the commuting bounded approximation property and the bounded projection approximation property.

Willis's counterexample [1992, pp. 100, 102] shows that the bounded compact approximation property does not imply the approximation property. According to Casazza/Jarchow [1996, p. 357], the reverse implication fails as well.

5.7.4.7 The fact that the classical Banach spaces $L_p(M,\mathscr{M},\mu)$ and $C_b(M)$ have the metric approximation property seems to be a part of the folklore; see Dunford/Pettis [1940, p. 371]. The first explicit proof is contained in [GRO$_1$, Chap. I, p. 185] and [PAR$^{\Sigma}_{53}$, exposé 15, pp. 1–2].

5.7.4.8 Almost all Banach spaces without some approximation property are highly artificial. The only exception, namely $\mathfrak{L}(l_2)$, was discovered by Szankowski [1981]; see also Pisier's lecture at the Séminaire Bourbaki [1979a]. The situation is open for $\mathfrak{L}(l_1)$ and the Hardy space H_∞.

5.7.4.9 In view of $\iota(T) \le \nu(T) \le \nu^\circ(T)$, the inequality

(BAP$_1$) $\qquad \nu^\circ(T) \le c\,\iota(T)$ for all $T \in \mathfrak{F}(Y,X)$

implies

(AP$_1$) $\qquad \nu^\circ(T) \le c\,\nu(T)$ for all $T \in \mathfrak{F}(Y,X)$

and

(B) $\qquad \nu(T) \le c\,\iota(T)$ for all $T \in \mathfrak{F}(Y,X)$.

However, **(AP$_1$)** is equivalent to

(AP$_2$) $\qquad \nu^\circ(T) = \nu(T)$ for all $T \in \mathfrak{F}(Y,X)$.

Hence

$$\textbf{(BAP}_1\textbf{)} \Leftrightarrow \textbf{(B)} + \textbf{(AP}_1\textbf{)}.$$

Unfortunately, we have only little information about Banach spaces X such that **(B)** holds for arbitrary Banach spaces Y. Nevertheless, there is a sufficient condition that could be called the c-bounded *weakly compact* approximation property.

5.7.4.10 Grothendieck discovered conditions that guarantee a much stronger property than **(B)**; see [GRO$_1$, Chap. I, pp. 132–135], 5.7.3.13, and 6.3.18.2.

If X is reflexive or a separable dual, then $\mathfrak{N}(Y,X)$ and $\mathfrak{I}(Y,X)$ coincide isometrically for every Banach space Y. The same conclusion follows from the weaker assumptions that X has the Radon–Nikodym property and is 1-complemented in its bidual.

This result has a striking corollary:

For reflexive Banach spaces or separable duals, the approximation property implies the metric approximation property.

5.7.4.11 The bounded approximation property is stable under passing to equivalent norms, but the constant c may change. Unfortunately, we do not know any renorming process that improves the constant. Of course, the final aim would be to get $c=1$.

Johnson [1972, p. 309] showed that every space with the commuting bounded approximation property admits a renorming under which it even has the metric approximation property. Casazza/Kalton [1989, pp. 52–53] obtained a converse:

Every separable Banach space with the metric approximation property also has the commuting bounded approximation property.

Hence, in the separable setting, the commuting bounded approximation property can be viewed as an isomorphic version of the metric approximation property.

5.7.4.12 Consider c_0 under any equivalent norm. Then the corresponding dual is separable and has the approximation property. Hence l_1 has the metric approximation property with respect to any dual norm, and the same is true for *all* renormings of c_0; see below. On the other hand, it seems to be unknown whether the metric approximation property of l_1 is preserved under *all* renormings.

5.7.4.13 Grothendieck [GRO$_1$, Chap. I, pp. 167, 180]:

Si X^* satisfait à la condition d'approximation, X y satisfait aussi.

Si X^* satisfait à la condition d'approximation métrique, il en est de même de X.

Finally, Johnson [1971, p. 482] proved a corresponding result for the c-bounded approximation property.

In all cases, the converse implication may fail. However, we have a remarkable theorem due to Figiel/Johnson [1973, p. 198]:

If X has the c-bounded approximation property for every equivalent norm, then X^* has the $2c(4c+1)$-bounded approximation property.

This is a supplement to an earlier result of Johnson [1972, p. 308]:

If X has the metric approximation property for every equivalent norm, then X^* has the metric approximation property as well.

5.7.4.14 In *The Scottish Book* [MAUL•, Problem 153], the following question was raised by Mazur (November 6, 1936):

Does every continuous real-valued function K on $[0,1]\times[0,1]$ have the property that for every $\varepsilon>0$, there exist points $s_1,\dots,s_n$ and $t_1,\dots,t_n$ in $[0,1]$ as well as constants $c_1,\dots,c_n$ such that

$$\Big|K(s,t)-\sum_{i=1}^{n}c_iK(s,t_i)K(s_i,t)\Big|\le\varepsilon \quad \text{if } 0\le s,t\le 1. \tag{5.7.4.14.a}$$

Without reference to *The Scottish Book*, Grothendieck asked the same question. However, he went further by showing that a negative answer would even provide a negative answer to the approximation problem; [GRO_1, Chap. I, pp. 170–171]. I stress that Grothendieck made this observation only around 1953 with the help of techniques that were not available in 1936.

In a letter to me, Pełczyński remembers that Mazur claimed several times to know that an affirmative answer to his question would imply the approximation property for all Banach spaces; see also Pełczyński [1973, стр. 68] and [1979, pp. 243–244]. Unfortunately, no information about Mazur's proof has been handed down. Nevertheless, the offered prize–a live goose–indicates that Mazur realized the far-reaching consequences of his problem very well.

Maybe, Mazur's condition (5.7.4.14.a) was suggested by considerations like these:

By the Weierstrass approximation theorem, every continuous kernel K defined on the unit square induces an approximable and therefore compact operator K from $C[0,1]^*$ into $C[0,1]$. Take the canonical factorization

$$K: C[0,1]^* \xrightarrow{\ Q\ } C[0,1]^*/N(K) \xrightarrow{\ K_0\ } \overline{M(K)} \xrightarrow{\ J\ } C[0,1].$$

Since the passage from K to K_0 preserves compactness, it is natural to ask whether K_0 can be approximated by finite rank operators. The answer is based on the following observation: $\overline{M(K)}$ coincides with the closed linear span in $C[0,1]$ of all functions $y_t: s\mapsto K(s,t)$. Similarly, $\overline{M(K^*)}$ is the closed linear span in $C[0,1]$ of all functions $x_s^*: t\mapsto K(s,t)$, where $C[0,1]$ is viewed as a subspace of $C[0,1]^{**}$. Hence assuming that $\overline{M(K)}$ and $\overline{M(K^*)}$ have the approximation property, we get the required approximation of K_0, even in the strong sense of (5.7.4.14.a).

An affirmative answer to Mazur's conjecture would have implied a remarkable consequence. Suppose that (5.7.4.14.a) is satisfied for a kernel K with

$$\int_0^1 K(s,\xi)K(\xi,t)\,d\xi=0 \quad \text{for } 0\le s,t\le 1.$$

Then

$$\Big|\int_0^1 K(\xi,\xi)\,d\xi-\sum_{i=1}^{n}c_i\int_0^1 K(s_i,\xi)K(\xi,t_i)\,d\xi\Big|\le\varepsilon$$

implies

$$\int_0^1 K(\xi,\xi)\,d\xi=0.$$

In other words: if K is viewed as a nuclear operator on $C[0,1]$ or $L_1[0,1]$, then it follows from $K^2 = O$ that $\operatorname{trace}(K) = 0$. Surprisingly, this is not true for all continuous kernels. Hence Mazur's conjecture fails.

In the setting of nuclear operators on l_1, there is a discrete analogue. "Strange" infinite matrices are the most convenient tools for constructing spaces without the approximation property; see [LIND_1^+, p. 87] and [PIE_3, p. 138]. The "only" thing one needs is a matrix (τ_{hk}) with the following properties:

$$\sum_{h=1}^{\infty} \sup_k |\tau_{hk}| < \infty, \quad \sum_{i=1}^{\infty} \tau_{hi}\tau_{ik} = 0 \quad \text{for } h,k = 1,2,\dots \text{, and} \quad \sum_{i=1}^{\infty} \tau_{ii} = 1.$$

Then

$$T : (\xi_k) \mapsto \Big(\sum_{k=1}^{\infty} \tau_{hk}\xi_k \Big)$$

defines an operator $T \in \mathfrak{N}(l_1)$ such that $T^2 = O$ and $\operatorname{trace}(T) = 1$. Subject to these conditions, it follows that the quotient $l_1/N(T)$ cannot possess the approximation property; see [PIE_4, pp. 213–214].

5.7.4.15 According to the winged words

Thy wish is father … to that thought,
(William Shakespeare, *Henry IV*, Part II)

many functional analysts believed that the approximation problem would be solved in the affirmative. Stimulated by Cohen's contributions to set theory, it was also taken into account that **(AP)** could have the same character as the continuum hypothesis or the axiom of choice.

Here is Grothendieck's opinion; [GRO_1, Chap. II, p. 135]:

On peut espérer, même si le problème d'approximation se solvait par la négative dans le cas général (*ce qui me semble probable*) *que toute forme bilinéaire faiblement compact sur $E \times F$ soit adhérente à $E' \times F'$ pour la topologie de la convergence bicompacte. Cela impliquerait que tout espace de Banach réflexif satisfait à la condition d'approximation, donc même à la condition d'approximation métrique.*

Lindenstrauss [1970a, p. 169] summarized the above as follows:

CONJECTURE 1. *Not every Banach space has the approximation property.*

CONJECTURE 2. *All reflexive Banach spaces have the approximation property.*

He added: *We feel that these may be true.*

In his thesis, finished in the pre-Enflo era, Figiel [1973, p. 191] observed that these conjectures are contradictory: if all reflexive Banach spaces have the approximation property, then all Banach spaces have the approximation property too. This follows from the fact that every compact operator $T : X \to Y$ factors through a reflexive space W such that A and B are compact: $T \xrightarrow{A} W \xrightarrow{B} Y$.

The sensation came in May 1972 when Enflo discovered a counterexample to the approximation property. Subsequently, his construction was considerably simplified: Davie [1973], Pełczyński/Figiel [1973], and Szankowski [1978].

When proofreading his paper mentioned above, Figiel [1973, p. 208] wrote:

Enflo's result has completely changed many opinions about the approximation property.

5.7.4.16 Thirty years before Enflo, Bohnenblust [1941, p. 64] had expressed the opinion that the basis problem may have a negative answer. Letting

$$p_n(X) := \inf\{\|P\| : \text{projection on } X \text{ such that } \operatorname{rank}(P) = n\},$$

he stated:

If separable Banach spaces X exist for which the $p_n(X)$ are not bounded, the open question, whether or not every separable Banach space admits a basis, will be answered negatively.

In a next step, Singer [1963, pp. 548–549], [SIN_1, pp. 109–110], made the hypothesis that there is a sequence of N-dimensional spaces E_N for which $p_n(E_N)$ increases to ∞ in a suitable way. Then gluing together these little pieces, a counterexample could be found.

Though the two authors were not able to carry out their program, they paved the way for the final solution of the basis problem.

Only in [1983] did Pisier discover an infinite-dimensional space X_{Pis} satisfying Bohnenblust's condition in its strongest form; see 7.4.2.4. Indeed, $p_n(X_{\text{Pis}}) \asymp \sqrt{n}$. A few month earlier, Szarek [1983, p. 154] constructed a sequence of $2n$-dimensional spaces E_{2n} such that $p_n(E_{2n}) \geq c\sqrt{n}$ for some constant $c > 0$.

5.7.4.17 Life in Banach spaces with certain approximation properties is much easier. Thus the effect of Enflo's counterexample can be described as the banishment from Paradise.

Finally, I summarize the troubles of the real world:

- There are separable Banach spaces without bases.
- There are compact operators that cannot be approximated by finite rank operators.
- There are trace class operators without traces.
- There are nuclear operators whose nuclear norms differ from their integral norms.
- There are non-nuclear operators with nuclear duals.
- There are nuclear operators with non-vanishing traces, but having no eigenvalues $\lambda \neq 0$.

6

Modern Banach Space Theory – Selected Topics

Due to lack of space, it is impossible to present in this chapter all branches of modern Banach space theory. Luckily enough, the *Handbook of the Geometry of Banach Spaces* (almost 2000 pages) covers most of these gaps. In particular, the reader is advised to consult the survey articles about the classical spaces $L_p(M, \mathscr{M}, \mu)$ and $C(K)$ written by Alspach/Odell [$\text{JOHN}_1^{\cup}$, pp. 123–159], Johnson/Schechtman [$\text{JOHN}_1^{\cup}$, pp. 837–870], Koldobsky/König [$\text{JOHN}_1^{\cup}$, pp. 899–939], and Rosenthal [$\text{JOHN}_2^{\cup}$, pp. 1547–1602].

The selection of topics reflects my personal interest and knowledge. By no means should any conclusions be drawn about the significance of specific subjects that are or are not included.

6.1 Geometry of Banach spaces

This section is mainly devoted to the *local theory* of Banach spaces. I hope that the following quotations will replace a non-existent precise definition.

Rutovitz [1965, p. 241]
It may be that little progress will be made with some Banach space problems until more has been learned about finite-dimensional spaces.

Pełczyński/Rosenthal [1975, p. 263]
Localization refers to obtaining quantitative finite-dimensional formulations of infinite-dimensional results.

Tomczak-Jaegermann [TOM, p. 5]
A property (*of Banach spaces or of operators acting between them*) *is called local if it can be defined by a quantitative statement or inequality concerning a finite number of vectors or finite-dimensional subspaces.*

Lindenstrauss/Milman [1993, p. 1151]
Actually the name "local theory" is applied to two somewhat different topics:
(a) *The quantitative study of n-dimensional normed spaces as $n \to \infty$.*
(b) *The relation of the structure of an infinite-dimensional space and its finite-dimensional subspaces.*

6.1.1 Banach–Mazur distances, projection, and basis constants

In order to compare the asymptotic behavior of positive scalar sequences (α_n) and (β_n), the following notation will be used:

$$\alpha_n \preceq \beta_n \quad \text{means that} \quad \alpha_n \leq c\beta_n \quad \text{for} \quad n = 1, 2, \ldots \text{ and some constant } c > 0.$$

In other words, a *"big Oh"* relation in the sense of Landau holds: $\alpha_n = O(\beta_n)$. We write $\alpha_n \asymp \beta_n$ if $\alpha_n \preceq \beta_n$ and $\beta_n \preceq \alpha_n$.

6.1.1.1 As already mentioned in 4.9.1.4, the **Banach–Mazur distance** of n-dimensional Banach spaces E_n and F_n is defined by

$$d(E_n, F_n) = \min\Big\{ \|U\| \|U^{-1}\| : E_n \underset{U^{-1}}{\overset{U}{\rightleftarrows}} F_n \Big\},$$

and for spaces E_n, F_n, and G_n, we have a multiplicative triangle inequality:

$$d(E_n, G_n) \leq d(E_n, F_n) d(F_n, G_n).$$

By identification of isometric copies, the collection of all n-dimensional Banach spaces becomes a compact metric space, the so-called **Minkowski compactum**, which will be denoted by M_n. The name **Banach–Mazur compactum** is also common.

6.1.1.2 Using Auerbach bases of E_n and F_n, we get $d(E_n, F_n) \leq n^2$; see also 6.1.1.5. Thus M_n is bounded. In the early 1950s, Mazur mentioned in his seminar at Warsaw University that M_n is compact. At this time, Pełczyński was a student of the fifth year, and he remembers, *I proved it myself without any problem.*

In 1987, Mityagin made the (unpublished) observation that M_n is contractible.

At the *International Topology Conference in Moscow* (1996), Fabel showed that M_2 is an absolute retract. Subsequently, Antonyan [1998] generalized Fabel's result to arbitrary dimensions: every continuous map from a closed subspace of a metric space into M_n extends continuously to the whole space. An unpleasant priority dispute was carried out in TopCom 2, #3; see `http://at.yorku.ca/t/o/p/c/32.htm` and `http://at.yorku.ca/t/o/p/c/33.htm`.

It seems to be unknown whether M_h and M_k are non-homeomorphic for $h \neq k$. Further topological properties of the Minkowski compactum were obtained by Antonyan [2000].

6.1.1.3 The asymptotic behavior of the Banach–Mazur distance between the spaces l_p^n was determined by Gurariĭ/Kadets/Matsaev [1966]:

$$d(l_p^n, l_q^n) \asymp \begin{cases} n^{|1/p-1/q|} & \text{if } 1 \leq p, q \leq 2 \text{ or } 2 \leq p, q \leq \infty, \\ \max\{n^{1/p-1/2}, n^{1/2-1/q}\} & \text{if } \quad 1 \leq p \leq 2 \leq q \leq \infty, \\ \max\{n^{1/2-1/p}, n^{1/q-1/2}\} & \text{if } \quad 1 \leq q \leq 2 \leq p \leq \infty. \end{cases}$$

Tomczak-Jaegermann [1978, pp. 307–308] proved the same result for the non-commutative n^2-dimensional analogues $\mathfrak{S}_p(l_2^n)$ and $\mathfrak{S}_q(l_2^n)$.

6.1.1.4 John [1948, p. 203] showed that given any norm f on $\mathbb{R}^n$, there exists a positive definite quadratic form Q such that

$$\sqrt{\tfrac{1}{n}Q(x)} \le f(x) \le \sqrt{Q(x)} \quad \text{for all } x \in \mathbb{R}^n.$$

In other words, we have $d(E_n, l_2^n) \le \sqrt{n}$. Since $d(l_\infty^n, l_2^n) = \sqrt{n}$, this result is sharp.

The simplest proof of **John's theorem** is due to Kwapień; see 6.3.19.3. Another approach can be found in 6.3.4.15.

Kadets/Snobar [1971] made an important observation. Carefully looking at John's proof, they extracted a significant lemma:

Let E_n be any n-dimensional Banach space. Then there exist an inner product $(x|y)$, positive coefficients $c_1, \dots, c_N$, and elements $u_1, \dots, u_N \in E_n$ such that

(1) If $|||x||| := \sqrt{(x|x)}$, then $|||x||| \le \|x\|$ for $x \in E_n$.
(2) $\|u_k\| = |||u_k||| = 1$ for $n = 1, \dots, N$.
(3) $\sum_{k=1}^N c_k (x|u_k)(u_k|y) = (x|y)$ for $x, y \in E_n$.

It can be arranged that $N \le \frac{n(n+1)}{2}$ (real case) and $N \le n^2$ (complex case).

For the functional $u_k^* : x \mapsto (x|u_k)$, we have $\|u_k^*\| = 1$. Rewriting (3) in the form

$$I_{E_n} = \sum_{k=1}^N c_k u_k^* \otimes u_k \quad \text{yields} \quad n = \text{trace}(I_{E_n}) = \sum_{k=1}^N c_k.$$

In the language of geometry, $\{x \in E_n : |||x||| \le 1\}$ is the *"ellipsoid of minimal volume"* containing the original unit ball of E_n; see 6.1.11.2.

In the *Collected Papers* [JOHN$^{\bowtie}$, Vol. II, pp. 638–639] we find a commentary of Kuhn, which states that John's article [1948]

has long since passed into the classical literature of nonlinear programming ... .

Ironically, its striking impact on the theory of Banach spaces and convex bodies is only mentioned in passing.

6.1.1.5 The estimates $d(E_n, l_2^n) \le \sqrt{n}$ and $d(F_n, l_2^n) \le \sqrt{n}$ imply that

$$d(E_n, F_n) \le n \quad \text{for } E_n, F_n \in \mathsf{M}_n.$$

Hence the diameter of the n^{th} Minkowski compactum does not exceed n. The question whether this order of growth is best possible was a long standing open problem. We owe its solution to Gluskin [1981b]; details are described in 7.3.1.8. Since Gluskin's approach is based on stochastic techniques, his proof yields only the existence of spaces E_n and F_n such that

$$d(E_n, F_n) \ge cn \quad \text{for } n = 1, 2, \dots \text{ and some constant } c > 0.$$

No concrete examples are known.

6.1.1.6 Let E be a finite-dimensional Banach space. If E is a subspace of a larger Banach space X, then one refers to

$$\lambda(E,X) := \inf\Big\{\|P\| : P \text{ is a projection from } X \text{ onto } E\Big\}$$

as the **(relative) projection constant** of E with respect to X.

In particular, E can be viewed as a subspace of l_∞. Since l_∞ has the metric extension property, it follows that $\lambda(E,X) \le \lambda(E,l_\infty)$. This inequality helps to explain why

$$\lambda(E) := \lambda(E,l_\infty)$$

is called the **(absolute) projection constant** of E.

6.1.1.7 The term "projection constant" goes back to Grünbaum [1960]. By means of John's theorem, he proved on p. 464 that

$$\lambda(E_n) \le d(E_n,l_2^n)\lambda(l_2^n) \le c_n n$$

for every n-dimensional space E_n, the constants c_n being strictly less than 1.

The decisive breakthrough was achieved by Kadets/Snobar [1971]. Using the lemma stated in 6.1.1.4, they obtained the estimate $\lambda(E_n) \le \sqrt{n}$, which provides the best possible order. Nowadays, the **Kadets–Snobar theorem** is just a simple corollary of the theory of 2-summing operators; see 6.3.19.7.

6.1.1.8 To simplify matters, the following results are formulated only for *real* spaces.

Grünbaum [1960, p. 452] (upper estimate) and Rutovitz [1965, p. 249] computed the projection constant

$$\lambda(l_2^n) = \frac{2\Gamma\big(\frac{n+2}{2}\big)}{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{n+1}{2}\big)}.$$

In view of

$$\lim_{n\to\infty} \frac{\lambda(l_2^n)}{\sqrt{n}} = \sqrt{\frac{2}{\pi}} < 1,$$

the order of growth of $\lambda(l_2^n)$ is as large as possible. However, it remained an open question whether there exist n-dimensional spaces E_n such that

$$\lim_{n\to\infty} \frac{\lambda(E_n)}{\sqrt{n}} = 1.$$

König [1985] gave an affirmative answer for an increasing sequence of natural numbers. On the other hand, König/Lewis [1987] proved the surprising fact that $\lambda(E_n)$ is strictly smaller than $\sqrt{n}$ for *all* n-dimensional spaces E_n. Later on, König/Tomczak-Jaegermann [1994, p. 255] obtained the estimate

$$\lambda(E_n) \le \frac{2+(n-1)\sqrt{n+2}}{n+1} = \sqrt{n} - 1/\sqrt{n} + O\big(\tfrac{1}{n}\big).$$

The upper bound is attained at least for $n = 2,3,7,23$.

In [1990, p. 815], König/Tomczak-Jaegermann had constructed Banach spaces E_n such that $\lambda(E_n) \geq \sqrt{n} - 2/\sqrt{n} + 2/n$ for $n = 4, 4^2, 4^3, \ldots$. Thus the growth of $\max \lambda(E_n)$ is "almost" known.

In my opinion, the estimates above are less important than the method of their proofs, since we learn what spaces with a maximal projection constant look like:

Search for a system of *equiangular* vectors $x_1, \ldots, x_N$ on the unit sphere of l_2^n. This means that

$$\|x_k|l_2^n\| = 1 \quad \text{and} \quad \big|(x_h|x_k)\big| = \alpha \quad \text{whenever } h \neq k.$$

Then the polyhedron

$$B := \Big\{x \in \mathbb{R}^n : \big|(x|x_k)\big| \leq 1 \text{ for } k = 1, \ldots, N\Big\}$$

is the closed unit ball of the required space. For $n=2$ we get the regular hexagon and for $n=3$ the dodecahedron. There are interesting connections with coding theory and packings on spheres.

6.1.1.9 The asymptotic behavior of $\lambda(l_p^n)$ was determined by Grünbaum [1960, p. 452] ($p=1$), Rutovitz [1965, p. 252] ($2 \leq p \leq \infty$), and Gordon [1968, p. 296] ($1 \leq p < 2$):

$$\lambda(l_p^n) \asymp \begin{cases} n^{1/2} & \text{if } 1 \leq p < 2, \\ n^{1/p} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

6.1.1.10 Let $0 \leq \alpha \leq \frac{1}{2}$. Then the following properties are equivalent for any Banach space X:

(BM$_\alpha$) There exists a constant $c \geq 1$ such that for every n-dimensional subspace E, $d(E, l_2^n) \leq cn^\alpha$.

(P$_\alpha$) There exists a constant $c \geq 1$ such that for every n-dimensional subspace E, we can find a projection P from X onto E with $\|P\| \leq cn^\alpha$.

(HP$_\alpha$) There exists a constant $c \geq 1$ such that for every n-dimensional subspace E, we can find a projection P from X onto E with $\lambda_2(P) \leq cn^\alpha$. Here λ_2 denotes the Hilbertian ideal norm; see 6.3.5.5.

All inequalities are supposed to hold uniformly for $n = 1, 2, \ldots$, with possibly different constants.

This remarkable theorem is due to König/Retherford/Tomczak-Jaegermann [1980, p. 111] and Pietsch [1991a, pp. 66–67]. Some steps of its proof are decisively based on results about Grothendieck numbers and eigenvalue distributions of nuclear operators; see 6.1.11.4 and 6.4.4.4.

In the limiting case $\alpha = 0$, the properties **(BM$_0$)**, **(P$_0$)**, and **(HP$_0$)** characterize isomorphic copies of Hilbert spaces. On the other hand, *all* Banach spaces are obtained for $\alpha = 1/2$. Note that **(BM$_{1/2}$)** is just John's theorem, and the modern proof of the Kadets–Snobar theorem implies **(HP$_{1/2}$)**.

This line of research was opened by Lewis [1978, p. 210], who showed that L_p spaces possess the properties **(BM$_\alpha$)**, **(P$_\alpha$)**, and **(HP$_\alpha$)** with $\alpha = |1/p - 1/2|$. Tomczak-Jaegermann [1980, pp. 273, 280] proved the same result for the Schatten–von Neumann classes $\mathfrak{S}_p(H)$.

6.1.1.11 The **basis constant** $\mathrm{bc}(E)$ of a finite-dimensional Banach space E is defined to be the smallest $c \ge 1$ with the following property; compare with 5.6.1.6:

There exists a basis $\{x_1, \dots, x_n\}$ of E such that

$$\Big\| \sum_{k=1}^{m} \xi_k x_k \Big\| \le c \Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| \quad \text{if } \xi_1, \dots, \xi_n \in \mathbb{K} \text{ and } 1 \le m \le n.$$

It follows from John's theorem that $\mathrm{bc}(E_n) \le \sqrt{n}$ if $\dim(E_n) = n$. This result is asymptotically the best possible. Indeed, Szarek [1983, p. 154] produced a sequence of n-dimensional Banach spaces E_n such that $\mathrm{bc}(E_n) \ge c_0 \sqrt{n}$. In other words, we have a finite-dimensional version of Enflo's counterexample to the basis problem.

6.1.1.12 Pełczyński [1971, pp. 241–242] proved that given $\varepsilon > 0$, every finite-dimensional Banach space E is 1-complemented in a finite-dimensional Banach space F with $\mathrm{bc}(F) \le 1+\varepsilon$; see also Johnson/Rosenthal/Zippin [1971, p. 503]. Therefore, in view of Szarek's result above, the inequality $\mathrm{bc}(E) \le \lambda(E,F)\mathrm{bc}(F)$ is not always true.

It seems to be unknown whether $\mathrm{bc}(E) \le \lambda(E)$ for every finite-dimensional Banach space E.

6.1.2 Dvoretzky's theorem

6.1.2.1 The celebrated Dvoretzky theorem on almost spherical sections goes back to the question whether a Banach space in which absolute and unconditional convergence of series coincide must be finite-dimensional.

An affirmative answer was given by the **Dvoretzky–Rogers theorem**, the proof of which is based on a classical lemma [1950, pp. 193–194]:

Every n^2-dimensional Banach space contains normalized elements $x_1, \dots, x_n$ such that

$$\Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| \le 3 \Big(\sum_{k=1}^{n} |\xi_k|^2 \Big)^{1/2} \quad \text{for} \quad \xi_1, \dots, \xi_n \in \mathbb{R}.$$

Since any orthonormal system in a Hilbert space has this property, the preceding result means that in every infinite-dimensional Banach space we can find arbitrarily large systems $\{x_1, \dots, x_n\}$ that simulate a weak form of orthonormality.

6.1.2.2 The ideas of Dvoretzky/Rogers were further developed by Grothendieck. In [1956c, p. 108], he formulated the following conjecture:

Pour n and ε donnés, tout espace de Banach E de dimension assez grande contient un sous-espace isomorphe à ε près à l'espace de Hilbert de dimension n.

Dvoretzky, MR 20, # 1195, acted as the reviewer of Grothendieck's paper mentioned above, and he solved the problem in [1959], [1960, p. 156]. A gap in his proof was filled by Figiel [1972b].

Dvoretzky's theorem, [1960, p. 156]:

Let $k \geq 2$ be an integer and $0 < \varepsilon < 1$. Then there exists an $N = N(k, \varepsilon)$ so that every normed space having more than N dimensions has a k-dimensional subspace whose distance from the k-dimensional Hilbert space is less than ε.

Grothendieck's formulation *"F soit isomorphe à ε près à l_2^n"* means that $d(F, l_2^n) \leq 1+\varepsilon$ (see 6.1.3.1), while Dvoretzky uses the distance $\log d(F, l_2^n)$.

6.1.2.3 Most proofs of Dvoretzky's theorem are based on stochastic arguments. So a probability is required. In his original approach, Dvoretzky used the Haar measure on the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^N$. Next, Milman [1971b] employed the rotation invariant measure on the unit sphere of $\mathbb{R}^N$. His main tool was the so-called *concentration of measure phenomenon*, described for the first time in [LÉVY$_1$, 2nd edition, p. 214]; see also [LED] and Milman [1988$^\bullet$]. A detailed presentation of Milman's approach is given in [MIL$^+$]. Further simplifications are due to Szankowski [1974] and Figiel [1976b]. My favorite is the Maurey/Pisier proof of Dvoretzky's theorem by *Gaussian methods* which can be found in Pisier's Varenna lectures [1986a, Chap. 1] as well as in [PIS$_2$, Chap. 4]. "Combinatorial" proofs were supplied by Tzafriri [1974] and Krivine [1976, p. 2].

For additional information, the competent surveys of Lindenstrauss [1990] and Milman [1992$^\bullet$] are strongly recommended.

6.1.2.4 Next, I discuss the quantitative aspects of Dvoretzky's theorem.

Given any finite-dimensional Banach space E and $\varepsilon > 0$, we denote by $k(E, \varepsilon)$ the largest natural number k for which there exists a k-dimensional subspace E_k of E with $d(E_k, l_2^k) \leq 1+\varepsilon$. Dvoretzky's theorem asserts that $k(E, \varepsilon) \to \infty$ as $\dim(E) \to \infty$. More precisely,

$$k(E, \varepsilon) \geq c\varepsilon^2(1+\log n) \quad \text{whenever} \quad \dim(E) = n,$$

where $c > 0$ is a constant. This lower estimate was obtained step by step:

$$k(E,\varepsilon) \geq c \begin{cases} \varepsilon^2 \sqrt{\frac{1+\log n}{1+\log(1+\log n)}} & : \text{ Dvoretzky [1960],} \\ \frac{\varepsilon^2}{|\log \varepsilon|}(1+\log n) & : \text{ Milman [1971b],} \\ \varepsilon^2(1+\log n) & : \text{ Gordon [1988], Schechtman [1988].} \end{cases}$$

Since $k(E,\varepsilon)=n$ for $1+\varepsilon \geq d(E,l_2^n)$, considerations of $k(E,\varepsilon)$ make sense only if ε is not too big.

6.1.2.5 The preceding result can be improved for finite-dimensional Banach spaces with a specific structure.

Let $\varepsilon > 0$ be fixed. Then Figiel/Lindenstrauss/Milman [1977, p. 64] showed that

$$k(l_p^n,\varepsilon) \asymp \begin{cases} n & \text{if } 1 \leq p \leq 2, \\ n^{2/p} & \text{if } 2 \leq p < \infty, \\ 1+\log n & \text{if } \quad p = \infty. \end{cases}$$

6.1.2.6 Let π_γ denote the γ-summing norm, which is defined in 6.3.6.15. With the help of this quantity, Dvoretzky's theorem can be stated as follows:

$$c\varepsilon^2 \pi_\gamma(E)^2 \leq k(E,\varepsilon) \leq (1+\varepsilon)^2 \pi_\gamma(E)^2. \tag{6.1.2.6.a}$$

Indeed, this is just a shorthand version of Theorem 1.3 in Pisier's Varenna lectures [1986a, pp. 170–173]; see also Theorem 4.4 and Proposition 4.6 in [PIS$_2$].

The theory of γ-summing operators was developed by Linde/Pietsch [1974]. Their theorem on p. 456 implies that

$$\pi_\gamma(l_p^n) \asymp \begin{cases} n^{1/2} & \text{if } 1 \leq p \leq 2, \\ n^{1/p} & \text{if } 2 \leq p < \infty, \\ \sqrt{1+\log n} & \text{if } \quad p = \infty. \end{cases}$$

Compare with 6.1.2.5.

6.1.2.7 Using the fact that π_γ is an ideal norm, one easily obtains

$$\sqrt{n} \leq d(E_n,l_2^n)\pi_\gamma(E_n) \quad \text{and} \quad n \leq d(E_n,l_2^n)\pi_\gamma(E_n)\pi_\gamma(E_n^*)$$

for every n-dimensional Banach space E_n. Via (6.1.2.6.a), the formulas above pass into

$$k(E_n,\varepsilon) \geq c\varepsilon^2 \frac{n}{d(E_n,l_2^n)^2} \quad \text{and} \quad k(E_n,\varepsilon)k(E_n^*,\varepsilon) \geq c^2\varepsilon^4 \frac{n^2}{d(E_n,l_2^n)^2}.$$

Both inequalities were proved for the first time by Figiel/Lindenstrauss/Milman [1977, pp. 61–62].

6.1.2.8 An immediate consequence of (6.1.2.6.a) says that

$$k(E,\varepsilon_1) \leq k(E,\varepsilon_2) \leq \left(\frac{1+\varepsilon_2}{c\varepsilon_1}\right)^2 k(E,\varepsilon_1) \quad \text{if } 0 < \varepsilon_1 < \varepsilon_2.$$

In other words, the asymptotic behavior of $k(E,\varepsilon)$ as $\dim(E) \to \infty$ is the same for all fixed $\varepsilon > 0$.

6.1.2.9 The following counterpart of 6.1.2.5 was established by Figiel/Lindenstrauss/Milman [1977, p. 66]:

$$k\big(\mathfrak{S}_p(l_2^n),\varepsilon\big) \asymp \begin{cases} n^2 & \text{if } 1 \le p \le 2, \\ n^{1+2/p} & \text{if } 2 \le p \le \infty. \end{cases}$$

6.1.2.10 Though the next results are closely related to Dvoretzky's theorem, their proofs are based on quite different techniques.

Milman's **quotient of subspace theorem** asserts that for $\frac{1}{2} \le \alpha < 1$, every n-dimensional Banach space contains subspaces E and F such that

$$F \subset E, \quad m := \dim(E/F) \ge \alpha n \quad \text{and} \quad d(E/F, l_2^m) \le c\,\frac{|\log(1-\alpha)|}{1-\alpha},$$

where c is a universal constant; see Milman [1985].

6.1.2.11 The space l_1^{2n} can be written in the form

$$l_1^{2n} = E_n \oplus E_n^{\perp} \quad \text{such that } d(E_n, l_2^n) \le 4e \text{ and } d(E_n^{\perp}, l_2^n) \le 4e,$$

where the n-dimensional subspaces E_n and $E_n^{\perp}$ are orthogonal with respect to the inner product given on l_2^{2n}.

This result is implicitly contained in a paper of Kashin [1977]. The explicit version goes back to Szarek [1978], who also coined the term **Kashin decomposition**; see [PIS$_2$, pp. 94–97]. Szarek's proof has led to the concept of *volume ratio*, which will be discussed in Subsection 6.1.11.

6.1.3 Finite representability, ultraproducts, and spreading models

6.1.3.1 In [1956c, p. 109], Grothendieck defined the following concept:

L'espace normé E a un type linéaire inférieur à celui d'un espace normé F, si on peut trouver un $M > 0$ fixe tel que tout sous-espace de dimension finie E_1 de E soit isomorphe "à M près" à un sous-espace F_1 de F (i.e. il existe une application linéare biunivoque de E_1 sur F_1, de norme ≤ 1, dont l'application inverse a une norme $\le 1+M$); et que E a un type métrique inférieur à celui de F, si le condition précédente est satisfaite pour tout $M > 0$.

Independently of Grothendieck's idea, James [1972a, p. 169], [1972c, p. 896] invented the same definition in his theory of superreflexivity:

A normed linear space X being **finitely representable** *in a normed linear space Y means that for each finite-dimensional subspace X_n of X and each number $\lambda > 1$, there is an isomorphism T_n of X_n into Y for which*

$$\lambda^{-1}\|x\| \le \|T_n(x)\| \le \lambda\|x\| \quad \text{if } x \in X_n.$$

He used the term **crudely finitely representable** in order to indicate that the inequality above is supposed to hold only for sufficiently large λ's.

6.1.3.2 Phrased in the terminology above, Dvoretzky's theorem says that l_2 is finitely representable in every infinite-dimensional Banach space.

6.1.3.3 Let $\mathbb{P}$ be a property of Banach spaces. Following Brunel/Sucheston [1975, p. 79], we say that X has the property **super-$\mathbb{P}$** if any X_0 that is finitely representable in X has property $\mathbb{P}$. Super properties are stable with respect to isometries as well as under the formation of closed subspaces and ultrapowers (see below).

The historical root of this concept and also the most important example is superreflexivity. Pisier [1975a, p. 329] observed that

$$\textit{superreflexivity} = \textit{super Radon–Nikodym property}.$$

6.1.3.4 In [1969, p. 332], Lindenstrauss/Rosenthal discovered a striking result *which shows that all Banach spaces are "locally reflexive."* More precisely, every bidual X^{**} is finitely representable in the original space X. Therefore, on the local level, we always have reflexivity.

A slightly stronger version was found by Johnson/Rosenthal/Zippin [1971, p. 493]. Following their proposal, one speaks of the **principle of local reflexivity**:

Let X be a Banach space (regarded as a subspace of X^{**}), let E and F be finite dimensional subspaces of X^{**} and X^*, respectively, and let $\varepsilon > 0$. Then there exists a one-to-one operator $T : E \to X$ with $T(x) = x$ for all $x \in X \cap E$, $f(Te) = e(f)$ for all $e \in E$ and $f \in F$, and $\|T\|\,\|T^{-1}\| < 1+\varepsilon$.

The original proof was based on a separation theorem of Klee [1951]. Subsequently, Dean [1973] deduced the required result from $\mathfrak{L}(E,X^{**}) = \mathfrak{L}(E,X)^{**}$ with the help of Helly's lemma 3.4.3.3. An even more elementary approach is due to Stegall [1980].

6.1.3.5 Let $(X_i)_{i\in\mathbb{I}}$ be any family of Banach spaces, and fix an ultrafilter $\mathscr{U}$ on the index set $\mathbb{I}$. With every bounded family (x_i) such that $x_i \in X_i$ for $i \in \mathbb{I}$ we associate the equivalence class

$$(x_i)^{\mathscr{U}} = \big\{\, (x_i^\circ) : x_i^\circ \in X_i,\ (x_i^\circ) \text{ bounded},\ \mathscr{U}\text{-}\lim_i \|x_i - x_i^\circ\| = 0 \,\big\}.$$

Defining the norm of $(x_i)^{\mathscr{U}}$ by $\|(x_i)^{\mathscr{U}}\| := \mathscr{U}\text{-}\lim_i \|x_i\|$ yields the Banach space $(X_i)^{\mathscr{U}}$.

If the family $(T_i)_{i\in\mathbb{I}}$ with $T_i \in \mathfrak{L}(X_i,Y_i)$ is bounded, then $(T_i)^{\mathscr{U}} : (x_i)^{\mathscr{U}} \to (T_i x_i)^{\mathscr{U}}$ gives an operator from $(X_i)^{\mathscr{U}}$ into $(Y_i)^{\mathscr{U}}$ such that $\|(T_i)^{\mathscr{U}}\| = \mathscr{U}\text{-}\lim_i \|T_i\|$.

These new objects are referred to as **ultraproducts**. If $X_i = X$, $Y_i = Y$, and $T_i = T$, then we speak of **ultrapowers**, denoted by $X^{\mathscr{U}}$, $Y^{\mathscr{U}}$, and $T^{\mathscr{U}}$, respectively.

6.1.3.6 Ultraproducts of L_p spaces appeared for the first time in a paper of Bretagnolle/Dacunha-Castelle/Krivine [1966, pp. 246–247]. The aim of these authors was to give a local characterization of subspaces of L_p. With the help of this new tool, they were able to avoid a lengthy compactness argument that had been used in the case $p = 2$ by Lindenstrauss [1963a, pp. 246–247] and Joichi [1966, p. 424]; see 6.1.4.3. However, only in [1972] did Dacunha-Castelle/Krivine introduce ultraproducts of Banach spaces as a mathematical concept in its own right; see also Stern [1974].

A few years later, Pietsch [1974b] adapted this notion to operators. The most important applications to the theory of operator ideals are due to Heinrich [1980a], [HEIN].

Nowadays, ultraproducts have become an indispensable tool of Banach space theory. The classical references are the surveys of Heinrich [1980a] and [SIMS].

6.1.3.7 Passing from Banach spaces to their ultraproducts may preserve additional structures. For example, if the X_i's are Hilbert spaces, then so is $(X_i)^{\mathscr{U}}$. The same conclusion holds for Banach lattices and Banach algebras. In particular, we know from Dacunha-Castelle/Krivine [1972, p. 326] that the property of being some $L_p(M,\mathscr{M},\mu)$ or some $C(K)$ is stable under the formation of ultraproducts. However, ultrapowers of reflexive spaces need not be reflexive; see 6.1.9.3.

For further details, the reader is referred to Heinrich [1980a, p. 76].

6.1.3.8 The close connection between finite representability and ultraproducts was established by Henson/Moore [1974a, p. 135] and Stern [1976, p. 100]:

A Banach space X_0 is finitely representable in a Banach space X if and only if X_0 is isometric to a subspace of some ultrapower of X.

Another theorem of Henson/Moore [1974a, p. 138] and Stern [1976, p. 106] says that the bidual X^{**} of every Banach space X is metrically isomorphic to a 1-complemented subspace of some ultrapower $X^{\mathscr{U}}$.

6.1.3.9 Henson/Moore presented their results in terms of **non-standard hulls**. This concept, which goes back to Luxemburg [1969a, p. 82], is closely related to that of an ultrapower. Details can be found in surveys of Henson/Moore [1983] and Henson/Iovino [2002]. The reports of these authors also outline the connection with model theory and first order logic.

6.1.3.10 Letting

$$\big\langle (x_i^*)^{\mathscr{U}}, (x_i)^{\mathscr{U}} \big\rangle := \mathscr{U}\text{-}\lim_i \langle x_i^*, x_i\rangle$$

yields a duality between $(X_i^*)^{\mathscr{U}}$ and $(X_i)^{\mathscr{U}}$. In this way, $(X_i^*)^{\mathscr{U}}$ becomes a subspace of $[(X_i)^{\mathscr{U}}]^*$. Examples show that the inclusion $(X_i^*)^{\mathscr{U}} \subseteq [(X_i)^{\mathscr{U}}]^*$ may be strict. In the case of ultrapowers, Henson/Moore [1974a, p. 137] and Stern [1978, p. 235] proved that $(X^*)^{\mathscr{U}} = (X^{\mathscr{U}})^*$ for all ultrafilters $\mathscr{U}$ if and only if X is superreflexive.

6.1.3.11 The following result was obtained by Stern [1978, pp. 235–239] and Kürsten [1978, стр. 64]; its final version is due to Heinrich [HEIN, p. 8].

Principle of local duality for ultraproducts:

Let E and F be finite-dimensional subspaces of $[(X_i)^{\mathscr{U}}]^*$ and $(X_i)^{\mathscr{U}}$, respectively, and let $\varepsilon > 0$. Then there is a one-to-one operator $T : E \to (X_i^*)^{\mathscr{U}}$ with $Tx^* = x^*$ for all $x^* \in (X_i^*)^{\mathscr{U}} \cap E$, $\langle x^*, Tx\rangle = \langle x^*, x\rangle$ for all $x^* \in E$ and $x \in F$, and $\|T\|\,\|T^{-1}\| < 1+\varepsilon$.

Hence $[(X_i)^{\mathscr{U}}]^*$ is finitely representable in $(X_i^*)^{\mathscr{U}}$.

6.1.3.12 First, but unsatisfactory, attempts to extend the concept of finite representability from spaces to operators were made by Beauzamy [1976, p. 110], [1977, pp. 221–222]. In a next step, Heinrich [HEIN, pp. 7–8] suggested a "better" definition. However, he could not avoid an unpleasant $\varepsilon > 0$. Only much later was Pietsch [1999] able to eliminate this troublemaker. His approach will be described next.

6.1.3.13 For every operator $T \in \mathfrak{L}(X,Y)$, let

$$\mathfrak{L}(T|E,F) := \big\{BTA : \|A : E \to X\| \le 1,\ \|B : Y \to F\| \le 1\big\},$$

where E and F are arbitrary finite-dimensional Banach spaces. This set may have a quite complicated shape. For instance, $\mathfrak{L}(I_K|E,F) = \big\{x^* \otimes y : \|x^*\| \le 1,\ \ \|y\| \le 1\big\}$ looks like a hedgehog.

The norm-closed hull of $\mathfrak{L}(T|E,F)$ is given by

$$\overline{\mathfrak{L}(T|E,F)} = \bigcap_{\varepsilon>0}(1+\varepsilon)\mathfrak{L}(T|E,F).$$

This crucial formula was proved by Johnson; see Pietsch [1999, pp. 278–279].

It easily follows from the principle of local reflexivity and the principle of local duality for ultraproducts that

$$\overline{\mathfrak{L}(T^{**}|E,F)} = \overline{\mathfrak{L}(T|E,F)} \quad \text{and} \quad \overline{\mathfrak{L}(T^{\mathscr{U}}|E,F)} = \overline{\mathfrak{L}(T|E,F)}.$$

6.1.3.14 Now we are ready to give the desired definition:

The operator $T_0 \in \mathfrak{L}(X_0,Y_0)$ is said to be **finitely representable** in the operator $T \in \mathfrak{L}(X,Y)$ if $\overline{\mathfrak{L}(T_0|E,F)} \subseteq \overline{\mathfrak{L}(T|E,F)}$ for every pair of finite-dimensional Banach spaces E and F.

6.1.3.15 Consider the index set $\mathbb{I}$ formed by all pairs $i = (M,N)$, where M is a finite-dimensional subspace M of X and N is a finite-codimensional closed subspace of Y. Put $X_i := M$ and $Y_i := Y/N$. Fix any ultrafilter $\mathscr{U}$ on $\mathbb{I}$ that contains all final sections $\{i \in \mathbb{I} : M \supseteq M_0, N \subseteq N_0\}$.

Obviously,

$$J_X^{\mathscr{U}} : x \mapsto (x_i)^{\mathscr{U}} \quad \text{with} \quad x_i := \begin{cases} \mathsf{o} & \text{if } x \notin M, \\ x & \text{if } x \in M, \end{cases}$$

yields a metric injection from X into $(X_i)^{\mathscr{U}}$. The counterpart is a metric surjection $Q_Y^{\mathscr{U}}$ from $(Y_i)^{\mathscr{U}}$ onto Y^{**}. Given $(\widehat{y}_i)^{\mathscr{U}} \in (Y_i)^{\mathscr{U}}$, we choose a bounded family (y_i) in Y such that y_i belongs to the equivalence class $\widehat{y}_i \in Y_i = Y/N$. In view of Alaoglu's theorem, $(K_Y y_i)$ converges to some element in Y^{**}, with respect to the weak* topology and along the ultrafilter $\mathscr{U}$. Moreover, this limit does not depend on the special choice of (y_i). Therefore the definition

$$Q_Y^{\mathscr{U}} : (\widehat{y}_i)^{\mathscr{U}} \mapsto \mathscr{U}\text{-}\lim_i K_Y y_i$$

makes sense, and we necessarily end up in Y^{**} and not in Y. This fact explains why the canonical map K_Y from Y into Y^{**} occurs in the following factorization. Goldstine's theorem tells us that $Q_Y^{\mathscr{U}}$ is indeed a metric surjection.

According to Pietsch [1974b, p. 124], every operator $T \in \mathfrak{L}(X,Y)$ can be "almost" reconstructed from its *elementary parts* $Q_i T J_i = Q_N^Y T J_M^X$:

$$K_Y T : X \xrightarrow{J_X^{\mathscr{U}}} (X_i)^{\mathscr{U}} \xrightarrow{(Q_i T J_i)^{\mathscr{U}}} (Y_i)^{\mathscr{U}} \xrightarrow{Q_Y^{\mathscr{U}}} Y^{**}.$$

Here $J_i = J_M^X$ denotes the canonical embedding from $X_i = M$ into X, and $Q_i = Q_N^Y$ denotes the canonical surjection from Y onto $Y_i = Y/N$.

6.1.3.16 In order to prove the "only if" part of the next criterion, the indices $i = (M,N)$ used in the previous paragraph are replaced by $i = (M,N,\varepsilon)$ with $\varepsilon > 0$; see above and Pietsch [1999, p. 283].

The "if" part follows from $\overline{\mathfrak{L}(T^{\mathscr{U}}|E,F)} = \overline{\mathfrak{L}(T|E,F)}$.

The operator $T_0 \in \mathfrak{L}(X_0, Y_0)$ is finitely representable in the operator $T \in \mathfrak{L}(X,Y)$ if and only if there exists a factorization

$$K_{Y_0} T_0 : X_0 \xrightarrow{A} X^{\mathscr{U}} \xrightarrow{T^{\mathscr{U}}} Y^{\mathscr{U}} \xrightarrow{B} Y_0^{**}$$

such that $\|A\| = 1$ and $\|B\| = 1$.

Note that Heinrich [HEIN, p. 8] proved a similar criterion in which A is supposed to be a metric injection and B is supposed to be a metric surjection. Therefore his concept of finite representability is stronger.

6.1.3.17 Here are some further useful properties:

The operators T, T^{**}, and $T^{\mathscr{U}}$ are finitely representable in each other. Moreover, T_0 is finitely representable in T if and only if T_0^* is finitely representable in T^*.

6.1.3.18 Quite likely, the reader is waiting for a criterion that states that a Banach space X_0 is finitely representable in a Banach space X if and only if the identity map I_{X_0} is finitely representable in I_X. This is not true, for the following reason.

Originally, the concept of finite representability was designed for being applied to classes of spaces that are invariant under the formation of closed subspaces. Of course, one may also consider a dual, but less important, property: invariance under the formation of quotients; see Stern [1978, p. 242] and Heinrich [HEIN, p. 22].

In the setting of operators, the definition of finite representability does not take into account *injectivity* and *surjectivity*. However, a slight modification suffices to eliminate this "defect."

Let us say that T_0 is **finitely** (E, l_∞^n)**-representable** in T if $\overline{\mathfrak{L}(T_0|E,F)} \subseteq \overline{\mathfrak{L}(T|E,F)}$ is supposed to hold only in the special case that $F = l_\infty^n$ and $n = 1, 2, \ldots$. Then all is well.

A Banach space X_0 is finitely representable in a Banach space X if and only if the identity map I_{X_0} is finitely (E, l_∞^n)-representable in I_X.

The dual concept of **finite** (l_1^n, F)**-representability** is obtained by setting $E = l_1^n$, whereas F may be arbitrary.

6.1.3.19 Let (x_k) be a sequence in a Banach space X. A Banach sequence space $S(x_k)$, in which the unit sequences $e_1, e_2, \dots$ are fundamental, is said to be a **spreading model** generated by (x_k) if given $\varepsilon > 0$ and $n = 1, 2, \dots$, there is an i such that

$$\frac{1}{1+\varepsilon}\Big\|\sum_{k=1}^{n}\xi_k e_k\Big\| \le \Big\|\sum_{k=1}^{n}\xi_k x_{i_k}\Big\| \le (1+\varepsilon)\Big\|\sum_{k=1}^{n}\xi_k e_k\Big\|$$

for $\xi_1, \dots, \xi_n \in \mathbb{K}$ and $i < i_1 < \cdots < i_n$. Nowadays one even assumes that (e_k) is a basis of $S(x_k)$; see [JOHN$_1^\cup$, p. 136], [JOHN$_2^\cup$, pp. 1035, 1339–1340], and Odell [2002, p. 203]. Taking a sequence $x_k \xrightarrow{w} x \neq o$ shows that this requirement is indeed stronger; see 5.6.3.19.

The concept of a spreading model goes back to Brunel/Sucheston [1974, pp. 295–296], [1975, p. 83]. The name was coined by Beauzamy [1979, p. 359], motivated by the fact that the norm on $S(x_k)$ is *invariant under spreading*:

$$\Big\|\sum_{k=1}^{n}\xi_k e_k\Big\| = \Big\|\sum_{k=1}^{n}\xi_k e_{i_k}\Big\| \quad \text{for } \xi_1, \dots, \xi_n \in \mathbb{K} \text{ and } i_1 < \cdots < i_n.$$

According to Rosenthal [1976, p. 23], Tzafriri was the first to observe that spreading models could be a useful tool in Banach space theory. For example, $S(x_k)$ is finitely representable in X; see Brunel [1974, exposé 15, p. 5]. Therefore, when looking for a specific finite-dimensional subspace of X, it may happen that passing to a spreading model simplifies the search considerably. We also know that $S(x_k)$ always contains an unconditional basic sequence, namely $(e_{2k-1} - e_{2k})$; see Brunel/Sucheston [1975, pp. 86–88] and Krivine [1976, p. 7].

6.1.3.20 Spreading models were constructed by Brunel/Sucheston with the aid of Ramsey's theorem; see 7.3.2.4:

Every bounded sequence in a Banach space contains a subsequence (x_k) such that

$$L(\xi_k) := \lim_{i\to\infty}\Big\|\sum_{k=1}^{\infty}\xi_k x_{i_k}\Big\| \quad \text{with } i \le i_1 < i_2 < \cdots$$

exists for every scalar sequence (ξ_k) having finite support.

If (x_k) does not contain any norm-convergent subsequence, then $L(\xi_k)$ is a norm, and the required spreading model can be obtained by completion. Another approach via ultraproducts was developed by Krivine [1976, pp. 6–7].

6.1.3.21 Finally, I stress that spreading models are, implicitly or explicitly, the main tool for proving the famous **Krivine theorem** [1976, p. 16]; see also Rosenthal [1978a, pp. 216–217] and Lemberg [1981]:

Let (x_k) be any sequence whose linear span is infinite-dimensional. Then we find an exponent p with the following property $(1 \le p \le \infty)$:
Given $n = 1, 2, \dots$ and $\varepsilon > 0$, there exist blocks

$$u_i = \sum_{k_i \le k < k_{i+1}} \lambda_k x_k$$

such that

$$\|(\xi_i)|l_p^n\| \le \Big\| \sum_{i=1}^{n} \xi_i u_i \Big\| \le (1+\varepsilon)\|(\xi_i)|l_p^n\| \quad \text{for } \xi_1, \dots, \xi_n \in \mathbb{K}.$$

In the case $p = \infty$, it is known only that the conclusion above holds after a suitable permutation of (x_k).

6.1.3.22 A detailed presentation of the theory of spreading models is given in [BEAU$^+$] and [GUER].

6.1.4 $\mathscr{L}_p$-spaces

6.1.4.1 Let $1 \le p \le \infty$. A Banach space X is called an $\mathscr{L}_p$**-space** if there exists a constant $c \ge 1$ such that every finite-dimensional subspace E of X is contained in a finite-dimensional subspace F with $d(F, l_p^n) \le c$, where $n = \dim(F)$. This fundamental concept was invented by Lindenstrauss/Pełczyński [1968, p. 283].

Afterward, Lindenstrauss/Rosenthal [1969, p. 328] observed that $\mathscr{L}_p$-spaces automatically have a stronger property: besides $d(F, l_p^n) \le c$, one can arrange that there is a projection P from X onto F with $\|P\| \le c$. Of course, the constant $c \ge 1$ need not be the same.

6.1.4.2 Lindenstrauss/Rosenthal [1969, p. 330] proved that X is an $\mathscr{L}_p$-space if and only if X^* is an $\mathscr{L}_{p^*}$-space.

6.1.4.3 According to a theorem of Joichi [1966, p. 424], the $\mathscr{L}_2$-spaces are just the **Hilbertian** Banach spaces, which means that their topology can be obtained from an inner product. In other words, they are isomorphic to a Hilbert space.

6.1.4.4 The "only if" part of the following criterion is due to Lindenstrauss/Pełczyński [1968, p. 306], whereas Lindenstrauss/Rosenthal [1969, p. 329] supplied the "if" part.

Let $1 < p < \infty$ and $p \ne 2$. A Banach space is an $\mathscr{L}_p$-space if and only if it is isomorphic to a complemented and non-Hilbertian subspace of some $L_p(M, \mathscr{M}, \mu)$.

Note that $L_p[0,1]$ contains complemented subspaces that are isomorphic to l_2. Take, for example, the closed linear span of the Rademacher functions.

6.1.4.5 In the limiting cases, the situation is more complicated.

A Banach space X is an $\mathscr{L}_1$-space or an $\mathscr{L}_\infty$-space if and only if X^{**} is isomorphic to a complemented subspace of some $L_1(M,\mathscr{M},\mu)$ or $L_\infty(M,\mathscr{M},\mu)$, respectively.

Passing to the biduals is indeed necessary; see Lindenstrauss/Pełczyński [1968, p. 315] and Bourgain/Delbaen [1980, p. 163]. The latter constructed an infinite-dimensional $\mathscr{L}_\infty$-space that has the Schur property; see also [BOUR$_2$, Chap. III]. Since we know from 4.8.5.5 that $\mathfrak{V}(C(K),Y) = \mathfrak{W}(C(K),Y)$ for all Y, the Bourgain–Delbaen space cannot by a quotient of any $C(K)$.

6.1.4.6 If $1 < p < \infty$ and $p \neq 2$, then Lindenstrauss/Pełczyński [1968, pp. 315–317] proved that l_p, $l_p \oplus l_2$, $[l_p, l_2]$ and $L_p[0,1]$ are mutually non-isomorphic $\mathscr{L}_p$-spaces. Moreover, they asked whether these are the only separable and infinite-dimensional spaces of this kind. The answer is NO.

For $2 < p < \infty$, Rosenthal [1970, pp. 280–282] discovered two more examples. One of these consists of all sequences $x = (\xi_k)$ such that

$$\|x\| = \max\left\{ \Big(\sum_{k=1}^{\infty} |\xi_k|^p\Big)^{1/p}, \Big(\sum_{k=1}^{\infty} |\xi_k|^2 w_k^2\Big)^{1/2} \right\}$$

is finite. Here the weights $w_k > 0$ are supposed to satisfy the conditions

$$\lim_{k\to\infty} w_k = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} w_k^r = \infty, \quad \text{where } 1/r = 1/2 - 1/p.$$

Different sequences (w_k) yield isomorphic spaces.

In a next step, Schechtman [1975] constructed a sequence of mutually non-isomorphic, separable, and infinite-dimensional $\mathscr{L}_p$-spaces. Finally, Bourgain/Rosenthal/Schechtman [1981, p. 194] obtained a strictly increasing scale $\{X_\alpha^p\}_{\alpha<\omega_1}$ of such spaces, where ω_1 denotes the first uncountable ordinal; see also [BOUR$_2$, Chap. IV].

6.1.4.7 A countable set of mutually non-isomorphic, separable, and infinite-dimensional $\mathscr{L}_1$-spaces was discovered by Lindenstrauss [1970b], and Johnson/Lindenstrauss [1980] showed this set can be enlarged to get the cardinality $2^{\aleph_0}$.

The isometric and isomorphic classification of separable $C(K)$ spaces is already discussed in 4.9.3.5. According to Bourgain/Delbaen [1980, p. 176], there exist $2^{\aleph_0}$ mutually non-isomorphic, separable, and infinite-dimensional $\mathscr{L}_\infty$-spaces.

6.1.4.8 Next, I present a theorem that goes back to [LIND, pp. 11–12]. Its final version is due to Lindenstrauss/Rosenthal [1969, p. 336].

An $\mathscr{L}_\infty$-space Y is characterized by the **compact** extension property:

Every compact operator $T_0 : X_0 \to Y$ defined on a subspace of an arbitrary Banach space X admits a compact extension T; see 4.9.2.1.

6.1.5 Local unconditional structure

6.1.5.1 The **unconditional basis constant** $\mathrm{unc}(E)$ of a finite-dimensional Banach space E is defined to be the smallest $c \geq 1$ with the following property:

There exists a basis $\{x_1,\dots,x_n\}$ of E such that

$$\Big\| \sum_{k=1}^{n} \xi_k x_k \Big\| \leq c \Big\| \sum_{k=1}^{n} \eta_k x_k \Big\| \quad \text{if } |\xi_k| \leq |\eta_k|.$$

Some authors replace this inequality by (5.6.3.22.a).

The unconditional basis constant is larger than or equal to the basis constant defined in 6.1.1.11.

6.1.5.2 Obviously, $\mathrm{unbc}(l_p^n) = 1$ for $1 \leq p \leq \infty$. If $\dim(E_n) = n$, then it follows from $d(E_n, l_2^n) \leq \sqrt{n}$ that $\mathrm{unbc}(E_n) \leq \sqrt{n}$. This result is asymptotically the best possible. Indeed, Figiel/Kwapień/Pełczyński [1977, p. 1221] produced a sequence of n-dimensional Banach spaces E_n such that $\mathrm{unbc}(E_n) \geq c_0\sqrt{n}$.

6.1.5.3 The following concepts were invented by Dubinsky/Pełczyński/Rosenthal [1972, p. 624] and Gordon/Lewis [1974, pp. 35–36].

(DPR) A Banach space X has **local unconditional structure** if there exists a constant $c \geq 1$ such that every finite-dimensional subspace E of X is contained in a finite-dimensional subspace F with $\mathrm{unbc}(F) \leq c$.

(GL) A Banach space X has **local unconditional structure** if there exists a constant $c \geq 1$ such that the canonical embedding from every finite-dimensional subspace E into X admits a factorization $J_E^X : E \xrightarrow{A} L \xrightarrow{B} X$ through a finite-dimensional space L such that $\|B\|\mathrm{unbc}(L)\|A\| \leq c$.

Plainly, we have **(DPR)**$\Rightarrow$**(GL)**. But it is still unknown whether the two conditions are equivalent. This question had already been asked by Figiel/Johnson/Tzafriri [1975, pp. 396–397], who gave an affirmative answer for Banach lattices, but not for *proper* complemented subspaces. According to [PIS$_1$, p. 118], *the problem boils down to showing that a complemented subspace of a Banach lattice has l.u.st. in the sense of* **(DPR)**.

Stern [1978, p. 244] translated the dilemma above in the language of ultraproducts.

6.1.5.4 An elementary observation of Gordon/Lewis [1974, p. 36] says that every complemented subspace of a Banach space with an unconditional basis has local unconditional structure (in their sense).

The upshot is the following criterion; Figiel/Johnson/Tzafriri [1975, p. 399]:

A Banach space X has local unconditional structure in the sense of **(GL)** if and only if X^{**} is isomorphic to a complemented subspace of a Banach lattice.

This characterization helps to explain why most authors prefer the Gordon–Lewis approach. Another reason is the fact that the constant $\lambda_{\text{ust}}(X) := \inf c$ can be derived from an ideal norm; see 6.3.13.8. The infimum above is taken over all $c \geq 1$ for which condition **(GL)** is satisfied.

6.1.5.5 All $\mathscr{L}_p$-spaces and, in particular, the classical Bananch lattices $C(K)$ and $L_p(M, \mathscr{M}, \mu)$ have local unconditional structure. Counterexamples are the disk algebra $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$; see 6.7.12.19.

6.1.5.6 Gordon/Lewis [1974, p. 42] observed that

$$\lambda_{\text{ust}}\big(\mathfrak{S}_p(l_2^n)\big) \asymp n^{|1/p-1/2|}.$$

6.1.5.7 Pisier [1978a, p. 18] asked whether a Banach space must be Hilbertian if all of its subspaces have local unconditional structure. This problem is still open.

6.1.6 Banach spaces containing l_p^n's uniformly

6.1.6.1 An infinite-dimensional Banach space X **contains the spaces l_p^n uniformly** if there exists a sequence of n-dimensional subspaces E_n with $\sup_n d(E_n, l_p^n) < \infty$.

For spaces with this property, Krivine's theorem 6.1.3.21 implies that we can do much better; see Krivine [1976, p. 22] and [GUER, p. 103]:

Given $\varepsilon > 0$, there exist n-dimensional subspaces $E_{n,\varepsilon}$ with $d(E_{n,\varepsilon}, l_p^n) \leq 1+\varepsilon$ for $n = 1, 2, \dots$.

Hence the property above means that l_p is finitely representable in X.

6.1.6.2 The case $p=2$ is covered by Dvoretzky's theorem 6.1.2.2.

6.1.6.3 In [1964c, pp. 547–549], James showed that every Banach space that contains an isomorphic copy of l_1 or c_0 also contains copies X_ε such that $d(X_\varepsilon, l_1) \leq 1+\varepsilon$ or $d(X_\varepsilon, c_0) \leq 1+\varepsilon$, respectively. Due to the negative solution of the distortion problem, the corresponding conclusion fails for $1 < p < \infty$; see 7.4.4.4.

Carefully looking at the proof of James yields the following result; see also [TOM, p. 239]:

Let $c > 1$. Then every n^2-dimensional Banach space E_{n^2} with $d\big(E_{n^2}, l_1^{n^2}\big) < c^2$ contains an n-dimensional subspace E_n with $d\big(E_n, l_1^n\big) < c$. The same conclusion holds if $l_1^{n^2}$ and l_1^n are replaced by $l_\infty^{n^2}$ and l_∞^n, respectively.

6.1.6.4 In the remaining cases, where $1 < p < \infty$ and $p \neq 2$, a quantitative version of Krivine's theorem was obtained by Amir/Milman [1980], [1985]; see also [MIL$^+$, pp. 63–68]:

For $n = 1, 2, \dots$, $\varepsilon > 0$, $c > 1$, and $1 \leq p \leq \infty$, there exists a natural number $N(n, \varepsilon, c, p)$ such that every N-dimensional space E_N with $N \geq N(n, \varepsilon, c, p)$ and $d(E_N, l_p^N) \leq c$ contains an n-dimensional subspace E_n with $d(E_n, l_p^n) \leq 1+\varepsilon$.

6.1.6.5 The **spectrum** $\Sigma(X)$ of an infinite-dimensional Banach space consists of all exponents $1 \leq p \leq \infty$ for which X contains l_p^n's uniformly; see [SCHW$_3$, p. 57]. The inequality $d(E_n, l_p^n) \leq d(E_n, l_q^n) n^{|1/p-1/q|}$ implies that the spectrum is a closed subset of $[1,\infty]$. Dvoretzky's theorem says that $2 \in \Sigma(X)$. While $\Sigma(X) \cap [1,2]$ is always an interval, it seems likely that any bounded closed subset of $[2,\infty)$ containing 2 can be the spectrum of a suitable Banach space. We have $\Sigma(X) = [1,\infty]$ whenever $\infty \in \Sigma(X)$.

By 6.1.6.1, the spectrum of a Banach space is invariant under isomorphisms.

6.1.6.6 For the infinite-dimensional classical function spaces, we have

$$\Sigma(L_p) = [p,2] \text{ if } 1 \leq p \leq 2, \quad \Sigma(L_q) = \{2,q\} \text{ if } 2 \leq q < \infty, \quad \Sigma(L_\infty) = [1,\infty].$$

6.1.6.7 The concept of ***B*-convexity** was invented by Beck [1962] when he characterized Banach spaces X such that a *strong law of large numbers* holds for certain sequences of independent X-valued random variables; see Subsection 6.8.9. Beck's definition [1962, p. 329] reads as follows.

A Banach space X is said to have property (B) *if there exists an integer $n > 0$ and an $\varepsilon > 0$ such that any choice $x_1, x_2, \dots, x_n$ of elements from X with $\|x_k\| \leq 1$ gives us*

$$\| \pm x_1 \pm x_2 \pm \cdots \pm x_n \| < n(1-\varepsilon)$$

for some combination of the $+$ and $-$ signs.

6.1.6.8 The term *B*-convex was coined by Giesy, who observed that a Banach space has property (B) if and only if it does not contain l_1^n's uniformly. He also discovered the following elementary facts; see [1966, pp. 129–130]:

B-convexity is stable with respect to isomorphisms as well as under the formation of closed subspaces, quotients, and duals.

6.1.6.9 The example $[l_2, l_1^n]$, due to Beck, tells us that a reflexive space need not be *B*-convex. On the other hand, uniform convexity implies *B*-convexity; see Beck [1958]. Moreover, Giesy [1966, p. 142] was able to prove that every *B*-convex space with an unconditional basis is reflexive. As a consequence, he said on p. 144:

The principal outstanding conjecture about B-convex Banach spaces is that they are reflexive.

This problem remained open for many years. Only in [1974b], did James succeed in constructing a non-reflexive Banach space that is non-octahedral. This means that the Banach–Mazur distance of all 3-dimensional subspaces to l_1^3 is larger than a constant $c > 1$. Hence the class of *B*-convex spaces and the class of reflexive spaces are incomparable.

However, Johnson [1974, p. 237] showed that the concepts of *B*-convexity and superreflexivity coincide for spaces with local unconditional structure in the sense of Dubinsky/Pełczyński/Rosenthal. This result should be compared with 5.6.3.10.

6.1.7 Rademacher type and cotype, Gauss type and cotype

6.1.7.1 Recall from 5.6.4.6 that the k^{th} **Rademacher function** is defined by

$$r_k(t) := (-1)^{i-1} \quad \text{whenever } t \in \Big[\tfrac{i-1}{2^k}, \tfrac{i}{2^k}\Big) \text{ and } i = 1, \dots, 2^k.$$

From the stochastic point of view, $r_1, \dots, r_n$ are independent random variables taking the values $+1$ and -1 with probability $\frac{1}{2}$. This process models "coin tossing." An equivalent realization can be obtained as follows:

Let $\mathbb{E}^n$ denote the multiplicative group of all n-tuples $e = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_k = \pm 1$. Make $\mathbb{E}^n$ a probability space by using the normalized counting measure and put $r_k(e) = \varepsilon_k$.

With the notation above, we have

$$\bigg(\int_0^1 \Big|\sum_{k=1}^n \xi_k r_k(t)\Big|^s dt\bigg)^{1/s} = \bigg(\frac{1}{2^n}\sum_{e\in\mathbb{E}^n}\Big|\sum_{k=1}^n \varepsilon_k \xi_k\Big|^s\bigg)^{1/s} \quad \text{for } \xi_1, \dots, \xi_n \in \mathbb{K}.$$

This observation means that the left-hand integral is, in fact, "discrete."

6.1.7.2 For $0 < s < \infty$, there exist positive constants A_s and B_s such that

$$A_s\Big(\sum_{k=1}^n |\xi_k|^2\Big)^{1/2} \le \bigg(\int_0^1 \Big|\sum_{k=1}^n \xi_k r_k(t)\Big|^s dt\bigg)^{1/s} \le B_s\Big(\sum_{k=1}^n |\xi_k|^2\Big)^{1/2} \qquad (6.1.7.2.a)$$

whenever $\xi_1, \dots, \xi_n \in \mathbb{K}$ and $n = 1, 2, \dots$. This is the celebrated **Khintchine inequality**, which was independently discovered by several authors.

First of all, Khintchine [1923, p. 111] proved a *Hilfssatz* that implies the upper estimate:

Es seien $\gamma_1, \gamma_2, \dots, \gamma_n$ *reelle Konstanten, die nicht gleichzeitig verschwinden.* $E(\delta)$ *bedeute die Menge der zwischen* 0 *und* 1 *gelegenen Zahlen* t*, für welche*

$$|\gamma_1 r_1(t) + \gamma_2 r_2(t) + \dots + \gamma_n r_n(t)| > \delta > 0$$

ist. Dann ist ($\mathfrak{m}$ *bedeutet das Lebesguesche Maß*)

$$\mathfrak{m}E(\delta) < C\mathrm{e}^{-\frac{\delta^2}{2\sum_{k=1}^n \gamma_k^2}},$$

wo C *absolut konstant ist.*

A direct proof of the upper estimate, yielding the constant $\sqrt{\frac{s+2}{2}}$, was given by Paley/Zygmund [1930/32, Part I, p. 340]; see also [KAC$^+$, Chap. IV, § 5].

Using the upper estimate for the exponent 4, Littlewood [1930, pp. 170–171] derived the lower ones, for $0 < s < 2$, from

$$\|f|L_2\| \le \|f|L_s\|^{1-\theta}\|f|L_4\|^{\theta}, \quad \text{where } 1/2 = (1-\theta)/s + \theta/4.$$

6.1.7.3 In what follows, we assume that the **Khintchine constants** A_s and B_s are chosen as large as possible and as small as possible, respectively. Tomaszewski (unpublished) observed that the values of A_s and B_s are the same for real and complex scalars; a suitable reference is [PIE$^+$, p. 71]. In view of Parseval's equation, we have $A_s = 1$ if $s \geq 2$ and $B_s = 1$ if $s \leq 2$. The remaining cases were treated by Haagerup [1982]. As a result of sophisticated computations, he obtained that

$$A_s = \begin{cases} 2^{1/2-1/s} & \text{if } 0 < s \leq s_0, \\ \sqrt{2}\left[\dfrac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{1}{2})}\right]^{1/s} & \text{if } s_0 \leq s \leq 2, \end{cases} \quad \text{and} \quad B_s = \sqrt{2}\left[\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{1}{2})}\right]^{1/s} \text{ if } 2 \leq s < \infty.$$

Here $s_0 = 1.8474\ldots$ is the unique solution of $\Gamma\big(\frac{s+1}{2}\big) = \Gamma\big(\frac{3}{2}\big)$ in $(0,2)$.

The most important case, $A_1 = \frac{1}{\sqrt{2}}$, had already been settled by Szarek [1976]. A very elegant approach is due to Latała/Oleszkiewicz [1994].

6.1.7.4 In view of Khintchine's inequality, the expressions

$$\left(\int_0^1 \Big|\sum_{k=1}^n \xi_k r_k(t)\Big|^s dt\right)^{1/s}$$

can be estimated against each other for different exponents s. Results of Kahane [1964], [KAH, pp. 18–20] imply the same in the vector-valued setting: $x_1,\ldots,x_n \in X$. Actually, given $0 < s_1 < s_2 < \infty$, besides

$$\left(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|^{s_1} dt\right)^{1/s_1} \leq \left(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|^{s_2} dt\right)^{1/s_2},$$

we have the **Khintchine–Kahane inequality**

$$\left(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|^{s_2} dt\right)^{1/s_2} \leq K\left(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|^{s_1} dt\right)^{1/s_1}, \qquad (6.1.7.4.a)$$

where $K \geq 1$ depends only on s_1 and s_2; see also Pisier [1977].

6.1.7.5 In contrast to the previous result, both parts of the Khintchine inequality (6.1.7.2.a) may fail if the scalars $\xi_1,\ldots,\xi_n$ are replaced by elements $x_1,\ldots,x_n$ of an arbitrary Banach space. This fact gave rise to the following definitions.

A Banach space X is said to have **Rademacher type** p if for some/all s with $0 < s < \infty$, there exists a constant $c \geq 1$ (depending on s) such that

$$\left(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|^{s} dt\right)^{1/s} \leq c\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p} \qquad (6.1.7.5.a)$$

whenever $x_1,\ldots,x_n \in X$ and $n = 1,2,\ldots$.

Similarly, **Rademacher cotype** q means that

$$\Big(\sum_{k=1}^{n}\|x_k\|^q\Big)^{1/q} \leq c\bigg(\int_0^1\Big\|\sum_{k=1}^{n}x_k r_k(t)\Big\|^s dt\bigg)^{1/s} \qquad (6.1.7.5.b)$$

with the usual modification for $q=\infty$.

It follows from the Khintchine–Kahane inequality that these definitions do not depend on the special choice of s. The most suitable exponents are $s=p$ (type) and $s=q$ (cotype) as well as $s=2$.

The concepts above are non-trivial only if $1<p\leq 2$ and $2\leq q<\infty$. Obviously, every Banach space has type 1 and cotype ∞.

6.1.7.6 For completeness, I add a trivial result:

If $1\leq p_2\leq p_1\leq 2$, then Rademacher type p_1 implies Rademacher type p_2.

If $2\leq q_1\leq q_2\leq\infty$, then Rademacher cotype q_1 implies Rademacher cotype q_2.

6.1.7.7 It is easy to check what happens for the classical spaces:

Every Banach space L_p with $1<p\leq 2$ is of Rademacher type p and Rademacher cotype 2.

Every Banach space L_q with $2\leq q<\infty$ is of Rademacher type 2 and Rademacher cotype q.

The same results hold for the Schatten–von Neumann class $\mathfrak{S}_p(H)$; see 6.1.9.12. Lorentz spaces were treated by Creekmore [1981].

6.1.7.8 The property "Rademacher type p" is stable under the formation of closed subspaces and quotients. Concerning "Rademacher cotype q," we have stability only for closed subspaces. In passing to quotients, "Rademacher cotype 2" may get lost: c_0 is a quotient of l_1.

6.1.7.9 What follows is one of the deepest results from the geometry of Banach spaces.

In a first step, Maurey/Pisier [1973] proved that a Banach space either has some finite Rademacher cotype or contains l_∞^n's uniformly.

Next, Pisier [1973b] characterized B-convexity by showing that a Banach space either has non-trivial Rademacher type or contains l_1^n's uniformly.

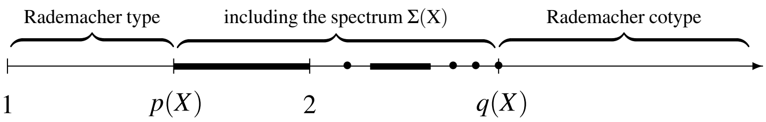
The statements above are limiting cases of the **Maurey–Pisier theorem** that says that for every Banach space X,

$$\sup\{p : X \text{ has Rademacher type } p\} = \min\{p : X \text{ contains } l_p^n\text{'s uniformly}\},$$

$$\inf\{q : X \text{ has Rademacher cotype } q\} = \max\{q : X \text{ contains } l_q^n\text{'s uniformly}\}.$$

These numbers are denoted by $p(X)$ and $q(X)$, respectively.

The general situation looks as follows:



First-hand comments on the history of this theorem can be found in a survey of Maurey [2003b]. The famous Maurey/Pisier paper [1976] was submitted to Studia Mathematica in February 1975. Its original version contained only a weaker result. Then the authors learned about Krivine's theorem 6.1.3.21. Applying these new ideas, they obtained a significant improvement, which was later appended on p. 85. A simplified approach is due to Milman/Sharir [1979]; see also [MIL$^+$, Chap. 13].

6.1.7.10 If a Banach space has Rademacher type for some $1 < p \leq 2$, then it does not contain l_1^n's uniformly, which implies that it cannot contain l_∞^n's, either. Thus the space must be of Rademacher cotype for some $2 \leq q < \infty$. In summary, every space with non-trivial Rademacher type also has finite Rademacher cotype. A quantitative version of this result is due to König/Tzafriri [1981, p. 92], who proved that one may take $q = 2 + (2c)^{p^*}$, where c is the *type constant* appearing in (6.1.7.5.a) with $s = 2$; see also 6.1.8.7.

6.1.7.11 Obviously, if X has Rademacher type p, then X^* has Rademacher cotype p^*. The example $l_1 = c_0^*$ shows that the converse implication may fail. This defect was the reason why Hoffmann-Jørgensen [HOF-J, p. 3] defined two kinds of cotype. The first one was designed in order to get full duality, while the second one, which he called "weak" cotype, is just our present concept of cotype.

6.1.7.12 Luckily, excluding the spaces l_1^n yields satisfactory duality relations. A striking theorem of Pisier says that X has Rademacher type p if and only if X^* is B-convex and has Rademacher cotype p^*; the roles of X and X^* can be interchanged.

Pisier's proof is based on a new concept. According to Maurey/Pisier [1976, p. 88], a Banach space X is said to be ***K*-convex** if the **Rademacher projection**

$$[R,X] : f \mapsto \sum_{k=1}^{\infty} r_k(s) \int_0^1 \boldsymbol{f}(t) r_k(t)\,dt$$

is continuous on $\big[L_2[0,1),X\big]$. Equivalently, there must exist a constant $c \geq 1$ such that

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(s) \int_0^1 \boldsymbol{f}(t) r_k(t)\,dt \right\|^2 ds \right)^{1/2} \leq c \left(\int_0^1 \|\boldsymbol{f}(t)\|^2 dt \right)^{1/2} \qquad (6.1.7.12.a)$$

whenever $\boldsymbol{f} \in L_2[0,1) \otimes X$ and $n = 1,2,\ldots$. Obviously, K-convexity is carried over from a Banach space to its dual and conversely. Moreover, it follows that X has Rademacher type p if X^* is K-convex and has Rademacher cotype p^*; the roles of X and X^* can be interchanged. Pisier [1981, p. 144] remarked that *the reason for the choice of the letter K remains a well kept secret*.

Using the Beurling–Kato theorem 5.3.3.3 from the theory of analytic semi-groups, Pisier [1982, p. 384] was able to show that

$$\textit{B-convexity} \;=\; \textit{K-convexity}.$$

6.1.7.13 According to Pełczyński/Rosenthal [1975, p. 284], a Banach space X is called **locally π-Euclidean** if there exist a constant $c \ge 1$ and numbers $N(k,\varepsilon)$ depending on $k = 1,2,\dots$ and $\varepsilon > 0$ such that every $N(k,\varepsilon)$-dimensional subspace contains a c-complemented k-dimensional subspace E_k with $d(E_k, l_2^k) \le 1+\varepsilon$.

The most important examples are L_p spaces with $1 < p < \infty$. In a next step, Figiel/Tomczak-Jaegermann [1979, p. 166] proved that all K-convex spaces have the property above. Pisier [1982, p. 388] verified the reverse implication. Hence

$$\textit{K-convex} \;=\; \textit{locally } \pi\textit{-Euclidean}.$$

6.1.7.14 The standard **Gaussian measure** on the real line is defined by

$$\gamma(B) := \frac{1}{\sqrt{2\pi}} \int_B \mathrm{e}^{-|t|^2/2}\,dt \quad \text{for } B \in \mathscr{B}\mathrm{orel}(\mathbb{R}).$$

Then

$$\Big(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big| \sum_{k=1}^{n} t_k \xi_k \Big|^2 d\gamma(t_1)\cdots d\gamma(t_n) \Big)^{1/2} = \Big(\sum_{k=1}^{n} |\xi_k|^2 \Big)^{1/2}. \quad (6.1.7.14.a)$$

6.1.7.15 A counterpart of the Khintchine–Kahane inequality (6.1.7.4.a) says that the expressions

$$\Big(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^{n} t_k x_k \Big\|^s d\gamma(t_1)\cdots d\gamma(t_n) \Big)^{1/s}$$

are equivalent whenever $0 < s < \infty$. This result is based on the work of Fernique [1970] and Landau/Shepp [1970]; for comprehensive historical comments see [LED$^+$, p. 87].

6.1.7.16 A Banach space X is said to have **Gauss type** p if for some/all s with $0 < s < \infty$, there exists a constant $c \ge 1$ (depending on s) such that

$$\Big(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^{n} t_k x_k \Big\|^s d\gamma(t_1)\cdots d\gamma(t_n) \Big)^{1/s} \le c \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}$$

whenever $x_1,\dots,x_n \in X$ and $n = 1,2,\dots$.

Similarly, **Gauss cotype** q means that

$$\Big(\sum_{k=1}^{n} \|x_k\|^q \Big)^{1/q} \le c \Big(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^{n} t_k x_k \Big\|^s d\gamma(t_1)\cdots d\gamma(t_n) \Big)^{1/s}.$$

6.1.7.17 The concepts of Gauss type 2 and Rademacher type 2 were invented by Dubinsky/Pełczyński/Rosenthal [1972, pp. 637, 641]. These authors used the term **subquadratic Gaussian** or **Rademacher average**. They observed that subquadratic Rademacher average implies subquadratic Gaussian average. Later on, Hoffmann-Jørgensen [HOF-J, p. 59] showed that both properties coincide; see also Pisier [1973a]. This result holds even for $1 < p \le 2$:

$$\textit{Rademacher type } p \;=\; \textit{Gauss type } p.$$

6.1.7.18 According to Pisier [1973a],

$$\left(\int_0^1 \Big\| \sum_{k=1}^n x_k r_k(t) \Big\|^s dt\right)^{1/s} \leq \sqrt{\frac{\pi}{2}} \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k x_k \Big\|^s d\gamma(t_1)\cdots d\gamma(t_n)\right)^{1/s}.$$

Substituting the unit vectors $e_1,\dots,e_n \in l_\infty^n$ yields

$$\left(\int_0^1 \Big\| \sum_{k=1}^n e_k r_k(t) \Big\|^s dt\right)^{1/s} = 1$$

and

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k e_k \Big\|^s d\gamma(t_1)\cdots d\gamma(t_n)\right)^{1/s} \asymp \sqrt{1+\log n}.$$

Therefore an estimate in the converse direction cannot hold in arbitrary Banach spaces. The situation improves when l_∞^n is excluded. Maurey/Pisier [1976, p. 68] showed that a Banach space X does not contain l_∞^n's uniformly if and only if there is a constant $c \geq 1$ such that

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k x_k \Big\|^2 d\gamma(t_1)\cdots d\gamma(t_n)\right)^{1/2} \leq c \left(\int_0^1 \Big\| \sum_{k=1}^n x_k r_k(t) \Big\|^2 dt\right)^{1/2}$$

whenever $x_1,\dots,x_n \in X$ and $n = 1,2,\dots$. Combining these results gives

Rademacher cotype q = Gauss cotype q.

6.1.7.19 The Gaussian version of type and cotype is the main tool for proving **Kwapień's theorem** [1972b, p. 586]:

A Banach space is isomorphic to a Hilbert space if and only if it has Gauss/Rademacher type 2 and Gauss/Rademacher cotype 2.

In my opinion this is one of the most beautiful results of Banach space theory.

6.1.7.20 Pisier [1980, pp. 32, 36] proved a remarkable supplement to Kwapień's theorem:

A Banach space X *with the approximation property* is isomorphic to a Hilbert space if and only if X and X^* have Gauss/Rademacher cotype 2.

This solves a problem posed by Maurey [MAU, p. 152]. The famous Pisier space shows that assuming the approximation property is indeed crucial; see 7.4.2.4.

6.1.7.21 Let $0 < p \leq 2$. Then there exists a unique Borel measure on the real line such that

$$\mathrm{e}^{-|s|^p} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st} d\mu_p(t) \quad \text{for } s \in \mathbb{R}.$$

The **stable laws** μ_p were invented by Lévy [1923]; see also [LÉVY$_2$, pp. 254–263] and [LÉVY$_3$, p. 221].

For $p=2$, we just get a Gaussian measure

$$d\mu_2(t) = \frac{1}{\sqrt{4\pi}} e^{-|t|^2/4}\, dt.$$

Since this case is already treated in 6.1.7.14, we may assume that $0<p<2$.

If $0<s<p<2$, then the absolute moments

$$c_{ps} = \left(\int_{\mathbb{R}} |t|^s d\mu_p(t) \right)^{1/s} = 2 \left[\frac{\Gamma\left(\frac{p-s}{p}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1}{2}\right)} \right]^{1/s}$$

exist, and we have

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big| \sum_{k=1}^n t_k \xi_k \Big|^s d\mu_p(t_1)\cdots d\mu_p(t_n) \right)^{1/s} = c_{ps} \Big(\sum_{k=1}^n |\xi_k|^p \Big)^{1/p}, \quad (6.1.7.21.a)$$

an analogue of (6.1.7.14.a).

To the best of my knowledge, formula (6.1.7.21.a) goes back to Herz [1963, pp. 670–671] and Bretagnolle/Dacunha-Castelle/Krivine [1966, pp. 238–239]; see also Rosenthal [1973, p. 355]. Kadets [1958a, стр. 95] was the first who used stable laws in order to show that L_s contains *isomorphic* copies of l_p. Instead of (6.1.7.21.a), he proved only a two-sided inequality; see Kadets [1958a, стр. 97].

6.1.7.22 As in the case of Rademacher functions and Gaussian measures, the expressions

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k x_k \Big\|^s d\mu_p(t_1)\cdots d\mu_p(t_n) \right)^{1/s}$$

are equivalent for all s with $0<s<p<2$, but fixed p. This follows from a general theorem of Hoffmann-Jørgensen; see [HOF-J, p. 47].

6.1.7.23 A Banach space X is said to have **stable type** p if for some/all s with $0<s<p<2$, there exists a constant $c\geq 1$ (depending on s) such that

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k x_k \Big\|^s d\mu_p(t_1)\cdots d\mu_p(t_n) \right)^{1/s} \leq c \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p}$$

whenever $x_1,\dots,x_n \in X$ and $n=1,2,\dots$.

Pisier [1973a] proved that stable type p implies Rademacher type p. On the other hand, Rademacher type p yields only stable type $p_0<p<2$. This result is sharp, since l_p has Rademacher type p, but fails to have stable type p.

6.1.7.24 Maurey [1972b] also defined **stable cotype** p by requiring that

$$\Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p} \leq c \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k x_k \Big\|^s d\mu_p(t_1)\cdots d\mu_p(t_n) \right)^{1/s}$$

whenever $x_1,\dots,x_n \in X$ and $n=1,2,\dots$. Ironically, it turned out that this condition is satisfied for *all* Banach spaces and $0<s<p<2$; see Maurey [1972c]. In the limiting case $p=2$ one gets Gauss cotype 2.

6.1.7.25 Banach spaces of Rademacher **equal-norm** type p and cotype q are defined by the property that (6.1.7.5.a) and (6.1.7.5.b), respectively, hold for all finite families of elements $x_1,\dots,x_n \in X$ such that $\|x_1\| = \cdots = \|x_n\|$.

This modification was used for the first time by James. In [1978, p. 2], he presented a result of Pisier that asserts that the concepts of equal-norm type 2 and ordinary type 2 coincide. The same is true for cotype 2. On the other hand, Tzafriri [1979, p. 33] constructed Tsirelson-like spaces that show that for exponents different from 2, the new concepts are weaker than the old ones; see also [CASA$^+$, p. 107].

Banach spaces of *weak* type p and cotype q are discussed in 6.1.11.7 and 6.3.14.8.

6.1.7.26 The concepts of type and cotype, created in 1972, have several independent roots.

(1) In January 1972, Kwapień submitted his famous paper

Isomorphic characterizations of inner product spaces
by orthogonal series with vector valued coefficients

to Studia Mathematica. He concludes with the following acknowledgment: *The author is deeply indebted to professor A. Pełczyński whom the paper owes his existence.*

(2) The same volume of Studia Mathematica contains a paper of Dubinsky/Pełczyński/Rosenthal [1972] in which the concept of Gauss type 2 is introduced; see 6.1.7.17. These investigations were inspired by problems from the theory of p-summing operators. Details will be discussed in 6.3.8.4.

(3) In February 1972, Rosenthal submitted his paper

On subspaces of L^p

to the Annals of Mathematics. On p. 356, he used a condition of the form

$$\int_0^1 \Big\| \sum_{k=1}^n x_k f_k(t) \Big\| dt \le c \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p}.$$

The fixed functions $f_1, f_2, \dots \in L_1[0,1)$ are chosen such that

$$\int_0^1 \Big| \sum_{k=1}^n \xi_k f_k(t) \Big| dt = \Big(\sum_{k=1}^n |\xi_k|^p \Big)^{1/p}$$

whenever $\xi_1,\dots,\xi_n \in \mathbb{R}$ and $n = 1,2,\dots$.

(4) In October 1972, Hoffmann-Jørgensen presented his well-known preprint

Sums of independent Banach space valued random variables,

in which the names *"type"* and *"cotype"* appeared for the first time. His definitions were stated in terms of probability theory, [HOF-J, p. 55]:

A Banach space X is of type p if the series $\sum_{k=1}^\infty x_k r_k(t)$ converges almost surely whenever $\sum_{k=1}^\infty \|x_k\|^p < \infty$. On the other hand, if the almost sure convergence of

$\sum_{k=1}^{\infty} x_k r_k(t)$ implies $\sum_{k=1}^{\infty} \|x_k\|^q < \infty$, then X is said to be of (original version: weak) cotype q.

The reader is warned that the attribute "weak" may have quite different meanings; see 6.1.7.11, 6.1.11.7 and 6.3.14.8.

The studies of Hoffmann-Jørgensen were based on the earlier work of Nordlander [1961], Ito/Nisio [1968], and the first edition of Kahane's book, [KAH].

(5) After these preliminaries the French team entered the scene. Maurey [2003b, p. 1304] emphasizes the significant role of Kwapień, who visited Paris in 1971 and 1972, *just before all this started.* At the beginning, Maurey and Pisier preferred to use stable laws instead of Rademacher functions. However, Maurey [2003b, p. 1303] says that *later on, it has been universally admitted that Rademacher type is easier to work with, and the notion of stable type p essentially disappeared, except for* $p=2$.

6.1.8 Fourier type and cotype, Walsh type and cotype

6.1.8.1 In [1933c, pp. 178–179], Bochner made an observation that gave rise to the theory of Fourier type and cotype:

Es sei nun $\boldsymbol{f}(t)$ *eine summierbare abstrakte Funktion mit der Periode* 2π*. Für jedes ganze n existiert das Integral*

$$\gamma_n(\boldsymbol{f}) := \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{f}(t)\mathrm{e}^{-\mathrm{i}nt}\,dt,$$

und die formale Reihe

$$\sum_{-\infty}^{+\infty} \gamma_n(\boldsymbol{f})\mathrm{e}^{\mathrm{i}nt}$$

besitzt viele Eigenschaften der Fourierreihe einer komplexen Funktion. Nur eine Eigenschaft besitzt sie nicht: es braucht nicht die Besselsche Ungleichung

$$\sum_{-\infty}^{+\infty} \|\gamma_n(\boldsymbol{f})\|^2 \le \frac{1}{2\pi}\int_0^{2\pi} \|\boldsymbol{f}(t)\|^2\,dt$$

zu bestehen.

Bochner presented a simple counterexample. Let $f \in L_1(\mathbb{T})$ be any scalar-valued function such that $\big(\gamma_n(f)\big) \notin l_2$. Since the Fourier coefficients of the continuous L_1-valued function $\boldsymbol{f} : t \mapsto f(\xi+t)$ are given by $\gamma_n(\boldsymbol{f}) = \gamma_n(f)\mathrm{e}^{\mathrm{i}n\xi}$, we have $\|\gamma_n(\boldsymbol{f})\| = |\gamma_n(f)|$.

6.1.8.2 The **Hausdorff–Young theorem** says that the Fourier transform

$$F_{\mathrm{our}} : f(t) \mapsto \widehat{f}(s) := \int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}st}\,dt$$

defines an operator from $L_p(\mathbb{R})$ into $L_{p^*}(\mathbb{R})$ whenever $1 \le p \le 2$. Hence the question arises, *What happens in the case of vector-valued functions?*

According to Peetre [1969, p. 18], a Banach space X is said to be of **Fourier type** p if there exists a constant $c \geq 1$ such that

$$\left(\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \boldsymbol{f}(t) \mathrm{e}^{-\mathrm{i}st} dt \right\|^{p^*} ds \right)^{1/p^*} \leq c \left(\int_{\mathbb{R}} \|\boldsymbol{f}(t)\|^p dt \right)^{1/p} \tag{6.1.8.2.a}$$

for all infinitely differentiable and rapidly decreasing X-valued functions $\boldsymbol{f}$ on $\mathbb{R}$.

Instead of (6.1.8.2.a) one may require that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{k=1}^n x_k \mathrm{e}^{\mathrm{i}ks} \right\|^{p^*} ds \right)^{1/p^*} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}, \tag{6.1.8.2.b}$$

or

$$\left(\frac{1}{n} \sum_{h=1}^n \left\| \sum_{k=1}^n \exp\left(2\pi\mathrm{i}\tfrac{hk}{n}\right) x_k \right\|^{p^*} \right)^{1/p^*} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}. \tag{6.1.8.2.c}$$

These inequalities are supposed to hold for any choice of $x_1, \dots, x_n \in X$ and $n = 1, 2, \dots$. Real versions are

$$\left(\frac{2}{\pi} \int_0^{\pi} \left\| \sum_{k=1}^n x_k \cos ks \right\|^{p^*} ds \right)^{1/p^*} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}, \tag{6.1.8.2.d}$$

and

$$\left(\frac{2}{\pi} \int_0^{\pi} \left\| \sum_{k=1}^n x_k \sin ks \right\|^{p^*} ds \right)^{1/p^*} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}. \tag{6.1.8.2.e}$$

The equivalence of these conditions (with possibly different constants $c \geq 1$) was certainly known to Kwapień [1972b, p. 591]; see also [PIE$^+$, pp. 286, 289, 308].

6.1.8.3 Modifying the inequalities in the previous paragraph yields the concept of **Fourier cotype** q with $2 \leq q \leq \infty$. For example, letting $q = p^*$, we get the following counterpart of (6.1.8.2.c):

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq c \left(\frac{1}{n} \sum_{h=1}^n \left\| \sum_{k=1}^n \exp\left(2\pi\mathrm{i}\tfrac{hk}{n}\right) x_k \right\|^{q^*} \right)^{1/q^*}.$$

However, in contrast to the setting of Rademacher functions, nothing new is obtained:

Fourier cotype q = Fourier type q^*;

see Kwapień [1972b, p. 593].

6.1.8.4 The following results are due to Peetre [1969, pp. 20–21]:

- *Soit X reflexif. Alors X et sont dual X^* sont au même temps de type p.*
- *Tout espace de Banach est de type 1.*
- *Tout espace de Hilbert est de type 2.*
- *L'espace L_p, $1 \leq p \leq 2$, est de type p.*
- *L'espace L_p, $2 \leq p \leq \infty$, est de type p^*.*

Obviously, Peetre was not aware of the fact that "having Fourier type p" is a local property. Moreover, he could not know the principle of local reflexivity. This explains why reflexivity is needed in the proof of his duality theorem.

An interpolation theorem of Peetre [1969, p. 19] yields that
Fourier type p_2 implies Fourier type p_1 if $1 \le p_1 \le p_2 \le 2$.

6.1.8.5 The main problem of the theory of Fourier type was mentioned in a footnote of Peetre's paper, [1969, p. 20, footnote [2])]:

Nous ne connaissons aucun espace de type 2 qui ne soit pas un espace de Hilbert.

The answer was given by the following version of **Kwapień's theorem**, [1972b, p. 592]:

A Banach space is isomorphic to a Hilbert space if and only if it has Fourier type 2.

A little bit earlier, Vági [1969, p. 307] proved the preceding result for Banach spaces with an unconditional basis.

6.1.8.6 Kwapień [1972b, p. 589] had already shown that every Banach space with Fourier type 2 also has Rademacher type 2. Nowadays, we know that

Fourier type p implies Rademacher type p and Rademacher cotype p^*.

This folklore result follows from the *principle of contraction*; see [KAH, 2nd edition, pp. 20–21].

Conclusions in the converse direction are more complicated:

Let $1 < p \le 2 \le q < \infty$, and define r by $1/r - 1/2 = 1/p - 1/q$. If $1 < r_0 < r < 2$, then

Rademacher type p and Rademacher cotype q imply Fourier type r_0.

Bourgain [1985, p. 117] showed that this conclusion fails for $r_0 > r$, and the case $r_0 = r$ seems to be open; see also [PIE$^+$, pp. 314–315]. On the other hand, Pisier [1979b, p. 275] had observed that for Banach lattices, the above result holds even when $r = \min\{p, q^*\}$.

6.1.8.7 A very deep theorem was obtained by Bourgain [1981b, p. 174], [1982a, pp. 255–256], [1988, p. 239]:

If a Banach space X has Rademacher type p with $1 < p \le 2$, then it must be of some Fourier cotype q with $2 \le q < \infty$.

By analogy to 6.1.7.10, the required exponent q depends on the Rademacher type constant c of X. More precisely, we may take $q > (c_0 c)^{p^*}$, where c_0 is a universal constant; see Bourgain [1988, p. 241].

6.1.8.8 A similar theory can be developed if the trigonometric system is replaced by the Walsh system. The common feature in both cases is the fact that we deal with characters on a compact abelian group. Nobody knows whether the resulting concepts are different. Some details concerning this problem can be found in [PIE$^+$, pp. 341–343].

6.1.9 Superreflexivity, Haar type and cotype

A classical theorem says that every uniformly convex space is reflexive; see 5.5.2.4. Subsequently, Day [1941] observed that there are *reflexive Banach spaces not isomorphic to uniformly convex spaces.* Thus the problem arose to characterize such Banach spaces that admit uniformly convex renormings.

6.1.9.1 One may say that the theory of superreflexivity was born at a symposium at Baton Rouge (1967) when James [1972a, pp. 159–160, 169] introduced the *finite tree properties* and the concept of *finite representability*. On p. 173 of his article, he wrote:

A property of type P_1 [read: finite tree property] *was originally of interest because it seemed a natural candidate for a condition equivalent to nonisomorphism to a space with a uniformly convex unit ball.*

Though this statement sounds pessimistic, James's idea turned out to be the right one. Indeed, Enflo [1972, p. 281] proved that

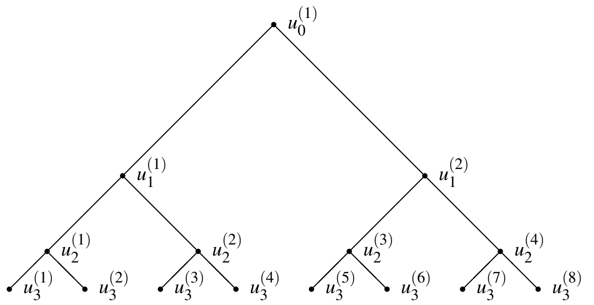
a Banach space can be given an equivalent uniformly convex norm if and only if it does not have the finite tree property.

6.1.9.2 By a dyadic **tree** of length n in a Banach space X we mean a collection of elements $\{u_k^{(i)}\}$ with $i = 1, \dots, 2^k$ and $k = 0, \dots, n$ such that

$$u_k^{(i)} = \tfrac{1}{2}\big(u_{k+1}^{(2i-1)} + u_{k+1}^{(2i)}\big).$$

Clearly, a tree of length n is determined by its n^{th} level $\big(u_n^{(1)}, \dots, u_n^{(2^n)}\big)$.

The following picture shows a tree of length 3:



A Banach space is said to have the **finite tree property** if for every n and ε in $(0,1)$, we can find a tree of length n in the closed unit ball such that

$$\tfrac{1}{2}\big\|u_{k+1}^{(2i-1)} - u_{k+1}^{(2i)}\big\| \geq \varepsilon.$$

An equivalent property is obtained if one replaces *"every* ε" by *"some* ε."

The basic example of such a tree is obtained by taking the unit vectors $(e_1, \dots, e_{2^n})$ in $l_1^{2^n}$ as the n^{th} level. Hence a Banach space that contains l_1^n's uniformly has the finite tree property.

6.1.9.3 The name *super-reflexive* (in this text: superreflexive) was coined by James in the title of [1972b]. In [1972c, p. 896], he gave the definition that is used nowadays:

A **super-reflexive** *Banach space is a Banach space X which has the property that no non-reflexive Banach space is finitely representable in X.*

At this time, James [1972a, pp. 170–171] already knew that a Banach space is superreflexive if and only if it does not have the finite tree property. Consequently, Enflo's result takes the following form:

a Banach space can be given an equivalent uniformly convex norm if and only if it is superreflexive.

A more elegant, but also more sophisticated, criterion for superreflexivity requires that all ultrapowers of X be reflexive; see Henson/Moore [1974a, p. 137] and Heinrich [1980a, p. 85].

The reflexive space $[l_2, l_1^n]$ fails to be superreflexive, since it has the finite tree property..

6.1.9.4 The next theorem was proved by James [1972c, p. 897]:

Superreflexivity is invariant under isomorphisms. A Banach space X is superreflexive if and only if X^* is superreflexive.

Of course, we also have invariance in passing to closed subspaces and quotients.

6.1.9.5 Let $\mathscr{D}_k$ denote the algebra on $[0,1)$ generated by the dyadic intervals

$$\Delta_k^{(i)} := \Big[\tfrac{i-1}{2^k}, \tfrac{i}{2^k}\Big) \quad \text{with } i = 1, \dots, 2^k.$$

Given any tree $\{u_k^{(i)}\}$ of length n, we put

$$\boldsymbol{f}_k(t) := u_k^{(i)} \quad \text{if } t \in \Delta_k^{(i)}.$$

Then $\boldsymbol{f}_k$ is the conditional expectation of $\boldsymbol{f}_n$ with respect to $\mathscr{D}_k$; see 6.8.11.1. This observation induced Pisier [1975a, p. 329] to make the following remark:

The starting point of our work is to notice that the finite tree property can be translated in terms of martingales.

Following Pisier [1975a, p. 327], we refer to $(\boldsymbol{f}_0, \dots, \boldsymbol{f}_n)$ as an X-valued **Walsh–Paley martingale** of length n.

If $\boldsymbol{f}_n$ is written as an X-valued **Haar polynomial** (see 5.6.4.9)

$$\boldsymbol{f}_n = x_0 + \sum_{k=0}^{n-1} \sum_{i=1}^{2^k} x_k^{(i)} \chi_k^{(i)},$$

then the **martingale differences** take the form

$$\boldsymbol{d}_0 := \boldsymbol{f}_0 = x_0 \quad \text{and} \quad \boldsymbol{d}_k := \boldsymbol{f}_k - \boldsymbol{f}_{k-1} = \sum_{i=1}^{2^{k-1}} x_{k-1}^{(i)} \chi_{k-1}^{(i)} \quad \text{for } k = 1, 2, \dots .$$

6.1.9.6 Let $1 < p \le 2$. A Banach space X is said to have **Haar type** p if there exists a constant $c \ge 1$ such that

$$\left(\int_0^1 \left\| \sum_{k=0}^{n} \boldsymbol{d}_k(t) \right\|^p dt \right)^{1/p} \le c \left(\sum_{k=0}^{n} \int_0^1 \| \boldsymbol{d}_k(t) \|^p \, dt \right)^{1/p} \qquad (6.1.9.6.a)$$

for every X-valued Walsh–Paley martingale of arbitrary length n.

Similarly, **Haar cotype** q with $2 \le q < \infty$ means that

$$\left(\sum_{k=0}^{n} \int_0^1 \| \boldsymbol{d}_k(t) \|^q \, dt \right)^{1/q} \le c \left(\int_0^1 \left\| \sum_{k=0}^{n} \boldsymbol{d}_k(t) \right\|^q dt \right)^{1/q}. \qquad (6.1.9.6.b)$$

In his Varenna lectures, Pisier [1986a, pp. 221–222] used the terms **martingale type** and **martingale cotype** or just M-*type* and M-*cotype*.

6.1.9.7 If X has the finite tree property, then we can find Walsh–Paley martingales such that

$$\|\boldsymbol{f}_n(t)\| \le 1 \quad \text{and} \quad \|\boldsymbol{d}_1(t)\| \ge \tfrac{1}{2}, \ \dots\ , \|\boldsymbol{d}_n(t)\| \ge \tfrac{1}{2} \quad \text{for } t \in [0,1).$$

Hence X does not have any Haar cotype q, which means that spaces with some Haar cotype must be superreflexive. The same is true for Haar type p.

6.1.9.8 I stress that, in contrast to the case of Rademacher type and cotype, there is an immaculate duality between Haar type and cotype:

A Banach space X has Haar type p if and only if X^* has Haar cotype p^*.

A Banach space X has Haar cotype q if and only if X^* has Haar type q^*.

6.1.9.9 Pisier's main result [1975a, p. 337] gives a quantitative refinement of Enflo's theorem:

A Banach space has Haar type p if and only if it admits a p-smooth renorming.

A Banach space has Haar cotype q if and only if it admits a q-convex renorming.

In the right-hand case, the desired q-convex norm is obtained by

$$|||x||| := \inf \left(c^q \int_0^1 \|\boldsymbol{f}_n(t)\|^q dt - \sum_{k=1}^{n} \int_0^1 \| \boldsymbol{d}_k(t) \|^q \, dt \right)^{1/q},$$

the infimum being taken over all X-valued Walsh–Paley martingales with $\boldsymbol{f}_0(t) = x$.

6.1.9.10 In view of the preceding criteria, a theorem of Pisier [1975a, p. 340] can be restated as follows; see 5.5.2.10:

Every superreflexive space has Haar type p and Haar cotype q for some $p > 1$ and some $q < \infty$.

6.1.9.11 Another corollary of 6.1.9.9 says that for $1 \le p_1 \le p_2 \le 2$, Haar type p_2 implies Haar type p_1. Unfortunately, I do not know any elementary proof that directly uses the defining inequality (6.1.9.6.a); see also [PIE$^+$, pp. 367–369].

6.1.9.12 It follows from 5.5.2.7 that

every Banach space L_p with $1<p\le 2$ is of Haar type p and of Haar cotype 2.

every Banach space L_q with $2\le q<\infty$ is of Haar type 2 and of Haar cotype q.

Thanks to a theorem of Tomczak-Jaegermann [1974, p. 168] about the moduli of smoothness and convexity, the same results hold for the Schatten–von Neumann classes.

6.1.9.13 The concept of Haar type is quite different from those of Fourier and Walsh type.

If $2\le q<q_0<\infty$, then L_{q_0} has Haar type 2, but fails to have Fourier or Walsh type q^*.

Pisier/Xu [1987, p. 189] constructed a non-reflexive Banach space X such that $d(E_n,l_2^n)\le c(1+\log n)$ for all n-dimensional subspaces; see 6.6.5.6. Hence X has Fourier and Walsh type whenever $1<p<2$, but fails to have any Haar type.

6.1.9.14 Substituting $\boldsymbol{d}_0:=\boldsymbol{o}$ and $\boldsymbol{d}_k:=x_k r_k$ into (6.1.9.6.a) and (6.1.9.6.b) shows that Haar type p implies Rademacher type p, and Haar cotype q implies Rademacher cotype q. Since James [1978, pp. 3, 8] constructed a non-reflexive space X of Rademacher type 2, the type and cotype properties associated with the Haar system are much stronger than that associated with the Rademacher system; see also the Pisier–Xu example mentioned above.

6.1.10 *UMD* spaces = *HT* spaces

6.1.10.1 Following Burkholder [1981a, p. 997], we refer to X as a *UMD* **space** if there exists a constant $c\ge 1$ such that

$$\left(\int_0^1 \left\| \sum_{k=0}^n \varepsilon_k \boldsymbol{d}_k(t)\right\|^2 dt\right)^{1/2} \le c\left(\int_0^1 \left\| \sum_{k=0}^n \boldsymbol{d}_k(t)\right\|^2 dt\right)^{1/2} \qquad (6.1.10.1.a)$$

for every X-valued Walsh–Paley martingale of arbitrary length n and for any choice of signs $\varepsilon_k=\pm 1$. As shown by Wenzel [1997c], it suffices to take $\varepsilon_k=(-1)^k$. Of course, the required constant may be different.

The letters *UMD* abbreviate the lengthy term *"unconditionality property for martingale differences."* For the first time, *"espaces de Banach vérifiant la propriété d'inconditionnalité des différences de martingales"* were considered by Maurey [1974c, exposé 2, pp. 11–12]. He already knew that the exponent 2 in (6.1.10.1.a) can be replaced by any p with $1<p<\infty$.

Maurey further observed that the class of *UMD* spaces contains all L_p's whenever $1<p<\infty$. It is stable under the formation of closed subspaces, quotients, and duals.

Being a *UMD* space is a superproperty. We also know from Maurey [1974c, exposé 2, p. 11] that *UMD* spaces are superreflexive. On the other hand, Pisier [1975b, p. 10] constructed superreflexive non-*UMD* spaces.

6.1.10.2 According to Burkholder [1981a, p. 1003], [1981b, pp. 35–36],

a Banach space X is said to be ζ**-convex** *if there is a symmetric biconvex function ζ on $X \times X$ such that $\zeta(o,o) > 0$ and $\zeta(x,y) \le \|x+y\|$ if $\|x\| \le 1 \le \|y\|$.*

In this context, symmetry and biconvexity mean that $\zeta(x,y) = \zeta(y,x)$ is separately convex in both variables x and y.

The function ζ determines the underlying norm up to equivalence; see Burkholder [1981a, p. 1009].

Burkholder [1981b, p. 38] also showed that a Banach space X is isometric to a Hilbert space if and only if there exists a function ζ with $\zeta(o,o) = 1$. In this case, one has $\zeta(x,y) = 1 + (x|y)$ (real scalars) and $\zeta(x,y) = 1 + \mathrm{Re}(x|y)$ (complex scalars).

Burkholder's main result [1981a, p. 1003] says that

$$\textit{UMD property} \;=\; \zeta\textit{-convexity}.$$

It seems to me that the concept of a ζ-convex space is not based on geometric intuition; its invention was rather motivated by considerations about martingales.

6.1.10.3 The classical Marcel Riesz theorem asserts that for $1 < p < \infty$, the **Hilbert transform**

$$H_{\mathrm{ilb}} : f(t) \mapsto \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{s-t}\, dt$$

defines an operator from $L_p(\mathbb{R})$ into itself. The right-hand integral exists almost everywhere in the sense of Cauchy's principal value.

We know from Pichorides [1972] that

$$\|H_{\mathrm{ilb}} : L_p(\mathbb{R}) \to L_p(\mathbb{R})\| = \begin{cases} \tan \frac{\pi}{2p} & \text{if } 1 < p \le 2, \\ \cot \frac{\pi}{2p} & \text{if } 2 \le p < \infty. \end{cases}$$

6.1.10.4 The work of Bochner/Taylor [1938, pp. 934–935] had implicitly shown that the preceding result does not extend to functions taking their values in arbitrary Banach spaces: "pathological cases" such as l_1 and l_∞ must be excluded. Therefore the natural question arose to identify the correct class of spaces. This problem was solved by the efforts of Burkholder [1983, p. 283] (in collaboration with McConnell) and Bourgain [1983b, p. 165]:

$$\textit{UMD property} \;=\; \textit{HT property}.$$

We say that X is an HT **space** if there exists a constant $c \ge 1$ such that

$$\|H_{\mathrm{ilb}}\boldsymbol{f}|L_2(\mathbb{R})\| \le c\|\boldsymbol{f}|L_2(\mathbb{R})\| \quad \text{for } \boldsymbol{f} = \textstyle\sum_{k=1}^{n} f_k x_k \in L_2(\mathbb{R}) \otimes X. \qquad (6.1.10.4.a)$$

The same concept results if $[L_2(\mathbb{R}), X]$ is replaced by $[L_p(\mathbb{R}), X]$ with $1 < p < \infty$. One may also use the *periodic* Hilbert transform on $L_p(\mathbb{T})$ or the *discrete* Hilbert transform on $l_p(\mathbb{Z})$; see [PIE$^+$, pp. 269–280].

More information can be found in Burkholder's impressive surveys [1986, 2001]; another reference is [PIE$^+$, Chap. 8].

6.1.10.5 Results about convolution operators from $[L_p(\mathbb{R}),X]$ into $[L_p(\mathbb{R}),Y]$ were independently obtained by Benedek/Calderón/Panzone [1962, pp. 357–358] and Schwartz [1961, p. 788].

6.1.11 Volume ratios and Grothendieck numbers

When dealing with volumes, I consider only the real case. This is, however, no serious restriction, since every n-dimensional *complex* linear space can be viewed as a $2n$-dimensional *real* linear space.

Every n-dimensional real linear space E is algebraically isomorphic to $\mathbb{R}^n$. Hence the Lebesgue measure on $\mathbb{R}^n$ induces a translation invariant measure vol_n on E, which is unique up to a positive factor.

6.1.11.1 Let B be the closed unit ball obtained from some norm on $\mathbb{R}^n$. Then the polar B° with respect to the canonical inner product can be identified with the closed unit ball obtained from the dual norm.

Santaló [1949] proved that $\mathrm{vol}_n(B)\mathrm{vol}_n(B^\circ)$ attains its maximal value for the closed unit ball of l_2^n. This is **Santaló's inequality**:

$$\mathrm{vol}_n(B)\mathrm{vol}_n(B^\circ) \le \mathrm{vol}_n(B_{l_2^n})\mathrm{vol}_n(B_{l_2^n}^\circ) = \frac{\pi^n}{\Gamma\big(\frac{n+2}{2}\big)^2}.$$

A conjecture of Mahler [1939] says that the minimum is attained for the closed unit balls of l_1^n or l_∞^n, which are dual to each other:

$$\frac{4^n}{\Gamma(n+1)} = \mathrm{vol}_n(B_{l_1^n})\mathrm{vol}_n(B_{l_1^n}^\circ) \overset{?}{\le} \mathrm{vol}_n(B)\mathrm{vol}_n(B^\circ).$$

This problem is still unsettled. However, Bourgain/Milman [1987, p. 320] succeeded in proving a slightly weaker conclusion: there exists a universal constant $c>0$ such that

$$c^n\,\mathrm{vol}_n(B_{l_2^n})\mathrm{vol}_n(B_{l_2^n}^\circ) \le \mathrm{vol}_n(B)\mathrm{vol}_n(B^\circ).$$

Combining the preceding results gives us the two-sided **Santaló inequality**:

$$c \le \left(\frac{\mathrm{vol}_n(B)}{\mathrm{vol}_n(B_{l_2^n})}\,\frac{\mathrm{vol}_n(B^\circ)}{\mathrm{vol}_n(B_{l_2^n}^\circ)}\right)^{1/n} \le 1. \tag{6.1.11.1.a}$$

So far, I have not seen any concrete value of c, which can be at most $\frac{2}{\pi}$.

The history of the lower estimate is presented in [PIS$_2$, pp. 124–125]. Though the subject belongs to convex geometry, the proof is based on techniques from the local theory of Banach spaces.

6.1.11.2 By an **ellipsoid** of an n-dimensional Banach space E we mean a subset $\mathscr{E} = U\big(B_{l_2^n}\big)$, where $U : l_2^n \to E$ is an isomorphism. It follows from the proof of John's theorem that there exist two specific samples, which are uniquely determined; see Danzer/Laugwitz/Lenz [1957] and Zaguskin [1958].

$\mathscr{E}_E^{\max}$: the **ellipsoid of maximal volume** contained in the unit ball B_E,

$\mathscr{E}_E^{\min}$: the **ellipsoid of minimal volume** containing the unit ball B_E.

According to Busemann [1950, pp. 159–160], the latter ellipsoid was discovered by Löwner, who, however, did not publish his result; see Netuka [1993•]. Therefore the term **Löwner ellipsoid** is also in use.

6.1.11.3 Based on preliminary work of Kashin [1977] and Szarek [1978], the concept of **volume ratio** was introduced by Szarek/Tomczak-Jaegermann [1980, pp. 367–368].

For any n-dimensional Banach space E, we let

$$\mathrm{vr}(E) := \left(\frac{\mathrm{vol}_n(B_E)}{\mathrm{vol}_n(\mathscr{E}_E^{\max})}\right)^{1/n} \quad \text{and} \quad \mathrm{vr}^*(E) := \left(\frac{\mathrm{vol}_n(\mathscr{E}_E^{\min})}{\mathrm{vol}_n(B_E)}\right)^{1/n}.$$

These ratios are equivalent by duality, since (6.1.11.1.a) implies that

$$c\,\mathrm{vr}^*(E) \le \mathrm{vr}(E^*) \quad \text{and} \quad c\,\mathrm{vr}(E^*) \le \mathrm{vr}^*(E).$$

Because of $\mathscr{E}_E^{\max} \subseteq B_E \subseteq \sqrt{n}\mathscr{E}_E^{\max}$, we have $\mathrm{vr}(E) \le \sqrt{n}$. Analogously, $\mathrm{vr}^*(E) \le \sqrt{n}$.

6.1.11.4 The n^{th} **Grothendieck number** of a Banach space X is defined by

$$\Gamma_n(X) := \sup\Big\{ \big|\det\big(\langle x_h, x_k^*\rangle\big)\big|^{1/n} : x_1,\dots,x_n \in B_X,\ \ x_1^*,\dots,x_n^* \in B_{X^*}\Big\}.$$

This quantity was first considered by Grothendieck [1956a, pp. 346–348] in his theory of abstract Fredholm determinants. He inferred from Hadamard's inequality that $\Gamma_n(X) \le \sqrt{n}$. However, it seems that Grothendieck did not realize the geometric significance of $\Gamma_n(X)$. The situation changed only in the late 1980s, thanks to the work of Pisier [1988, p. 576], Pajor/Tomczak-Jaegermann [1989], and in particular, Geiss [1990a, 1990b]. Among other results, Geiss [1990b, p. 328] proved that

$$c_0\Gamma_n(E) \le \mathrm{vr}(E)\mathrm{vr}^*(E) \le c_1\Gamma_n(E) \tag{6.1.11.4.a}$$

for every n-dimensional Banach space E, where c_0 and c_1 are universal constants. He also studied Grothendieck numbers of operators; see 6.3.13.10.

6.1.11.5 Let $0 < \alpha \le \frac{1}{2}$. Then the list of properties given in 6.1.1.10 can be enlarged by the following condition; see Geiss [1990a, p. 76] and Pietsch [1991a, pp. 66–67].

(G$_\alpha$) There exists a constant $c \ge 1$ such that $\Gamma_n(X) \le cn^\alpha$.

6.1.11.6 We have

$$\mathrm{vr}(l_p^n) \asymp \begin{cases} 1 & \text{if } 1 \le p \le 2, \\ n^{1/2-1/p} & \text{if } 2 \le p \le \infty, \end{cases} \qquad \mathrm{vr}^*(l_p^n) \asymp \begin{cases} n^{1/p-1/2} & \text{if } 1 \le p \le 2, \\ 1 & \text{if } 2 \le p \le \infty, \end{cases}$$

and

$$\Gamma_n(l_p^n) \asymp n^{|1/2-1/p|};$$

see Szarek/Tomczak-Jaegermann [1980, p. 371] and Geiss [1990a, p. 68].

6.1.11.7 The following definitions go back to Milman/Pisier [1986, p. 140] and Pisier [1988, pp. 550–551], respectively.

A Banach space X has **weak type** 2 if there exists a constant $c \ge 1$ such that $\mathrm{vr}^*(E) \le c$ for every finite-dimensional subspace E.

A Banach space X has **weak cotype** 2 if there exists a constant $c \ge 1$ such that $\mathrm{vr}(E) \le c$ for every finite-dimensional subspace E.

A comprehensive presentation of the theory of *weak* type and *weak* cotype is given in [PIS_2, Chaps. 10 and 11].

6.1.11.8 According to Kwapień's theorem, a Banach space is Hilbertian if and only if it has Rademacher type 2 and cotype 2. Thus it is natural to call a Banach space **weakly Hilbertian** if it has weak type 2 and weak cotype 2. Pisier [1988, p. 552] preferred the term **weak Hilbert space**.

In view of (6.1.11.4.a), weak Hilbert spaces are characterized by the property that the sequence of their Grothendieck numbers is bounded; see Pisier [1988, p. 576]. Note that this is just condition $(\mathbf{G}_0)$ as defined in 6.1.11.5. A number of further criteria can be found in [PIS_2, Chap. 12]; see also 6.4.4.4. Johnson/Pisier [1991] characterized weak Hilbert spaces by a *proportional uniform approximation property*.

Johnson observed that weak Hilbert spaces are superreflexive; see [PIS_2, p. 217].

6.1.11.9 In order to make the preceding definitions meaningful, one must show that the weak concepts are indeed "weaker" than the original ones. The required examples are Tsirelson spaces; see [PIS_2, Chap. 13] and [CASA^+, pp. 125–158].

Nevertheless, it seems to me that the "weak" theory has remained a little bit esoteric, since the following problem posed by Pisier [1988, p. 553] is still unsolved:

It would be worthwhile to exhibit more examples (*perhaps more "canonical" ones in some sense*) *of weak Hilbert spaces.*

6.2 *s*-Numbers

In the theory of *s*-numbers, one associates with every operator T various kinds of scalar sequences $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. The main purpose is to classify operators by the behavior of $s_n(T)$ as $n \to \infty$. The operator ideals obtained in this way will be discussed in Subsection 6.3.3.

6.2.1 *s*-Numbers of operators on Hilbert space

6.2.1.1 Let

$$Tx = \sum_{n=1}^{\infty} \tau_n (x|u_n) v_n \quad \text{for } x \in H$$

be the **Schmidt representation** 4.10.1.4 of a compact operator $T \in \mathfrak{K}(H)$ with $\operatorname{rank}(T) = \infty$. Then $Tu_n = \tau_n v_n$ and $T^\star v_n = \tau_n u_n$ imply

$$T^\star T u_n = \tau_n^2 u_n \quad \text{and} \quad TT^\star v_n = \tau_n^2 v_n;$$

see Schmidt [1907a, p. 461]. Obviously, we may arrange that $\tau_1 \geq \tau_2 \geq \cdots > 0$. Then the **minimax principle** yields

$$\tau_n = \min \Big\{ \max \big\{ \|Tx\| : x \in N^\perp, \ \|x\| = 1 \big\} \Big\}, \tag{6.2.1.1.a}$$

where the minimum is taken over all subspaces N of H such that $\dim(N) < n$. This result goes back to Fischer [1905, p. 249] and Courant [1920, p. 19].

Another characterization is due to Allakhverdiev [1957, стр. 33–34] and, independently, to Fiedler/Pták [1962, pp. 3805–3806]:

$$\tau_n = \min \|T - A\|. \tag{6.2.1.1.b}$$

Here the minimum is taken over all finite rank operator $A \in \mathfrak{F}(H)$ such that $\operatorname{rank}(A) < n$.

6.2.1.2 If “min” and “max” are replaced by “inf” and “sup,” respectively, the right-hand sides of (6.2.1.1.a) and (6.2.1.1.b) make sense and coincide for arbitrary operators. Thus

$$s_n(T) := \inf_{\operatorname{rank}(A) < n} \|T - A\| = \inf_{\dim(N) < n} \Big\{ \sup \big\{ \|Tx\| : x \in N^\perp, \ \|x\| = 1 \big\} \Big\}$$

is defined for *all* $T \in \mathfrak{L}(H)$.

Smithies [1937, p. 257] referred to $s_n(T)$ as the n^{th} *singular value* of T. Nowadays, one uses the name ***s*-number**, which was probably coined in [GOH$_3^+$, p. 26].

For a while, the n^{th} *s*-number of a compact operator was defined as the n^{th} eigenvalue of $|T| = \sqrt{T^\star T}$. The underlying enumeration of eigenvalues will be described in 6.4.1.1.

6.2.1.3 Let $t = (\tau_n)$ be a scalar sequences such that $\tau_1 \geq \tau_2 \geq \cdots \geq 0$. Then the n^{th} *s*-number of the **diagonal operator** $D_t : (\xi_n) \mapsto (\tau_n \xi_n)$ on the Hilbert space l_2 is given by $s_n(D_t : l_2 \to l_2) = \tau_n$.

6.2.2 Axiomatics of s-numbers

6.2.2.1 The decisive step in extending the concept of s-numbers to operators on Banach spaces was made by Pietsch [1974a] when he looked for an axiomatic characterization in the Hilbert space setting; see also [PIE$_2$, p. 41].

Let $\boldsymbol{s} : T \mapsto \big(s_n(T)\big)$ be a rule that assigns to every operator a scalar sequence such that the following conditions are satisfied:

(SN$_1$) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0.$

(SN$_2$) $s_n(S+T) \leq s_n(S) + \|T\|$ for $X \xrightarrow{S,T} Y$ and $n = 1,2,\ldots$.

(SN$_3$) $s_n(BTA) \leq \|B\| s_n(T) \|A\|$ for $X_0 \xrightarrow{A} X \xrightarrow{T} Y \xrightarrow{B} Y_0$ and $n = 1,2,\ldots$.

(SN$_4$) $s_n(Id : l_2^n \to l_2^n) = 1$ for $n = 1,2,\ldots$.

(SN$_5$) $s_n(T) = 0$ whenever $\operatorname{rank}(T) < n$.

We refer to $s_n(T)$, or $s_n(T : X \to Y)$, as the n^{th} **s-number** of the operator T.

It is an easy exercise to show that the s-numbers of operators $T \in \mathfrak{L}(H)$ are uniquely determined by these properties. However, this is not so for operators on Banach spaces. Here we have a large variety of s-numbers that make our life more interesting.

6.2.2.2 In view of **(SN$_3$)** and **(SN$_4$)**, the converse of **(SN$_5$)** holds as well:

$$s_n(T) = 0 \quad \text{if and only if} \quad \operatorname{rank}(T) < n.$$

6.2.2.3 Conditions **(SN$_2$)** and **(SN$_3$)** can be strengthened as follows:

(SN$_2^\bullet$) $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$ for $X \xrightarrow{S,T} Y$ and $m,n = 1,2,\ldots$.

(SN$_3^\bullet$) $s_{m+n-1}(ST) \leq s_m(S) s_n(T)$ for $X \xrightarrow{T} Y \xrightarrow{S} Z$ and $m,n = 1,2,\ldots$.

If this is so, then the s-numbers are called **additive** and **multiplicative**, respectively.

According to (Ky) Fan [1951, p. 764], the s-numbers of operators on Hilbert spaces enjoy both properties.

6.2.2.4 It follows from **(SN$_2$)** that $|s_n(T) - s_n(T_0)| \leq \|T - T_0\|$. Hence the s-numbers are continuous in the norm topology of $\mathfrak{L}(X,Y)$.

6.2.3 Examples of s-numbers

6.2.3.1 For $T \in \mathfrak{L}(X,Y)$ and $n = 1,2,\ldots$, the n^{th} **approximation number** is defined by

$$a_n(T) := \inf\Big\{ \|T - A\| : A \in \mathfrak{F}(X,Y),\ \operatorname{rank}(A) < n \Big\}.$$

In the setting of Banach spaces, this concept was invented by Pietsch [1963c, p. 429]. The a_n's are additive and multiplicative s-numbers. It even turns out that the approximation numbers are the *largest* s-numbers.

6.2.3.2 Applying the principle of local reflexivity, Hutton [1974] proved that

$$a_n(T^*) = a_n(T) \quad \text{for all } T \in \mathfrak{K}.$$

She also observed that the above equation may fail in the non-compact case:

$$a_2(Id : l_1 \to c_0) = 1 \quad \text{and} \quad a_2(Id : l_1 \to l_\infty) = 1/2.$$

On the other hand, Edmunds/Tylli [1986, p. 236] showed that

$$a_n(T^*) \le a_n(T) \le 5a_n(T^*) \quad \text{for all } T \in \mathfrak{L}.$$

6.2.3.3 In order to ensure that the right-hand expression in (6.2.1.1.a) also makes sense in the setting of Banach spaces, we replace $N^\perp$ by a finite-codimensional closed subspace M. This leads to the definition of the n^{th} **Gelfand number**:

$$c_n(T) := \inf\Big\{ \|TJ_M^X\| : \operatorname{cod}(M) < n \Big\}$$

for $T \in \mathfrak{L}(X,Y)$ and $n = 1, 2, \ldots$. Recall from 4.9.1.6 that J_M^X denotes the natural injection from M into X. The c_n's are additive and multiplicative s-numbers.

Gelfand widths of bounded subsets were introduced by Tikhomirov [1965]. The naming is justified by the fact that their definition goes back to a proposal of Gelfand.

6.2.3.4 Ironically, only the second of the following quotations became relevant for the theory of s-numbers.

Keller [1931, p. 751]:

Sei S^0 ein beliebiger Punkt der [beschränkten und abgeschlossenen] *Menge $\mathfrak{M}$. Angenommen, wir hätten die Punkte $S^0, S^1, S^2, \ldots, S^{n-1}$ schon definiert. Der von diesen Punkten aufgespannte $(n-1)$-dimensionale Raum möge $\mathfrak{R}^{n-1}$ heißen. Es sei nun S^n ein Punkt der Menge $\mathfrak{M}$, dessen Abstand von $\mathfrak{R}^{n-1}$ einen denkbar größten Wert d_n hat* [existence?].

...

Eine beschänkte Menge $\mathfrak{M}$ ist dann und nur dann kompakt, wenn die Folge der d_n gegen 0 *konvergiert.*

Kolmogoroff [1936, pp. 107, 110]:

Die Menge aller Funktionen $\varphi = c_1\varphi_1 + \cdots + c_n\varphi_n$ mit festen Funktionen $\varphi_1, \ldots, \varphi_n$ bildet einen n-dimensionalen linearen Unterraum Φ_n des Raumes R aller in Betracht kommenden Funktionen. $E_n(f)$ ist die Entfernung des Punktes f von der Menge Φ_n. Für eine Klasse F von Funktionen f bezeichnen wir weiter mit $E_n(F)$ die obere Grenze von $E_n(f)$ für alle f aus F. Die Grösse $E_n(F)$ ist also ein natürliches Mass der Abweichung der Menge F von dem linearen Raum Φ_n.

Wir stellen jetzt eine neue Aufgabe: bei gegebenen F und n durch die Wahl von Funktionen $\varphi_1, \ldots, \varphi_n$ das Minimum von $E_n(F)$ zu erreichen.

Die untere Grenze $D_n(F)$ der Grössen $E_n(F)$ kann man, folglich, als die n-te **Breite** *der Menge F bezeichnen.*

6.2.3.5 The preceding problem has led to the concept of **Kolmogorov numbers**:

$$d_n(T) := \inf\Big\{ \|Q_N^Y T\| : \dim(N) < n \Big\}.$$

Here Q_N^Y denotes the natural surjection from Y onto Y/N. I stress that $d_n(T)$ is just the $(n-1)^{\text{th}}$ Kolmogorov width of $T(B_X)$. The d_n's are additive and multiplicative s-numbers; see Novoselskiĭ [1964].

6.2.3.6 Note that

$$c_n(T : H \to Y) = a_n(T : H \to Y) \quad \text{and} \quad d_n(T : X \to H) = a_n(T : X \to H).$$

6.2.3.7 The following properties of s-numbers were defined by Pietsch [1974a, pp. 206–208]; see also [PIE$_2$, pp. 42–44].

s-Numbers are called **injective** if

$$s_n(JT) = s_n(T) \quad \text{for} \quad X \xrightarrow{T} Y \xrightarrow{J} Y_0,$$

where J is any metric injection. Injectivity means that $s_n(T)$ does not depend on the size of the target space Y. The Gelfand numbers are the *largest* injective s-numbers.

s-Numbers are called **surjective** if

$$s_n(TQ) = s_n(T) \quad \text{for} \quad X_0 \xrightarrow{Q} X \xrightarrow{T} Y,$$

where Q is any metric surjection. Surjectivity means that $s_n(T)$ does not depend on the size of the source space X. The Kolmogorov numbers are the *largest* surjective s-numbers.

6.2.3.8 Let $T \in \mathfrak{L}(X,Y)$. Then

$$\begin{aligned} c_n(T) &= a_n(T) \quad \text{if } Y \text{ has the metric extension property,} \\ d_n(T) &= a_n(T) \quad \text{if } X \text{ has the metric lifting property.} \end{aligned}$$

Given X and Y, we may choose a metric surjection Q from some $l_1(\mathbb{I})$ onto X and a metric injection J from Y into some $l_\infty(\mathbb{I})$. With the help of these mappings, the Gelfand and Kolmogorov numbers can be related to the approximation numbers:

$$c_n(T) = a_n(JT) \quad \text{and} \quad d_n(T) = a_n(TQ).$$

6.2.3.9 Gelfand and Kolmogorov numbers are dual to each other:

$$c_n(T^*) = d_n(T) \quad \text{for all } T \in \mathfrak{K} \qquad \text{and} \quad d_n(T^*) = c_n(T) \quad \text{for all } T \in \mathfrak{L}.$$

In view of $d_2(Id : l_1 \to c_0) = 1$ and $c_2(Id : l_1 \to l_\infty) = 1/2$, the left-hand formula may fail in the non-compact case.

6.2.3.10 An operator T is compact if and only if $c_n(T) \searrow 0$ or $d_n(T) \searrow 0$. The speed of this convergence can be used as a measure of compactness.

6.2.3.11 Instead of **(SN$_4$)**, the quantities a_n, c_n, and d_n satisfy a stronger condition that was originally used in the definition of s-numbers:

(SN$_4^\bullet$) $s_n(I_X) = 1$ whenever $\dim(X) \geq n$.

However, the s-numbers introduced below show that **(SN$_4^\bullet$)** is too restrictive; see Lubitz [1982, p. 19].

6.2.3.12 If $l_2 \xrightarrow{A} X \xrightarrow{T} Y \xrightarrow{B} l_2$, then

$$a_n(BTA) = s_n(BTA) \leq \|B\| s_n(T) \|A\|$$

for all s-numbers. Hence $h_n(T) \leq s_n(T)$, where

$$h_n(T) := \sup\Big\{a_n(BTA) : \quad \|A : l_2 \to X\| \leq 1, \quad \|B : Y \to l_2\| \leq 1\Big\}$$

is the n^{th} **Hilbert number**. This concept was invented by Bauhardt [1977], who also showed that the h_n's are additive s-numbers and $h_n(T^*) = h_n(T)$.

6.2.3.13 Following Pietsch [1980b], one may split the preceding definition into two parts. This idea yields the **Weyl numbers**

$$x_n(T) := \sup\Big\{a_n(TA) : \quad \|A : l_2 \to X\| \leq 1\Big\}$$

and their dual counterparts, the **Chang numbers**,

$$y_n(T) := \sup\Big\{a_n(BT) : \quad \|B : Y \to l_2\| \leq 1\Big\}.$$

Indeed, $x_n(T^*) = y_n(T)$ and $y_n(T^*) = x_n(T)$ for all operators T.

The x_n's and y_n's are additive and multiplicative s-numbers. Moreover, x_n is injective, while y_n is surjective.

The naming of *Weyl* and *Chang* numbers was suggested by Pietsch in [1980b, p. 152] and [PIE$_4$, p. 95], respectively. Though Chang and Weyl worked only in Hilbert spaces, they paved the way for all of the further progress.

6.2.3.14 Finally, I compare the s-numbers defined above. All inequalities hold for arbitrary operators and $n = 1, 2, \ldots$. Looking at diagonal operators shows that in the setting of Banach spaces, we have indeed six different objects.

Obviously,

$$h_n(T) \leq x_n(T) \leq c_n(t) \leq a_n(T) \quad \text{and} \quad h_n(T) \leq y_n(T) \leq d_n(t) \leq a_n(T).$$

Estimates in the converse direction, however, are more interesting:

$$a_n(T) \le 2\sqrt{n}c_n(T), \qquad a_n(T) \le 2\sqrt{n}d_n(T),$$
$$c_{2n-1}(T) \le 2\mathrm{e}\sqrt{n}\Big(\prod_{k=1}^n x_k(T)\Big)^{1/n}, \qquad d_{2n-1}(T) \le 2\mathrm{e}\sqrt{n}\Big(\prod_{k=1}^n y_k(T)\Big)^{1/n},$$
$$x_{2n-1}(T) \le \sqrt{n}\Big(\prod_{k=1}^n h_k(T)\Big)^{1/n}, \qquad y_{2n-1}(T) \le \sqrt{n}\Big(\prod_{k=1}^n h_k(T)\Big)^{1/n};$$

see Pietsch [1974a, p. 218], [1980b, p. 159], and [PIE$_4$, pp. 115–117].

The factor $\sqrt{n}$ is always necessary. Concerning the first line, an example was constructed by K.-D. Kürsten [1977]. Nowadays, one knows that

$$c_n(Id : l_2^{2n} \to l_\infty^{2n}) \asymp 1, \quad \text{while} \quad d_n(Id : l_2^{2n} \to l_\infty^{2n}) \asymp \tfrac{1}{\sqrt{n}};$$

see 6.2.5.1 and 6.2.5.2. Further examples are obtained by looking at diagonal limit orders as presented in 6.2.5.3.

Combining the previous results yields an estimate of the approximation numbers by the Hilbert numbers:

$$a_{4n-3}(T) \le cn^{3/2}\Big(\prod_{k=1}^n h_k(T)\Big)^{1/n}.$$

However, it seems likely that $n^{3/2}$ can be replaced by n.

6.2.4 Entropy numbers

6.2.4.1 The following concept was introduced by Pontryagin/Schnirelman [1932, p. 156] for characterizing the dimension of compact metric spaces:

Soit donné un espace métrique compact F. Nous le recouvrons par un système d'un nombre fini d'ensembles fermés de diamètres ne surpassant pas ε. Ceci est possible d'après le théorème de Heine–Borel. Nous pouvons définir le nombre minimum $N_F(\varepsilon)$ de tels ensembles nécessaire pour recouvrir F.

Pontryagin/Schnirelman referred to $N_F(\varepsilon)$ as the *fonction de volume de l'espace F relative à la métrique donnée*. Kolmogorov [1956] observed that it is more convenient to work with the **ε-entropy**, $H_F(\varepsilon) := \log N_F(\varepsilon)$. In a next step, Brudnyĭ/Timan [1959] passed to the inverse function; see also Mityagin/Pełczyński [1966, p. 369] and Triebel [1970, p. 102].

6.2.4.2 The concept above has led to the following definition:
For $T \in \mathfrak{L}(X,Y)$ and $n = 1,2,\dots$, the n^{th} (outer) **entropy number** $e_n(T)$ is the infimum of all $\varepsilon > 0$ such that $T(B_X)$ can be covered by 2^{n-1} balls $y + \varepsilon B_Y$.

Of course, 2^{n-1} may be replaced by any natural number. However, Carl/Pietsch [1976, pp. 21–23] observed that the *dyadic* definition guarantees that the e_n's are "almost" s-numbers. Indeed, properties **(SN$_1$)**, ... , **(SN$_4$)** are fulfilled. In addition, entropy numbers are additive and multiplicative. Remarkably, **(SN$_5$)** does not hold: $e_n(Id : \mathbb{R} \to \mathbb{R}) = \frac{1}{2^{n-1}}$.

6.2.4.3 The (outer) entropy numbers fail to be injective. However, this defect can be circumvented by passing to the (inner) **entropy numbers** $e_n^\circ(T)$ that are defined as the suprema of all $\rho \geq 0$ such that there exist more than 2^{n-1} elements $x_1, \dots, x_N \in B_X$ with $\|Tx_h - Tx_k\| > 2\rho$ whenever $h \neq k$. The difference between both quantities is inessential. Indeed, we have $e_n^\circ(T) \leq e_n(T) \leq 2e_n^\circ(T)$. According to [CARL$^+$, p. 125], the factor 2 cannot be removed.

6.2.4.4 An operator T is compact if and only if $e_n(T) \searrow 0$. As in 6.2.3.10, the speed of this convergence can be used as a measure of compactness.

6.2.4.5 The following inequality goes back to Carl [1981a, p. 295]; its present form is taken from [CARL$^+$, p. 100]:

$$e_n(T) \leq c_p \left(\frac{1}{n} \sum_{k=1}^{n} a_k(T)^p \right)^{1/p} \quad \text{for } T \in \mathfrak{L} \text{ and } p > 0.$$

The same result holds when a_k is replaced by c_k or d_k.

In the converse direction, Carl [1982a, p. 766] proved that

$$x_n(T) \leq 2\sqrt{n}e_n(T) \quad \text{and} \quad y_n(T) \leq 2\sqrt{n}e_n(T) \quad \text{for } T \in \mathfrak{L},$$

and in [PIE$_3$, p. 176] we find the estimates

$$c_n(T) \leq ne_n(T) \quad \text{and} \quad d_n(T) \leq ne_n(T) \quad \text{for } T \in \mathfrak{L}.$$

6.2.4.6 In terms of entropy numbers, Schauder's theorem 2.6.4.6 says that

$$e_n(T^*) \to 0 \quad \text{if and only if} \quad e_n(T) \to 0.$$

We may ask whether a quantitative version of this equivalence holds; see [PIE$_2$, p. 38]. The problem

Compare the asymptotic behavior of $e_n(T^*)$ *and* $e_n(T)$ *as* $n \to \infty$

has become one of the most interesting challenges of operator theory. A full solution is still missing.

By means of the polar decomposition, it follows immediately that $e_n(T^*) = e_n(T)$ for all operators between Hilbert spaces. Since this simple formula may fail for operators in Banach spaces, one had to look for weaker *"duality conjectures."*

(EC$_1$) There exist constants $c_p \geq 1$ depending on $p > 0$ such that

$$\left(\sum_{k=1}^{n} e_k(T^*)^p \right)^{1/p} \leq c_p \left(\sum_{k=1}^{n} e_k(T)^p \right)^{1/p} \quad \text{for } T \in \mathfrak{L}(X,Y) \text{ and } n = 1, 2, \dots .$$

(EC$_2$) There exist a natural number k and a constant $c \geq 1$ such that

$$e_{kn}(T^*) \leq ce_n(T) \quad \text{for } T \in \mathfrak{L}(X,Y) \text{ and } n = 1, 2, \dots .$$

Obviously, **(EC$_2$)** implies **(EC$_1$)**.

Next, I present the most important partial results.

(EC$_1$) holds for all operators $T \in \mathfrak{L}(X,Y)$ if at least one of the spaces X and Y is

Hilbertian	B-convex
Tomczak-Jaegermann [1987]	Bourgain/Pajor/Szarek/Tomczak-Jaegermann [1987].

(EC$_2$) holds for all operators $T \in \mathfrak{L}(X,Y)$ if at least one of the spaces X and Y is

Hilbertian	B-convex
Artstein/Milman/Szarek [2004]	Artstein/Milman/Szarek/Tomczak-Jaegermann [2004].

It seems to be unknown whether **(EC$_2$)** could be true for $k=1$.

6.2.4.7 The term "entropy" is borrowed from information theory; see [KOL$^{\bowtie}$, Vol. III, pp. 231–239] and Tikhomirov [1963$^\bullet$, стр. 59–63].

Suppose that a certain point x_0 in a compact metric space X is known to somebody who will answer our questions just by "yes" or "no." How many questions are needed to determine x_0 up to a prescribed deviation $\varepsilon > 0$?

The following strategy may be applied:
Cover X by 2^n balls of radius $\varepsilon > 0$. Divide these balls into two parts A_1 and B_1 such that each has 2^{n-1} members.

1st question: Does x_0 belong to any ball from part A_1?

If the answer is *yes*, then divide A_1 into two parts A_2 and B_2 of equal size.	If the answer is *no*, then divide B_1 into two parts A_2 and B_2 of equal size.

2nd question: Does x_0 belong to any ball from part A_2?

In the n^{th} step, we arrive at an ε-ball that contains x_0.

6.2.5 *s*-Numbers of diagonal operators

Having found the underlying definitions, it was easy to develop the abstract theory of *s*-numbers. However, as not unusual in mathematics, treating concrete cases turns out to be quite difficult. This phenomenon also occurs when we try to determine or estimate the value of specific approximation numbers, Gelfand numbers, etc. In general, this is a hopeless task. Positive exceptions are

- identity operators $Id : l_p^n \to l_q^n$,
- diagonal operators $D_t : l_p \to l_q$ with $t = (\tau_n)$ and $\tau_1 \geq \tau_2 \geq \cdots \geq 0$,
- embedding operators $Id : W_p^\sigma \to L_q$.

In what follows, some results along these lines will be presented. Further information can be found in Subsections 6.3.10 and 6.7.8.

As usual, given positive scalar sequences (α_n) and (β_n), we write $\alpha_n \preceq \beta_n$ or $\alpha_n = O(\beta_n)$ if there exists a constant $c > 0$ such that $\alpha_n \leq c\beta_n$ for $n = 1, 2, \ldots$. The symbol $\alpha_n \asymp \beta_n$ means that $\alpha_n \preceq \beta_n$ and $\beta_n \preceq \alpha_n$.

6.2.5.1 Let $n=1,\dots,m$. Stechkin [1954] showed that

$$a_n(Id : l_1^m \to l_2^m) = d_n(Id : l_1^m \to l_2^m) = \sqrt{\tfrac{m-n+1}{m}},$$

and Pietsch [1974a, p. 215] proved the formula

$$a_n(Id : l_p^m \to l_q^m) = c_n(Id : l_p^m \to l_q^m) = d_n(Id : l_p^m \to l_q^m) = (m-n+1)^{1/q-1/p}$$

whenever $1 \le q \le p \le \infty$. More generally, we have

$$a_n(D_t : l_p \to l_q) = c_n(D_t : l_p \to l_q) = d_n(D_t : l_p \to l_q) = \Big(\sum_{k=n}^{\infty} \tau_k^r\Big)^{1/r},$$

where $1/r := 1/q - 1/p$.

6.2.5.2 The case $p=2$ and $q=\infty$ turned out to be the key problem for determining the asymptotic behavior of $d_n(l_p^m, l_q^m)$ for $1 \le p \le q \le \infty$.

Using Gaussian sums, Ismagilov [1974, стр. 167] showed that

$$d_n(l_2^m, l_\infty^m) \preceq \frac{m^{1/4}}{n^{1/2}}.$$

The real breakthrough was achieved by Kashin. In a first step [1974, p. 304], he obtained an upper estimate of $d_n(l_1^m, l_\infty^m)$. His main result is contained in [1977, p. 318]:

$$d_n(l_2^m, l_\infty^m) \preceq \frac{\big(1+\log\frac{m}{n}\big)^{3/2}}{n^{1/2}}.$$

Next, Gluskin [1981a, p. 166] established a lower estimate:

$$d_n(l_2^m, l_\infty^m) \succeq \frac{1}{n^{1/2}}.$$

The concluding result was obtained by Garnaev/Gluskin [1984, p. 200], who determined the exact asymptotic behavior:

$$d_n(l_2^m, l_\infty^m) \asymp \frac{\big(1+\log\frac{m}{n}\big)^{1/2}}{n^{1/2}}.$$

I add an interesting generalization due to Carl/Pajor [1988, p. 487], namely

$$d_n(T) \le c\,\frac{\big(1+\log\frac{m}{n}\big)^{1/2}}{n^{1/2}}\|T\| \quad \text{for all } T \in \mathfrak{L}(H, l_\infty^m).$$

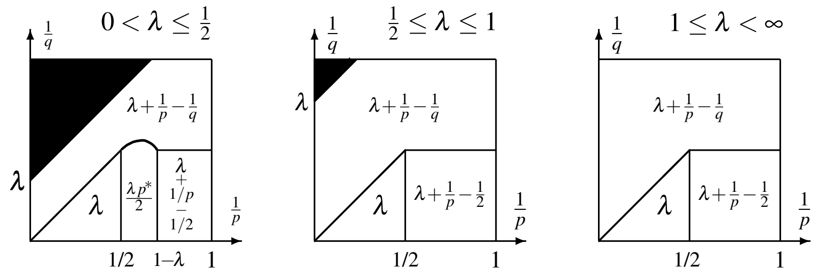
6.2.5.3 The **diagonal operator** $D_\lambda : (\xi_n) \mapsto (\frac{1}{n^\lambda}\xi_n)$ acts from l_p into l_q whenever $\lambda > \lambda_0 := (1/q - 1/p)_+$ and $1 \le p, q \le \infty$.

Given s-numbers $s : T \mapsto \big(s_n(T)\big)$, the **diagonal limit order** is defined by

$$\rho_{\text{diag}}(\lambda, p, q|s) := \sup\Big\{\rho \ge 0 : s_n(D_\lambda : l_p \to l_q) \preceq \tfrac{1}{n^\rho}\Big\}.$$

Using the variables $\frac{1}{p}$ and $\frac{1}{q}$, we present this limit order with the help of diagrams. The value $\rho_{\text{diag}}(\lambda, p, q|s)$ is undefined in the black areas $\lambda \le \lambda_0$.

6.2.5.4 Thanks to the work of Kashin, Gluskin, and Garnaev/Gluskin described in 6.2.5.2, for the **Gelfand numbers** we have



In the left-hand diagram, the curved line connecting $(\frac{1}{2}, \frac{1}{2})$ and $(1-\lambda, \frac{1}{2})$ is given by the equation

$$\frac{1}{q} = \lambda + \frac{1}{p} - \frac{\lambda}{2-2/p}.$$

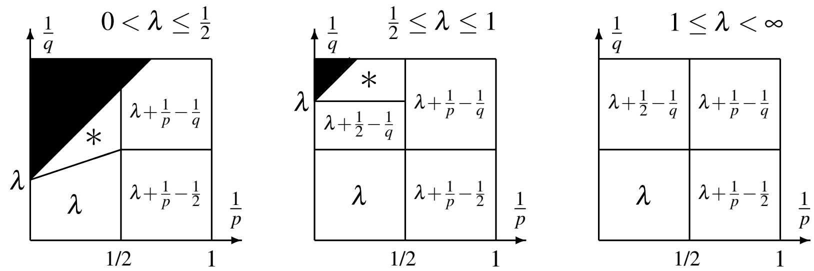
Dual diagrams hold for the Kolmogorov numbers:

$$d_n(D_\lambda : l_p \to l_q) = c_n(D_\lambda : l_{q^*} \to l_{p^*}).$$

6.2.5.5 Based on preliminary results of Carl/Pietsch [1978] and Höllig [1979], the limit order of the **approximation numbers** was completely determined by Gluskin [1983, стр. 181]. He showed that

$$a_n(D_\lambda : l_p \to l_q) \asymp \max\{c_n(D_\lambda : l_p \to l_q), d_n(D_\lambda : l_p \to l_q)\}.$$

6.2.5.6 The following diagrams for the **Weyl numbers** are due to Pietsch [1980b] and Lubitz [1982, p. 30]:



Here $*$ stands for $\frac{p}{2}(\lambda + \frac{1}{p} - \frac{1}{q})$.

6.2.5.7 Further diagrams can be found in a survey of R. Linde [1985].

6.2.5.8 In contrast to the complicated diagrams above, the situation becomes quite simple for the entropy numbers. Indeed, Carl [1981b, pp. 141–144] proved that

$$e_n(D_\lambda : l_p \to l_q) \asymp n^{-(\lambda+1/p-1/q)}$$

for any choice of $1 \le p, q \le \infty$ and $\lambda > \lambda_0$. Curiously enough, the corresponding result for embedding operators $Id : W_p^\sigma \to L_q$ was obtained much earlier; see 6.7.8.15.

6.2.6 s-Numbers versus widths

Finally, I discuss the relation between

s-numbers and widths (поперечники).

A short glance at the references (литература) in [TIK] or in Ismagilov's survey [1977] shows that Soviet mathematicians had to work in considerable isolation during the time of the Cold War. As a sad consequence, both theories were developed more or less separately.

Following Kolmogorov [1936], the Russian school was mainly interested in measuring the degree of compactness of subsets. However, the operator theoretic point of view turned out to be much more flexible. In particular, I have in mind the striking applications to eigenvalue distributions.

The "danger" of mixing the two aspects is demonstrated by the process of creating the concept of Gelfand numbers for operators $T : X \to Y$. In a first attempt, Triebel [1970, p. 92] used the Gelfand widths of $T(B_X)$:

$$\tilde{c}_n(T) := \inf_{y_1^*,\dots,y_{n-1}^* \in Y^*} \Big\{ \sup \big\{ \|Tx\| : \|x\| \le 1,\ \langle Tx, y_k^* \rangle = 0 \text{ for } k < n \big\} \Big\}.$$

This approach can be modified as follows:

$$c_n(T) := \inf_{x_1^*,\dots,x_{n-1}^* \in X^*} \Big\{ \sup \big\{ \|Tx\| : \|x\| \le 1,\ \langle x, x_k^* \rangle = 0 \text{ for } k < n \big\} \Big\}.$$

Of course, $c_n(T) \le \tilde{c}_n(T)$. But $c_n(T)$ may be considerably smaller than $\tilde{c}_n(T)$. Take for example the finite rank operators $l_1 \xrightarrow{Q_n} l_2^n \xrightarrow{Id} l_\infty^n$, where Q_n is any metric surjection. Then the quotients $\tilde{c}_n/c_n$ behave like $\sqrt{n}$ as $n \to \infty$. By the way, the difference between $c_n(T)$ and $\tilde{c}_n(T)$ was stressed for the first time in [PINK, p. 30]: *minor though it may be*. Therefore one had to make a choice, and $c_n(T)$ became the clear winner. The most convincing reason is that the "injective" and "surjective" quantity $\tilde{c}_n(T)$ does not satisfy all properties required in the definition of s-numbers; see 6.2.3.7.

6.3 Operator ideals

6.3.1 Ideals on Hilbert space

This subsection contains only a small portion of the *theory of norm ideals on Hilbert space*, which has been the main source of inspiration for analogous considerations in the Banach space setting. Further information can be found in the monographs [SCHA$_1$], [SCHA$_2$], [GOH$_3^+$], [RIN], and [SIM].

Throughout, $H = l_2$ denotes the infinite-dimensional separable Hilbert space.

6.3.1.1 The theory of operator ideals was born at the beginning of the 1940s with the following observation of Calkin [1941, p. 839]:

The ring $\mathscr{B}$ of bounded everywhere defined operators in Hilbert space contains non-trivial two-sided ideals. This fact, which has escaped all but oblique notice in the development of the theory of operators, is of course fundamental from the point of view of algebra and ... .

6.3.1.2 The theory of operator ideals on Hilbert spaces is based on the famous **Calkin theorem**, [1941, p. 841]:

Let $\mathscr{J}$ be an arbitrary ideal in $\mathscr{B}$. Then either $\mathscr{J} = \mathscr{B}$ or $\mathscr{J} \subseteq \mathscr{T}$,
where $\mathscr{T}$ denotes the ideal of compact (totally continuous) operators.

Furthermore, Calkin [1941, p. 842] discovered that the answer to the question whether an operator T belongs to a given ideal depends only on its *characteristic sequence*, which, in modern notation, is just $\big(s_n(T)\big)$.

6.3.1.3 The non-zero terms of any null sequence $t = (\tau_k)$ can be arranged in such a way that

$$|\tau_{\pi(1)}| \geq |\tau_{\pi(2)}| \geq \cdots .$$

If there are only finitely many $\tau_k \neq 0$, then the sequence is completed by adding zeros. One refers to $(\tau_{\pi(n)})$ as the **non-increasing rearrangement** of (τ_k); see 6.6.4.1. The n^{th} term $\tau_n^* := |\tau_{\pi(n)}|$ is given by

$$s_n(t) = \inf_{|\mathbb{F}|<n} \sup_{k \notin \mathbb{F}} |\tau_k|.$$

I stress the fact that the right-hand expression, which makes sense even for bounded sequences, is an analogue of the singular number $s_n(T)$ with $T \in \mathfrak{L}(H)$.

6.3.1.4 An ideal $\mathfrak{a}$ in the commutative ring $\mathfrak{l}_\infty$ is called **symmetric** or **rearrangement-invariant** if

$$x \in \mathfrak{l}_\infty,\ \ t \in \mathfrak{a} \ \text{ and } \ s_n(x) \leq s_n(t) \ \text{ for } \ n = 1, 2, \ldots \quad \text{imply } \ x \in \mathfrak{a}.$$

There is a one-to-one correspondence between operator ideals $\mathfrak{A}(H)$ in $\mathfrak{L}(H)$ and symmetric sequence ideals $\mathfrak{a}$ in $\mathfrak{l}_\infty$:

$\mathfrak{A}(H)$ consists of all operators $T \in \mathfrak{L}(H)$ such that $\big(s_n(T)\big) \in \mathfrak{a}$. Conversely, $\mathfrak{a}$ is the collection of all sequences $t = (\tau_k) \in \mathfrak{l}_\infty$ for which the associated diagonal operator $D_t : (\xi_k) \mapsto (\tau_k \xi_k)$ belongs to $\mathfrak{A}(l_2)$.

In the form above, this result was stated by Garling [1967, p. 119]. However, its essence goes back to Calkin [1941, p. 848], who considered only those sequences $t = (\tau_k) \in \mathfrak{a}$ for which $\tau_1 \geq \tau_2 \geq \cdots \geq 0$.

6.3.1.5 We have just seen that ideal theory in the non-commutative ring $\mathfrak{L}(H)$ can be reduced to ideal theory in the commutative ring $\mathfrak{l}_\infty$. Although this looks like a simplification, the situation remains hopeless: any attempt to classify symmetric sequence ideals is doomed to failure. Therefore we restrict our attention to the most important special cases.

Given $0<p<\infty$ and $0<q<\infty$, the **Lorentz ideal** $\mathfrak{l}_{p,q}$ consists of all sequences $t=(\tau_k)$ for which

$$\|t|\mathfrak{l}_{p,q}\| := \left(\sum_{n=1}^{\infty} n^{q/p-1} s_n(t)^q\right)^{1/q}$$

is finite. In the limiting case $q=\infty$, we let

$$\|t|\mathfrak{l}_{p,\infty}\| := \sup_n n^{1/p} s_n(t).$$

Compare with (6.6.4.1.a) and (6.6.4.1.b). Obviously, $\mathfrak{l}_p=\mathfrak{l}_{p,p}$.

6.3.1.6 The operator ideals $\mathfrak{S}_p(H)$ associated with the classical sequence ideals $\mathfrak{l}_p$ were studied by Schatten/von Neumann [1948, p. 580]; see 4.10.1.5. Almost 20 years later, Triebel [1967, pp. 277–279] invented the Lorentz ideals $\mathfrak{S}_{p,q}(H)$.

In a fundamental paper, von Neumann [1937] deals with the finite-dimensional setting. Via trace duality, he verified that

$$\|T|\mathfrak{S}_p\| := \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{1/p}$$

satisfies the triangle inequality for $1\le p<\infty$. The extension to H is straightforward. The case $0<p<1$ turned out to be more complicated: the p-triangle inequality was obtained by Rotfeld [1967, стр. 96] and McCarthy [1967, p. 263]. An elegant approach is due to Oloff [1972, p. 107].

In summary, we can say that $\mathfrak{S}_{p,q}(H)$ is a quasi-Banach ideal, sometimes even a Banach ideal, over H as defined according to 6.3.2.1 and 6.3.2.4.

6.3.1.7 Based on Weyl's technique [1949], Horn [1950, p. 375] proved that

$$\prod_{k=1}^{n} s_k(ST) \le \prod_{k=1}^{n} s_k(S) \prod_{k=1}^{n} s_k(T) \quad \text{for } S,T\in\mathfrak{L}(H). \tag{6.3.1.7.a}$$

The same paper of Weyl contains an analytical lemma:

Let $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_n\ge 0$ and $\beta_1\ge\beta_2\ge\cdots\ge\beta_n\ge 0$ such that

$$\alpha_1\le\beta_1,\ \ \alpha_1\alpha_2\le\beta_1\beta_2,\ \ \dots,\ \ \alpha_1\cdots\alpha_n\le\beta_1\cdots\beta_n.$$

Then

$$\alpha_1^p+\cdots+\alpha_n^p\le\beta_1^p+\cdots+\beta_n^p \quad \text{for } 0<p<\infty.$$

Combining the previous results yields a Hölder type inequality:

$$\left(\sum_{k=1}^{n} s_k(ST)^r\right)^{1/r} \le \left(\sum_{k=1}^{n} s_k(S)^p\right)^{1/p} \left(\sum_{k=1}^{n} s_k(T)^q\right)^{1/q} \quad \text{with } 1/r=1/p+1/q.$$

Hence $S\in\mathfrak{S}_p(H)$ and $T\in\mathfrak{S}_q(H)$ imply

$$ST\in\mathfrak{S}_r(H) \quad \text{and} \quad \|ST|\mathfrak{S}_r\|\le\|S|\mathfrak{S}_p\|\|T|\mathfrak{S}_q\|.$$

Chang [1947, p. 185] had earlier proved that $ST\in\mathfrak{S}_1(H)$ whenever $S,T\in\mathfrak{S}_2(H)$.

6.3.1.8 (Ky) Fan [1951, p. 762] showed that

$$\sum_{k=1}^{n} s_k(T) = \sup \sum_{k=1}^{n} \big|(Tx_k|y_k)\big|, \tag{6.3.1.8.a}$$

where the supremum is taken over all orthonormal families $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in H. As an immediate consequence, he obtained an additive counterpart of (6.3.1.7.a):

$$\sum_{k=1}^{n} s_k(S+T) \le \sum_{k=1}^{n} s_k(S) + \sum_{k=1}^{n} s_k(T) \quad \text{for } S,T \in \mathfrak{L}(H).$$

6.3.1.9 Let $\mathfrak{a}$ be a symmetric sequence ideal that is complete with respect to a quasi-norm $\|\cdot|\mathfrak{a}\|$. In this context, **symmetry** means that

$$x \in \mathfrak{l}_\infty,\ t \in \mathfrak{a}, \text{ and } s_n(x) \le s_n(t) \text{ for } n=1,2,\dots \quad \text{imply} \quad x \in \mathfrak{a} \text{ and } \|x|\mathfrak{a}\| \le \|t|\mathfrak{a}\|.$$

Easy manipulations show that

$$\|T|\mathfrak{A}\| := \big\|\big(s_n(T)\big)\big|\mathfrak{a}\big\|$$

defines a quasi-norm on $\mathfrak{A}(H)$. However, if $\|\cdot|\mathfrak{a}\|$ satisfies the quasi-triangle inequality (3.2.5.2.a) with the constant Q, then we get for $\|\cdot|\mathfrak{A}\|$ the larger constant $2Q^2$. So far, nobody was able to improve this result without additional assumptions. The situation is even worse: we do not know whether completeness carries over from $\mathfrak{a}$ to $\mathfrak{A}(H)$.

6.3.1.10 In order to overcome the trouble above, a special kind of symmetric Banach sequence ideal was invented.

Let us say that $\mathfrak{a}$ has the **majorization property** if

$$x \in \mathfrak{l}_\infty,\ t \in \mathfrak{a}, \text{ and } \sum_{k=1}^{n} s_k(x) \le \sum_{k=1}^{n} s_k(t) \text{ for } n=1,2,\dots \quad \text{imply} \quad x \in \mathfrak{a} \text{ and } \|x|\mathfrak{a}\| \le \|t|\mathfrak{a}\|.$$

Calderón [1966, p. 280] used this property in his characterization of interpolation spaces; see 6.6.7.9. The name “свойства мажорантности” was proposed by Russu [1969, стр. 84] when he defined the analogous concept for normed operator ideals $\mathfrak{A}(H)$. The latter author also produced the first symmetric Banach sequence ideal without the majorization property; see Russu [1969, стр. 87].

6.3.1.11 Now we are perfectly happy.

The rules given in 6.3.1.4 yield a one-to-one correspondence $\mathfrak{A}(H) \leftrightarrow \mathfrak{a}$ between Banach operator ideals with the majorization property (defined analogously) and Banach sequence ideals with the majorization property; see [PIE$_3$, pp. 209–211].

This result is mainly based on the following formula, which, thanks to the majorization property, can easily be deduced from (6.3.1.8.a):

$$\|T|\mathfrak{A}\| = \sup \big\|\big((Tx_k|y_k)\big)\big|\mathfrak{a}\big\|,$$

the supremum being taken over all orthonormal sequences $(x_1,x_2,\dots)$ and $(y_1,y_2,\dots)$ in H.

6.3.1.12 Next, I present a construction that yields two kinds of symmetric Banach sequence ideals with the majorization property; see Schatten/von Neumann [1948, pp. 568–573, 577–578] and also [SCHA$_1$, pp. 84–110]. The reader should note the "local" nature of the following considerations. This means that everything could be formulated in terms of sequences with finite support and operators of finite rank.

Let Φ denote a **symmetric gauge function**. In modern terminology, this is a norm defined for all real sequences having only a finite number of non-zero coordinates. In addition, it is assumed that

$$\Phi(1,0,0,\dots)=1 \quad \text{and} \quad \Phi(\varepsilon_1\tau_{\pi(1)},\varepsilon_2\tau_{\pi(2)},\dots)=\Phi(\tau_1,\tau_2,\dots)$$

for $\varepsilon_k=\pm 1$ and any permutation π of the natural numbers.

Then all bounded sequences $t=(\tau_k)$ for which

$$\|t|\mathfrak{s}_\Phi|:=\sup_n \Phi(\tau_1,\dots,\tau_n,0,0,\dots)$$

is finite constitute a symmetric Banach sequence ideal, denoted by $\mathfrak{s}_\Phi$. A closed subideal $\mathfrak{s}_\Phi^\circ$ is obtained by taking all sequences that can be approximated with respect to the norm $\|\cdot|\mathfrak{s}_\Phi\|$ by sequences with finite support.

Combining results from Mityagin [1965, стр. 479–480] and Russu [1969] yields that $\mathfrak{s}_\Phi$ and $\mathfrak{s}_\Phi^\circ$ have the majorization property. Hence, in view of 6.3.1.11, the corresponding operator ideals $\mathfrak{S}_\Phi(H)$ and $\mathfrak{S}_\Phi^\circ(H)$ are Banach ideals.

The majorization property of $\mathfrak{s}_\Phi$ and $\mathfrak{s}_\Phi^\circ$ also follows from a lemma that goes back to Markus [1964, стр. 98]:

Fix any vector $x=(\xi_1,\dots,\xi_n)$, and denote by $\Gamma(x)$ the convex hull of the vectors $(\varepsilon_1\xi_{\pi(1)},\dots,\varepsilon_n\xi_{\pi(n)})$, where $\varepsilon_k=\pm 1$ and π ranges over all permutations of $\{1,\dots,n\}$. Then y is contained in $\Gamma(x)$ if and only if

$$\sum_{k=1}^m s_k(y)\le\sum_{k=1}^m s_k(x) \quad \text{for } m=1,\dots,n. \tag{6.3.1.12.a}$$

Conditions of the form (6.3.1.12.a) have a long history; see [HARD$^+$, pp. 45, 49].

6.3.1.13 Let $\mathfrak{a}$ be any symmetric Banach sequence ideal. Then

$$\Phi(\tau_1,\tau_2,\dots):=\|(\tau_1,\tau_2,\dots)|\mathfrak{a}\|$$

defines a symmetric gauge function such that $\mathfrak{s}_\Phi^\circ\subseteq\mathfrak{a}\subseteq\mathfrak{s}_\Phi$. Therefore, following Mityagin/Shvarts [1964, стр. 89], symmetric Banach sequence ideals of the form $\mathfrak{s}_\Phi^\circ$ and $\mathfrak{s}_\Phi$ are called **minimal** and **maximal**, respectively.

For many concrete Φ's, we have $\mathfrak{s}_\Phi^\circ=\mathfrak{s}_\Phi$. This happens, in particular, for

$$\Phi_p(\tau_1,\tau_2,\dots):=\Bigg(\sum_{n=1}^\infty|\tau_k|^p\Bigg)^{1/p} \quad \text{with } 1\le p<\infty.$$

If $\mathfrak{s}_\Phi^\circ$ is strictly smaller than $\mathfrak{s}_\Phi$ and $\mathfrak{s}_\Phi \neq \mathfrak{l}_\infty$, then there always exist intermediate ideals $\mathfrak{a}$ (промежуточные идеалы) such that $\mathfrak{s}_\Phi^\circ \subset \mathfrak{a} \subset \mathfrak{s}_\Phi$. A first example was given by Mityagin [1964, стр. 820], and Russu [1969, стр. 81] treated the general case.

6.3.2 Basic concepts of ideal theory on Banach spaces

The modern notion of an *operator ideal* was invented once it became clear that restriction to the ring $\mathfrak{L}(X)$ of operators on a fixed Banach space X is too limiting. Rather, in contrast to the Hilbert space setting, operators between different Banach spaces must be taken into account. The basic concepts go back to Pietsch [1968a]. A first *Zusammenfassung* [PIE$_2$] was given in 1972, while the monograph *Operator Ideals* [PIE$_3$] appeared in 1978 (Eastern edition) and 1980 (Western edition).

6.3.2.1 To begin with, I repeat from 2.6.6.1 the definition of an **operator ideal**. This is a subclass of operators,

$$\mathfrak{A} := \bigcup_{X,Y} \mathfrak{A}(X,Y),$$

that has the following properties:

(OI$_0$) $x^* \otimes y \in \mathfrak{A}(X,Y)$ for $x^* \in X^*$ and $y \in Y$.
(OI$_1$) $S+T \in \mathfrak{A}(X,Y)$ for $S,T \in \mathfrak{A}(X,Y)$.
(OI$_2$) $BTA \in \mathfrak{A}(X_0,Y_0)$ for $A \in \mathfrak{L}(X_0,X)$, $T \in \mathfrak{A}(X,Y)$, and $B \in \mathfrak{L}(Y,Y_0)$.

Condition **(OI$_0$)** implies that $\mathfrak{A}$ contains non-zero operators, and it follows that the finite rank operators form the smallest ideal, denoted by $\mathfrak{F}$.

For brevity, I will write $\mathfrak{A}(X)$ instead of $\mathfrak{A}(X,X)$. Obviously, $\mathfrak{A}(X)$ is an ideal in the ring $\mathfrak{L}(X)$.

Operator ideals are considered over the class of *all* Banach spaces, unless the contrary is explicitly stated. I stress the importance of the extreme case in which $\mathfrak{A}(X)$ is given only for a fixed Banach space X. In particular, one may take the separable infinite-dimensional Hilbert space H. This yields the concept of an **operator ideal over** H already used in 4.10.1.6 and in the previous subsection.

6.3.2.2 An **ideal norm** defined on an operator ideal $\mathfrak{A}$ is a rule α that assigns to every $T \in \mathfrak{A}$ a non-negative number $\alpha(T)$ such that the following conditions are satisfied:

(IN$_0$) $\alpha(x^* \otimes y) = \|x^*\| \, \|y\|$ for $x^* \in X^*$ and $y \in Y$.
(IN$_1$) $\alpha(S+T) \leq \alpha(S) + \alpha(T)$ for $S,T \in \mathfrak{A}(X,Y)$.
(IN$_2$) $\alpha(BTA) \leq \|B\| \, \alpha(T) \, \|A\|$ for $A \in \mathfrak{L}(X_0,X)$, $T \in \mathfrak{A}(X,Y)$ and $B \in \mathfrak{L}(Y,Y_0)$.

Since **(IN$_2$)** implies that

$$\alpha(\lambda T) = |\lambda| \alpha(T) \quad \text{for } T \in \mathfrak{A}(X,Y) \text{ and } \lambda \in \mathbb{K},$$

the function $T \to \alpha(T)$ is indeed a norm on every component $\mathfrak{A}(X,Y)$. Moreover, $\|T\| \leq \alpha(T)$. For greater clarity, I will sometimes write $\alpha(T : X \to Y)$ instead of $\alpha(T)$.

6.3.2.3 Some of the following considerations require a concept that is more general than that of an ideal norm; see for example 6.3.3.5 and 6.5.1.3.

We call α an **ideal quasi-norm** if the triangle inequality is replaced by a quasi-triangle inequality:

$$\alpha(S+T) \leq Q\,[\alpha(S)+\alpha(T)] \quad \text{for } S,T \in \mathfrak{A}(X,Y).$$

Here the constant $Q \geq 1$ does not depend on the underlying spaces X and Y. We use the name **ideal** p**-norm** whenever $\alpha(S+T)^p \leq \alpha(S)^p + \alpha(T)^p$ with $0 < p \leq 1$. In this case, the quasi-triangle inequality holds for $Q := 2^{1/p-1}$. Conversely, given some ideal quasi-norm α, the classical renorming process described in 3.2.5.4 yields an equivalent ideal p-norm.

6.3.2.4 A **Banach ideal** is an ideal $\mathfrak{A}$ equipped with an ideal norm α such that all components $\mathfrak{A}(X,Y)$ are complete with respect to the associated topology. **Quasi-Banach** and p**-Banach ideals** are defined analogously.

Of course, every closed ideal is a Banach ideal with respect to the usual operator norm.

It easily turns out that the ideal quasi-norm of a quasi-Banach ideal is unique up to equivalence. Consequently, we may refer to a quasi-Banach ideal $\mathfrak{A}$ without specifying the underlying quasi-norm. In view of this observation, some authors (including me) prefer the more suggestive notation $\|\cdot|\mathfrak{A}\|$ instead of α. In what follows, I will use both variants.

As a matter of fact, the definition of a concrete quasi-Banach ideal automatically leads to a *natural* ideal quasi-norm, under which it becomes complete. However, there are cases in which one and the same ideal can be obtained in different ways, and then we have, indeed, *"die Qual der Wahl."*

There are ideals that in no way can be made a quasi-Banach ideal; see 6.3.21.7. The most prominent example is $\mathfrak{F}$, the ideal of finite rank operators.

6.3.2.5 There are several methods to produce new operator ideals from old.

The **closed hull** $\overline{\mathfrak{A}}$ consists of all operators that can be approximated, with respect to the operator norm, by a sequence of operators from $\mathfrak{A}$. Enflo's counterexample tells us that $\overline{\mathfrak{F}} \neq \mathfrak{K}$; see Subsections 5.7.4 and 7.4.2.

6.3.2.6 The **dual ideal** is defined by $\mathfrak{A}^{\text{dual}} := \{T : T^* \in \mathfrak{A}\}$, and every quasi-norm on $\mathfrak{A}$ yields a **dual** quasi-norm $\|T|\mathfrak{A}^{\text{dual}}\| := \|T^*|\mathfrak{A}\|$ on $\mathfrak{A}^{\text{dual}}$. Schauder's theorem 2.6.4.6 says that $\mathfrak{K}^{\text{dual}} = \mathfrak{K}$, and Gantmacher [1940, p. 301] proved that $\mathfrak{W}^{\text{dual}} = \mathfrak{W}$. According to Hutton [1974], we have $\overline{\mathfrak{F}}^{\text{dual}} = \overline{\mathfrak{F}}$. The two ideals $\mathfrak{V}^{\text{dual}}$ and $\mathfrak{V}$ are incomparable. Figiel/Johnson [1973, p. 199] showed the strict inclusion $\mathfrak{N} \subset \mathfrak{N}^{\text{dual}}$.

6.3.2.7 The **injective hull** $\mathfrak{A}^{\text{inj}}$ consists of all operators $T \in \mathfrak{L}(X,Y)$ that become members of $\mathfrak{A}$ by extending the codomain (target) Y:

$$JT \in \mathfrak{A}(X,Y_0) \quad \text{for some metric injection } J : Y \to Y_0.$$

Similarly, the **surjective hull** $\mathfrak{A}^{\text{sur}}$ consists of all operators $T\in\mathfrak{L}(X,Y)$ that become members of $\mathfrak{A}$ by lifting the domain (source) X:

$$TQ\in\mathfrak{A}(X_0,Y) \quad \text{for some metric surjection } Q: X_0\to X.$$

Due to the extension and lifting properties we may take $J: Y\to l_\infty(\mathbb{I})$ and $Q: l_1(\mathbb{I})\to X$ with an appropriate index set.

Whenever $\mathfrak{A}$ is a quasi-Banach ideal, then so are $\mathfrak{A}^{\text{inj}}$ and $\mathfrak{A}^{\text{sur}}$ by letting

$$\|T|\mathfrak{A}^{\text{inj}}\| := \|JT|\mathfrak{A}\| \quad \text{and} \quad \|T|\mathfrak{A}^{\text{sur}}\| := \|TQ|\mathfrak{A}\|,$$

respectively.

An ideal $\mathfrak{A}$ is said to be **injective** if $\mathfrak{A}=\mathfrak{A}^{\text{inj}}$ and **surjective** if $\mathfrak{A}=\mathfrak{A}^{\text{sur}}$. The same terminology is used for ideal quasi-norms as well as for quasi-Banach ideals whenever we have $\|\cdot|\mathfrak{A}^{\text{inj}}\|=\|\cdot|\mathfrak{A}\|$ and $\|\cdot|\mathfrak{A}^{\text{sur}}\|=\|\cdot|\mathfrak{A}\|$, respectively.

Formulated in the language of cross norms, the concepts of injectivity and surjectivity appeared in Grothendieck's *Résumé* [1956b, pp. 25–28] as *injectivité à gauche* and *injectivité à droite*, respectively. The corresponding concepts for operator ideals were studied by Stephani [1970], [1973].

6.3.2.8 The Banach ideal of nuclear operators (see 5.7.3.1) is neither injective nor surjective. We have $\mathfrak{N}(H)=\mathfrak{S}_1(H)$, but $\mathfrak{N}^{\text{inj}}(H)=\mathfrak{N}^{\text{sur}}(H)=\mathfrak{S}_2(H)$.

An operator $T\in\mathfrak{L}(X,Y)$ belongs to $\mathfrak{N}^{\text{inj}}$ if and only if there exist functionals $x_1^*,x_2^*,\ldots\in X^*$ such that $\sum_{k=1}^\infty \|x_k^*\|<\infty$ and

$$\|Tx\|\le\sum_{k=1}^\infty|\langle x,x_k^*\rangle| \quad \text{for } x\in X.$$

Originally, those operators had been called **quasi-nuclear**; see Pietsch [1966].

Operators $T\in\mathfrak{L}(X,Y)$ belonging to $\mathfrak{N}^{\text{sur}}$ were characterized by Stephani [1973, pp. 86–87]:

The closed unit ball of X is mapped into a **nuclear** subset of Y, which means that there exist $y_1,y_2,\ldots\in Y$ such that $\sum_{k=1}^\infty\|y_k\|<\infty$ and

$$T(B_X)\subseteq\left\{\sum_{k=1}^\infty\lambda_k y_k : |\lambda_k|\le 1\right\}.$$

The reader should take into account that the preceding concept of a nuclear subset differs from that proposed by Grothendieck [GRO$_1$, Chap. II, p. 22].

6.3.2.9 The **product** $\mathfrak{B}\circ\mathfrak{A}$ of ideals $\mathfrak{A}$ and $\mathfrak{B}$ is the collection of all operators

$$T: X\xrightarrow{A\in\mathfrak{A}} Y\xrightarrow{B\in\mathfrak{B}} Z.$$

The n^{th} power of an ideal $\mathfrak{A}$ is defined by induction, $\mathfrak{A}^{n+1}:=\mathfrak{A}^n\circ\mathfrak{A}$.

In the case of quasi-Banach ideals, $\mathfrak{B} \circ \mathfrak{A}$ is complete with respect to the quasi-norm

$$\|T|\mathfrak{B} \circ \mathfrak{A}\| := \inf\Big\{\|B|\mathfrak{B}\|\|A|\mathfrak{A}\| : T = BA\Big\},$$

where the infimum ranges over all factorizations just described. The product of a p-Banach ideal and a q-Banach ideal becomes an r-Banach ideal with the exponent $1/r = 1/p + 1/q$.

A classical result, which goes back to Grothendieck [GRO_1, Chap. I, p. 112], says that $\mathfrak{K} \circ \mathfrak{K} = \mathfrak{V} \circ \mathfrak{W} = \mathfrak{K}$. The proof of $\overline{\mathfrak{F}} \circ \overline{\mathfrak{F}} = \overline{\mathfrak{F}}$ is an easy exercise.

6.3.2.10 The (left-hand) **quotient** $\mathfrak{B}^{-1} \circ \mathfrak{A}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is the largest operator ideal $\mathfrak{Q}$ such that $\mathfrak{B} \circ \mathfrak{Q} \subseteq \mathfrak{A}$. More precisely, $\mathfrak{B}^{-1} \circ \mathfrak{A}(X,Y)$ consists of all $T \in \mathfrak{L}(X,Y)$ such that $BT \in \mathfrak{A}(X,Y_0)$ for every $B \in \mathfrak{B}(Y,Y_0)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are quasi-Banach ideals, then the quotient is complete with respect to the quasi-norm

$$\|T|\mathfrak{B}^{-1} \circ \mathfrak{A}\| := \sup\Big\{\|BT|\mathfrak{A}\| : \|B|\mathfrak{B}\| \leq 1\Big\}.$$

A (right-hand) **quotient** $\mathfrak{A} \circ \mathfrak{B}^{-1}$ can be defined analogously. The two concepts were invented by Puhl [1977]; see also Reisner [1979, p. 510].

The members of the quotient $\mathfrak{W}^{-1} \circ \mathfrak{V}$ could be called **Dunford–Pettis operators**; see 4.8.5.3.

6.3.2.11 First examples of genuine quasi-Banach operator ideals were discovered by Grothendieck [GRO_1, Chap. II, p. 3]. For $0 < p \leq 1$, he considered operators that can be written in the form

$$T = \sum_{k=1}^{\infty} x_k^* \otimes y_k \quad \text{such that} \quad \sum_{k=1}^{\infty} \|x_k^*\|^p \|y_k\|^p < \infty.$$

These **opérateurs de Fredholm de puissance p.ème sommable** constitute the *smallest* p-Banach ideal $\mathfrak{F}_p$, the underlying ideal p-norm being given by

$$\varphi_p(T) := \inf\left(\sum_{k=1}^{\infty} \|x_k^*\|^p \|y_k\|^p\right)^{1/p},$$

where the infimum ranges over all admissible representations as described above. Obviously, $\mathfrak{F}_1$ is just the Banach ideal of nuclear operators. This is the reason why operators in $\mathfrak{F}_p$ are often called ***p*-nuclear**. Then instead of $\mathfrak{F}_p$ the symbol $\mathfrak{N}_p$ is used. Oloff [1972, p. 106] showed that $\mathfrak{F}_p(H) = \mathfrak{S}_p(H)$ and $\varphi_p = \sigma_p$.

A straightforward generalization yields the ideals $\mathfrak{F}_{p,q}$ for $0 < p < 1$ and $0 < q \leq \infty$.

6.3.2.12 In order to get $\mathfrak{F}_p \circ \mathfrak{F}_q \subseteq \mathfrak{F}_r$, Grothendieck [$\mathrm{GRO}_1$, Chap. II, p. 10] posed the strong and unnatural assumption $1/r = \max\{1/p + 1/q - 3/2, 1\}$. His conjecture that the same holds for $1/r = 1/p + 1/q - 1/2$ looks too optimistic. Quite likely, the truth lies in the middle: $1/r = 1/p + 1/q - 1$. Unfortunately, no progress has been achieved during the past 50 years. In this connection, it would be helpful to know the asymptotic behavior of $\|Id : l_1^n \to l_1^n|\mathfrak{F}_p\|$ as $n \to \infty$.

6.3.3 Ideals associated with s-numbers

6.3.3.1 The definition of $\mathfrak{S}_{p,q}(H)$ from 6.3.1.6 can be extended to the Banach space setting. For any choice of additive s-numbers we get the quasi-Banach ideals $\mathfrak{L}_{p,q}^{(s)}$, the corresponding quasi-norms being given by

$$\|T|\mathfrak{L}_{p,q}^{(s)}\| := \left(\sum_{n=1}^{\infty} n^{q/p-1} s_n(T)^q\right)^{1/q} \quad \text{if} \quad 0 < q < \infty$$

and

$$\|T|\mathfrak{L}_{p,\infty}^{(s)}\| := \sup_n n^{1/p} s_n(T) \quad \text{if} \quad q = \infty.$$

As in the case of sequence spaces, I will write $\mathfrak{L}_p^{(s)}$ instead of $\mathfrak{L}_{p,p}^{(s)}$.

The collection of all operators T with $\lim_{n\to\infty} s_n(T) = 0$ is a closed ideal.

6.3.3.2 Here are the most important examples:

$$\begin{tikzcd}
& \begin{array}{c}\mathfrak{L}_{p,q}^{\text{hilb}}\\ \text{(Hilbert numbers)}\end{array} & \\
\begin{array}{c}\mathfrak{L}_{p,q}^{\text{weyl}}\\ \text{(Weyl numbers)}\end{array} \arrow[ur] & & \begin{array}{c}\mathfrak{L}_{p,q}^{\text{chang}}\\ \text{(Chang numbers)}\end{array} \arrow[ul] \\
\begin{array}{c}\mathfrak{L}_{p,q}^{\text{gel}}\\ \text{(Gelfand numbers)}\end{array} \arrow[u] & & \begin{array}{c}\mathfrak{L}_{p,q}^{\text{kol}}\\ \text{(Kolmogorov numbers)}\end{array} \arrow[u] \\
& \begin{array}{c}\mathfrak{L}_{p,q}^{\text{app}}\\ \text{(approximation numbers)}\end{array} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The arrows point from the smaller ideals to the larger ones; see 6.2.3.14.

Letting $1/p_0 = 1/p + 1/2$, we have

$$\begin{array}{lll}
\mathfrak{L}_{p_0,q}^{\text{hilb}} \subset \mathfrak{L}_{p,q}^{\text{weyl}}, & \mathfrak{L}_{p_0,q}^{\text{weyl}} \subset \mathfrak{L}_{p,q}^{\text{gel}}, & \mathfrak{L}_{p_0,q}^{\text{gel}} \subset \mathfrak{L}_{p,q}^{\text{app}}, \\
\mathfrak{L}_{p_0,q}^{\text{hilb}} \subset \mathfrak{L}_{p,q}^{\text{chang}}, & \mathfrak{L}_{p_0,q}^{\text{chang}} \subset \mathfrak{L}_{p,q}^{\text{kol}}, & \mathfrak{L}_{p_0,q}^{\text{kol}} \subset \mathfrak{L}_{p,q}^{\text{app}}.
\end{array}$$

These inclusions easily follow from the non-trivial part of 6.2.3.14 and Carleman's inequality; [HARD$^+$, p. 249]:

$$\sum_{n=1}^{\infty} (\alpha_1 \cdots \alpha_n)^{1/n} \leq \mathrm{e} \sum_{n=1}^{\infty} \alpha_n \quad \text{whenever } \alpha_1, \alpha_2, \ldots \geq 0.$$

6.3.3.3 In [1963c, p. 436], Pietsch showed that $\mathfrak{L}_1^{\text{app}} \subset \mathfrak{N}$. On the other hand, $\mathfrak{N}(l_\infty, l_1)$ is not contained in $\mathfrak{L}_p^{\text{app}}(l_\infty, l_1)$ for any finite exponent p. The situation improves for specific spaces; see 6.1.11.4 and Pietsch [1991b, pp. 160–161]:

Let $\Gamma_n(X) \le cn^\alpha$ and $\Gamma_n(Y) \le cn^\beta$ with $0 \le \alpha, \beta \le \frac{1}{2}$. Then

$$\mathfrak{N}(X,Y) \subseteq \mathfrak{L}_{p,\infty}^{\text{app}}(X,Y), \quad \text{where } 1/p = 1-(\alpha+\beta).$$

Let X be arbitrary and $\Gamma_n(Y) \le cn^\beta$ with $0 \le \beta \le 1/2$. Then

$$\mathfrak{N}(X,Y) \subseteq \mathfrak{L}_{q,\infty}^{\text{app}} \circ \mathfrak{P}_2(X,Y) \subseteq \mathfrak{L}_{p,\infty}^{\text{weyl}}(X,Y), \quad \text{where } 1/p = 1/2+1/q = 1-\beta.$$

6.3.3.4 A generalization of $\mathfrak{L}_1^{\text{app}} \subset \mathfrak{N}$ yields

$$\mathfrak{L}_{p,q}^{\text{app}} \subset \mathfrak{F}_{p,q} \quad \text{for } 0 < p < 1 \text{ and } 0 < q \le \infty.$$

The basic idea of the proof is described in 6.3.20.6.

We also have a result in the converse direction, namely

$$\mathfrak{F}_{p_0,q} \subset \mathfrak{L}_{p,q}^{\text{app}} \quad \text{with } 1/p_0 = 1/p+1,$$

which is an immediate consequence of the fact that

$$D_t \in \mathfrak{L}_{p,q}^{\text{app}}(l_\infty, l_1) \quad \text{if and only if} \quad t \in l_{p_0,q};$$

see 6.2.5.1 and [PIE$_4$, p. 110].

6.3.3.5 Surprisingly, $\mathfrak{L}_p^{\text{app}}$ fails to be a Banach ideal for $1 \le p < \infty$. In this case, $\|\cdot|\mathfrak{L}_p^{\text{app}}\|$ satisfies the quasi-triangle inequality with $Q = 2^{1/p}$. Hence there exists an equivalent ideal p_0-norm, where $1/p_0 = 1/p+1$. Moreover, it follows from

$$\mathfrak{F}_{p_0,\infty}(l_\infty, l_1) \not\subseteq \mathfrak{L}_p^{\text{app}}(l_\infty, l_1)$$

that this is the best possible result. The same conclusions hold for $\mathfrak{L}_p^{\text{gel}}$ and $\mathfrak{L}_p^{\text{kol}}$. It seems that renormings of the ideals $\mathfrak{L}_p^{\text{weyl}}$, $\mathfrak{L}_p^{\text{chang}}$, and $\mathfrak{L}_p^{\text{hilb}}$ have not been studied.

6.3.3.6 The following theorem is due to Pietsch [1980a, p. 16], [PIE$_4$, p. 84]:

An operator $T \in \mathfrak{L}(X,Y)$ belongs to $\mathfrak{L}_{p,q}^{\text{app}}$ if and only if it can be written in the form

$$T = \sum_{k=0}^{\infty} A_k \quad \text{with } A_k \in \mathfrak{F}(X,Y) \text{ and } \operatorname{rank}(A_k) \le 2^k \tag{6.3.3.6.a}$$

such that

$$\sum_{k=0}^{\infty} \big[2^{k/p}\|A_k\|\big]^q < \infty \quad \text{if } 0 < q < \infty \quad \text{and} \quad \sup_k 2^{k/p}\|A_k\| < \infty \quad \text{if } q = \infty.$$

Moreover, an equivalent ideal quasi-norm is obtained by

$$\|T|\mathfrak{L}_{p,q}^{\text{app}}\|_0 := \alpha_{p,q} \inf_k \big\|(2^{k/p}\|A_k\|)\big|l_q\big\|,$$

where the infimum ranges over all representations (6.3.3.6.a). The norming factor

$$\alpha_{p,q} := \begin{cases} 1 & \text{if } 0 < q \le 1, \\ \|(2^{-k/p})|l_{q^*}\| & \text{if } 1 < q \le \infty. \end{cases}$$

ensures condition **(IN$_0$)** from 6.3.2.2.

The preceding criterion is an analogue of a well-known representation theorem in the theory of Besov spaces; see 6.7.3.11.

6.3.3.7 Obviously, the finite rank operators are dense in $\mathfrak{L}_{p,q}^{\text{app}}$. Probably, this is not so for $\mathfrak{L}_{p,q}^{\text{gel}}$ and $\mathfrak{L}_{p,q}^{\text{kol}}$. Since

$$x_n(Id : l_1 \to l_2) = \tfrac{1}{\sqrt{n}}, \qquad y_n(Id : l_2 \to l_\infty) = \tfrac{1}{\sqrt{n}}, \quad \text{and} \quad h_n(Id : l_1 \to l_\infty) = \tfrac{1}{n},$$

the ideals $\mathfrak{L}_{2,\infty}^{\text{weyl}}$, $\mathfrak{L}_{2,\infty}^{\text{chang}}$, and $\mathfrak{L}_{1,\infty}^{\text{hilb}}$ contain non-compact operators. On the other hand, it follows from Dvoretzky's theorem 6.1.2.2 that operators T with $x_n(T)\to 0$ are strictly singular; see Pietsch [1980b, p. 161].

6.3.3.8 The ideals $\mathfrak{L}_{p,q}^{\text{app}}$ and $\mathfrak{L}_{p,q}^{\text{hilb}}$ are symmetric, by 6.2.3.2 and 6.2.3.12:

$$\big(\mathfrak{L}_{p,q}^{\text{app}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{app}} \quad \text{and} \quad \big(\mathfrak{L}_{p,q}^{\text{hilb}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{hilb}}.$$

Moreover, in view of 6.2.3.9 and 6.2.3.13,

$$\big(\mathfrak{L}_{p,q}^{\text{gel}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{kol}} \quad \text{and} \quad \big(\mathfrak{L}_{p,q}^{\text{kol}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{gel}},$$

$$\big(\mathfrak{L}_{p,q}^{\text{weyl}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{chang}} \quad \text{and} \quad \big(\mathfrak{L}_{p,q}^{\text{chang}}\big)^{\text{dual}} = \mathfrak{L}_{p,q}^{\text{weyl}}.$$

6.3.3.9 If the underlying s-numbers are multiplicative, then

$$\mathfrak{L}_{p,u}^{(s)} \circ \mathfrak{L}_{q,v}^{(s)} \subseteq \mathfrak{L}_{r,w}^{(s)} \quad \text{with} \quad 1/r = 1/p + 1/q \quad \text{and} \quad 1/w = 1/u + 1/v.$$

This is, in particular, true for the Gelfand, Kolmogorov, Weyl, and Chang numbers. With the help of 6.3.3.6 it can be shown that in the case of the approximation numbers even equality holds; see Pietsch [1980a, p. 17].

6.3.3.10 Let X be a Banach space on which we can find a bounded sequence of projections P_n with $\text{rank}(P_n) = n$. Then the scale $\big\{\mathfrak{L}_p^{\text{app}}(X)\big\}_{p>0}$ increases strictly. Thus the question arises whether there exists a Banach space X such that $\mathfrak{L}_p^{\text{app}}(X) = \mathfrak{L}_q^{\text{app}}(X)$ for some pair $p<q$. In this (unlikely!) case, we would have $\mathfrak{L}_p^{\text{app}}(X) = \overline{\mathfrak{F}}(X)$ and $p \geq 2$. According to our present knowledge, the only candidate that comes to mind is the Pisier space discussed in 7.4.2.4.

6.3.3.11 Let $\mathfrak{A}$ be any quasi-Banach operator ideal. Then the n^{th} **$\mathfrak{A}$-approximation number** of $T \in \mathfrak{A}(X,Y)$ is given by

$$a_n(T|\mathfrak{A}) := \inf\Big\{ \|T - A|\mathfrak{A}\| : A \in \mathfrak{F}(X,Y),\ \text{rank}(A) < n \Big\}.$$

The basic properties of these generalized s-numbers are similar to those of the original ones. In particular, one may extend the definition from 6.3.3.1 to obtain the quasi-Banach ideals $\mathfrak{A}_{p,q}^{\text{app}}$; see Pietsch [1981b, p. 80].

For the ideal $\mathfrak{S}_2(H)$, this concept goes back to Schmidt [1907a, pp. 467–470], who showed that

$$a_n(T|\mathfrak{S}_2) = \bigg(\sum_{k=n}^{\infty} s_k(T)^2\bigg)^{1/2}.$$

In the setting of finite matrices, Schmidt's result (together with the Schmidt representation) was rediscovered by numerical mathematicians; see Eckart/Young [1936, p. 216], [1939, p. 119], and Householder/Young [1938, pp. 166–167].

Further information can be found in [PIE$_4$, pp. 100–107]. For example,

$$\mathfrak{L}^{\text{gel}}_{p_0,q} \subseteq (\mathfrak{P}_2)^{\text{app}}_{p,q} \subseteq \mathfrak{L}^{\text{weyl}}_{p_0,q} \quad \text{if } 1/p_0 = 1/p + 1/2,$$

where $\mathfrak{P}_2$ denotes the ideal of 2-summing operators defined in 6.3.6.2. The advantage of the intermediate ideal $(\mathfrak{P}_2)^{\text{app}}_{p,q}$ is that the finite rank operators are dense.

6.3.3.12 Applying the definition from 6.3.3.1 to the entropy numbers yields the injective and surjective ideals $\mathfrak{L}^{\text{ent}}_{p,q}$, which are complete with respect to the ideal quasi-norms

$$\|T|\mathfrak{L}^{\text{ent}}_{p,q}\| := \varepsilon_{p,q}\bigg(\sum_{n=1}^{\infty} n^{q/p-1} e_n(T)^q\bigg)^{1/q} \quad \text{if } 0 < q < \infty$$

and

$$\|T|\mathfrak{L}^{\text{ent}}_{p,\infty}\| := \varepsilon_{p,\infty} \sup_n n^{1/p} e_n(T) \quad \text{if } q = \infty.$$

The norming factors $\varepsilon_{p,q}$ are different in the real and the complex cases. In order to get injective and surjective ideal quasi-norms we have to use the *inner* entropy numbers; see 6.2.4.3.

The ideals $\mathfrak{L}^{\text{ent}}_{p,q}$ were invented by Triebel [1970, p. 103]. Their definition via *dyadic* entropy numbers is due to Carl/Pietsch [1976, p. 24]. For a comprehensive presentation the reader is referred to [CARL$^+$].

6.3.3.13 The inequalities from 6.2.4.5 imply that

$$\mathfrak{L}^{\text{gel}}_{p,q} + \mathfrak{L}^{\text{kol}}_{p,q} \subset \mathfrak{L}^{\text{ent}}_{p,q} \quad \text{and} \quad \mathfrak{L}^{\text{ent}}_{p_0,q} \subset \mathfrak{L}^{\text{gel}}_{p,q} \cap \mathfrak{L}^{\text{kol}}_{p,q}, \quad \text{where } 1/p_0 = 1/p + 1.$$

It seems likely that the right-hand inclusion even holds if $1/p_0 = 1/p + 1/2$.

For the Hilbert space components, we have

$$\mathfrak{L}^{\text{ent}}_{p,q}(H) = \mathfrak{S}_{p,q}(H).$$

6.3.4 Local theory of quasi-Banach ideals and trace duality

6.3.4.1 With every operator ideal $\mathfrak{A}$, one may associate its **minimal kernel** and its **maximal hull**:

$$\mathfrak{A}^{\min} := \overline{\mathfrak{F}} \circ \mathfrak{A} \circ \overline{\mathfrak{F}} \quad \text{and} \quad \mathfrak{A}^{\max} := \overline{\mathfrak{F}}^{-1} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}^{-1},$$

respectively. If $\mathfrak{A}$ is a quasi-Banach ideal, then so are $\mathfrak{A}^{\min}$ and $\mathfrak{A}^{\max}$.

Obviously, $\mathfrak{A}^{\min} \subseteq \mathfrak{A} \subseteq \mathfrak{A}^{\max}$. In general, the two inclusions are strict and the corresponding ideal quasi-norms may not be equivalent on $\mathfrak{A}^{\min}$. However, if X^* and Y have the metric approximation property, then

$$\|T|\mathfrak{A}^{\max}\| = \|T|\mathfrak{A}^{\min}\| \quad \text{for } T \in \mathfrak{A}^{\min}(X,Y);$$

see Schwarz [1975, p. 310].

An ideal is said to be **minimal** if $\mathfrak{A} = \mathfrak{A}^{\min}$ and **maximal** if $\mathfrak{A} = \mathfrak{A}^{\max}$. For every closed $\mathfrak{A}$, we have $\mathfrak{A}^{\min} = \overline{\mathfrak{F}}$ and $\mathfrak{A}^{\max} = \mathfrak{L}$.

If a quasi-Banach ideal $\mathfrak{A}$ is minimal, then the finite rank operators are dense with respect to $\|\cdot|\mathfrak{A}\|$. On the other hand, it follows from [DEF$^+$, pp. 329, 413] that the converse implication may fail.

Minimal and maximal quasi-Banach ideals were considered for the first time in [PIE$_2$, pp. 79–80].

6.3.4.2 For the ideals $\mathfrak{N}$ (nuclear operators 5.7.3.2) and $\mathfrak{I}$ (integral operators 5.7.3.10), we have $\mathfrak{N} = \mathfrak{N}^{\min} = \mathfrak{I}^{\min}$ and $\mathfrak{I} = \mathfrak{I}^{\max} = \mathfrak{N}^{\max}$; see [PIE$_3$, pp. 113–117].

Surprisingly, the ideals $\mathfrak{L}_p^{\mathrm{app}}$ with $0 < p < \infty$ are simultaneously minimal and maximal. The minimality can be deduced from 6.3.3.6, while the maximality is proved in [PIE$_3$, p. 193].

6.3.4.3 The following characterization holds only for p-Banach ideals. However, this is not a serious limitation, since every quasi-Banach ideal admits an equivalent ideal p-norm.

An operator $T \in \mathfrak{L}(X,Y)$ belongs to $\mathfrak{A}^{\min}$ if and only if it can be written in the form

$$T = \sum_{k=1}^{\infty} B_k T_k A_k \quad \text{such that} \quad \sum_{k=1}^{\infty} \|B_k\|^p \|T_k|\mathfrak{A}\|^p \|A_k\|^p < \infty, \tag{6.3.4.3.a}$$

where $A_k \in \mathfrak{L}(X,E_k)$, $T_k \in \mathfrak{L}(E_k,F_k)$ and $B_k \in \mathfrak{L}(F_k,Y)$ with finite-dimensional spaces E_k and F_k. Moreover,

$$\|T|\mathfrak{A}^{\min}\| := \|T|\overline{\mathfrak{F}} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}\| = \inf \left(\sum_{k=1}^{\infty} \|B_k\|^p \|T_k|\mathfrak{A}\|^p \|A_k\|^p \right)^{1/p},$$

the infimum being taken over all representations (6.3.4.3.a).

6.3.4.4 Let $\mathfrak{A}$ be any quasi-Banach ideal.

An operator $T \in \mathfrak{L}(X,Y)$ belongs to $\mathfrak{A}^{\max}$ if and only if

$$\|T|\mathfrak{A}^{\max}\| := \|T|\overline{\mathfrak{F}}^{-1} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}^{-1}\| = \sup \left\{ \|BTA|\mathfrak{A}\| : {\|A:E \to X\| \le 1 \atop \|B:Y \to F\| \le 1} \right\}$$

is finite, where E and F range over all finite-dimensional Banach spaces.

6.3.4.5 Next, I explain the local character of the ideals $\mathfrak{A}^{\min}$ and $\mathfrak{A}^{\max}$.

An ideal quasi-norm α is called **elementary** if it is defined only for all operators between finite-dimensional spaces.

Straightforward modifications of the preceding criteria show that every elementary ideal quasi-norm α generates a minimal quasi-Banach ideal $\mathfrak{S}_\alpha^\circ$ and a maximal quasi-Banach ideal $\mathfrak{S}_\alpha$; see also 6.3.1.13.

If $\mathfrak{A}$ is any quasi-Banach ideal such that the restriction of $\|\cdot|\mathfrak{A}\|$ to elementary operators coincides with α, then $\mathfrak{A}^{\min} = \mathfrak{S}_\alpha^\circ$ and $\mathfrak{A}^{\max} = \mathfrak{S}_\alpha$.

6.3.4.6 The **regular hull** $\mathfrak{A}^{\mathrm{reg}}$ consists of all operators $T \in \mathfrak{L}(X,Y)$ such that $K_Y T \in \mathfrak{A}(X,Y^{**})$, where K_Y is the canonical injection from Y into Y^{**}; see 2.2.3. If $\mathfrak{A}$ is a quasi-Banach ideal, then $\mathfrak{A}^{\mathrm{reg}}$ is complete with respect to the ideal quasi-norm $\|T|\mathfrak{A}^{\mathrm{reg}}\| := \|K_Y T|\mathfrak{A}\|$.

An ideal $\mathfrak{A}$ with $\mathfrak{A} = \mathfrak{A}^{\mathrm{reg}}$ is called **regular**. Every dual ideal $\mathfrak{A}^{\mathrm{dual}}$ has this property.

If $\mathfrak{A}$ is regular, then $\mathfrak{A} \subseteq \mathfrak{A}^{\mathrm{dual}}$ implies $\mathfrak{A} = \mathfrak{A}^{\mathrm{dual}}$. Hence $\mathfrak{N} \subset \mathfrak{N}^{\mathrm{dual}}$ tells us that the nuclear operators form a non-regular ideal.

6.3.4.7 Next, I discuss the relationship between several stability properties of quasi-Banach ideals $\mathfrak{A}$.

Stability with respect to finite representability means that every operator T_0 that is finitely representable in some $T \in \mathfrak{A}$ also belongs to $\mathfrak{A}$ and $\|T_0|\mathfrak{A}\| \leq \|T|\mathfrak{A}\|$.

Stability with respect to ultrapowers means that $T \in \mathfrak{A}$ implies $T^{\mathscr{U}} \in \mathfrak{A}$ and $\|T^{\mathscr{U}}|\mathfrak{A}\| \leq \|T|\mathfrak{A}\|$ for every ultrafilter $\mathscr{U}$.

Stability with respect to ultraproducts means that $T_i \in \mathfrak{A}$ and $\sup_{i\in\mathbb{I}} \|T_i|\mathfrak{A}\| < \infty$ imply $(T_i)^{\mathscr{U}} \in \mathfrak{A}$ and $\|(T_i)^{\mathscr{U}}|\mathfrak{A}\| \leq \mathscr{U}\text{-}\lim_i \|T_i|\mathfrak{A}\|$ for every ultrafilter $\mathscr{U}$.

$$
\begin{array}{ccccc}
\boxed{\begin{array}{c}\text{stable with respect to}\\ \text{ultrapowers}\end{array}} & + & \boxed{\text{regular}} & = & \boxed{\begin{array}{c}\text{stable with respect to}\\ \text{finite representability}\end{array}}\\
\Uparrow & & & & \Uparrow\\
\boxed{\begin{array}{c}\text{stable with respect to}\\ \text{ultraproducts}\end{array}} & + & \boxed{\text{regular}} & = & \boxed{\text{maximal}}
\end{array}
$$

The first "equation" is an immediate consequence of 6.1.3.16 and 6.1.3.17, while the second one was proved by Pietsch [1974b, p. 128] and Kürsten [1978]; see also Heinrich [1980a, p. 92].

Note that the property of being stable with respect to finite representability is much weaker than maximality. Indeed, $\mathfrak{L}$ is the only maximal closed ideal, whereas $\overline{\mathfrak{F}}$, $\mathfrak{K}$, and many other closed ideals are ultrapower stable and regular; see Pietsch [1974b, pp. 129–131].

6.3.4.8 Quite often, results from geometry of Banach spaces have corollaries in the theory of operator ideals. Take, for example, the following facts:

Phillips [1940] and Sobczyk [1941a] discovered that c_0 fails to be complemented in its bidual l_∞; see 4.9.1.10.

Henson/Moore [1974b, p. 283] showed that $(l_\infty)^{\mathscr{U}}$ does not have the extension property for any non-trivial ultrafilter on $\mathbb{N}$.

Conclusion: the ideal of operators that factor through l_∞ is neither regular nor stable with respect to ultrapowers.

6.3.4.9 The formulas

$$\mathfrak{K}(H)^* = \mathfrak{S}_1(H), \quad \mathfrak{S}_1(H)^* = \mathfrak{L}(H), \quad \text{and} \quad \mathfrak{S}_p(H)^* = \mathfrak{S}_{p^*}(H) \ \text{ with } \ 1<p<\infty,$$

which go back to von Neumann [1937], are based on **trace duality**:

$$\langle S,T\rangle = \text{trace}(ST).$$

Extending these considerations to the Banach space setting appeared as a natural task. In [1943, pp. 203, 208] and [1946, p. 73], Schatten invented the concept of an *associate cross norm*; see 6.3.11.3. The translation into the language of Banach ideals was carried out by Pietsch [1971] and Schwarz [1973].

In Banach spaces without the (metric) approximation property, the situation is extremely unpleasant, and non-reflexivity may cause some trouble as well. Therefore Grothendieck [1956b, pp. 8–10] made the wise decision to start with the finite-dimensional case (*espaces normés numériques*).

Given any elementary ideal norm α, the **adjoint** ideal norm is defined by

$$\alpha^*(T) := \sup\big\{|\,\text{trace}(ST)| : \alpha(S : F \to E) \le 1\big\} \quad \text{for } T \in \mathfrak{L}(E,F).$$

Then $\alpha^{**} = \alpha$, and

$\mathfrak{L}(E,F)$ equipped with the ideal norm α^*

is the dual of

$\mathfrak{L}(F,E)$ equipped with the ideal norm α.

6.3.4.10 The preceding definition can be extended to the infinite-dimensional setting. The **adjoint** $\mathfrak{A}^*$ of a Banach ideal $\mathfrak{A}$ consists of all $T \in \mathfrak{L}(X,Y)$ for which there exists a constant $c \ge 0$ such that

$$|\,\text{trace}(ASBT)| \le c\|A\|\alpha(S)\|B\| \quad \text{whenever} \quad \begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle A\in\mathfrak{F}}\uparrow & & \downarrow{\scriptstyle B\in\mathfrak{F}} \\ X_0 & \xleftarrow[S\in\mathfrak{F}]{} & Y_0 \end{array} .$$

Then $\mathfrak{A}^*$ becomes a Banach ideal under the ideal norm $\|T|\mathfrak{A}^*\| := \inf c$.

Using the terminology of 6.3.4.5, we may write $\mathfrak{A}^* = \mathfrak{S}_{\alpha^*}$. Moreover, $\mathfrak{A} \subseteq \mathfrak{A}^{**}$ and $\mathfrak{A}^* = \mathfrak{A}^{***}$. The reader should realize the analogy with the theory of perfect sequence spaces; see [KÖT$_1$, § 30].

6.3.4.11 The formula $(\mathfrak{L}_p^{\text{app}})^* = \mathfrak{L}_{p^*}^{\text{app}}$ does not make sense, since $\mathfrak{L}_p^{\text{app}}$ fails to be a Banach ideal.

6.3.4.12 The wish to represent the functionals on $\mathfrak{A}(X,Y)$ was the reason for inventing the adjoint of a Banach ideal $\mathfrak{A}$. However, the desired representation theorem holds only under specific conditions.

Given $\varphi \in \mathfrak{A}(X,Y)^*$, there exists an operator $T \in \mathfrak{L}(Y,X^{**})$ such that

$$\varphi(x^* \otimes y) = \langle x^*, Ty \rangle \quad \text{for } x^* \in X^* \text{ and } y \in Y.$$

In order to extend this formula to finite rank operators $S \in \mathfrak{F}(X,Y)$, we first observe that ST is undefined. Luckily, this obstacle can be circumvented by viewing S^{**} as an operator from X^{**} into Y, denoted by S^π. Then

$$\varphi(S) = \text{trace}(S^\pi T) \quad \text{for } S \in \mathfrak{F}(X,Y). \tag{6.3.4.12.a}$$

If the finite rank operators are dense in $\mathfrak{A}(X,Y)$, then (6.3.4.12.a) holds also for all $S \in \mathfrak{A}(X,Y)$. Moreover, it follows that $T \in \mathfrak{A}^*(Y,X^{**})$ and $\|T|\mathfrak{A}^*\| \leq \|\varphi\|$.

Next, we assume that X^* and Y have the metric approximation property. Then every operator $T \in \mathfrak{A}^*(Y,X^{**})$ defines a functional $\varphi : S \mapsto \text{trace}(S^\pi T)$ on $\mathfrak{A}(X,Y)$ with $\|\varphi\| \leq \|T|\mathfrak{A}^*\|$. Therefore the correspondence $\varphi \to T$ yields the isometry

$$\mathfrak{A}(X,Y)^* = \mathfrak{A}^*(Y,X^{**}).$$

This result goes back to Schwarz [1973, p. 305]; see also [PIE$_3$, pp. 133–134].

6.3.4.13 Let E be finite-dimensional. Then, for every Banach space X, the isometries

$$\mathfrak{L}(E,X)^* = \mathfrak{N}(X,E) \quad \text{and} \quad \mathfrak{N}(X,E)^* = \mathfrak{L}(E,X^{**})$$

hold. Hence

$$\mathfrak{L}(E,X)^{**} = \mathfrak{L}(E,X^{**}). \tag{6.3.4.13.a}$$

These formulas belong to the folklore of the theory of tensor products; see Schatten [1946, pp. 75–78], [SCHA$_1$, pp. 40, 51], and Grothendieck [1956b, p. 13]. Operator-theoretic approaches were given by Schwarz [1973, p. 298], and Dean [1973, p. 148]. The latter discovered an elementary proof of the principle of local reflexivity based on (6.3.4.13.a).

6.3.4.14 Finally, I present a useful lemma that is due to Lewis [1979, p. 19]:

Let E_n and F_n be n-dimensional Banach spaces. Then for every ideal norm α there exists an isomorphism $U : E_n \to F_n$ such that $\alpha(U) = 1$ and $\alpha^*(U^{-1}) = n$.

6.3.4.15 As a first consequence of the previous lemma, we find an isomorphism $U : l_1^n \to E_n$ such that

$$U : (\xi_k) \mapsto \sum_{k=1}^{n} \xi_k x_k \quad \text{with} \quad \|U\| = \max_{1 \le k \le n} \|x_k\| = 1$$

and

$$U^{-1} : x \mapsto \big(\langle x, x_k^* \rangle\big) \quad \text{with} \quad \nu(U^{-1}) = \sum_{k=1}^{n} \|x_k^*\| = n.$$

Hence (x_k) is an Auerbach basis; see 5.6.2.4. Moreover, we learn from [PIS_2, p. 33] that $U(B_{l_2^n})$ is the ellipsoid of maximal volume contained in the unit ball B_{E_n}. It also follows that $\pi_2(U^{-1}) = \sqrt{n}$; see 6.3.6.2 for the definition of π_2. These observations provide us with a new approach to John's theorem 6.1.1.4.

6.3.4.16 Another application of Lewis's lemma yields an operator $U : l_2^n \to E_n$ such that $\pi_\gamma(U) = \sqrt{n}$ and $\pi_\gamma^*(U^{-1}) = \sqrt{n}$, where π_γ denotes the γ-summing norm defined in 6.3.6.15. Then

$$d_k(U) \le c\sqrt{\tfrac{n}{k}} \quad \text{and} \quad c_k(U^{-1}) \le c\, K^{\text{gauss}}(E_n)\sqrt{\tfrac{n}{k}}.$$

Here c is the universal constant from 6.3.6.19, while $K^{\text{gauss}}(E_n)$ denotes the Gaussian K-convexity constant of E_n; see Figiel/Tomczak-Jaegermann [1979, pp. 164–168].

Since $K^{\text{gauss}}(E_n)$ may increase to infinity with n, the right-hand inequality above is of limited significance. However, using a tricky interpolation argument, Pisier [1989, p. 115] was able to prove the existence of isomorphisms $U_\alpha : l_2^n \to E_n$ such that

$$d_k(U_\alpha) \le c_\alpha \left(\tfrac{n}{k}\right)^\alpha \quad \text{and} \quad c_k(U_\alpha^{-1}) \le c_\alpha \left(\tfrac{n}{k}\right)^\alpha,$$

where the constants c_α depend only on $\alpha \in \left(\frac{1}{2}, 1\right)$. It seems to be open whether the same holds for $\alpha = 1/2$; see [PIS_2, p. 123]. This powerful lemma provides a new approach to the two-sided Santaló inequality and the quotient of subspace theorem; see Pisier [1989, pp. 126, 129] and [PIS_2, pp. 116–123].

6.3.5 *p*-Factorable operators

6.3.5.1 Let $1 \le p \le \infty$. An operator $T \in \mathfrak{L}(X,Y)$ is called ***p*-factorable** if there exists a factorization

$$K_Y T : X \xrightarrow{A \in \mathfrak{L}} L_p \xrightarrow{B \in \mathfrak{L}} Y^{**}$$

with some $L_p = L_p(M, \mathscr{M}, \mu)$. The underlying measure space may depend on T. In the limiting case $p = \infty$, it is possible to use $C(K)$ instead of $L_\infty(M, \mathscr{M}, \mu)$.

The concept above is due to Kwapień [1972a, p. 217], who showed that these operators form a maximal Banach ideal $\mathfrak{L}_p$ under the norm

$$\lambda_p(T) := \inf \|B\| \|A\|,$$

the infimum being taken over all possible factorizations. The maximality follows from the stability with respect to ultraproducts; see 6.1.3.7 and 6.3.4.7.

By definition, **strictly** p**-factorable** operators $T \in \mathfrak{L}(X,Y)$ even factor as

$$T : X \xrightarrow{A\in\mathfrak{L}} L_p \xrightarrow{B\in\mathfrak{L}} Y.$$

In the case $p=2$ nothing new is obtained. On the other hand, for $p=1$ and $p=\infty$ the latter concept is different; see 6.1.4.5 and Reĭnov [1982, pp. 133–134]. It seems likely that the term *“strictly”* is also justified whenever $1<p<\infty$ and $p\neq 2$; but this is a long-standing open problem already posed by Kwapień [1972a, p. 224].

6.3.5.2 The minimal kernel $\mathfrak{L}_p^{\min}$ consists of all operators

$$T : X \xrightarrow{A\in\overline{\mathfrak{F}}} l_p \xrightarrow{B\in\overline{\mathfrak{F}}} Y,$$

and an elementary ideal norm is obtained by letting

$$\lambda_p(T) := \inf \|B\|\|A\|,$$

where the infimum ranges over all factorizations

$$T : E \xrightarrow{A\in\mathfrak{F}} l_p^n \xrightarrow{B\in\mathfrak{F}} F \quad \text{with } n \geq \operatorname{rank}(T).$$

Then $\mathfrak{L}_p^{\min} = \mathfrak{S}_{\lambda_p}^{\circ}$ and $\mathfrak{L}_p = \mathfrak{S}_{\lambda_p}$, in the terminology of 6.3.4.5.

6.3.5.3 Obviously, $\mathfrak{L}_p^{\mathrm{dual}} = \mathfrak{L}_{p^*}$ for $1\leq p\leq\infty$.

It follows from Khintchine’s inequality that l_2 is complemented in $L_p[0,1]$ for $1<p<\infty$. Hence $\mathfrak{L}_2 \subset \mathfrak{L}_p$. Apart from this inclusion, $\mathfrak{L}_p$ and $\mathfrak{L}_q$ are incomparable whenever $p\neq q$.

6.3.5.4 Next, I present the ideal version of a fundamental criterion that goes back to Lindenstrauss/Pełczyński [1968, p. 313]; see also Maurey [1974b, p. 329].

An operator $T\in\mathfrak{L}(X,Y)$ belongs to $\mathfrak{L}_p^{\mathrm{inj}}$ if and only if there is a constant $c\geq 0$ with the following property:

For any choice of finite families $x_1,\dots,x_n \in X$ and $x_1^\circ,\dots,x_n^\circ \in X$ that satisfy the inequality

$$\Big(\sum_{k=1}^n \big|\langle x_k^\circ, x^*\rangle\big|^p\Big)^{1/p} \leq \Big(\sum_{k=1}^n \big|\langle x_k, x^*\rangle\big|^p\Big)^{1/p} \quad \text{whenever } x^*\in X^*, \tag{6.3.5.4.a}$$

we have

$$\Big(\sum_{k=1}^n \|Tx_k^\circ\|^p\Big)^{1/p} \leq c\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p}.$$

In this case, $\lambda_p^{\mathrm{inj}}(T) = \inf c$.

6.3.5.5 Of particular interest is the ideal $\mathfrak{L}_2$ formed by all operators that factor through a Hilbert space.

Termed as *applications hilbertiennes*, **Hilbertian operators** were invented by Grothendieck, who observed in his *Résumé* [1956b, pp. 40–41] that $\mathfrak{L}_2$ is injective, surjective, and symmetric. He formulated his results in terms of $\otimes$-norms; the translation into the language of operator ideals is due to Lindenstrauss/Pełczyński [1968, p. 292]. One refers to λ_2 as the **Hilbertian ideal norm**.

6.3.5.6 Kwapień [1972b, p. 588] observed that in the case $p=2$, the relation (6.3.5.4.a) is equivalent to the existence of an (n,n)-matrix (a_{hk}) with

$$x_h^\circ = \sum_{k=1}^{n} a_{hk} x_k \quad \text{and} \quad \big\|(a_{hk})|l_2^n \to l_2^n\big\| \le 1.$$

This led him to his famous characterization of inner product spaces. The final outcome reads as follows; see [PIS_1, p. 25]:

An operator $T \in \mathfrak{L}(X,Y)$ is Hilbertian if and only if we can find a constant $c \ge 0$ such that

$$\bigg(\sum_{h=1}^{n} \Big\| \sum_{k=1}^{n} a_{hk} T x_k \Big\|^2\bigg)^{1/2} \le c\Big(\sum_{k=1}^{n} \|x_k\|^2\Big)^{1/2}$$

for any finite family $x_1,\dots,x_n \in X$ and any (a_{hk}) with $\big\|(a_{hk})|l_2^n \to l_2^n\big\| \le 1$. An extreme point argument tells us that it is enough if (a_{hk}) ranges over all orthogonal matrices (real scalars) or all unitary matrices (complex scalars).

6.3.6 *p*-Summing operators

6.3.6.1 For $0<p<\infty$ and any Banach space X, we let

$$[l_p,X] := \big\{(x_k) : \big(\|x_k\|\big) \in l_p\big\} \text{ and } [w_p,X] := \big\{(x_k) : \big(\langle x_k, x^*\rangle\big) \in l_p \text{ if } x^* \in X^*\big\}.$$

Sequences with the required properties are called **absolutely p-summable** and **weakly p-summable**, respectively. If $1 \le p < \infty$, then $[l_p,X]$ becomes a Banach space under the norm

$$\|(x_k)|l_p\| := \bigg(\sum_{k=1}^{\infty} \|x_k\|^p\bigg)^{1/p}.$$

The same is true for $[w_p,X]$ with respect to

$$\|(x_k)|w_p\| := \sup_{\|x^*\|\le 1} \bigg(\sum_{k=1}^{\infty} |\langle x_k, x^*\rangle|^p\bigg)^{1/p}.$$

The right-hand supremum is finite because of the closed graph theorem. In the case $0<p<1$ we get p-Banach spaces.

Obviously, $[l_p,X] \subseteq [w_p,X]$. Generalizing the famous Dvoretzky–Rogers theorem 6.1.2.1, Grothendieck [1956c, p. 101] observed that equality holds only when X is finite-dimensional. This phenomenon has led to the theory of p-summing operators.

6.3.6.2 Let $0 < q \le p < \infty$. An operator $T \in \mathfrak{L}(X,Y)$ is said to be **absolutely** (p,q)**-summing** or just (p,q)**-summing** if it takes weakly q-summable sequences (x_k) of X to absolutely p-summable sequences (Tx_k) of Y. This means that $\boldsymbol{T} : (x_k) \mapsto (Tx_k)$ defines a linear map from $[w_q, X]$ into $[l_p, X]$ that is bounded in view of the closed graph theorem. Hence there exists a constant $c \ge 0$ such that

$$\left(\sum_{k=1}^{n} \|Tx_k\|^p\right)^{1/p} \le c \sup_{\|x^*\| \le 1} \left(\sum_{k=1}^{n} |\langle x_k, x^* \rangle|^q\right)^{1/q}$$

for every finite family $x_1, \ldots, x_n \in X$. Even more is true: the preceding inequality characterizes (p,q)-summing operators.

For $1 \le p < \infty$, the collection of all (p,q)-summing operators, denoted by $\mathfrak{P}_{p,q}$, is a maximal and injective Banach ideal under the norm

$$\pi_{p,q}(T) := \inf c = \big\| \boldsymbol{T} : [w_q, X] \to [l_p, Y] \big\|.$$

Most interesting is the case $p = q$. Then the operators are called p**-summing**, and we write $\mathfrak{P}_p$ instead of $\mathfrak{P}_{p,p}$. The p-summing norm is simply denoted by π_p.

6.3.6.3 Absolutely 1-summing operators were introduced by Grothendieck as *applications semi-intégrales à droite*; see [GRO$_1$, pp. 160–161]. In his *Résumé* [1956b, p. 33] he used the name *applications préintégrales droites*. The decisive breakthrough happened at the ICM 1966 in Moscow. In a short communication Pietsch sketched his theory of absolutely p-summing operators, whereas Mityagin/Pełczyński [1966, p. 371] defined the concept of a (p,q)-absolutely summing operator.

The first result in this area was the Orlicz theorem 5.1.1.6, which says that the identity map of L_p with $1 \le p \le 2$ is $(2,1)$-summing. Therefore Banach spaces X with $I_X \in \mathfrak{P}_{2,1}$ are said to have the **Orlicz property**.

6.3.6.4 The 1-parameter scale of the ideals $\mathfrak{P}_p$ is strictly increasing for $p \ge 1$. On the other hand, a deep theorem of Maurey [1972a], or [MAU, p. 75], says that $\mathfrak{P}_p = \mathfrak{P}_0$ for $0 < p < 1$. Here $\mathfrak{P}_0$ denotes the ideal of 0**-absolutely summing** operators, which, according to Kwapień [1970b, pp. 193–194], are defined by the following property:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \min\big(\|Tx_k\|, 1\big) \le \varepsilon$$

whenever the elements $x_1, \ldots, x_n \in X$ satisfy the condition

$$\sup_{\|x^*\| \le 1} \left\{ \frac{1}{n} \sum_{k=1}^{n} \min\big(|\langle x_k, x^* \rangle|, 1\big) \right\} \le \delta.$$

In the case of two parameters, Kwapień [1968a, p. 328] proved that

$$\mathfrak{P}_{p_0,q_0} \subseteq \mathfrak{P}_{p_1,q_1} \quad \text{if } p_0 \le p_1,\, q_0 \le q_1 \text{ and } 1/p_0 - 1/q_0 = 1/p_1 - 1/q_1.$$

6.3.6.5 The theory of p-summing operators is based on a crucial criterion due to Pietsch [1967, p. 341].

Recall that B_{X^*}, the closed unit ball of X^*, is a compact Hausdorff space in the weak* topology. Hence, thanks to the Riesz representation theorem, positive linear functionals on $C(B_{X^*})$ are just given by regular Borel measures.

An operator $T \in \mathfrak{L}(X,Y)$ is p-summing if and only if there exists a constant $c \geq 0$ and a (regular) Borel probability μ on B_{X^*} such that

$$\|Tx\| \leq c\left(\int_{B_{X^*}} |\langle x,x^*\rangle|^p d\mu(x^*)\right)^{1/p} \quad \text{for all} \quad x\in X. \tag{6.3.6.5.a}$$

In this case, $\pi_p(T)$ is the infimum, and even the minimum, of all constants $c \geq 0$ for which such a measure can be found.

In order to make the support of μ small, we may pass to any weakly* compact subset W of B_{X^*} such that $\|x\| = \sup_{x^*\in W} |\langle x,x^*\rangle|$. The most important examples are the spaces $C(K)$. Then the set of all Dirac measures $\delta_\xi : f \mapsto f(\xi)$ with $\xi \in K$ has the required property.

The main tool for proving (6.3.6.5.a) is the Hahn–Banach theorem. One may also use (Ky) Fan's minimax theorem; see 5.4.5.5 and [PIE$_3$, p. 232].

6.3.6.6 If (r_k) denotes the sequence of Rademacher functions, then $\rho : t \mapsto \big(r_k(t)\big)$ defines a Borel measurable map from $[0,1]$ into the closed unit ball of $l_\infty = l_1^*$, equipped with the weak* topology. Look at the image of Lebesgue's measure under ρ to see that Khintchine's inequality

$$\left(\sum_{k=1}^n |\xi_k|^2\right)^{1/2} \leq \sqrt{2}\int_0^1 \left|\sum_{k=1}^n \xi_k r_k(t)\right| dt$$

and Parseval's formula

$$\left(\sum_{k=1}^n |\xi_k|^2\right)^{1/2} = \left(\int_0^1 \left|\sum_{k=1}^n \xi_k r_k(t)\right|^2 dt\right)^{1/2}$$

are particular instances of (6.3.6.5.a). So $\pi_1(Id : l_1 \to l_2) \leq \sqrt{2}$ and $\pi_2(Id : l_1 \to l_2) = 1$. In the real case, we even have $\pi_1(Id : l_1 \to l_2) = \sqrt{2}$. The complex case can be treated by using the Steinhaus variables instead of the Rademacher functions. Then, according to Sawa [1986], the analogue of Khintchine's inequality holds with a smaller constant. A quick look at the underlying proof gives $\pi_1(Id : l_1 \to l_2) = \sqrt{\frac{4}{\pi}}$.

The preceding example tells us that the ideals $\mathfrak{P}_p$ contain non-compact operators.

6.3.6.7 The next theorem is an immediate consequence of (6.3.6.5.a).

Operators $T \in \mathfrak{P}_p(X,Y)$ are characterized by the property that they can be factorized as follows:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & Y \arrow[r, "J"] & Y_0 \\ C(K) \arrow[rr, "Id"] & & L_p(K,\mu) \arrow[u, "B"'] \end{tikzcd} \tag{6.3.6.7.a}$$

where $J : Y \to Y_0$ is a suitable metric injection and $\mu(K) = 1$.

6.3.6.8 Since p-summing operators factor through reflexive spaces, they are weakly compact: $\mathfrak{P}_p \subset \mathfrak{W}$. Moreover, in view of (6.3.6.5.a), the Lebesgue convergence theorem implies that weak null sequences are mapped to norm null sequences. Hence p-summing operators are completely continuous: $\mathfrak{P}_p \subset \mathfrak{V}$. Combining these facts yields another proof of the Dvoretzky–Rogers theorem: $\mathfrak{P}_p \circ \mathfrak{P}_p \subset \mathfrak{V} \circ \mathfrak{W} = \mathfrak{K}$.

6.3.6.9 The diagram (6.3.6.7.a) gets simpler for $p = 2$. In this case, the injection J is superfluous.

Operators $T \in \mathfrak{L}(X,Y)$ admitting a factorization of the form

$$T : X \xrightarrow{A} C(K) \xrightarrow{T_0} H \xrightarrow{B} Y$$

were already considered in Grothendieck's *Résumé* [1956b, p. 36]. Only later, it turned out that these *applications prohilbertiennes gauches*, also called *applications ayant la propriété de prolongement hilbertien*, are just the 2-summing operators. Most importantly: π_2 proved to be the *natural* ideal norm on $\mathfrak{P}_2$.

Grothendieck [1956b, p. 65] also knew that operators of the form $C \to H \to C \to H$ are nuclear. Hence $\mathfrak{P}_2 \circ \mathfrak{P}_2 \subset \mathfrak{N}$. Pietsch [1963b, p. 89], [1967, p. 348] completed this result by adding the inequality $\nu(ST) \le \pi_2(S)\pi_2(T)$.

6.3.6.10 A deep theorem of Pietsch [1967, p. 346] says that

$$\mathfrak{P}_p \circ \mathfrak{P}_q \subset \mathfrak{P}_r \quad \text{and} \quad \pi_r(ST) \le \pi_p(S)\pi_q(T)$$

if $1/r := 1/p + 1/q \le 1$. Pisier [1975c] observed that, apart from $1 \le p < q^* < 2$, the left-hand inclusion also holds for $0 < r < 1$. In the exceptional case he constructed a counterexample.

6.3.6.11 A technique described in 6.3.20.6 yields $\mathfrak{L}^{\text{app}}_{p,1} \subset \mathfrak{P}_{p,2}$, which in turn implies $\mathfrak{L}^{\text{weyl}}_{p,1} \subset \mathfrak{P}_{p,2}$.

6.3.6.12 Let $T \in \mathfrak{L}(l_2, X)$ and $\varepsilon > 0$. Then there exists an orthonormal sequence (u_n) such that $a_n(T) \le (1+\varepsilon)\|Tu_n\|$ for $n = 1, 2, \dots$. This useful lemma was implicitly contained in a preprint of Lewis [1983, p. 86], which circulated beginning in 1978 but was published only much later.

With the help of Pisier, König [1980c, p. 308] proved the following generalization of Lewis's original result; see Lewis [1983, p. 85]:

$$n^{1/p}a_n(T) \le \pi_{p,2}(T) \quad \text{for } T \in \mathfrak{P}_{p,2}(l_2, X).$$

In a next step, Pietsch [1980b, p. 157] inferred that

$$n^{1/p}x_n(T) \le \pi_{p,2}(T) \quad \text{for } T \in \mathfrak{P}_{p,2}(X, Y).$$

Hence $\mathfrak{P}_{p,2} \subset \mathfrak{L}^{\text{weyl}}_{p,\infty}$.

6.3.6.13 The first composition formula for $(p,2)$-summing operators is due to Maurey/Pełczyński [1976], who proved that

$$\mathfrak{P}_{p_1,2} \circ \cdots \circ \mathfrak{P}_{p_n,2} \subset \mathfrak{K}$$

if $1/p := 1/p_1 + \cdots + 1/p_n > 1/2$. According to König/Retherford/Tomczak-Jaegermann [1980, p. 104], the same condition ensures that

$$\mathfrak{P}_{p_1,2} \circ \cdots \circ \mathfrak{P}_{p_n,2} \subset \mathfrak{P}_2.$$

Finally, Pietsch [1980b, p. 158] combined the formulas $\mathfrak{L}^{\text{weyl}}_{p,1} \subset \mathfrak{P}_{p,2} \subset \mathfrak{L}^{\text{weyl}}_{p,\infty}$ and $\mathfrak{L}^{\text{weyl}}_{p,\infty} \circ \mathfrak{L}^{\text{weyl}}_{q,\infty} \subset \mathfrak{L}^{\text{weyl}}_{r,\infty}$ to get

$$\mathfrak{P}_{p,2} \circ \mathfrak{P}_{q,2} \subset \mathfrak{P}_{r,2} \quad \text{if } 1/p + 1/q > 1/r \text{ and } r \ge 2.$$

6.3.6.14 Let $0 < p, r, s < \infty$ and $1/p \le 1/r + 1/s$. An operator $T \in \mathfrak{L}(X,Y)$ is said to be **absolutely** (p,r,s)**-summing** or just (p,r,s)**-summing** if there exists a constant $c \ge 0$ such that

$$\Big(\sum_{k=1}^{n} \big|\langle Tx_k, y_k^*\rangle\big|^p\Big)^{1/p} \le c \sup_{\|x^*\| \le 1} \Big(\sum_{k=1}^{n} |\langle x_k, x^*\rangle|^r\Big)^{1/r} \sup_{\|y\| \le 1} \Big(\sum_{k=1}^{n} |\langle y_k^*, y\rangle|^s\Big)^{1/s}$$

for any choice of $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in Y^*$. Thus we get a maximal ideal $\mathfrak{P}_{p,r,s}$ equipped with the quasi-norm $\pi_{p,r,s}(T) := \inf c$. More information can be found in a paper of Lapresté [1976] and in [PIE$_3$, pp. 228–229]. In particular, we have

$$\mathfrak{P}^{\text{dual}}_{p,r,s} = \mathfrak{P}_{p,s,r}.$$

Following an idea of Simons [1972, p. 344], one can obtain this formula by applying the principle of local reflexivity.

The definition above, which goes back to [PIE$_2$, p. 107], was not given just for the sake of generalizing the limiting cases $\mathfrak{P}_p = \mathfrak{P}_{p,p,\infty}$ and $\mathfrak{P}_{p,q} = \mathfrak{P}_{p,q,\infty}$. On the contrary, already in [1970, p. 62], Pietsch observed that $\mathfrak{P}_{p,2,2}$ with $1 \le p < \infty$ is the largest ideal that coincides with $\mathfrak{S}_p$ on Hilbert spaces.

Moreover, $\mathfrak{P}_{1,p,p^*}$ with $1 \le p \le \infty$ was studied by Cohen [1973], who referred to its members as *p-nuclear*. In order to avoid confusion with p-nuclearity in the sense of Persson–Pietsch, some authors use the term *Cohen* p-nuclear operators;

see Gordon/Lewis/Retherford [1972, p. 351], [1973, p. 91]. The significance of $\mathfrak{P}_{1,p,p^*}$ stems from the fact that

$$\mathfrak{P}_{1,p,p^*}^* = \mathfrak{L}_{p^*} \quad \text{and} \quad \mathfrak{L}_p^* = \mathfrak{P}_{1,p^*,p}.$$

These formulas occur in the proof of a striking result of Kwapień [1972a, pp. 218–219], namely

$$\mathfrak{L}_p^* = \mathfrak{P}_p^{\text{dual}} \circ \mathfrak{P}_{p^*}.$$

In the setting of Hilbert spaces, it follows from $\mathfrak{S}_1(H) = \mathfrak{L}_2^*(H) = \mathfrak{P}_2^{\text{dual}}(H) \circ \mathfrak{P}_2(H)$ and $\mathfrak{S}_2(H) = \mathfrak{P}_2(H) = \mathfrak{P}_2^{\text{dual}}(H)$ that every trace class operator is the product of two Hilbert–Schmidt operators. Hence we have returned to the starting point of Schatten/von Neumann described in 4.10.1.3.

6.3.6.15 An operator $T \in \mathfrak{L}(X,Y)$ is said to be **Gauss-summing**, or just **γ-summing**, if there exists a constant $c \geq 0$ such that

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^n t_k T x_k \Big\|^2 d\gamma(t_1) \cdots d\gamma(t_n) \right)^{1/2} \leq c \sup_{\|x^*\| \leq 1} \left(\sum_{k=1}^n |\langle x_k, x^* \rangle|^2 \right)^{1/2}$$

for every finite family $x_1, \ldots, x_n \in X$.

Replacing the Gaussian variables by the Rademacher functions, one obtains the concept of a **Rademacher-summing** or **ρ-summing** operator:

$$\left(\int_0^1 \Big\| \sum_{k=1}^n T x_k r_k(t) \Big\|^2 dt \right)^{1/2} \leq c \sup_{\|x^*\| \leq 1} \left(\sum_{k=1}^n |\langle x_k, x^* \rangle|^2 \right)^{1/2}.$$

In [DIE$_1^+$, p. 234] these operators are called **almost summing**.

It seems to be a folklore result that the two definitions yield one and the same maximal Banach ideal, denoted either by $\mathfrak{P}_\gamma$ or by $\mathfrak{P}_\rho$. The corresponding norms π_γ and π_ρ are obtained as the least constants for which the required inequalities hold. We know from [DIE$_1^+$, pp. 239–242] that

$$\pi_\gamma \leq \pi_\rho \leq \sqrt{\tfrac{\pi}{2}}\, \pi_\gamma.$$

6.3.6.16 The theory of γ-summing operators was developed by Linde/Pietsch [1974], who proved, among others results, that $\mathfrak{P}_p \subset \mathfrak{P}_\gamma$. One has $\mathfrak{P}_2(X,Y) = \mathfrak{P}_\gamma(X,Y)$ for all Banach spaces X if and only if Y is of Rademacher/Gauss cotype 2. The ideals $\mathfrak{P}_\gamma$ and $\mathfrak{P}_{p,2}$ with $p > 2$ are incomparable.

6.3.6.17 As indicated in 6.1.2.6, the γ-summing norm plays a significant role in the local theory of Banach spaces. This fact was first observed by Figiel/Tomczak-Jaegermann [1979, p. 156], who introduced the quantity

$$\ell(T) = \left(\int_{l_2^n} \|Tx\|^2 d\gamma^{(n)}(x) \right)^{1/2} \quad \text{for } T \in \mathfrak{L}(l_2^n, X),$$

where $\gamma^{(n)}$ denotes the standard Gaussian probability measure on l_2^n. However, according to Linde/Pietsch [1974, p. 450], this is just $\pi_\gamma(T)$.

6.3.6.18 It follows from König [1980c, p. 313] and Pietsch [1980b, p. 157] that

$$(1+\log n)^{1/2} x_n(T) \le c\,\pi_\gamma(T) \quad \text{for } n=1,2,\ldots \text{ and } T \in \mathfrak{P}_\gamma,$$

where $c>0$ is a universal constant. Hence γ-summing operators are strictly singular.

The inequality above cannot be improved, since $D_t : (\xi_k) \mapsto \big((1+\log k)^{-1/2}\xi_k\big)$ is γ-summing on l_∞; see Linde/Pietsch [1974, p. 459].

6.3.6.19 A deep result of Pajor/Tomczak-Jaegermann [1986, p. 639] says that

$$n^{1/2} d_n(T) \le c\pi_\gamma(T) \quad \text{for } n=1,2,\ldots \text{ and } T \in \mathfrak{L}(l_2,X),$$

where $c>0$ is a universal constant; see also Gordon [1987, p. 89] and [PIS$_2$, pp. 75–80].

6.3.7 *p*-Nuclear and *p*-integral operators

6.3.7.1 Let $1 \le p < \infty$. An operator $T \in \mathfrak{L}(X,Y)$ is called ***p*-nuclear** if it can be written in the form

$$T = \sum_{k=1}^{\infty} x_k^* \otimes y_k \quad \text{with } (x_k^*) \in [l_p, X^*] \text{ and } (y_k) \in [w_{p^*}, Y]. \tag{6.3.7.1.a}$$

These operators form the Banach ideal $\mathfrak{N}_p$, the ideal norm being given by

$$\nu_p(T) := \inf \|(x_k^*)|l_p\|\,\|(y_k)|w_{p^*}\|, \tag{6.3.7.1.b}$$

where the infimum extends over all p-nuclear representations. The limiting case $p=1$ yields the ideal of nuclear operators; see 5.7.3.1, and for $p=\infty$ we assume that $(x_k^*) \in [c_0, X^*]$ and $(y_k) \in [w_1, Y]$.

Operators $T \in \mathfrak{N}_p(X,Y)$ are characterized by the property that they admit a factorization

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & Y \\ l_\infty \arrow[r, "D_t"'] & l_p \arrow[u, "B"'] \end{tikzcd} \tag{6.3.7.1.c}$$

where $D_t : (\xi_k) \mapsto (\tau_k \xi_k)$ is generated from a sequence $t = (\tau_k) \in l_p$.

The concept of a p-nuclear operator was introduced by Persson/Pietsch [1969, p. 21]. However, their definition goes back to earlier work of Saphar [1970] and Chevet [1968, 1969] about the tensor norms g_p. Saphar [1966, pp. 125, 132] had already treated the case g_2. In this way, he obtained the *applications de Hilbert–Schmidt à gauche*.

6.3.7.2 For finite rank operators, an ideal norm ν_p° is defined by taking the infimum in (6.3.7.1.b) just over all *finite* representations; compare with 5.7.3.7. Then $\nu_p(T) \le \nu_p^\circ(T)$, and equality holds for operators between Minkowski spaces. This phenomenon gave rise to the introduction of various kinds of approximation properties depending on a parameter $1 \le p \le \infty$; see Reĭnov [1982]. In the case $p=1$ we get the classical concepts, and for $p=2$ everything is fine in *all* Banach spaces; see Pietsch [1968b, p. 242].

It turns out that $\mathfrak{N}_p$ is the minimal Banach ideal associated with the elementary ideal norm ν_p°; see 6.3.4.5.

6.3.7.3 Trace duality yields $\mathfrak{N}_p^* = \mathfrak{P}_{p^*}$. In order to preserve this formula in the limiting case $p=1$, we let $\mathfrak{P}_\infty := \mathfrak{L}$.

For Banach spaces with the approximation property, the formula $\mathfrak{N}_p^* = \mathfrak{P}_{p^*}$ was proved by Persson/Pietsch [1969, p. 50]; the general case is due to Pietsch [1971, p. 202].

6.3.7.4 In order to determine the maximal hull of $\mathfrak{N}_p$, one needs a "continuous" counterpart of (6.3.7.1.c).

An operator $T \in \mathfrak{L}(X,Y)$ is called ***p*-integral** if there exists a diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & Y \arrow[r, "K_Y"] & Y^{**} \\ C(K) \arrow[rr, "Id"] & & L_p(K,\mu) \arrow[u, "B"'] \end{tikzcd} \qquad (6.3.7.4.a)$$

with $\mu(K)=1$. The lower arrow may be replaced by $L_\infty(M,\mathscr{M},\mu) \xrightarrow{Id} L_p(M,\mathscr{M},\mu)$, where $(M,\mathscr{M},\mu)$ is a suitable probability space, $\mu(M)=1$.

Pietsch [1971, pp. 203–205] referred to these operators as *regulär-p-integral*, and he showed that they form a maximal Banach ideal $\mathfrak{I}_p$ under the ideal norm

$$\iota_p(T) := \inf \|B\|\|A\|.$$

As usual, the infimum is taken over all possible factorizations.

Originally, Persson/Pietsch [1969, p. 27] defined p-integral operators in such a way that they can be characterized by a diagram without the injection $K_Y : Y \to Y^{**}$. Nowadays, those operators are said to be **strictly *p*-integral**.

Sophisticated examples of Reĭnov [1982, p. 133] show that, apart from the case $p=2$, one indeed gets smaller ideals.

The following formulas explain why requiring a factorization (6.3.7.4.a) yields the "better" concept

$$\mathfrak{N}_p^{\max} = \mathfrak{I}_p \quad \text{and} \quad \mathfrak{P}_p^* = \mathfrak{I}_{p^*}.$$

The 1-integral operators are just the integral operators defined in 5.7.3.9, and in the limiting case $p=\infty$ we get $\mathfrak{I}_\infty = \mathfrak{L}_\infty$.

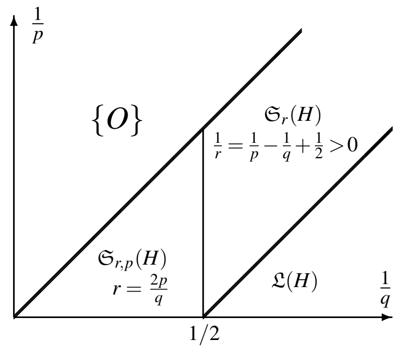
6.3.7.5 The diagrams (6.3.6.7.a) and (6.3.7.4.a) imply that $\mathfrak{I}_p^{\text{inj}} = \mathfrak{P}_p$; see Persson/Pietsch [1969, p. 40].

6.3.7.6 The fruitful interplay between ideal theory and harmonic analysis is indicated by the following example due to Pełczyński [1969b, p. 67]. Given any subset Λ of $\mathbb{Z}$, we denote by $C^{\Lambda}(\mathbb{T})$ and $L_p^{\Lambda}(\mathbb{T})$ the closed subspaces of $C(\mathbb{T})$ and $L_p(\mathbb{T})$, respectively, that are spanned by all characters e^{int} with $n \in \Lambda$. Then the p-summing embedding map $Id : C^{\Lambda}(\mathbb{T}) \to L_p^{\Lambda}(\mathbb{T})$ is p-integral if and only if $L_p^{\Lambda}(\mathbb{T})$ is complemented in $L_p(\mathbb{T})$. Thus $\mathfrak{I}_p$ with $p \neq 2$ is properly contained in $\mathfrak{P}_p$, since there are Λ's failing the property above; see [ROS, p. 36]. On the other hand, we have $\mathfrak{I}_2 = \mathfrak{P}_2$.

6.3.8 Specific components of operator ideals

Different operator ideals $\mathfrak{A}$ and $\mathfrak{B}$ may have components $\mathfrak{A}(X,Y)$ and $\mathfrak{B}(X,Y)$ that coincide for specific Banach spaces X and Y.

6.3.8.1 Of particular interest is the problem to determine the Hilbert space component of a given ideal. Thanks to the efforts of several authors, we know that $\mathfrak{P}_{p,q}(H)$ coincides with a Schatten–von Neumann ideal as indicated in the following diagram:



Pietsch [1967, p. 340] and Pelczyński [1967, p. 355] showed that for all p's, the p-summing operators are just the Hilbert–Schmidt operators. In [1968a, p. 334], Kwapień stated a theorem that summarizes the basic results about $\mathfrak{P}_{p,q}(H)$; most of them were attributed to Mityagin (unpublished). Finally, Bennett [1976, p. 30] and Bennett/Goodman/Newman [1975] settled the open case $2 < q < p < \infty$ by rejecting the conjecture that $\mathfrak{P}_{p,q}(H) = \mathfrak{S}_r(H)$ with $r = 2p/q$. In order to get the correct formula, namely $\mathfrak{P}_{p,q}(H) = \mathfrak{S}_{r,p}(H)$, they used probabilistic techniques. This was a real breakthrough; see 7.3.1.9.

I stress the fact that the study of the ideal $\mathfrak{P}_{p,q}$, which is defined via classical sequence spaces, automatically leads to the concept of a Lorentz sequence space.

6.3.8.2 The formula $\mathfrak{P}_p(H) = \mathfrak{S}_2(H)$ for $1 \le p < \infty$ was extended by Kwapień [1970a, p. 109]:

$$\mathfrak{P}_q(X,l_v) = \mathfrak{P}_2(X,l_v) \quad \text{for all Banach spaces } X,\ 1 \le v \le 2 \text{ and } 2 < q < \infty,$$

$$\mathfrak{P}_p(l_u,Y) = \mathfrak{P}_2(l_u,Y) \quad \text{for all Banach spaces } Y,\ 1 \le u \le 2 \text{ and } 1 \le p < 2.$$

Obviously, l_u and l_v may be replaced by function spaces L_u and L_v, respectively.

Maurey [MAU, p. 90] showed that if Y has cotype 2, then

$$\mathfrak{P}_q(X,Y)=\mathfrak{P}_2(X,Y) \quad \text{for all Banach spaces } X \text{ and } 2<q<\infty.$$

Trace duality yields that if X has cotype 2, then

$$\mathfrak{P}_p(X,Y)=\mathfrak{P}_2(X,Y) \quad \text{for all Banach spaces } Y \text{ and } 1<p<2.$$

The latter formula holds even for $0<p<2$; see [MAU, p. 116]. Combining the previous results, we infer that the component $\mathfrak{P}_p(X,Y)$ does not depend on $0<p<\infty$ whenever X and Y have cotype 2.

6.3.8.3 Let X be any Banach space and $0<p<\infty$. Then, according to Maurey [MAU, p. 114],

$$\mathfrak{P}_{p_0}(X,Y)=\mathfrak{P}_p(X,Y) \quad \text{for all Banach spaces } Y \text{ and } \textit{some } p_0<p$$

implies that

$$\mathfrak{P}_{p_0}(X,Y)=\mathfrak{P}_p(X,Y) \quad \text{for all Banach spaces } Y \text{ and } \textit{every } p_0<p.$$

Obviously, it is enough when Y ranges over all $C(K)$'s, even c_0 or l_∞ will do.

Let $P(X)$ denote the set of all p's with the above property. Then, for every infinite-dimensional Banach space X, we either have $P(X)=\big(0,q(X)^*\big)$ or $P(X)=(0,2]$; see [MAU, p. 113]. Here $q(X)$ denotes the infimum of all q's for which the identity map of X is $(q,1)$-summing. Only in [1976, p. 54] did Maurey/Pisier realize that $q(X)$ coincides with the cotype index discussed in 6.1.7.9.

Note that $(0,1)\subseteq P(X)\subseteq(0,2]$. The case $P(X)=(0,2]$ certainly occurs when X has cotype 2; see the diagram below. The other limiting case, namely $P(X)=(0,1)$, is equivalent to the property that X contains l_∞^n's uniformly; see [MAU, p. 115].

6.3.8.4 In the course of time, it became fashionable to state Maurey's results in a version that is obtained by trace duality.

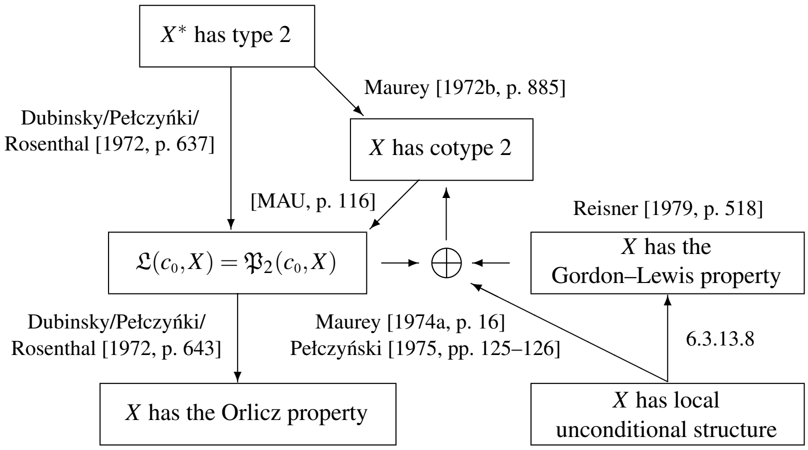
Let $Q(X)$ denote the set of all q's such that $\mathfrak{L}(c_0,X)=\mathfrak{P}_q(c_0,X)$. Then the elements $p\in P(X)\cap(1,\infty)$ and $q\in Q(X)$ are related by $1/p+1/q=1$, which means that the p's less than or equal to 1 are getting lost.

In the case $2<q(X)<\infty$, we have $Q(X)=\big(q(X),\infty\big)$. This formula extends an earlier result of Kwapień [1970b, p. 199]:

$$\mathfrak{L}(L_\infty,L_{q_0})=\mathfrak{P}_q(L_\infty,L_{q_0}) \;\text{ if }\; 2<q_0<q<\infty \;\text{ and }\; \mathfrak{L}(L_\infty,L_{q_0})\neq\mathfrak{P}_{q_0}(L_\infty,L_{q_0}).$$

The set $Q(X)$ turns out to be empty if and only if X contains l_∞^n's uniformly; see Maurey/Pisier [1976, pp. 66–67]. According to Maurey [1973, p. 3], we have $Q(L_\Phi)=(2,\infty)$ for the Orlicz space L_Φ associated with $\Phi(u)=u^2\log(1+u)$; see Subsection 6.7.14.

Finally, I present a diagram that illustrates the situation when $Q(X) = [2, \infty)$; see also 6.3.14.4:



The symbol $\oplus$ in the diagram above means that the left-hand property *plus* one of the right-hand properties imply that X has cotype 2.

Talagrand [1992, p. 31] constructed a sophisticated Banach lattice that disproved the original conjecture that the Orlicz property of X implies $\mathfrak{L}(c_0, X) = \mathfrak{P}_2(c_0, X)$ or even cotype 2; see [MAU, p. 151]. The challenging question whether X has cotype 2 *if* and only if $\mathfrak{L}(c_0, X) = \mathfrak{P}_2(c_0, X)$ is still open.

6.3.9 Grothendieck's theorem

When "lazy" mathematicians say that Theorem A and Theorem B are equivalent, then they just want to stress the fact that Theorem A and Theorem B are hard to prove, but can be deduced from each other quite easily. In this sense, the open mapping theorem and the closed graph theorem are equivalent; compare with 2.5.2 and 7.5.20. Further examples will follow now.

6.3.9.1 One of the most profound results in Banach space theory is

Grothendieck's théorème fondamental
de la théorie métrique des produits tensoriels;

[1956b, p. 59]. Translated into the language of operator ideals, its original version says that the identity map of any Hilbert space belongs to the injective and surjective hull of $\mathfrak{I}$. The following equivalent formulations can be found on pp. 59–60 of the *Résumé*:

$$\mathfrak{L}(L_1, L_2) = \mathfrak{P}_1(L_1, L_2) \qquad (6.3.9.1.a)$$

and

$$\mathfrak{L}_1 \circ \mathfrak{L}_\infty = \mathfrak{L}_2^*. \qquad (6.3.9.1.b)$$

The formula (6.3.9.1.a) splits into two parts, namely

$$\mathfrak{L}(L_1,L_2)=\mathfrak{P}_2(L_1,L_2) \tag{6.3.9.1.c}$$

and

$$\mathfrak{P}_2(L_1,L_2)=\mathfrak{P}_1(L_1,L_2). \tag{6.3.9.1.d}$$

One refers to (6.3.9.1.c) as the **"little" Grothendieck theorem**, and (6.3.9.1.d) was already discussed in 6.3.8.2. Equivalent versions of the little Grothendieck theorem say that

$$\mathfrak{L}_1(H)=\mathfrak{S}_2(H) \quad \text{and/or} \quad \mathfrak{L}_\infty(H)=\mathfrak{S}_2(H) \quad \text{for any Hilbert space } H;$$

see Grothendieck [1956b, p. 55].

6.3.9.2 Grothendieck's theorem belongs to the local theory of Banach spaces, since it can be formulated in terms of inequalities that contain only a finite number of elements. The point is that these inequalities hold uniformly for all $n=1,2,\dots$. Let us agree once for all that the appearing constants are chosen as small as possible.

In his theory of $\otimes$-norms, Grothendieck [1956b, pp. 72–73] considered five universal constants, and he remarked:

It n'est d'auilleurs pas certains que ces constantes soient les mêmes dans la "théorie réelle" et la "théorie complexe".

Since this conjecture seems to be true, we sometimes distinguish the two cases by adding the superscripts $^{\mathbb{R}}$ and $^{\mathbb{C}}$.

6.3.9.3 The finite-dimensional version of the *little* Grothendieck theorem says that there exists a constant K_g (originally: $\sqrt{\sigma}$) such that

$$\pi_2(T:l_\infty^n\to l_2^n)\le K_g\|T:l_\infty^n\to l_2^n\|.$$

The exact values are known:

$$K_g^{\mathbb{R}}=\sqrt{\tfrac{\pi}{2}}=1.253\dots \quad \text{and} \quad K_g^{\mathbb{C}}=\sqrt{\tfrac{4}{\pi}}=1.128\dots .$$

The real constant was determined by Grothendieck [1956b, p. 51], whereas his conclusion in the complex case contained a slip of the pen.

6.3.9.4 There are further constants such that

$$\lambda_2(T:l_\infty^n\to l_1^n)\le K_H\|T:l_\infty^n\to l_1^n\|,$$

$$\pi_2(T:l_\infty^n\to l_1^n)\le K_M\|T:l_\infty^n\to l_1^n\|,$$

and

$$\lambda_2^*(T:l_\infty^n\to l_1^n)\le K_G\,\|T:l_\infty^n\to l_1^n\|.$$

Here K_G (originally: h) is the "famous" **Grothendieck constant** from p. 59 of the *Résumé*, and the number K_H (originally: k) can be found on p. 61. The constant K_M does not appear in Grothendieck's work. It is just the $(2,1)$-mixing norm of the identity map of l_1, in the sense of [PIE$_3$, p. 279]. Apart from the trivial inequality $K_H \le K_M \le K_G$, our knowledge about these constants is very poor. We have

$$K_g^2 \le K_G \le K_g K_M \quad \text{and} \quad K_g \le K_M \le K_g K_H.$$

6.3.9.5 The main steps in finding upper and lower bounds for the real and complex Grothendieck constant are indicated in the following diagram:

folklore Davie (unpublished) Grothendieck [1956b, p. 59]

$$\tfrac{4}{\pi} = 1.273\ldots < 1.338 < K_G^{\mathbb{C}} < 1.405 < 1.527 < \tfrac{\pi}{2} = 1.570\ldots < K_G^{\mathbb{R}} < 1.783 < 2.261 < 2.302$$

Haagerup [1987, p. 217] Krivine [1979, p. 17]

Pisier [1978c, p. 412] Rietz [1974, p. 275]

Grothendieck [1956b, p. 59]

Adapting the technique of Rietz to the complex case, Kaijser [1973, p. 10] obtained the estimate $K_G^{\mathbb{C}} < 1.607$. However, this was not enough to imply $K_G^{\mathbb{C}} < K_G^{\mathbb{R}}$. One had to wait until Pisier's paper [1978c]. Some information on the unpublished work of Davie can be found in an article of König [1991, pp. 196–197].

6.3.9.6 The fourth Grothendieck constant K_L (originally: ℓ) is defined by

$$(\lambda_1 \circ \lambda_\infty)^* \le K_L\, \lambda_1 \circ \lambda_\infty;$$

see [1956b, p. 61]. In the finite-dimensional setting, we get

$$(\lambda_1 \circ \lambda_\infty)^*(T : l_\infty^n \to l_1^n) \le K_L \|T : l_\infty^n \to l_1^n\|.$$

A symmetric version reads as follows:

$$|\operatorname{trace}(ASBT)| \le K_L \|A\| \|S\| \|B\| \|T\| \quad \text{whenever} \quad \begin{tikzcd} l_\infty^n \arrow[r, "T"] & l_1^n \arrow[d, "B"] \\ l_1^n \arrow[u, "A"] & l_\infty^n \arrow[l, "S"'] \end{tikzcd}.$$

Apart from $K_H \le K_L$, it seems to be unknown how K_L is placed among the constants $K_H \le K_M \le K_G$. Grothendieck knew that $K_L \le K_G K_H$. We also have $K_L \le K_M^2$.

6.3.9.7 Grothendieck himself conjectured that his fifth constant ρ, which was defined by $\lambda_2 \le \rho \lambda_2^*$, is superfluous; see [1956b, p. 47]:

Il n'est d'ailleurs pas exclu qu'on ait même $\rho = 1$.

It follows from Kwapień's formula $\lambda_2^* = \pi_2^{\text{dual}} \circ \pi_2$ that this is indeed so; see 6.3.6.14.

6.3.9.8 Grothendieck's sketchy *"Démonstration du théorème fondamental"* can be found on pp. 62–63 of his *Résumé*. The next step was done in a seminal paper of Lindenstrauss/Pełczyński [1968, pp. 279–280], who gave a formulation in terms of inequalities:

Let $\{a_{i,j}\}_{i,j=1,2,\dots,N}$ be a real-valued matrix and let M be a positive number such that

$$\Big|\sum_{i,j=1}^{N} a_{i,j} t_i s_j\Big| \leq M$$

for every real $\{t_i\}_{i=1}^N$ and $\{s_j\}_{j=1}^N$ satisfying $|t_i| \leq 1$ and $|s_j| \leq 1$. Then for arbitrary vectors $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ in a real inner product space

$$\Big|\sum_{i,j=1}^{N} a_{i,j}(x_i, y_j)\Big| \leq K_G M \sup_i \|x_i\| \sup_j \|y_j\|,$$

where K_G is the Grothendieck constant.

Therefore many people speak of **Grothendieck's inequality**.

Further proofs are contained in all papers that produce upper estimates of K_G, as presented in 6.3.9.5. Last but not least, I mention the elegant Pełczyński/Wojtaszczyk approach described in 6.7.12.22.

6.3.10 Limit order of operator ideals

6.3.10.1 The **diagonal operator** $D_\lambda : (\xi_n) \mapsto (\frac{1}{n^\lambda}\xi_n)$ acts from l_p into l_q whenever $\lambda > \lambda_0 := (1/q - 1/p)_+$ and $1 \leq p, q \leq \infty$.

According to [PIE$_2$, p. 126], the **diagonal limit order** of a quasi-Banach ideal $\mathfrak{A}$ is defined by

$$\lambda_{\text{diag}}(p,q|\mathfrak{A}) := \inf\big\{\, \lambda \geq 0 : D_\lambda \in \mathfrak{A}(l_p, l_q) \,\big\}.$$

König [1974, Part I, p. 53] observed that

$$\big|\lambda_{\text{diag}}(p_1,q_1|\mathfrak{A}) - \lambda_{\text{diag}}(p_0,q_0|\mathfrak{A})\big| \leq \big|1/p_1 - 1/p_0\big| + \big|1/q_1 - 1/q_0\big|.$$

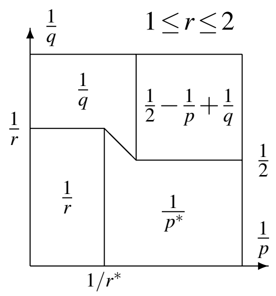
Hence $\lambda_{\text{diag}}(p,q|\mathfrak{A})$ is a continuous function of $1/p$ and $1/q$ on $[0,1]\times[0,1]$.

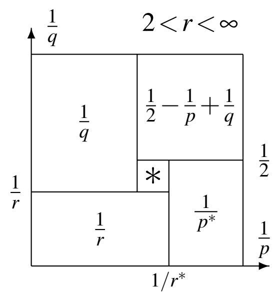
Though this function only reflects simple properties of simple operators between simple spaces, it provides useful information about the underlying ideal. The close connection with the theory of function spaces is discussed in 6.7.8.10.

6.3.10.2 The diagonal limit order is determined by the behavior of $\|Id : l_p^m \to l_q^m|\mathfrak{A}\|$ as $m \to \infty$. More precisely, König [1974, Part II, p. 67] showed that

$$\lambda_{\text{diag}}(p,q|\mathfrak{A}) = \inf\Big\{\lambda \geq 0 : \|Id : l_p^m \to l_q^m|\mathfrak{A}\| \preceq m^\lambda\Big\}.$$

6.3.10.3 The limit order $\lambda(p,q|\mathfrak{P}_r)$, viewed as a function of $1/p$ and $1/q$, is illustrated in the following diagrams, [PIE$_2$, pp. 312–313]:





Here $*$ stands for $\frac{1}{r}+\frac{(1/r^*-1/p)(1/q-1/r)}{1/2-1/r}$.

6.3.10.4 The limit order of $\mathfrak{P}_{r,1}$ was determined by Carl/Maurey/Puhl [1978, pp. 214, 216]. Their work is based, inter alia, on a theorem of Bennett [1973, pp. 21–22] and Carl [1974, pp. 356, 358], which says that the embedding $Id:l_p \to l_q$ is $(r,1)$-summing, where

$$1/r = \begin{cases} 1/2+1/p-1/q & \text{if } 1 \le p \le q \text{ and } 1 \le q \le 2, \\ \quad 1/p & \text{if } 1 \le p \le q \text{ and } 2 \le q \le \infty. \end{cases}$$

The special cases $Id:l_p \to l_p$ and $Id:l_1 \to l_2$ were earlier studied by Orlicz 5.1.1.6 and Grothendieck 6.3.9.1.

6.3.10.5 The limit order $\lambda_{\text{diag}}(p,q|\mathfrak{L}_r^{(s)})$ is a logarithmically convex, non-decreasing and continuous function of $1/r$ for any choice of additive s-numbers $\boldsymbol{s} : T \mapsto \big(s_n(T)\big)$. More precisely, the following situation occurs.

Letting $r_0 := \inf\big\{r \;:\; D_\lambda \in \mathfrak{L}_r^{(s)}(l_p,l_q) \text{ if } \lambda > \lambda_0 := (1/q-1/p)_+\big\}$, we can prove that the function

$$\tfrac{1}{r} \to \lambda = \lambda_{\text{diag}}(p,q|\mathfrak{L}_r^{(s)})$$

takes the constant value λ_0 on $\big(0,\frac{1}{r_0}\big]$ and is strictly increasing on $\big[\frac{1}{r_0},\infty\big)$. The inverse function has the form

$$\lambda \to \tfrac{1}{r} = \rho_{\text{diag}}(\lambda,p,q|\boldsymbol{s}) \quad \text{for } \lambda > \lambda_0,$$

where $\rho_{\text{diag}}(\lambda,p,q|\boldsymbol{s})$ is the limit order of the underlying s-numbers; see 6.2.5.3. In this way, the diagrams presented in 6.2.5.4 and 6.2.5.6 pass into corresponding diagrams for $\lambda_{\text{diag}}(p,q|\mathfrak{L}_r^{\text{gel}})$ and $\lambda_{\text{diag}}(p,q|\mathfrak{L}_r^{\text{weyl}})$, respectively.

6.3.11 Banach ideals and tensor products

6.3.11.1 As usual, $E = E^{**}$ and $F = F^{**}$ denote finite-dimensional spaces. Then, identifying the formal expressions

$$\sum_{k=1}^{n} x_k^* \otimes y_k \quad \text{and} \quad \sum_{k=1}^{n} x_k \otimes y_k$$

with the operators

$$x \mapsto \sum_{k=1}^{n} \langle x, x_k^* \rangle y_k \quad \text{and} \quad x^* \mapsto \sum_{k=1}^{n} \langle x^*, x_k \rangle y_k,$$

we get

$$E^* \otimes F = \mathfrak{L}(E,F) \quad \text{and} \quad E \otimes F = \mathfrak{L}(E^*,F),$$

respectively. This process establishes a one-to-one correspondence between

tensor norms and **elementary ideal norms**.

In view of this observation, the symbols $\alpha, \beta, \ldots$ may stand for both.

6.3.11.2 The relationship between minimal Banach ideals, maximal Banach ideals, and elementary ideal norms, described in 6.3.4.5, carries over to this new setting:

$$\begin{tikzcd} & \alpha \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{minimal} \quad \mathfrak{S}_\alpha^\circ \arrow[rr, leftrightarrow] & & \mathfrak{S}_\alpha \quad \text{maximal} \end{tikzcd}$$

Grothendieck [1956b, p. 16] referred to the members of $\mathfrak{S}_\alpha^\circ$ as *applications α-nucléaire*, while operators in $\mathfrak{S}_\alpha$ were called *α-applications* or *applications de type α*; see [1956b, p. 12]. The term *α-integral* is also in use. In the case of the largest elementary ideal norm $\nu = \iota$ one gets the usual nuclear and integral operators.

The *Grothendieck ideals* in the Urbana lecture notes [LOTZ, p. 36] are just the maximal Banach ideals.

6.3.11.3 Grothendieck [1956b, pp. 9–10, 28] invented six methods how to obtain new tensor norms from old ones. Here are the "idealistic" counterparts, namely

$$\begin{aligned}
\alpha^{\text{dual}}(T) &:= \alpha(T^*),\\
\alpha^*(T) &:= \sup\big\{\, |\text{trace}(ST)| : \alpha(S : F \to E) \le 1 \,\big\},\\
\alpha^{\text{inj}}(T) &:= \sup\big\{\, \alpha(BT) : \|B : F \to l_\infty^n\| \le 1,\ n = 1,2,\ldots \,\big\},\\
\alpha^{\text{sur}}(T) &:= \sup\big\{\, \alpha(TA) : \|A : l_1^n \to E\| \le 1,\ n = 1,2,\ldots \,\big\},\\
\alpha^{\text{ext}}(T) &:= \inf\big\{\, \alpha(T_0)\|A\| : T = T_0 A,\ \ E \xrightarrow{A} l_\infty^n \xrightarrow{T_0} F,\ n \ge \text{rank}(T) \,\big\},\\
\alpha^{\text{lift}}(T) &:= \inf\big\{\, \|B\|\alpha(T_0) : T = BT_0,\ \ E \xrightarrow{T_0} l_1^n \xrightarrow{B} F,\ n \ge \text{rank}(T) \,\big\}.
\end{aligned}$$

The following "dictionary" provides the relationship to Grothendieck's original notation; see also [DEF$^+$, pp. 263–265] and Diestel/Fourie/Swart [2002/03, Part 5, pp. 495–497]:

${}^{\mathrm{t}}\alpha$	α'	$({}^{\mathrm{t}}\alpha)'$	$/\alpha$	$\alpha\backslash$	$\backslash\alpha$	$\alpha/$
α^{dual}	$(\alpha^{\mathrm{dual}})^*$	α^*	α^{sur}	α^{inj}	α^{ext}	α^{lift}

Unfortunately, the terminology is chaotic. Grothendieck referred to ${}^{\mathrm{t}}\alpha$ as the $\otimes$-norm *transposée*. He called α' the $\otimes$-norm *duale*, while Schatten [1946, p. 73] used the term *associate*; see also [SCHA$_1$, p. 26] and [SCHA$_2$, p. 55]. On the other hand, we have the *dual* ideal norm α^{dual} and the *adjoint* ideal norm α^*. The latter terms were coined by myself: *mea culpa, mea culpa, mea maxima culpa*. My excuse: I wanted to say that π_2 is self-adjoint.

6.3.11.4 In his *Résumé* [1956b, p. 35], Grothendieck proposed a crucial definition:

Considérons le plus petit ensemble Φ de $\otimes$-normes qui contient la $\otimes$-norme fondamentale $\vee$, et est stable par les opérations $\alpha \to \alpha'$, ${}^{\mathrm{t}}\alpha$, $/\alpha$.

The set Φ contains, inter alia, the following $\otimes$-norms:

$\vee$	$\backslash\!\vee$	$\vee\!/$	$\backslash\!\vee\!/$	$/\!\wedge\!\backslash$	$/\!\wedge$	$\wedge\!\backslash$	$\wedge$
$\lVert\cdot\rVert$	λ_∞	λ_1	$\lambda_1\circ\lambda_\infty$	$(\lambda_1\circ\lambda_\infty)^*$	π_1^{dual}	π_1	ν

Further members can be found in 6.3.11.6.

6.3.11.5 In a subsequent conjecture, Grothendieck said on p. 39:

Il est bien probable que l'ensemble Φ lui-même est infini.

This problem is a special case of the following question.

Consider the semi-group generated by the operations

$$a:\alpha\mapsto\alpha^*,\quad d:\alpha\mapsto\alpha^{\mathrm{dual}},\quad i:\alpha\mapsto\alpha^{\mathrm{inj}},\quad j:\alpha\mapsto\alpha^{\mathrm{sur}}, x:\alpha\mapsto\alpha^{\mathrm{ext}},\quad y:\alpha\mapsto\alpha^{\mathrm{lift}}.$$

In view of $j = did$, $x = aia$, and $y = aja = adida$, it suffices to work with a, d, and i. We have

$$a^2 = \mathbf{1},\quad d^2 = \mathbf{1},\quad i^2 = i,\quad ad = da,\quad aiai = iaia,\quad didi = idid.$$

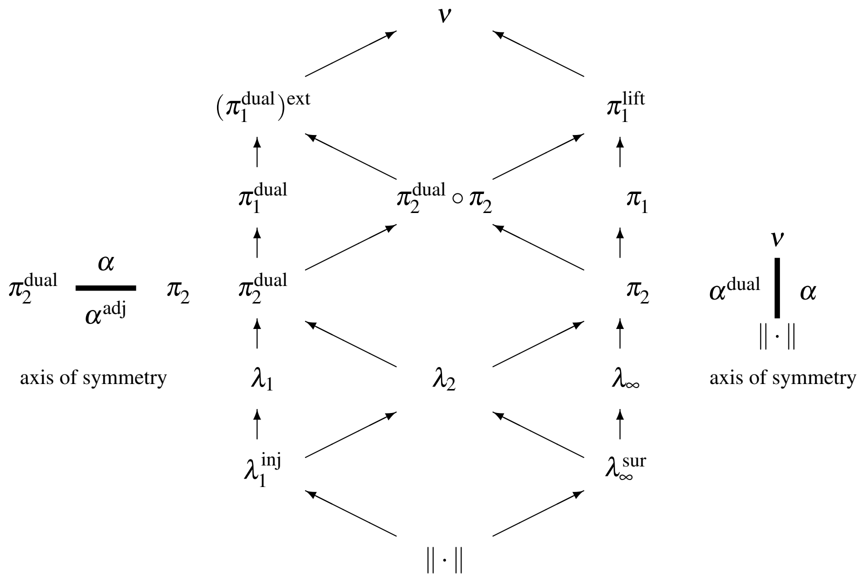
Further relations seem to be unknown. To the best of my knowledge, no attention has been paid to the problem whether the semi-group is finite. This sin of omission is not so aggravating because of the result presented in the next paragraph.

6.3.11.6 Given elementary ideal norms α and β, we write $\alpha \preceq \beta$ if there exists a constant $c \geq 1$ such that

$$\alpha(T) \leq c\beta(T) \quad \text{for all } T\in\mathfrak{L}(E,F) \text{ and arbitrary spaces } E \text{ and } F.$$

Equivalence $\alpha \asymp \beta$ means that $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

Following Grothendieck [1956b, p. 35], *on appelle $\otimes$-norme* **naturelle** *une $\otimes$-norme équivalente à une $\otimes$-norme de l'ensemble* Φ. Then Φ splits into 14 equivalence classes.



Note that the nuclear norm ν and the integral norm ι coincide for operators between finite-dimensional spaces. The $\otimes$-norm version of the table above is due to Grothendieck [1956b, p. 37], and a mixture can be found in [DEF$^+$, p. 362].

The *théorème fondamentale de la théorie métrique des produits tensoriels* 6.3.9.1 tells us that

$$\pi_2 \asymp \lambda_2 \circ \lambda_\infty \quad \text{and} \quad \pi_2^{\text{dual}} \circ \pi_2 \asymp \lambda_1 \circ \lambda_\infty .$$

These relations ensure that one gets only a finite number of equivalence classes. Hence Grothendieck's theorem is indeed a *théorème fondamentale*.

The arrows point from the smaller ideal norms to the larger ones, in the sense of the preordering $\preceq$. Grothendieck conjectured that there are no further relations as indicated above. However, he was not able to decide whether π_1 dominates λ_1; see [1956b, p. 72]. This problem was solved by Gordon/Lewis [1974, p. 40], who showed that $\lambda_1 \not\preceq \pi_1$. Nowadays, we know from [PEŁ, pp. 24–25] that $Id : A(\mathbb{T}) \to H_1(\mathbb{T})$ is 1-summing but fails to be L_1-factorable.

6.3.11.7 As described in Subsection 5.7.2, Schatten/von Neumann were the first to study tensor products of Banach spaces. Subsequently, Schatten published *A Theory of Cross-Spaces* (1950).

In his thesis [GRO$_1$, Introduction, p. 4], Grothendieck wrote:

Ce travail est assez peu lié à celui de R. Schatten sur les produits tensoriels d'espaces de Banach.

The bibliography of the *Résumé* also contains Schatten's book, [SCHA_1]. However, not a single result of Schatten/von Neumann is used. Quite likely, because of the bad scientific communication in the postwar period, the French mathematician Grothendieck did not know about the American approach when he started his own research.

6.3.11.8 Reading Grothendieck's works is indeed a hard job. Therefore it took many years until his ideas became popular. The situation was changed by Lindenstrauss/Pełczyński [1968, p. 275], who stressed the following point of view:

Though the theory of tensor products has its intrinsic beauty we feel that the results of Grothendieck and their corollaries can be more clearly presented without the use of tensor products.

The pioneering work stressing this new aspect was Schatten's monograph *Norm Ideals of Completely Continuous Operators* (1960). The famous book of Gohberg/Kreĭn on *Non-Self-Adjoint Operators in Hilbert Space* (1965) inspired Pietsch to write a counterpart in the setting of Banach spaces; see [PIE_2] and [PIE_3]. The revised theory of nuclear locally convex spaces [PIE_1] and the study of absolutely summing operators had already shown that there exists a very handy alternative to tensor products: **operator ideals**.

The state of the art around 1980 was bewailed in a survey of Gilbert/Leih [1980, p. 183]:

Consequently, most of the subsequent development, with the exception perhaps of the work of Saphar and Chevet, has been in operator-theoretic terms with the Operator Ideal Theory taking the place of tensor products. This notes attempt to remedy the situation.

In their monograph *Tensor Norms and Operator Ideals*, Defant/Floret [DEF^+, p. 3] plead for a peaceful coexistence by saying:

We hope that we can convince the reader that both theories are more easily understood and also richer if one works with both simultaneously. It should become obvious that certain phenomena have their natural framework in tensor products and others in operator ideals.

A further attempt to pursue the theory of tensor products was made by Diestel/Fourie/Swart [2002/03] in a recent series of papers with the subtitle *Grothendieck's Résumé revisited.* Another reference is [RYAN].

Let us compare some *pros* and *cons*:

- The language of operator ideals is more appropriate for applications.
- Operators can be composed, which is not so easy for tensors and bilinear forms.
- Tensor product techniques yield minimal and maximal ideals, but there are plenty in between. This is true in particular when we consider closed ideals.

- The nuclear operators form the least ideal that can be obtained from a tensor norm. However, smaller ones are needed, for example in trace theory.
- Tensor products provide a deeper insight into trace duality.
- Tensor products proved to be an indispensable tool for the understanding of all phenomena related to the various approximation properties.
- As described in 5.7.2.10, tensor products can be used to represent spaces of X-valued functions:

$$[L_1(M,\mathscr{M},\mu),X] = L_1(M,\mathscr{M},\mu)\overset{\wedge}{\otimes}X \quad \text{and} \quad [C(K),X] = C(K)\overset{\vee}{\otimes}X.$$

6.3.12 Ideal norms computed with n vectors

6.3.12.1 Recall that the 2-summing norm $\pi_2(T)$ of an operator $T\in\mathfrak{P}_2(X,Y)$ is the least constant $c\geq 0$ such that

$$\Big(\sum_{k=1}^{n}\|Tx_k\|^2\Big)^{1/2} \leq c \sup_{\|x^*\|\leq 1}\Big(\sum_{k=1}^{n}|\langle x_k,x^*\rangle|^2\Big)^{1/2}$$

for any choice of $x_1,\dots,x_n\in X$. If $T\notin\mathfrak{P}_2(X,Y)$, then there is no constant that works for all n. However, for any *fixed* n we can always find some $c\geq 0$ such that the above inequality obtains as long as we consider no more than n members of X. Let $\pi_2^{(n)}(T)$ denote the infimum of such constants. What results is a non-decreasing sequence $\big(\pi_2^{(n)}(T)\big)$ with $\pi_2^{(n)}(T)\leq\sqrt{n}\,\|T\|$, whose growth measures just how non-2-summing the operator T is. All identity maps of infinite-dimensional spaces X are examples of the worst case: $\pi_2^{(n)}(X)=\sqrt{n}$.

6.3.12.2 In the Hilbert space setting, we obviously have

$$\pi_2^{(n)}(T) = \Big(\sum_{k=1}^{n}s_k(T)^2\Big)^{1/2}.$$

Hence $\pi_2(T_n)=\pi_2^{(n)}(T_n)$ whenever $\operatorname{rank}(T_n)\leq n$. This is no longer true in Banach spaces. However, Tomczak-Jaegermann [1979] proved a beautiful substitute, which says that

$$\pi_2(T_n)\leq 2\,\pi_2^{(n)}(T_n).$$

Szarek observed that the factor 2 can be replaced by $\sqrt{2}$; see [TOM, p. 171].

6.3.12.3 Similar results have been obtained for further operator ideals whose members are defined via inequalities. For example, König [1980d, p. 131] showed that

$$\pi_{q,2}(T_n)\leq c_q\,\pi_{q,2}^{(n)}(T_n) \quad \text{with} \quad c_q=\tfrac{c}{q-2}.$$

Defant/Junge [1992, p. 63] proved this estimate with a factor c_q, which remains bounded when $q\searrow 2$, namely $c_q\leq\sqrt{2}$.

6.3.12.4 In collaboration with Figiel, Pełczyński [1980, pp. 161–162] observed that an inequality of Tomczak-Jaegermann type does not hold for the 1-summing norm. The number of the required vectors grows exponentially,

$$\pi_1(T_n) \leq 3\,\pi_1^{(4^n)}(T_n).$$

6.3.12.5 Sequences of ideal norms defined on the whole of $\mathfrak{L}$ can be used to produce new operator ideals. For example, the condition

$$\pi_2^{(n)}(T) = o\big(\sqrt{n}\big)$$

yields a closed ideal whose members are strictly singular; see Pietsch [2002, p. 269].

6.3.12.6 The systematic use of so-called *gradations* of ideal norms was proposed by Pełczyński [1980, p. 165]. Further developments can be found in [TOM, Chap. 5] and [PIE$^+$, p. 30].

6.3.13 Operator ideals and classes of Banach spaces

In the present subsection, I show that the language of operator ideals offers manifold opportunities to describe specific properties of Banach spaces, giving this landscape a much more colorful view. Since this problematic has various different aspects, the following presentation is necessarily rather inhomogeneous.

6.3.13.1 By a **space ideal** A we mean a class of Banach spaces that is stable under passage to isomorphic copies, complemented subspaces, and finite direct sums. In addition, it is assumed that A contains all finite-dimensional spaces. This concept was invented by Stephani [1976].

Given any operator ideal $\mathfrak{A}$, the collection of all spaces X with $I_X \in \mathfrak{A}$ is a space ideal A.

6.3.13.2 In this text, operator ideals are denoted by gothic uppercase letters (Fraktur). From now on, the corresponding sans serifs will be used to denote the associated space ideals. Here are a few examples:

arbitrary operators	$\mathfrak{L} \to \mathsf{L}$	arbitrary spaces,
finite rank operators	$\mathfrak{F} \to \mathsf{F}$	finite-dimensional spaces,
completely continuous operators	$\mathfrak{V} \to \mathsf{V}$	spaces with the Schur property,
weakly compact operators	$\mathfrak{W} \to \mathsf{W}$	reflexive spaces,
Hilbertian operators	$\mathfrak{L}_2 \to \mathsf{L}_2$	isomorphic copies of Hilbert spaces,
$(2,1)$-summing operators	$\mathfrak{P}_{2,1} \to \mathsf{P}_{2,1}$	spaces with the Orlicz property.

6.3.13.3 Every space ideal A can be obtained from some operator ideal $\mathfrak{A}$. The least $\mathfrak{A}$ with this property consists of all operators that factor through a space $U \in \mathsf{A}$. That is, $T : X \to U \to Y$.

Different operator ideals may well yield the same space ideal. This is in particular true for $\mathfrak{A}$ and its closed hull $\overline{\mathfrak{A}}$; see [PIE$_3$, p. 65].

Recall from 5.2.3.7 that an operator ideal $\mathfrak{A}$ is called **proper** if it does not contain the identity map of any infinite-dimensional Banach space: $\mathfrak{A} \mapsto \mathsf{F}$. Here is a list of examples:

$$\mathfrak{F},\ \overline{\mathfrak{F}},\ \mathfrak{K},\ \dots\ ,\mathfrak{P}_p,\ \mathfrak{P}_{p,2},\ \mathfrak{P}_\gamma,\ \mathfrak{N}_p,\ \mathfrak{I}_p,\ \dots\ ,\mathfrak{L}_p^{\mathrm{app}},\ \dots\ ,\mathfrak{L}_p^{\mathrm{weyl}},\ \mathfrak{L}_p^{\mathrm{ent}},\ \dots\ .$$

6.3.13.4 An operator ideal is said to have the **factorization property** if every operator $T \in \mathfrak{A}$ factors through a space $U \in \mathsf{A}$. The most prominent example is the ideal of weakly compact operators; see 6.6.5.2. Further closed operator ideals with the factorization property can be found in a paper of Heinrich [1980b].

6.3.13.5 The space ideal $\mathsf{A}^{\mathrm{inj}}$ associated with the injective hull $\mathfrak{A}^{\mathrm{inj}}$ consists of all closed subspaces of members of A. Similarly, $\mathsf{A}^{\mathrm{sur}}$ is the collection of all quotients.

6.3.13.6 A Banach space X belongs to L_p if and only if X^{**} is isomorphic to a complemented subspace of some $L_p(M,\mathscr{M},\mu)$. Therefore, according to 6.1.4.4, we almost get the Lindenstrauss–Pełczyński–Rosenthal $\mathscr{L}_p$-spaces:

$$\mathsf{L}_1 = \mathscr{L}_1,\ \mathsf{L}_2 = \mathscr{L}_2,\ \mathsf{L}_\infty = \mathscr{L}_\infty,\ \text{but}\ \mathsf{L}_p = \mathscr{L}_p \cup \mathscr{L}_2 \quad \text{if } 1<p<\infty \text{ and } p\neq 2.$$

6.3.13.7 The following criteria are due to Kwapień [1972a, pp. 220–222]:

$$\begin{array}{lll}
Y \in \mathsf{L}_p & \text{if and only if } \mathfrak{P}_p^{\mathrm{dual}}(X,Y) \subseteq \mathfrak{I}_p(X,Y) & \text{for all Banach spaces } X,\\
Y \in \mathsf{L}_p^{\mathrm{inj}} & \text{if and only if } \mathfrak{P}_p^{\mathrm{dual}}(X,Y) \subseteq \mathfrak{P}_p(X,Y) & \text{for all Banach spaces } X,\\
Y \in \mathsf{L}_p^{\mathrm{sur}} & \text{if and only if } \mathfrak{I}_p^{\mathrm{dual}}(X,Y) \subseteq \mathfrak{I}_p(X,Y) & \text{for all Banach spaces } X,\\
Y \in (\mathsf{L}_p^{\mathrm{inj}})^{\mathrm{sur}} & \text{if and only if } \mathfrak{I}_p^{\mathrm{dual}}(X,Y) \subseteq \mathfrak{P}_p(X,Y) & \text{for all Banach spaces } X.
\end{array}$$

The case $p=2$ was earlier treated by Cohen [1970] (isometric characterization) and Kwapień [1970c] (isomorphic characterization).

6.3.13.8 Let $\mathsf{L}_{\mathrm{ust}}$ denote the ideal of all Banach spaces that have local unconditional structure in the sense of Gordon/Lewis. As stated in 6.1.5.4, spaces $X \in \mathsf{L}_{\mathrm{ust}}$ are characterized by the property that X^{**} is isomorphic to a complemented subspace of a Banach lattice.

In order to get a generating operator ideal $\mathfrak{L}_{\mathrm{ust}}$, it seems natural to consider the collection of all T's such that $K_Y T : X \xrightarrow{A} L \xrightarrow{B} Y^{**}$ with a suitable Banach lattice L; see [PIE$_3$, p. 328] and Reisner [1979, p. 507]. The corresponding ideal norm can be obtained in a canonical way, $\lambda_{\mathrm{ust}}(T) := \inf \|A\|\,\|B\|$.

According to Reisner [1979, p. 512], a Banach space X is said to have the **Gordon–Lewis property** if

$$\mathfrak{P}_1(X,Y) \subseteq \mathfrak{L}_1(X,Y) \quad \text{for all Banach spaces } Y,$$

which means that X belongs to the space ideal $\mathsf{P}_1^{-1} \circ \mathsf{L}_1$ associated with the operator ideal $\mathfrak{P}_1^{-1} \circ \mathfrak{L}_1$; see also 6.3.19.11.

A famous result of Gordon/Lewis [1974, p. 37] implies that $\mathfrak{P}_1 \circ \mathfrak{L}_{\text{ust}} \subseteq \mathfrak{L}_1$, or equivalently, $\mathfrak{L}_{\text{ust}} \subseteq \mathfrak{P}_1^{-1} \circ \mathfrak{L}_1$. Thus Banach spaces with local unconditional structure have the Gordon–Lewis property. However, Johnson/Lindenstrauss/Schechtman [1980] showed that the inclusion $\mathsf{L}_{\text{ust}} \subseteq \mathsf{P}_1^{-1} \circ \mathsf{L}_1$ is proper; see also Ketonen [1981, p. 35] and Borzyszkowski [1983, p. 268]. Further information about $\mathfrak{L}_{\text{ust}}$ and related operator ideals can be found in [PIE$_3$, pp. 323–331].

6.3.13.9 The basic problem of this subsection consists in finding operator ideals $\mathfrak{A}$ that yield a given space ideal A. Many new operator ideals have been discovered in this way. Though the map $\mathfrak{A} \mapsto \mathsf{A}$ is far from being one-to-one, in most cases there is a favorite candidate $\mathfrak{A}$.

6.3.13.10 Here is another example that illustrates the preceding leitmotif.

Let $0 < p \leq 2$. An operator $T \in \mathfrak{L}(X,Y)$ is called **weakly p-Hilbertian** if there exists a constant $c \geq 0$ such that

$$\big\| \big(\langle Tx_h, y_k^* \rangle \big) : l_2^n \to l_2^n \big| \mathfrak{S}_{p,\infty} \big\| \leq c \Big(\sum_{h=1}^{n} \|x_h\|^2 \Big)^{1/2} \Big(\sum_{k=1}^{n} \|y_k^*\|^2 \Big)^{1/2}$$

for any choice of $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in Y^*$; see Pietsch [1991a, pp. 56–60]. The collection of these operators is an ideal, denoted by $\mathfrak{H}_{p,\infty}$. It turns out that $\mathfrak{H}_{p,\infty}$ is proper for $0 < p < 1$, and in the limiting case $p = 2$ we have $\mathfrak{H}_{2,\infty} = \mathfrak{L}$.

As shown by Pietsch [1991a, p. 62], an operator is weakly p-Hilbertian if and only if $\Gamma_n(T) = O\big(n^{1-1/p}\big)$, where

$$\Gamma_n(T) := \sup \Big\{ \big| \det \big(\langle Tx_h, y_k^* \rangle \big) \big| : x_1, \dots, x_n \in B_X, \;\; y_1^*, \dots, y_n^* \in B_{Y^*} \Big\}$$

is the n^{th} **Grothendieck number** of the operator $T \in \mathfrak{L}(X,Y)$; see 6.1.11.4.

In the case $p = 1$, one gets the *weak-γ_2 operators* in the sense of Pisier [1988, p. 571], who was interested mainly in the associated space ideal $\mathsf{H}_{1,\infty}$ formed by the weak Hilbert spaces; see 6.1.11.8. Pisier [1988, p. 571] also observed that the summation operator Σ belongs to $\mathfrak{H}_{1,\infty}(l_1, l_\infty)$; see 6.3.16.4. Since Σ fails to be weakly compact, the ideal $\mathfrak{H}_{1,\infty}$ cannot have the factorization property.

If $1 < p \leq 2$ and $\alpha = 1 - 1/p$, then the members of $\mathsf{H}_{p,\infty}$ are characterized by the properties listed in 6.1.1.10 and 6.1.11.5.

6.3.13.11 An operator $T \in \mathfrak{L}(X,Y)$ is called **unconditionally summing** if it takes every weakly summable sequence

$$\sum_{k=1}^{\infty} \big| \langle x_k, x^* \rangle \big| < \infty \quad \text{for } x^* \in X^*$$

into a summable sequence (Tx_k). This concept was invented by Pełczyński [1962b], who used the attribute *"unconditionally converging."*

The collection of all unconditionally summing operators is an injective closed ideal, denoted by $\mathfrak{U}$. We have $\mathfrak{V} \subseteq \mathfrak{U}$ and $\mathfrak{W} \subseteq \mathfrak{U}$. While the first inclusion is trivial, the second one follows from the Orlicz–Pettis theorem 5.1.1.2.

6.3.13.12 The following criterion, which can be found in [PRZ$^+$, p. 270], extends the Bessaga–Pełczyński c_0-theorem 5.6.3.7:

Either an operator $T \in \mathfrak{L}(X,Y)$ is unconditionally summing or X contains a subspace X_0 isomorphic to c_0 such that $\|Tx\| \geq c\|x\|$ for all $x \in X_0$ and some $c > 0$.

Consequently, operators in $\mathfrak{U}$ could also be called *strictly c_0-singular*.

6.3.13.13 A Banach space X is said to have **Pełczyński's property** (V) if

$$\mathfrak{W}(X,Y) = \mathfrak{U}(X,Y) \quad \text{for all Banach spaces } Y.$$

This is obviously true for all reflexive spaces. Moreover, Pełczyński [1962b, p. 643] proved that the condition above holds for all $C(K)$'s, where K is any compact Hausdorff space.

The Schur property of l_1 implies that $\mathfrak{L}(l_1) = \mathfrak{U}(l_1) = \mathfrak{V}(l_1)$. Hence l_1 cannot possess Pełczyński's property.

6.3.13.14 A Banach space X has the **Grothendieck property** if

$$\langle x_n^*, x\rangle \to 0 \quad \text{for all } x \in X \quad \text{implies} \quad \langle x_n^*, x^{**}\rangle \to 0 \quad \text{for all } x^{**} \in X^{**}.$$

This is equivalent to the condition that

$$\mathfrak{L}(X,Y) = \mathfrak{W}(X,Y) \quad \text{for all } \textit{separable} \text{ Banach spaces } Y;$$

see Grothendieck [1953, p. 169]. It suffices to know that $\mathfrak{L}(X,c_0) = \mathfrak{W}(X,c_0)$.

All reflexive spaces satisfy the preceding requirement. Conversely, separable spaces with the Grothendieck property must be reflexive. Hence $C[0,1]$ is a typical counterexample.

On the other hand, Grothendieck himself proved that all $C(K)$'s defined over Stonean spaces K have "his" property; see [1953, p. 168]. This is true in particular for l_∞. Based on preliminary work of Ando [1961], Grothendieck's result was extended by Seever [1968, p. 272] to a larger class of compact Hausdorff spaces.

6.3.13.15 According to Diestel [1980, p. 40]:

It is a part of the folklore of the subject that for dual spaces property V already implies the Grothendieck property.

The space c_0 shows that this implication may fail without some additional assumption.

Diestel's question, *Do Grothendieck spaces have property V?* is still open.

6.3.13.16 A remarkable result was proved by Pfitzner [1994, p. 369]:

$C^\star$-algebras have property (V).

Summing up, we get the following picture:

The spaces c_0 and $\mathfrak{K}(l_2)$ have property (V) but not Grothendieck's property,
while
their biduals l_∞ and $\mathfrak{L}(l_2)$ have property (V) and Grothendieck's property.

6.3.14 Rademacher type and cotype, Gauss type and cotype

6.3.14.1 Recall from 6.1.7.5 that a Banach space is of Rademacher type p if there exists a constant $c \geq 1$ such that

$$\left(\int_0^1 \Big\| \sum_{k=1}^n x_k r_k(t) \Big\|^2 dt \right)^{1/2} \leq c \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p}$$

whenever $x_1, \dots, x_n \in X$ and $n = 1, 2, \dots$. Hence it is natural to say that an operator $T \in \mathfrak{L}(X,Y)$ has **Rademacher type** p if

$$\left(\int_0^1 \Big\| \sum_{k=1}^n T x_k r_k(t) \Big\|^2 dt \right)^{1/2} \leq c \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p}, \qquad (6.3.14.1.a)$$

where $c \geq 0$. This definition yields the injective, surjective, and maximal Banach ideal $\mathfrak{T}_p^{\text{rad}}$, whose ideal norm is given by $\|T|\mathfrak{T}_p^{\text{rad}}\| := \inf c$. Equivalent ideal quasi-norms are obtained by using on the left-hand side any exponent $0 < s < \infty$ instead of 2.

6.3.14.2 The Banach ideals $\mathfrak{C}_q^{\text{rad}}$ (**Rademacher cotype**), $\mathfrak{T}_p^{\text{gauss}}$ (**Gauss type**), and $\mathfrak{C}_q^{\text{gauss}}$ (**Gauss cotype**) are defined analogously. Operators of type and cotype already occurred in the early work of Maurey/Pisier; see [MAU, pp. 66–67] and [PAR$_{73}^{\Sigma}$, exposé 3, p. 1].

A straightforward adaptation of the proof of $\mathsf{T}_p^{\text{rad}} = \mathsf{T}_p^{\text{gauss}}$ gives $\mathfrak{T}_p^{\text{rad}} = \mathfrak{T}_p^{\text{gauss}}$. On the other hand, in contrast to $\mathsf{C}_q^{\text{rad}} = \mathsf{C}_q^{\text{gauss}}$ the inclusion $\mathfrak{C}_q^{\text{rad}} \subset \mathfrak{C}_q^{\text{gauss}}$ is strict; see [PIE$^+$, p. 191].

6.3.14.3 The operator $C_{1/2}$ considered in [PIE$^+$, p. 164] shows that unlike 6.1.7.10,

$$\mathfrak{T}_2^{\text{rad}} \not\subseteq \bigcup_{2 \leq q < \infty} \mathfrak{C}_q^{\text{rad}}.$$

Hence $\mathfrak{T}_p^{\text{rad}}$ cannot have the factorization property.

6.3.14.4 In the following diagram the arrows point from the smaller ideals to the larger ones; see also 6.3.8.4. All inclusions are strict.

$$\begin{tikzcd} \mathfrak{C}_2^{\text{gauss}} & & \mathfrak{P}_{2,1} \\ & \mathfrak{C}_2^{\text{rad}} \arrow[ul] \arrow[ur] & \\ & \mathfrak{P}_2 \arrow[u] & \end{tikzcd}$$

Next, I present some prototypes of operators that illustrate the situation described above:

$$\begin{array}{ll} Id : C[0,1] \to L_2[0,1] & \text{is 2-summing,} \\ Id : C[0,1] \to L_{u^2\log(1+u)}[0,1] & \text{is of Gauss cotype 2,} \\ Id : C[0,1] \to L_{2,1}[0,1] & \text{is } (2,1)\text{-summing;} \end{array}$$

see 6.3.6.7, Montgomery-Smith [1989, p. 124], Montgomery-Smith/Talagrand [1991, p. 188], Pisier [1986b, p. 117], Talagrand [1989, p. 513], [1992, p. 11]. The spaces $L_{u^2\log(1+u)}[0,1]$ and $L_{2,1}[0,1]$ as well as the ideals $\mathfrak{C}_2^{\text{gauss}}$ and $\mathfrak{P}_{2,1}$ are incomparable.

Unfortunately, I do not know a corresponding operator $Id : C[0,1] \to$? in the case of Rademacher cotype 2.

6.3.14.5 Up to slight modifications, the results presented in this paragraph are due to Beauzamy [1975].

For any *fixed n*, the ideal norm $\|T|\mathfrak{T}_2^{\text{rad}}\|_n$ is defined to be the least constant $c \geq 0$ such that

$$\left(\int_0^1 \Big\| \sum_{k=1}^n Tx_k r_k(t) \Big\|^2 dt\right)^{1/2} \leq c\Big(\sum_{k=1}^n \|x_k\|^2\Big)^{1/2}$$

whenever $x_1,\dots,x_n \in X$; see (6.3.14.1.a) and Pisier [1973b, p. 991]. It easily follows that $\|T\| \leq \|T|\mathfrak{T}_2^{\text{rad}}\|_n \leq \sqrt{n}\,\|T\|$.

We say that $T \in \mathfrak{L}(X,Y)$ has **Rademacher subtype** if

$$\|T|\mathfrak{T}_2^{\text{rad}}\|_n = o\big(\sqrt{n}\big).$$

These *opérateurs de type Rademacher* (Beauzamy's terminology) constitute the closed ideal $\mathfrak{T}^{\text{rad}}$, which is injective, surjective, and symmetric.

In view of the following dichotomy, operators of Rademacher subtype could also be called *B*-**convex**:

Either an operator $T \in \mathfrak{L}(X,Y)$ is of Rademacher subtype or there exist constants $a,b > 0$ such that for every n, we can find $x_1,\dots,x_n \in X$ satisfying

$$a\sum_{k=1}^n |\xi_k| \leq \Big\|\sum_{k=1}^n \xi_k Tx_k\Big\| \leq \|T\|\Big\|\sum_{k=1}^n \xi_k x_k\Big\| \leq b\sum_{k=1}^n |\xi_k| \quad \text{if } \xi_1,\dots,\xi_n \in \mathbb{K}.$$

Passing to Gaussian variables yields nothing new: $\mathfrak{T}^{\text{rad}} = \mathfrak{T}^{\text{gauss}}$.

6.3.14.6 There is an analogous theory of operators having **Rademacher subcotype**. In this case, we get the closed ideal $\mathfrak{C}^{\text{rad}}$, which is injective, but neither surjective nor symmetric.

Defining operators of **Gauss subcotype** by the tricky property

$$\|T|\mathfrak{C}_2^{\text{gauss}}\|_n = o\Big(\sqrt{\tfrac{n}{1+\log n}}\Big),$$

Hinrichs [2001a, p. 467] was able to prove that $\mathfrak{C}^{\mathrm{rad}} = \mathfrak{C}^{\mathrm{gauss}}$. In [1999b, p. 210], he characterized the operators T belonging to this ideal by

$$\rho(T|\mathcal{G}_n, \mathcal{R}_n) = o\Big(\sqrt{1+\log n}\Big),$$

where the (unpopular but suggestive) symbol $\rho(T|\mathcal{G}_n, \mathcal{R}_n)$ denotes the least constant $c \geq 0$ such that

$$\bigg(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big\| \sum_{k=1}^{n} t_k T x_k \Big\|^2 d\gamma(t_1) \cdots d\gamma(t_n)\bigg)^{1/2} \leq c \bigg(\int_0^1 \Big\| \sum_{k=1}^{n} x_k r_k(t) \Big\|^2 dt\bigg)^{1/2}$$

for $x_1, \ldots, x_n \in X$ and *fixed n*; see also [PIE$^+$, pp. 193–196].

6.3.14.7 We know from 6.1.7.9 that

$$\bigcup_{1<p\leq 2} \mathsf{T}_p^{\mathrm{rad}} = \mathsf{T}^{\mathrm{rad}} \quad \text{and} \quad \bigcup_{2\leq q<\infty} \mathsf{C}_q^{\mathrm{rad}} = \mathsf{C}^{\mathrm{rad}}.$$

However, in the setting of operators there occur strict inclusions:

$$\bigcup_{1<p\leq 2} \mathfrak{T}_p^{\mathrm{rad}} \subset \mathfrak{T}^{\mathrm{rad}} \quad \text{and} \quad \bigcup_{2\leq q<\infty} \mathfrak{C}_q^{\mathrm{rad}} \subset \mathfrak{C}^{\mathrm{rad}};$$

see [PIE$^+$, pp. 143, 159]. Consequently, the ideals $\mathfrak{T}^{\mathrm{rad}}$ and $\mathfrak{C}^{\mathrm{rad}}$ fail to have the factorization property.

6.3.14.8 Let $1 < p < 2 < q < \infty$. An operator $T \in \mathfrak{L}(X,Y)$ is said to have **weak Rademacher type** p and **weak Rademacher cotype** q if

$$\|T|\mathfrak{T}_2^{\mathrm{rad}}\|_n = O\big(n^{1/p-1/2}\big) \quad \text{and} \quad \|T|\mathfrak{C}_2^{\mathrm{rad}}\|_n = O\big(n^{1/2-1/q}\big),$$

respectively; see [PIE$^+$, pp. 139, 154].

In the limiting cases $p=2$ and $q=2$, we would get the usual concepts of Rademacher type 2 and cotype 2; compare with 6.1.11.7.

For Banach spaces and $1 < p < 2 < q < \infty$, the preceding definitions are due to Mascioni who showed in [1988, pp. 83, 85, 96, 98] that

equal-norm type p = weak type p and *equal-norm cotype q = weak cotype q*.

6.3.14.9 Replacing (6.1.7.12.a) by

$$\bigg(\int_0^1 \Big\| \sum_{k=1}^{n} r_k(s) \int_0^1 T\boldsymbol{f}(t) r_k(t)\, dt \Big\|^2 ds\bigg)^{1/2} \leq c \bigg(\int_0^1 \|\boldsymbol{f}(t)\|^2 dt\bigg)^{1/2}$$

yields the concept of a ***K*-convex operator**; see Pisier [1981, p. 139]. In this way, one gets the Banach ideal $\mathfrak{K}^{\mathrm{rad}}$. There is also a Gaussian version denoted by $\mathfrak{K}^{\mathrm{gauss}}$. Figiel/Tomczak-Jaegermann [1979, pp. 166–168] observed that $\mathsf{K}^{\mathrm{gauss}} = \mathsf{K}^{\mathrm{rad}}$. It seems to be unknown whether this equality extends to the setting of operators. At present, we know only the inclusion $\mathfrak{K}^{\mathrm{gauss}} \subseteq \mathfrak{K}^{\mathrm{rad}}$, which may be strict.

Obviously,

$$\mathfrak{K}^{\text{rad}} \circ (\mathfrak{C}_q^{\text{rad}})^{\text{dual}} \subseteq \mathfrak{T}_{q^*}^{\text{rad}} \quad \text{and} \quad \mathfrak{K}^{\text{gauss}} \circ (\mathfrak{C}_q^{\text{gauss}})^{\text{dual}} \subseteq \mathfrak{T}_{q^*}^{\text{gauss}}.$$

6.3.14.10 Let $\|T|\mathfrak{K}^{\text{rad}}\|_n$ denote the least constant $c \geq 0$ such that

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(s) \int_0^1 T\boldsymbol{f}(t) r_k(t) dt \right\|^2 ds\right)^{1/2} \leq c \left(\int_0^1 \|\boldsymbol{f}(t)\|^2 dt\right)^{1/2}$$

for $x_1, \ldots, x_n \in X$ and *fixed n*. Then $\|T\| \leq \|T|\mathfrak{K}^{\text{rad}}\|_n \leq \sqrt{n}\,\|T\|$.

By 6.3.14.9, an operator is K-convex if and only if

$$\|T|\mathfrak{K}^{\text{rad}}\|_n = O(1).$$

On the other hand, a deep theorem in [PIE$^+$, p. 224] tells us that an operator is of Rademacher subtype if and only if

$$\|T|\mathfrak{K}^{\text{rad}}\|_n = o\big(\sqrt{n}\big).$$

Both operator ideals $\mathfrak{K}^{\text{rad}}$ and $\mathfrak{T}^{\text{rad}}$ are different but generate one and the same space ideal. Even more can be said: a huge variety of further operator ideals lies in between, since we know from Hinrichs [1999a, p. 139] that for $0 \leq \alpha \leq 1/2$, there exist operators D_α with $\|D_\alpha|\mathfrak{K}^{\text{rad}}\|_n \asymp n^{1/2-\alpha}$.

6.3.15 Fourier type and cotype, Walsh type and cotype

6.3.15.1 Obvious modifications of (6.1.8.2.a), ... ,(6.1.8.2.e) yield the operator ideals $\mathfrak{T}_p^{\text{four}} = \mathfrak{C}_q^{\text{four}}$ (**Fourier type** $p =$ **Fourier cotype** q) with $1/p + 1/q = 1$.

6.3.15.2 Using Kahane's contraction principle, one easily gets $\mathfrak{T}_p^{\text{four}} \subset \mathfrak{T}_p^{\text{rad}}$; see [PIE$^+$, p. 313].

6.3.15.3 Kwapień's theorem says that Banach spaces of Fourier type 2 are Hilbertian: $\mathsf{T}_2^{\text{four}} = \mathsf{L}_2$. It is a long-standing open problem whether every operator of Fourier type 2 factors through a Hilbert space: $\mathfrak{T}_2^{\text{four}} \overset{?}{=} \mathfrak{L}_2$. Partial results of Hinrichs [2001b, p. 201] support the conjecture that this is not the case.

6.3.15.4 Let $\|T|\mathfrak{T}_2^{\text{four}}\|_n$ denote the least constant $c \geq 0$ such that

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left\| \sum_{k=1}^n T x_k \mathrm{e}^{\mathrm{i}ks} \right\|^{p^*} ds\right)^{1/p^*} \leq c \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}$$

for any choice of $x_1, \ldots, x_n \in X$ and *fixed n*. Then $\|T\| \leq \|T|\mathfrak{T}_2^{\text{four}}\|_n \leq \sqrt{n}\,\|T\|$.

Equivalent ideal norms are obtained by requiring the inequalities

$$\left(\frac{1}{n}\sum_{h=1}^{n}\left\|\sum_{k=1}^{n}\exp\left(2\pi i\tfrac{hk}{n}\right)Tx_k\right\|^{p^*}\right)^{1/p^*} \leq c\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p},$$

or

$$\left(\frac{2}{\pi}\int_0^{\pi}\left\|\sum_{k=1}^{n}Tx_k\cos ks\right\|^{p^*}ds\right)^{1/p^*} \leq c\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p},$$

or

$$\left(\frac{2}{\pi}\int_0^{\pi}\left\|\sum_{k=1}^{n}Tx_k\sin ks\right\|^{p^*}ds\right)^{1/p^*} \leq c\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p};$$

see [PIE$^+$, p. 286].

6.3.15.5 We say that $T\in\mathfrak{L}(X,Y)$ has **Fourier subtype** if

$$\|T|\mathfrak{T}_2^{\text{four}}\|_n = o\big(\sqrt{n}\big).$$

These operators form the closed, injective, surjective, and symmetric ideal $\mathfrak{T}^{\text{four}}$. The concept of **Fourier subcotype** yields nothing new.

6.3.15.6 A beautiful result of Hinrichs [1996, p. 232] says that for some constant $c>0$,

$$2^{-n/2}\|T|\mathfrak{T}_2^{\text{four}}\|_{2^n} \leq c\,n^{-1/2}\|T|\mathfrak{T}_2^{\text{rad}}\|_n.$$

Combined with $\|T|\mathfrak{T}_2^{\text{rad}}\|_n \leq c_0\|T|\mathfrak{T}_2^{\text{four}}\|_n$, this inequality implies that $\mathfrak{T}^{\text{rad}}=\mathfrak{T}^{\text{four}}$.

6.3.15.7 In analogy to 6.3.14.8, a theory of **weak Fourier type** p and **weak Fourier cotype** q can be developed; see [PIE$^+$, pp. 290, 301].

6.3.15.8 Mutatis mutandis, the previous theory remains true if the trigonometric functions are replaced by the Walsh functions. As above, we get $\mathfrak{T}^{\text{rad}}=\mathfrak{T}^{\text{walsh}}$ from Hinrichs's inequality [1996, p. 232].

Unfortunately, except for $\mathfrak{T}_2^{\text{four}}\subseteq\mathfrak{T}_2^{\text{walsh}}$, almost nothing is known about the relationship between $\mathfrak{T}_p^{\text{four}}$ and $\mathfrak{T}_p^{\text{walsh}}$; see Hinrichs [2001c, p. 324].

6.3.16 Super weakly compact operators, Haar type and cotype

6.3.16.1 With every operator ideal $\mathfrak{A}$ we associate the **superideal**

$$\mathfrak{A}^{\text{super}} := \big\{T : T^{\mathscr{U}}\in\mathfrak{A} \quad \text{for all ultrafilters } \mathscr{U}\ \big\}.$$

Obviously, $\mathfrak{A}^{\text{super}}\subseteq\mathfrak{A}$ and $(\mathfrak{A}^{\text{super}})^{\text{super}}=\mathfrak{A}^{\text{super}}$. If $\mathfrak{A}$ is closed, then so is $\mathfrak{A}^{\text{super}}$.

In the case of a regular ideal, $\mathfrak{A}^{\text{super}}$ consists of all operators T such that any operator T_0 that is finitely representable in T belongs to $\mathfrak{A}$.

The theory of superideals goes back to Heinrich [HEIN, p. 14], [1980a, p. 93], who used, however, a different concept of finite representability; see 6.1.3.12 and 6.1.3.16.

Please, notice that being a member of the space ideal $\mathsf{A}^{\text{super}}$ is not always a superproperty in the sense of 6.1.3.3. In order to get invariance under the formation of closed subspaces, one should assume injectivity; see also 6.1.3.18.

6.3.16.2 The following example was discovered by Heinrich [HEIN, p. 32]:

The ideal of compact operators is the superideal associated with the closed ideal of all operators having separable ranges.

6.3.16.3 In contrast to the preceding result, the weakly compact operators do not form a superideal. Therefore the members of the injective, surjective, symmetric, and closed ideal $\mathfrak{W}^{\text{super}}$ deserve their own name, **super weakly compact operators**. Obviously, the corresponding space ideal $\mathsf{W}^{\text{super}}$ consists of the superreflexive spaces.

6.3.16.4 In [1964d, p. 114], James proved that a Banach space X is non-reflexive if and only if for every $\theta \in (0,1)$, there exist $x_1, x_2, \ldots \in B_X$ and $x_1^*, x_2^*, \ldots \in B_{X^*}$ such that

$$\langle x_h^*, x_k \rangle = \theta \quad \text{if } h \le k \quad \text{and} \quad \langle x_h^*, x_k \rangle = 0 \quad \text{if } h > k.$$

It follows that the **summation operator**

$$\Sigma : (\xi_k) \mapsto \Big(\sum_{k=1}^{h} \xi_k \Big),$$

viewed as a map from l_1 into l_∞, factors through every non-reflexive X. Indeed, letting

$$A : (\xi_k) \mapsto \theta^{-1} \sum_{k=1}^{\infty} \xi_k x_k, \quad B : x \mapsto \big(\langle x_h^*, x \rangle \big), \quad \text{and} \quad V : (\eta_h) \mapsto (\eta_1 - \eta_{h+1})$$

we have

$$BA : (\xi_k) \mapsto \Big(\sum_{k=h}^{\infty} \xi_k \Big).$$

Hence $\Sigma = VBA$.

The previous considerations culminated in the following result of Lindenstrauss/Pełczyński [1968, p. 322]:

An operator $T \in \mathfrak{L}(X,Y)$ fails to be weakly compact if and only if it factors the summation operator, $\Sigma : l_1 \xrightarrow{A} X \xrightarrow{T} Y \xrightarrow{B} l_\infty$.

The **finite summation operators** $\Sigma_n : l_1^n \to l_\infty^n$ are defined in the same way as Σ. A counterpart of the preceding criterion tells us that an operator $T \in \mathfrak{L}(X,Y)$ fails to be super weakly compact if and only if it factors the Σ_n's uniformly, which means that $\Sigma_n : l_1^n \xrightarrow{A_n} X \xrightarrow{T} Y \xrightarrow{B_n} l_\infty^n$ and $\|A_n\|\|B_n\| \le c$ for $n = 1, 2, \ldots$ and some $c > 0$; see James [1972b, p. 410] and [PIE$^+$, p. 377].

The criteria above illustrate the different natures of $\mathfrak{W}^{\text{super}}$ and $\mathfrak{W}$. Whereas super weakly compact operators are characterized by a uniform *finite*-dimensional non-factorization property, in the classical case of weakly compact operators we have an *infinite*-dimensional condition.

6.3.16.5 Taking 5.5.2.1 as a pattern, Beauzamy [1976, p. 121] called an operator $T \in \mathfrak{L}(X,Y)$ **uniformly convex** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\Big\| \frac{x_0 + x_1}{2} \Big\| \le 1 - \delta \quad \text{whenever } \|x_0\| = \|x_1\| = 1 \text{ and } \|Tx_0 - Tx_1\| \ge \varepsilon.$$

In the case that this property can be achieved by a renorming of X, we refer to T as **uniformly convexifiable** (*uniformément convexifiant* in Beauzamy's terminology).

According to Beauzamy [1976, pp. 110–112], an operator is uniformly convexifiable if and only if it does not have the **finite tree property**; see 6.1.9.2:

There exists some $\varepsilon > 0$ such that for every n, we can find a dyadic tree of length n in the closed unit ball of X with

$$\tfrac{1}{2}\big\|Tu_{k+1}^{(2i-1)} - Tu_{k+1}^{(2i)}\big\| \geq \varepsilon.$$

Though Beauzamy used a different concept of finite representability (see 6.1.3.12), his final conclusion says that T is uniformly convexifiable if and only if all operators finitely representable in T are weakly compact. In other words, $\mathfrak{W}^{\text{super}}$ coincides with the ideal of uniformly convexifiable operators.

6.3.16.6 Replacing (6.1.9.6.a) by

$$\left(\int_0^1 \left\| \sum_{k=0}^{n} T\,\boldsymbol{d}_k(t) \right\|^p dt\right)^{1/p} \leq c \left(\sum_{k=0}^{n} \int_0^1 \|\boldsymbol{d}_k(t)\|^p\, dt\right)^{1/p} \qquad (6.3.16.6.a)$$

yields the concept of **Haar type** p. The corresponding ideal is denoted by $\mathfrak{T}_p^{\text{haar}}$.

It can be shown that an operator has Haar type p if and only if it is p**-smoothable**; see 6.1.9.9. For details the reader is referred to [PIE$^+$, pp. 399–411], where these operators are called *uniformly* p-smoothable.

6.3.16.7 Let $\|T|\mathfrak{T}_2^{\text{haar}}\|_n := \inf c$, the infimum being taken over all constants $c \geq 0$ such that (6.3.16.6.a) holds for $p = 2$ and all X-valued Walsh–Paley martingales of *fixed* length n. Then $\|T\| \leq \|T|\mathfrak{T}_2^{\text{haar}}\|_n \leq \sqrt{n+1}\|T\|$.

We say that $T \in \mathfrak{L}(X,Y)$ has **Haar subtype** if

$$\|T|\mathfrak{T}_2^{\text{haar}}\|_n = o\big(\sqrt{n}\big).$$

These operators form the closed ideal $\mathfrak{T}^{\text{haar}}$, which coincides with $\mathfrak{W}^{\text{super}}$; see [PIE$^+$, p. 428] and Wenzel [2002, p. 228].

6.3.16.8 We have the proper inclusions

$$\mathfrak{T}_2^{\text{haar}} \subset \mathfrak{T}_{p_0}^{\text{haar}} \subset \mathfrak{T}_{p_1}^{\text{haar}} \subset \bigcup_{1<p\leq 2} \mathfrak{T}_p^{\text{haar}} \subset \mathfrak{T}^{\text{haar}} \quad \text{if } 2 > p_0 > p_1 > 1.$$

6.3.16.9 There is a parallel theory that deals with the operator ideals $\mathfrak{C}_q^{\text{haar}}$ (**Haar cotype** q) and $\mathfrak{C}^{\text{haar}}$ (**Haar subcotype**). In view of the duality relations

$$(\mathfrak{C}_q^{\text{haar}})^{\text{dual}} = \mathfrak{T}_{q^*}^{\text{haar}}, \quad (\mathfrak{T}_p^{\text{haar}})^{\text{dual}} = \mathfrak{C}_{p^*}^{\text{haar}}, \quad (\mathfrak{C}^{\text{haar}})^{\text{dual}} = \mathfrak{T}^{\text{haar}}, \quad (\mathfrak{T}^{\text{haar}})^{\text{dual}} = \mathfrak{C}^{\text{haar}},$$

these concepts are only of aesthetic value. Nevertheless, I mention that an operator has Haar cotype q if and only if it is q**-convexifiable**; see 6.1.9.9.

6.3.16.10 By analogy to 6.3.14.8, a theory of **weak Haar type** p and **weak Haar cotype** q can be developed; see Wenzel [1997a] and [PIE$^+$, p. 367].

6.3.17 *UMD* operators and *HT* operators

6.3.17.1 Replacing (6.1.10.1.a) by

$$\left(\int_0^1 \left\|\sum_{k=0}^n \varepsilon_k T\boldsymbol{d}_k(t)\right\|^2 dt\right)^{1/2} \le c\left(\int_0^1 \left\|\sum_{k=0}^n \boldsymbol{d}_k(t)\right\|^2 dt\right)^{1/2} \qquad (6.3.17.1.a)$$

yields the concept of a *UMD* **operator**. We obtain the injective, surjective, symmetric, and maximal Banach ideal $\mathfrak{UMD}$ whose norm is given by $\|T|\mathfrak{UMD}\| := \inf c$. These basic facts were proved by (Martin) Defant [1989].

6.3.17.2 It seems that nobody has thought about the question how to define ζ-convex operators; see 6.1.10.2.

6.3.17.3 In view of (6.1.10.4.a), one refers to $T \in \mathfrak{L}(X,Y)$ as an HT **operator** if

$$\|H_{\mathrm{ilb}}(T\boldsymbol{f})|L_2(\mathbb{R})\| \le c\|\boldsymbol{f}|L_2(\mathbb{R})\| \quad \text{for } \boldsymbol{f} = \sum_{k=1}^n f_k x_k \in L_2(\mathbb{R})\otimes X.$$

The collection of these operators is an injective, surjective, symmetric, and maximal Banach ideal $\mathfrak{HT}$ with $\|T|\mathfrak{HT}\| := \inf c$; see (Martin) Defant [1989, p. 253].

6.3.17.4 A result of Wenzel [1994, p. 64] implies that $T \in \mathfrak{HT}(X,Y)$ if and only if there exists a constant $c \ge 0$ such that

$$\left(\int_0^\pi \left\|\sum_{k=1}^n Tx_k \sin kt\right\|^2 dt\right)^{1/2} \le c\left(\int_0^\pi \left\|\sum_{k=1}^n x_k \cos kt\right\|^2 dt\right)^{1/2} \qquad (6.3.17.4.a)$$

for any choice of $x_1,\dots,x_n \in X$. This characterization reflects the fact that the periodic Hilbert transform maps $\cos kt$ to $\sin kt$.

6.3.17.5 The Burkholder–Bourgain theorem tells as that the space ideals UMD and HT coincide. Unfortunately, nothing is known about the relationship between the operators ideals $\mathfrak{UMD}$ and $\mathfrak{HT}$.

6.3.17.6 Let $\|T|\mathfrak{UMD}\|_n$ denote the least constant $c \ge 0$ such that (6.3.17.1.a) holds for all X-valued Walsh–Paley martingales of *fixed* length n. Easy manipulations show that $\|T\| \le \|T|\mathfrak{UMD}\|_n \le (n+1)\|T\|$; see [PIE$^+$, p. 439].

Similarly, denote by $\rho(T|\mathcal{S}_n,\mathcal{C}_n)$ the least constant $c \ge 0$ such that (6.3.17.4.a) is satisfied for $x_1,\dots,x_n \in X$ and *fixed* n. Then $\|T\| \le \rho(T|\mathcal{S}_n,\mathcal{C}_n) \le c_0(1+\log n)\|T\|$ with a universal constant $c_0 > 0$; see [PIE$^+$, pp. 59, 261, 268].

It is a challenging problem to compare the asymptotic behavior of $\rho(T|\mathcal{S}_{2^n},\mathcal{C}_{2^n})$ with that of $\|T|\mathfrak{UMD}\|_n$. The most optimistic conjecture $\rho(T|\mathcal{S}_{2^n},\mathcal{C}_{2^n}) \asymp \|T|\mathfrak{UMD}\|_n$ seems to be improbable; the summation operator is a promising candidate for a counterexample. At present, we know only that

$$\rho(\Sigma : l_1 \to l_\infty|\mathcal{S}_{2^n},\mathcal{C}_{2^n}) \asymp n \quad \text{and} \quad \sqrt{n} \preceq \|\Sigma : l_1 \to l_\infty|\mathfrak{UMD}\|_n \preceq n.$$

A partial result taken from [PIE$^+$, p. 443] says that

$$\rho(T|\mathcal{S}_{2^n},\mathcal{C}_{2^n}) = o(n) \;\Rightarrow\; T \in \mathfrak{W}^{\mathrm{super}} \;\Rightarrow\; \|T|\mathfrak{UMD}\|_n = o(n).$$

6.3.18 Radon–Nikodym property: operator-theoretic aspects

6.3.18.1 Mimicking the characterization 5.1.4.8, an operator $T \in \mathfrak{L}(X,Y)$ is said to have the **Radon–Nikodym property** if given any finite measure space $(M, \mathscr{M}, \mu)$ and any $A \in \mathfrak{L}\big(L_1(M,\mathscr{M},\mu),X\big)$, there exists a bounded Bochner measurable Y-valued function $\boldsymbol{g}$ such that

$$TAf = \int_M f(t)\boldsymbol{g}(t)\,d\mu \quad \text{for } f \in L_1.$$

This concept was independently defined by Linde [1976] and Reĭnov [1975, 1978].

The collection of all Radon–Nikodym operators is a closed injective ideal, denoted by $\mathfrak{RN}$; see [PIE$_3$, pp. 337–341].

6.3.18.2 The following criterion is due to Linde [1976, p. 70] and Reĭnov [1975, стр. 529]. In the setting of spaces, it goes back to Diestel [1972, p. 1618].

An operator $T \in \mathfrak{L}(X,Y)$ has the Radon–Nikodym property if and only if for any compact Hausdorff space K,

$$A \in \mathfrak{I}\big(C(K),X\big) \quad \text{implies} \quad TA \in \mathfrak{N}\big(C(K),Y\big).$$

In this case, $\nu(TA) \le \|T\|\,\iota(A)$.

In order to ensure the Radon–Nikodym property, one needs to verify the criterion above only for $K = [0,1]$.

It follows that $\mathfrak{RN} \circ \mathfrak{I}^{\text{strict}} \subseteq \mathfrak{N}$, where $\mathfrak{I}^{\text{strict}}$ denotes the ideal of *strictly* integral operators; see 6.3.7.4.

6.3.18.3 Since every weakly compact operator factors through a reflexive Banach space, we have $\mathfrak{W} \subset \mathfrak{RN}$, the inclusion being proper. According to Heinrich [HEIN, p. 29], the difference between $\mathfrak{W}$ and $\mathfrak{RN}$ disappears when we pass to the associated superideals: $\mathfrak{W}^{\text{super}} = \mathfrak{RN}^{\text{super}}$; see also 6.1.3.3.

6.3.19 Ideal norms and parameters of Minkowski spaces

In what follows, E_n always denotes an n-dimensional Banach space. The values of specific constants will be given only in the real case.

6.3.19.1 Let α be any quasi-norm defined on an operator ideal $\mathfrak{A}$. If the identity map of a Banach space X belongs to $\mathfrak{A}$, then we write

$$\alpha(X) := \alpha(I_X) \quad \text{or} \quad \|X|\mathfrak{A}\| := \|I_X|\mathfrak{A}\|.$$

This metric quantity, which is defined for all finite-dimensional spaces, provides important information about the structure of the Banach space in question.

6.3.19.2 Some ideal norms depend only on the dimension of the underlying space. For example, we have $\nu(E_n)=n$ and $\pi_2(E_n)=\sqrt{n}$. Using an Auerbach basis, Ruston [1962] established the first formula; the second one is due to Garling/Gordon [1971, p. 356]. The simplest proof of $\pi_2(E_n)=\sqrt{n}$ goes back to Kwapień; see [PIE$_3$, p. 385].

6.3.19.3 Obviously, $\lambda_2(E_n)=d(E_n,l_2^n)$, and John's theorem 6.1.1.4 tells us that $\lambda_2(E_n)\le\sqrt{n}$. Conversely, $\lambda_2(E_n)\le\pi_2(E_n)=\sqrt{n}$ yields a proof of John's theorem.

6.3.19.4 In contrast to the 2-summing norm, $\pi_1(E_n)$ strongly reflects the properties of E_n. Gordon [1968, p. 296], [1969, p. 153] showed that

$$\pi_1(l_r^n)\asymp n^{1/2} \quad \text{if} \quad 1\le r\le 2 \quad \text{and} \quad \pi_1(l_r^n)\asymp n^{1/r^*} \quad \text{if} \quad 2\le r\le\infty.$$

Moreover, we know from Deschaseaux [1973, p. 1349] and Garling [1974, p. 413] that $\pi_1(E_n)=n$ if and only if E_n is isometric to l_∞^n.

6.3.19.5 Macphail [1947, p. 121] considered the infimum of the quotients

$$\frac{\sup\Big\{\Big\|\sum\limits_{k\in\mathbb{F}} x_k\Big\| : \mathbb{F}\subseteq\{1,\dots,n\}\Big\}}{\sum\limits_{k=1}^{n}\|x_k\|},$$

where $(x_1,\dots,x_n)$ ranges over all finite families in a given Banach space X. From Garling/Gordon [1971, p. 348], we know that the **Macphail constant** coincides with $\big[2\pi_1(E)\big]^{-1}$ for every real finite-dimensional space E, while it is zero in the infinite-dimensional case.

6.3.19.6 According to Garling/Gordon [1971, p. 348], a Banach space E is said to have **enough symmetries** if every operator $T\in\mathfrak{L}(E)$ that commutes with all isometries is a scalar multiple of the identity.

Assuming this property, the authors above showed that $\lambda_\infty(E_n)\,\pi_1(E_n)=n$. Actually, their reasoning yields $\alpha(E_n)\,\alpha^*(E_n)=n$ for any ideal norm α; see Gordon [1973, p. 53].

6.3.19.7 The (absolute) projection constant $\lambda(E)$, discussed in 6.1.1.6, is just the ∞-factorable norm $\lambda_\infty(E)$. Hence $\lambda(E_n)=\lambda_\infty(E_n)\le\pi_2(E_n)\le\sqrt{n}$, which proves the Kadets–Snobar theorem 6.1.1.7.

6.3.19.8 The asymptotic behavior of the Rademacher type and cotype constants is summarized in [TOM, pp. 15–16], $1<p\le 2\le q<\infty$:

$$\|l_r^n|\mathfrak{T}_p^{\mathrm{rad}}\|\asymp n^{1/r-1/p} \quad \text{if} \quad 1\le r\le p \quad \text{and} \quad \|l_r^n|\mathfrak{T}_p^{\mathrm{rad}}\|\asymp 1 \quad \text{if} \quad p\le r<\infty,$$

$$\|l_r^n|\mathfrak{C}_q^{\mathrm{rad}}\|\asymp 1 \quad \text{if} \quad 1\le r\le q \quad \text{and} \quad \|l_r^n|\mathfrak{C}_q^{\mathrm{rad}}\|\asymp n^{1/q-1/r} \quad \text{if} \quad q\le r<\infty.$$

The same results hold for $\mathfrak{T}_p^{\mathrm{gauss}}$ and $\mathfrak{C}_q^{\mathrm{gauss}}$.

In the limiting case $r = \infty$ the situation is different:

$$\|l_\infty^n|\mathfrak{T}_p^{\text{rad}}\| \asymp (1+\log n)^{1/p^*} \quad \text{and} \quad \|l_\infty^n|\mathfrak{T}_p^{\text{gauss}}\| \asymp (1+\log n)^{1/p^*},$$

but

$$\|l_\infty^n|\mathfrak{C}_q^{\text{rad}}\| \asymp n^{1/q} \quad \text{and} \quad \|l_\infty^n|\mathfrak{C}_q^{\text{gauss}}\| \asymp \frac{n^{1/q}}{(1+\log n)^{1/2}}.$$

6.3.19.9 Concerning the significance of $\pi_\gamma(E)$ in connection with Dvoretzky's theorem, the reader is referred to 6.1.2.6.

6.3.19.10 Obviously, the unconditional basis constant $\text{unc}(E)$ and the constant of local unconditional structure are related by $\lambda_{\text{ust}}(E) \le \text{unc}(E)$; see 6.1.5.1 and 6.1.5.4. It is unknown whether there exists a constant $c \ge 1$ such that $\text{unc}(E) \le c\lambda_{\text{ust}}(E)$ for all finite-dimensional Banach spaces E.

6.3.19.11 Two variants of the **Gordon–Lewis constant** have been considered:

$$\text{gl}(E) := \sup\left\{ \lambda_1(T : E \to X) : \pi_1(T : E \to X) \le 1 \right\},$$

where X ranges over all Banach spaces, and

$$\text{gl}_2(E) := \sup\left\{ \lambda_1(T : E \to l_2) : \pi_1(T : E \to l_2) \le 1 \right\}.$$

The first definition goes back to Reisner [1979, pp. 512–513], while the latter is taken from Figiel/Johnson [1980, p. 93]; see also [PIS_1, p. 106] and [TOM, p. 260]. Obviously, we have $\text{gl}_2(E) \le \text{gl}(E)$. However, I cannot see any reason why the two quantities should show the same behavior as $\dim(E) \to \infty$.

From Gordon/Lewis [1974, p. 37] we know that $\text{gl}(E) \le \lambda_{\text{ust}}(E)$.

6.3.20 Ideal norms of finite rank operators

The following estimates are supposed to hold for operators $T_n \in \mathfrak{L}(X,Y)$, between arbitrary Banach spaces X and Y, with $\text{rank}(T_n) = n$ and $n = 1, 2, \dots$.

6.3.20.1 Ideal quasi-norms α and β defined on different ideals $\mathfrak{A}$ and $\mathfrak{B}$ cannot be compared in general. The situation improves when the considerations are restricted to finite rank operators. Then we always find constants $c \ge 1$ and $\lambda \ge 0$ such that

$$\alpha(T_n) \le cn^\lambda \beta(T_n).$$

The infimum of all possible exponents λ may be used to measure the (non-symmetric) distance from α to β.

6.3.20.2 The first result along these lines seems to be some kind of the folklore. It was explicitly stated by Ruston only in [1962]:

$$\nu(T_n) \le n\|T_n\|.$$

The proof uses an Auerbach basis of the range of T_n. The same idea yields

$$\varphi_p(T_n) \le n^{1/p}\|T_n\|$$

if $0 < p \le 1$; see 6.3.2.11.

6.3.20.3 The inequality

$$\pi_2(T_n) \le \sqrt{n}\|T_n\|$$

follows from a result of Garling/Gordon [1971, p. 356], namely $\pi_2(E)=\sqrt{n}$ whenever $\dim(E)=n$, and a convexity argument gives

$$\pi_{q,2}(T_n) \le n^{1/q}\|T_n\| \quad \text{if } 2<q<\infty.$$

6.3.20.4 The main body of results is due to Lewis [1978, p. 210]:

$$\begin{array}{lll} \textbf{(1)} & \iota_p(T_n) \le n^{|1/2-1/p|}\pi_p(T_n) & \text{if } 1\le p\le\infty, \\ \textbf{(2)} & \pi_2(T_n) \le n^{1/2-1/q}\,\pi_q(T_n) & \text{if } 2\le q\le\infty, \\ \textbf{(3)} & \iota_p(T_n) \le n^{1/p-1/2}\,\iota_2(T_n) & \text{if } 1\le p\le 2. \end{array}$$

Combining **(2)** and **(3)** yields

$$\iota_p(T_n) \le n^{1/p-1/q}\pi_q(T_n) \qquad \text{if } 1\le p\le 2\le q\le\infty.$$

Moreover, at the cost of a constant, König/Retherford/Tomczak-Jaegermann [1980, p. 97] improved **(2)** as follows:

$$\pi_2(T_n) \le c_q n^{1/2-1/q}\,\pi_{q,2}(T_n) \quad \text{if } 2<q<\infty.$$

Inequalities between $\pi_{p,q}$ and $\pi_{r,s}$ were established by Carl [1980, p. 145].

6.3.20.5 It is worthwhile to point out some analogies with other fields.

If a sequence $x=(\xi_k)$ has at most n non-zero coordinates and $0<q<p\le\infty$, then Hölder's inequality implies that $\|x|l_q\| \le n^{1/q-1/p}\,\|x|l_p\|$.

Nikolskiĭ's inequality (6.7.8.5.a) says that with a universal constant $c>1$,

$$\|T|L_q\| \le c\,n^{1/p-1/q}\,\|T|L_p\| \quad \text{for } 1\le p<q\le\infty$$

and every trigonometric polynomial T of degree less than or equal to n.

6.3.20.6 As in the theory of function spaces, estimates of the above type combined with (6.3.3.6.a) can be used to prove embedding theorems. For instance, applying $\pi_2(A_k) \le 2^{k/2}\|A_k\|$ to a decomposition

$$T=\sum_{k=0}^{\infty} A_k \quad \text{with} \quad \operatorname{rank}(A_k)\le 2^k \quad \text{and} \quad \sum_{k=0}^{\infty} 2^{k/2}\|A_k\| < \infty$$

yields

$$\pi_2(T) \le \sum_{k=0}^{\infty} \pi_2(A_k) \le \sum_{k=0}^{\infty} 2^{k/2}\|A_k\|.$$

Hence $\mathfrak{L}_{2,1}^{\text{app}} \subseteq \mathfrak{P}_2$. One even gets $\mathfrak{L}_{2,1}^{\text{weyl}} \subseteq \mathfrak{P}_2$; see [PIE$_4$, p. 98].

6.3.20.7 As stated in [TOM, pp. 192–193], corresponding results hold for the type and cotype norms $\|\cdot|\mathfrak{T}_p^{\text{rad}}\|$, $\|\cdot|\mathfrak{T}_p^{\text{gauss}}\|$, and $\|\cdot|\mathfrak{C}_q^{\text{rad}}\|$. It seems that the case of $\|\cdot|\mathfrak{C}_q^{\text{gauss}}\|$ has not been treated so far; König/Tzafriri [1981] considered only identity maps of n-dimensional spaces.

6.3.21 Operator ideals and classes of locally convex linear spaces

6.3.21.1 Let p be a semi-norm on a linear space X, form the null space $N(p)$, and consider the quotient $X(p) := X/N(p)$ whose elements $x(p) := x + N(p)$ are equivalence classes. Denote by $\widetilde{X}(p)$ the Banach space obtained by completing $X(p)$ with respect to the norm $\|x(p)\| := p(x)$. For semi-norms p and q we write $p \preceq q$ if there exists a constant $c \geq 0$ such that $p(x) \leq cq(x)$ whenever $x \in X$. Then $N(q) \subseteq N(p)$, and $I_X(p,q) : x(q) \mapsto x(p)$ defines a canonical map from $X(q)$ onto $X(p)$ that admits a continuous extension $\widetilde{I}_X(p,q) : \widetilde{X}(q) \to \widetilde{X}(p)$.

6.3.21.2 A **locally convex space** is a linear space X together with a system $\mathscr{P}$ of semi-norms such that the following conditions are satisfied:

- If $p(x) = 0$ for all $p \in \mathscr{P}$, then $x = \mathsf{o}$.
- For $p_1 \in \mathscr{P}$ and $p_2 \in \mathscr{P}$ there exists $p \in \mathscr{P}$ with $p_1 \preceq p$ and $p_2 \preceq p$.

The sets $U(p,\varepsilon) := \{x \in X : p(x) < \varepsilon\}$ form a basis of convex zero neighborhoods of a Hausdorff topology on X; see also 3.3.1.3.

6.3.21.3 Recall from 6.3.13.1 that every operator ideal $\mathfrak{A}$ determines a class of Banach spaces $\mathsf{A} := \{X : I_X \in \mathfrak{A}\}$. This correspondence can be extended as follows:

We denote by A^{loc} the class of all locally convex linear spaces X with the property that for every $p \in \mathscr{P}$, there exists $q \in \mathscr{P}$ such that $p \preceq q$ and $\widetilde{I}(p,q) \in \mathfrak{A}$. Obviously, injectivity of $\mathfrak{A}$ implies that A^{loc} is stable in passing to subspaces, while surjectivity yields invariance under the formation of quotients. The relationship between X and X^* is more involved.

If $\mathfrak{A}$ and $\mathfrak{B}$ are operator ideals such that $\mathfrak{A}^n := \overbrace{\mathfrak{A} \circ \cdots \circ \mathfrak{A}}^{n \text{ times}} \subseteq \mathfrak{B}$ for some power n, then $\mathsf{A}^{loc} \subseteq \mathsf{B}^{loc}$.

6.3.21.4 The ideals $\mathfrak{N}_p$, $\mathfrak{I}_p$, $\mathfrak{P}_p$ with $1 \leq p < \infty$ and $\mathfrak{L}_p^{\text{app}}$ $\mathfrak{L}_p^{\text{gel}}$, $\mathfrak{L}_p^{\text{kol}}$, $\mathfrak{L}_p^{\text{weyl}}$, $\mathfrak{L}_p^{\text{chang}}$, and $\mathfrak{L}_p^{\text{ent}}$ with $0 < p < \infty$ generate one and the same class N^{loc}, whose members are the famous **nuclear spaces** invented in Grothendieck's thesis [GRO$_1$, Chap. II, p. 34]. The characterization of nuclearity via Kolmogorov widths and ε-entropy is due to Mityagin [1961].

6.3.21.5 An important example is the nuclear space C^∞ formed by all infinitely differentiable functions f defined on $\mathbb{R}^N$. An appropriate system of semi-norms is given by

$$p_{K,n}(f) := \sup\left\{ \left| \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_N}}{\partial x_N^{k_N}} f(x_1,\dots,x_N) \right| : \begin{array}{c} (x_1,\dots,x_N) \in K \\ k_1 + \cdots + k_N \leq n \end{array} \right\},$$

where K ranges over all compact subsets and $n = 0, 1, \dots$.

Taking only functions with compact support, we obtain the nuclear space $\mathscr{D}$, which carries a more complicated topology. Note that $\mathscr{D}^*$ is the space of distributions; see [SCHW$_1$].

6.3.21.6 In his autobiography [SCHW$^\bullet$, Engl. transl. pp. 283–284], Schwartz reports:

In 1952, just before leaving to spend the summer in Brazil, I proposed the following subject to Grothendieck: put a good topology on the tensor product $E \otimes F$ of locally convex spaces E and F.

At the end of July, in Brazil, I received a very disappointed letter from him: there were two locally convex topologies on $E \otimes F$, both equally natural and yet different!

Two weeks later, I received a new, triumphant letter: the two topologies coincide in the case of $\mathscr{D}^* \otimes F$. There is a new type of locally convex spaces, which he called "nuclear", such that for every F the topologies on $E \otimes F$ coincide. Everything became clear. The space $\mathscr{D}^*$ is nuclear. The word "nuclear" comes from the fact that if F is another $\mathscr{D}^*$ space, one rediscovers the kernel theorem which I had lectured on at the International Congress of Mathematicians in 1950.

6.3.21.7 Spaces of harmonic or analytic functions have a stronger property than nuclearity; they belong to the class associated with the *very small* ideal

$$\mathfrak{F}_0 := \bigcap_{0<p\leq 1} \mathfrak{F}_p,$$

which already occurred in [GRO$_1$, Chap. II, p. 9]. Concerning the definition of $\mathfrak{F}_p$, the reader is referred to 6.3.2.11.

It follows from 6.3.3.2 and 6.3.3.4 that

$$\mathfrak{F}_0 = \bigcap_{p>0} \mathfrak{L}_p^{(s)} = \bigcap_{p>0} \mathfrak{L}_{p\infty}^{(s)}$$

for any choice of additive s-numbers.

Note that $\mathfrak{F}_0$ can not be made a quasi-Banach ideal. However, it is complete with respect to a metrizable topology obtained from countably many ideal quasi-norms: $\varphi_1, \varphi_{1/2}, \varphi_{1/3}, \cdots$.

Operators in $\mathfrak{F}_0$ are sometimes called **strongly nuclear**; see [JAR, p. 443] and [PIE$_1$, pp. 138–141].

6.3.21.8 The ideal $\mathfrak{K}$ yields the class of **Schwartz spaces**; this naming, which was coined by Grothendieck [1954, p. 117], is an homage of a grateful pupil to his teacher.

6.4 Eigenvalue distributions

In this section, we are concerned with *complex* eigenvalues. Hence the underlying Banach spaces are supposed to be *complex* as well.

6.4.1 Eigenvalue sequences and classical results

6.4.1.1 Let $T \in \mathfrak{L}(X)$ be a Riesz operator; see 5.2.3.2. Then all points $\lambda \neq 0$ in the spectrum $\sigma(T)$ are eigenvalues with finite **algebraic multiplicities**

$$m(\lambda) := \dim \bigcup_{k=0}^{\infty} \big\{ x \in X : (\lambda I - T)^k x = \mathrm{o} \big\}.$$

The **eigenvalue sequence** $\big(\lambda_n(T)\big)$ is obtained by applying the following conventions:

(1) Every eigenvalue $\lambda \neq 0$ occurs $m(\lambda)$ times. This fact is described by saying that *eigenvalues are counted according to their algebraic multiplicities*.

(2) $|\lambda_1(T)| \geq \cdots \geq |\lambda_n(T)| \geq \cdots \geq 0$. In order to make the outcome unique, eigenvalues λ and μ with the same absolute value are arranged such that λ precedes μ whenever $0 \leq \arg(\lambda) < \arg(\mu) < 2\pi$.

(3) For operators that have only a finite number of non-zero eigenvalues, the sequence is completed by adding zeros.

6.4.1.2 The Riesz theory tells us that $\lambda_n(T) \to 0$ as $n \to \infty$. This is a qualitative statement. The theory of eigenvalue distributions deals with the quantitative question, *How fast does* $\big(\lambda_n(T)\big)$ *tend to zero?* The most convenient measures are expressed as follows:

$$\sup_n n^{1/p} |\lambda_n(T)| < \infty \quad \text{or} \quad \sum_{n=1}^{\infty} |\lambda_n(T)|^p < \infty \quad \text{for some } p > 0.$$

More generally, we say that an ideal $\mathfrak{A}$ is of **eigenvalue type** $l_{p,q}$ if for every Banach space X, all operators $T \in \mathfrak{A}(X)$ are Riesz and

$$\big\| \big(\lambda_n(T)\big) \big| l_{p,q} \big\| := \Big(\sum_{n=1}^{\infty} n^{q/p-1} |\lambda_n(T)|^q \Big)^{1/q} < \infty.$$

Being of **optimal** eigenvalue type $l_{p,q}$ means that the parameters p and q are chosen such that $l_{p,q}$ becomes as small as possible.

6.4.1.3 If a quasi-Banach ideal $\mathfrak{A}$ is of eigenvalue type $l_{p,q}$, then there exists a constant $c \geq 1$ (not depending on X) such that

$$\big\| \big(\lambda_n(T)\big) \big| l_{p,q} \big\| \leq c \|T|\mathfrak{A}\| \quad \text{whenever } T \in \mathfrak{A}(X).$$

6.4.1.4 The classical approach to eigenvalue distributions is based on infinite determinants. Thus it is not surprising that the first results are implicitly contained in Fredholm's paper [1903, p. 368]. Details will be discussed in Subsection 6.5.2.

A second and (in my opinion!) more elegant method goes back to Weyl [1949]. His idea has led to the development described in Subsections 6.4.2 and 6.4.3.

6.4.1.5 The determinant-free era began with **Schur's inequality**, [1909, p. 506]:

$$\sum_{n=1}^{\infty} |\lambda_n(K)|^2 \leq \int_a^b \int_a^b |K(s,t)|^2 ds dt.$$

Here $\lambda_n(K)$ denotes the n^{th} eigenvalue of the integral operator

$$K_{\text{op}} : f(t) \mapsto \int_a^b K(s,t) f(t) dt,$$

provided that the underlying space is chosen in a natural way. Of course, this terminology is quite sloppy. However, apart from some rare cases, no serious danger of confusion will arise. In the concrete situation, Schur [1909, footnote *) on p. 502] specified the analytical context as follows:

Um einen einfachen Fall vor Augen zu haben, beschränke ich mich im folgenden auf die Betrachtung stetiger Funktionen. Es würde aber genügen anzunehmen, daß die hier vorkommenden Funktionen gewissen leicht zu formulierenden Bedingungen der Integrabilität genügen.

6.4.1.6 In a next step, Lalesco [1915] considered the composition of two kernels:

$$K(s,t) = \int_a^b A(s,\xi) B(\xi,t) d\xi.$$

Without specifying the properties of A and B (continuity ?), he claimed that the eigenvalue sequence of K is summable.

6.4.2 Inequalities between s-numbers and eigenvalues

In this subsection, we let $0 < p < \infty$. Furthermore, it is understood that all inequalities hold uniformly for $n = 1, 2, \dots$.

6.4.2.1 As described in 6.2.1.2, we may assign with every operator $T \in \mathfrak{K}(H)$ the sequence of its singular values (eigenvalues in the sense of Schmidt). Still, in [1937, p. 257] Smithies confessed:

I have so far been unable to establish any direct connection between the orders of magnitude of the eigenvalues and the singular values when the kernel is not symmetric.

A first result along these lines was obtained by Chang, a Ph.D. student of Smithies. Using techniques from the theory of entire functions, he proved in [1949, pp. 352–359] that $\big(s_n(T)\big) \in l_p$ implies $\big(\lambda_n(T)\big) \in l_p$ for every Hilbert–Schmidt (integral) operator T.

6.4.2.2 The situation changed dramatically when Weyl published a short note [1949] on *Inequalities between the two kinds of eigenvalues of a linear transformation*. His main result is the **multiplicative Weyl inequality**, which says that

$$\prod_{k=1}^{n} |\lambda_k(T)| \leq \prod_{k=1}^{n} s_k(T). \tag{6.4.2.2.a}$$

A lemma stated in 6.3.1.7 yields the **additive Weyl inequality**

$$\sum_{k=1}^{n} |\lambda_k(T)|^p \leq \sum_{k=1}^{n} s_k(T)^p \quad \text{for } 0 < p < \infty.$$

6.4.2.3 The extension of Weyl's inequalities to Riesz operators on Banach spaces took almost 30 years. A preliminary result is due to Markus/Matsaev [1971, стр. 305]:

$$\left(\sum_{k=1}^{n} |\lambda_k(T)|^p \right)^{1/p} \leq c_p \left(\sum_{k=1}^{n} a_k(T)^p \log\left(1 + \tfrac{\|T\|}{a_k(T)}\right) \right)^{1/p}.$$

The decisive breakthrough was achieved by König [1978, p. 44], who proved that

$$\left(\sum_{k=1}^{n} |\lambda_k(T)|^p \right)^{1/p} \leq c_p \left(\sum_{k=1}^{n} a_k(T)^p \right)^{1/p}. \tag{6.4.2.3.a}$$

He also discovered that the best possible constants c_p in (6.4.2.3.a) are strictly larger than 1; see [KÖN, pp. 85–86]. The asymptotic behavior of c_p as $p \to 0$ is still unknown.

Subsequently, Johnson/König/Maurey/Retherford [1979, pp. 369–375] observed that (6.4.2.3.a) remains true for Gelfand and Kolmogorov numbers.

In view of John's theorem, (6.4.2.2.a) yields

$$\prod_{k=1}^{n} |\lambda_k(T)| \leq n^{n/2} \prod_{k=1}^{n} a_k(T).$$

Very recently, Hinrichs [2005] discovered "simple" operators on l_∞^n that show that the factor $n^{n/2}$ is indeed necessary. Thus a useful multiplicative Weyl inequality does not hold for operators on arbitrary Banach spaces.

The strongest result of this theory was obtained by Pietsch [1980b, p. 161]:

$$|\lambda_{2n-1}(T)| \leq \mathrm{e} \left(\prod_{k=1}^{n} x_k(T) \right)^{1/n};$$

see also [PIE$_4$, p. 156]. It seems to be unknown whether $|\lambda_{2n-1}(T)|$ can be replaced by $|\lambda_n(T)|$, possibly with a larger factor instead of e.

6.4.2.4 The relationship between eigenvalues and entropy numbers was discovered by Carl [1981a, p. 297]:

$$|\lambda_n(T)| \leq \sqrt{2}\, e_{n+1}(T).$$

Different proofs were given by Carl/Triebel [1980] and Carl [1982a].

6.4.2.5 Summarizing the previous results, we state:
The ideals $\mathfrak{L}_{p,q}^{\mathrm{app}}$, $\mathfrak{L}_{p,q}^{\mathrm{gel}}$, $\mathfrak{L}_{p,q}^{\mathrm{weyl}}$, and $\mathfrak{L}_{p,q}^{\mathrm{ent}}$ are of optimal eigenvalue type $l_{p,q}$.

6.4.3 Eigenvalues of p-summing operators

6.4.3.1 Operators $S \in \mathfrak{L}(X)$ and $T \in \mathfrak{L}(Y)$ are said to be **related** if there exist factors $A \in \mathfrak{L}(X,Y)$ and $B \in \mathfrak{L}(Y,X)$ such that

$$\begin{tikzcd} X \arrow[rr, "S"] \arrow[dr, "A"'] & & X \arrow[dr, "A"] & \\ & Y \arrow[ur, "B"] \arrow[rr, "T"'] & & Y \end{tikzcd} .$$

6.4.3.2 The **principle of related operators** says that $S \in \mathfrak{L}(X)$ and $T \in \mathfrak{L}(Y)$ have many spectral properties in common:

If S is a Riesz operator, then so is T. Moreover, both operators possess the same non-zero eigenvalues with the same algebraic multiplicities.

This elementary but useful observation is due to Pietsch [1963a, pp. 162–164]. However, in the case of finite matrices it goes back to Sylvester [1883, p. 267]:

It will be convenient to introduce here a notion, namely that of the **latent roots** *of a matrix – latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf. If from each term in the diagonal of a given matrix, λ be subtracted, the determinant to the matrix so modified will be a rational integer function of λ; the roots of this function are the latent roots of the matrix.*

The latent roots of the product of two matrices, it may be added, are the same in whichever order the factors be taken.

6.4.3.3 Here is a convincing application of the principle of related operators:

Every 2-summing $S : X \to X$ admits a factorization $S : X \xrightarrow{A} C \xrightarrow{Id} L_2 \xrightarrow{B} X$. Thus it is related to the Hilbert–Schmidt operator $T : L_2 \xrightarrow{B} X \xrightarrow{A} C \xrightarrow{Id} L_2$, and we get:

The ideal $\mathfrak{P}_2$ is of eigenvalue type l_2.

6.4.3.4 Let $1 \le p < 2$. Then the preceding result also holds for $\mathfrak{P}_p \subset \mathfrak{P}_2$. Moreover, in view of $\mathfrak{P}_p(H) = \mathfrak{S}_2(H)$, the eigenvalue type l_2 is optimal.

6.4.3.5 The case $2 < p < \infty$ turned out to be much more difficult. A first solution was given by Johnson/König/Maurey/Retherford [1979, pp. 355–362]:

The ideal $\mathfrak{P}_p$ is of optimal eigenvalue type l_p.

Later on, Pietsch [1986] discovered an elegant proof that is based on the tensor stability of the ideal $\mathfrak{P}_p$. This trick goes back to Russo [1977, p. 244].

6.4.3.6 Let $2 < p < \infty$. The following supplement of the preceding theorem is due to König/Retherford/Tomczak-Jaegermann [1980, pp. 93–97]:

The ideal $\mathfrak{P}_{p,2}$ is of optimal eigenvalue type $l_{p,\infty}$.

The technique used in the proof of this result inspired Pietsch [1980b] when he introduced the concept of Weyl numbers. Nowadays, the eigenvalue type of $\mathfrak{P}_{p,2}$ is obtained from the inequality $n^{1/p} x_n(T) \le \pi_{p,2}(T)$; see [KÖN, p. 81] or [PIE$_4$, p. 98].

6.4.3.7 For γ-summing operators, König [1980c, p. 313] proved the asymptotic formula $\lambda_n(T) = O\big((1+\log n)^{-1/2}\big)$, which is optimal in view of 6.3.6.18.

6.4.3.8 Let $1 \le p,q < \infty$, and fix any measure space $(M,\mathscr{M},\mu)$. A **Hille–Tamarkin kernel** is a measurable function K on $M \times M$ such that

$$\|K|[L_p,L_q]\| := \left(\left(\int_M \int_M |K(s,t))|^q d\mu(t)\right)^{p/q} d\mu(s)\right)^{1/p} \qquad (6.4.3.8.a)$$

is finite. For $p=q=2$, we get the famous **Hilbert–Schmidt kernels**.

Assigning to K the L_q-valued function $\boldsymbol{k} : s \mapsto K(s,\cdot)$, we may identify the collection of all Hille–Tamarkin kernels with $[L_p(M,\mathscr{M},\mu),L_q(M,\mathscr{M},\mu)]$, which is a Bochner space $[L_p(M,\mathscr{M},\mu),X]$ in the sense of 5.1.2.5.

Kernels satisfying condition (6.4.3.8.a) were introduced by Hille/Tamarkin [1931, p. 59], [1934, p. 446], while Benedek/Panzone [1961] studied $[L_p,L_q]$ as a Banach space; see also [KUP].

6.4.3.9 It easily turns out that for $K \in [L_p,L_q]$, the induced integral operator

$$K_{\mathrm{op}} : f(t) \mapsto \int_M K(s,t)f(t)\,d\mu(t)$$

is p-summing as a map from L_{q^*} into L_p.

In order to get an operator from a Banach space into itself, one needs additional properties. Assume, for example, that $q = p^*$. Then K_{op} acts on L_p. Moreover, 6.4.3.4 and 6.4.3.5 imply that

$$\big(\lambda_n(K)\big) \in l_2 \quad \text{if } 1 \le p \le 2 \quad \text{and} \quad \big(\lambda_n(K)\big) \in l_p \quad \text{if } 2 \le p < \infty.$$

This basic theorem is due to Johnson/König/Maurey/Retherford [1979, p. 361]. Two years earlier, Russo [1977, p. 242] had proved its hard part under the stronger condition that not only K, but also the transposed kernel, is of Hille–Tamarkin type $[L_p,L_{p^*}]$.

6.4.3.10 The following result is due to Carl [1982b, p. 405]:

Let $1/p+1/q>1$ and $1/r=\begin{cases} 1/p & \text{if } 1 \le q \le 2, \\ 1/p+1/q-1/2 & \text{if } 2 \le q < \infty. \end{cases}$

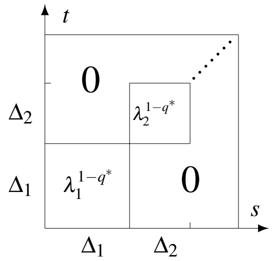
Then $A \in [l_p,l_q]$ defines an operator on l_p such that $\big(\lambda_n(A)\big) \in l_{r,p}$.

6.4.3.11 Let $K \in [L_p,L_q]$ with $1/p+1/q \le 1$. Then, for measure spaces with a finite total mass, K_{op} acts on L_p. Moreover,

$$\big(\lambda_n(K)\big) \in l_2 \quad \text{if } 2 \le q < \infty \quad \text{and} \quad \big(\lambda_n(K)\big) \in l_{q^*} \quad \text{if } 1 \le q \le 2.$$

For $1 < q < 2$, the preceding result of Carl as well as the refined version of the Hausdorff–Young theorem (6.6.7.2.b) suggests the question whether l_{q^*} can be replaced by a smaller Lorentz space $l_{q^*,w}$ with $w < q^*$. An elementary example shows that this is not the case.

Let (λ_n) be any positive decreasing sequence such that $\sum_{n=1}^{\infty} \lambda_n^{q^*} = 1$. Dividing $[0,1)$ into consecutive intervals Δ_n of length $\lambda_n^{q^*}$, a kernel K is obtained as follows:



Then $\|K|[L_p, L_q]\| = 1$, and the characteristic function of Δ_n is an eigenfunction associated with the eigenvalue λ_n.

6.4.4 Eigenvalues of nuclear operators

6.4.4.1 The basic result on eigenvalues of nuclear operators was discovered by Grothendieck [GRO$_1$, Chap. II, p. 16]:

The ideal $\mathfrak{N}$ is of eigenvalue type l_2.

Nowadays, we deduce this conclusion from the fact that every nuclear operator is related to a Hilbert–Schmidt operator.

6.4.4.2 A much better rate of convergence can be obtained if we consider nuclear operators on specific Banach spaces. Hilbert spaces are optimal: a modern version of Lalesco's theorem 6.4.1.6 says that the product of two Hilbert–Schmidt operators has a summable sequence of eigenvalues. Therefore we get $\big(\lambda_n(T)\big) \in l_1$ for all $T \in \mathfrak{S}_1(H)$.

On the other hand, Carleman [1918, p. 378] was able to construct a continuous 2π-periodic function for which the sequence of Fourier coefficients fails to be p-summable whenever $0 < p < 2$. Hence there are operators $T \in \mathfrak{N}\big(C(\mathbb{T})\big)$ such that $\big(\lambda_n(T)\big) \notin l_p$; see 6.7.10.7.

6.4.4.3 Johnson/König/Maurey/Retherford [1979, p. 376] observed the striking fact that a Banach space X is isomorphic to a Hilbert space if and only if $\big(\lambda_n(T)\big) \in l_1$ for all $T \in \mathfrak{N}(X)$.

6.4.4.4 The previous theorem suggests a modification of a question of Kac [1966], *Can one hear the shape of a drum?* Instead, we may ask, *Can one hear the structure of a Banach space?* This is indeed the case, since the properties **(BM$_\alpha$)**, **(P$_\alpha$)**, **(HP$_\alpha$)**, and **(G$_\alpha$)** considered in 6.1.1.10 and 6.1.11.5 are equivalent to

(E$_\alpha$) $\big(\lambda_n(T)\big) \in l_{p,\infty}$ for all $T \in \mathfrak{N}(X)$, where $\alpha = 1/p^*$, $0 < \alpha < 1/2$ and $1 < p < 2$.

The non-trivial implication **(G_α)⇒(E_α)** follows from

$$\Big(\prod_{k=1}^{n}\big|\lambda_k(T)\big|\Big)^{1/n} \le \frac{\mathrm{e}^2}{n}\,\Gamma_n(X)\lambda_2^*(T) \quad \text{for } T\in\mathfrak{L}_2^*(X),$$

a beautiful Weyl-type inequality that is due to Geiss [1990a, p. 75].

In the limiting case $\alpha=0$ and $p=1$, the condition **(E_0)** characterizes weak Hilbert spaces; see 6.1.11.8 and Pisier [1988, p. 563].

6.4.4.5 It is natural to consider the class of Banach spaces X such that $\big(\lambda_n(T)\big)\in l_p$ for all $T\in\mathfrak{N}(X)$. Unfortunately, no geometric or "idealistic" criterion seems to be known; see Pietsch [1991a, p. 62]. However, Pisier [1979b, p. 276] showed that for $1<r\le 2$ and $1/p=1/2+1/r^*$, all r-convex and r^*-concave Banach lattices have the property above. This is, in particular, true for L_r and L_{r^*}; see Johnson/König/Maurey/Retherford [1979, p. 366].

6.4.4.6 Let $0<p<1$, and look at 6.3.2.11 for the definition of $\mathfrak{F}_p$. Then we have the following result of König [1977]:
The ideal $\mathfrak{F}_p$ is of optimal eigenvalue type $l_{r,p}$ with $1/r=1/p-1/2$.

6.4.5 Operators of eigenvalue type $l_{p,q}$

6.4.5.1 Recall from 5.2.3.6 that the ideal $\mathfrak{R}_{\rm ad}$ consists of all operators $T\in\mathfrak{L}(X,Y)$ such that AT is a Riesz operator for every $A\in\mathfrak{L}(Y,X)$. Hence we may consider the eigenvalue sequence $\big(\lambda_n(AT)\big)$, and the question arises whether the definition

$$\mathfrak{L}_{p,q}^{\rm eig} := \Big\{T\in\mathfrak{R}_{\rm ad} : \big(\lambda_n(AT)\big)\in l_{p,q} \text{ if } X \underset{A}{\overset{T}{\rightleftarrows}} Y\Big\}$$

yields an ideal. Of course, $\mathfrak{L}_{p,q}^{\rm eig}(H)=\mathfrak{S}_{p,q}(H)$ for every Hilbert space H. However, the answer is negative for Banach spaces; see Pietsch [1982b, p. 362].

We have $Id\in\mathfrak{P}_2^{\rm dual}(l_2,c_0)$ and $D_t\in\mathfrak{P}_2(c_0,l_2)$ for $(\tau_k)\in l_2$. Hence the operators defined by

$$S:(x,y)\mapsto(\mathrm{o},x) \quad\text{and}\quad T:(x,y)\mapsto(D_t y,\mathrm{o})$$

belong to $\mathfrak{L}_2^{\rm eig}(l_2\oplus c_0)$. On the other hand, it follows from

$$S+T:\big(\tau_k^{1/2}e_k,e_k\big)\mapsto\big(\tau_k e_k,\tau_k^{1/2}e_k\big)$$

that $S+T$ has the eigenvalues $\big(\tau_k^{1/2}\big)$ with the corresponding eigenvectors $\big(\tau_k^{1/2}e_k,e_k\big)$. Consequently, in the worst case, $S+T$ is only a member of $\mathfrak{L}_4^{\rm eig}(l_2\oplus c_0)$.

6.4.5.2 The following example shows that the technique of cloning is not unknown in Banach space theory.

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DISTRIBUTION OF EIGENVALUES AND NUCLEARITY

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In this paper we shall use the terminology introduced in [6]. In particular, $\mathfrak{L}(E, F)$ denotes the set of all (bounded linear) operators from the Banach space E into the Banach space F. Since we are concerned with spectral properties of operators, all Banach spaces under consideration are supposed to be complex.

1. $\mathfrak{S}_p^{\mathrm{eig}}$-operators

Let $S \in \mathfrak{L}(E, E)$ and put

$$N(\lambda, S) := \bigcup_{k=1}^{\infty} \{x \in E\colon (\lambda I_E - S)^k x = 0\}.$$

Here I_E denotes the identity map of E. If $N(\lambda, S) \neq \{0\}$, then $\lambda \in \boldsymbol{C}$ (complex field) is called an *eigenvalue* of S and

$$\alpha(\lambda, S) := \dim N(\lambda, S)$$

is said to be its *algebraic multiplicity*.

Let $0 < p < \infty$. An operator $S \in \mathfrak{L}(E, F)$ is of *Riesz type* l_p if

$$\sum_{\lambda \in \boldsymbol{C}} \alpha(\lambda, LS)|\lambda|^p < \infty \quad \text{for all} \quad L \in \mathfrak{L}(F, E).$$

The class of these operators will be denoted by $\mathfrak{S}_p^{\mathrm{eig}}$.

Remark. If $S \in \mathfrak{S}_p^{\mathrm{eig}}(E, E)$, then we have

$$\sum_{\lambda \in \boldsymbol{C}} \alpha(\lambda, S)|\lambda|^p = \sum_{I} |\lambda_i(S)|^p,$$

where $(\lambda_i(S)\colon i \in I)$ is the (countable!) family of all eigenvalues $\lambda \neq 0$ repeated according to their (finite!) algebraic multiplicities.

In order to check the following result we need an elementary consequence of the spectral mapping theorem; [1], VII.3.19.

LEMMA. *Let $0 < p < \infty$ and $n = 1, 2, \ldots$ Then*

$$\sum_{\mu \in \boldsymbol{C}} \alpha(\mu, S^n)|\mu|^{p/n} = \sum_{\lambda \in \boldsymbol{C}} \alpha(\lambda, S)|\lambda|^p \quad \textit{for all } S \in \mathfrak{L}(E, E).$$

[361]

The main result stated on p. 364 of this article reads as follows.

THEOREM. *Let $0 < p < \infty$. If $\mathfrak{A}$ is an operator ideal such that $\mathfrak{A} \subseteq \mathfrak{S}_p^{\mathrm{eig}}$, then $\mathfrak{A}^{2n} \subseteq \mathfrak{N}$ whenever $n \geq p$.*

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On Eigenvalues of p–Riesz Operators

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SUMMARY

In this paper we consider the eigenvalues of p-Riesz Operators and establish a property of nuclearity in terms of it.

Notation :—Let L (E, F) denote the set of all continuous linear operators from the complex Banach space E into the complex Banach space F. Let S ∈ L (E, E) and put

$$N(\lambda, S) := \bigcup_{k=1}^{\infty} \{ x \in E : (\lambda I_E - S)^k x = 0 \}.$$

Here I_E denotes the identity operator on E. If $N(\lambda, S) \neq \{0\}$, then $\lambda \in C$ (Complex field) is called an eigenvalue of S and

$$\alpha(\lambda, S) := \dim N(\lambda, S)$$

is said to be its algebraic multiplicity.

Let $0 < p < \infty$. An operator $S \in L(E, F)$ is of Riesz type l_p if

$$\sum_{\lambda \in C} \alpha(\lambda, LS) \, |\lambda|^p < \infty \text{ for all } L \in L(F, E).$$

The class of these operators will be denoted by Sp.

REMARK :—

If $S \in Sp(E, E)$, then we have

$$\sum_{\lambda \in C} \alpha(\lambda, S) \, |\lambda|^p = \sum_{i \in I} |\lambda_i(S)|^p,$$

where $(\lambda_i(S) : i \in I)$ is the (countable) family of all eigenvalue $\lambda \neq 0$ repeated according to their (finite) algebraic multiplicities.

We are now prepared to prove.

The main result stated on p. 14 of this article reads as follows.

THEOREM: – Let $0 < p < \infty$. If A is an operator ideal such that $A \subseteq S_p$, then $A^{2n} \subseteq N$ whenever $n \geq p$.

6.5 Traces and determinants

The concepts of traces and determinants make sense in the real as well as in the complex setting. However, since Fredholm denominators are entire functions, the underlying Banach spaces are supposed to be *complex*.

Standard references are [GOH$_2^+$], [KÖN], and [PIE$_4$].

6.5.1 Traces

Some preliminary facts about traces can be found in 4.10.1.1, 4.10.1.3, and 5.7.3.8. In preparation of the following, the reader is recommended to consult these paragraphs. The Hilbert space case is nicely presented in [RETH].

6.5.1.1 Let

$$T=\sum_{k=1}^{n} x_k^*\otimes x_k$$

be any finite rank operator on a Banach space X. Then its trace is defined by

$$\operatorname{trace}(T):=\sum_{k=1}^{n}\langle x_k^*,x_k\rangle.$$

Looking at the properties of the functional $T\mapsto\operatorname{trace}(T)$, Pietsch [1981b, p. 63] proposed an axiomatic approach.

A **trace** on an operator ideal $\mathfrak{A}$ is a scalar-valued function $\tau: T\mapsto\tau(T)$, defined on all components $\mathfrak{A}(X)$, such that the following conditions are satisfied:

(T$_0$) $\tau(x^*\otimes x)=\langle x^*,x\rangle$ for $x^*\in X^*$ and $x\in X$.
(T$_1$) $\tau(S+T)=\tau(S)+\tau(T)$ for $S,T\in\mathfrak{A}(X)$.
(T$_2$) $\tau(\lambda T)=\lambda\tau(T)$ for $\lambda\in\mathbb{C}$ and $T\in\mathfrak{A}(X)$.
(T$_3$) $\tau(AT)=\tau(TA)$ for $T\in\mathfrak{A}(X,Y)$ and $A\in\mathfrak{L}(Y,X)$.

In particular, τ is linear on every $\mathfrak{A}(X)$. I stress the significance of condition **(T$_3$)**, which relates the values of τ on (possibly different) components $\mathfrak{A}(X)$ and $\mathfrak{A}(Y)$.

6.5.1.2 Let $\mathfrak{A}$ be a quasi-Banach operator ideal such that

$$|\operatorname{trace}(T)|\le c\|T|\mathfrak{A}\|\quad\text{for all } T\in\mathfrak{F}(X),$$

where the constant $c\geq 1$ does not depend on the Banach space X. If the finite rank operators are dense in all components $\mathfrak{A}(X)$, then the functional $T\mapsto\operatorname{trace}(T)$ admits a unique continuous extension: $\operatorname{trace}_{\mathfrak{A}}$. This is the usual and most elementary way to produce traces.

6.5.1.3 In view of property **(AP$_5$)** in 5.7.4.1, the extension process just described does not apply to the ideal of nuclear operators. In other words, there exists no continuous trace on $\mathfrak{N}$. Since $\mathfrak{N}$ is the smallest Banach ideal, we really need quasi-Banach ideals for developing a meaningful theory of traces.

The disaster just described can be avoided by restricting the ideal of nuclear operators to the class of all Banach spaces with the approximation property. Then

$$\operatorname{trace}_{\mathfrak{N}}(T) := \sum_{k=1}^{\infty} \langle x_k^*, x_k \rangle$$

does not depend on the special choice of the nuclear representation (5.7.3.1.a)

$$T = \sum_{k=1}^{\infty} x_k^* \otimes x_k.$$

In particular, if $K_{\mathrm{op}} \in \mathfrak{N}\big(C[a,b]\big)$ is induced by a continuous kernel K, then

$$\operatorname{trace}_{\mathfrak{N}}(K_{\mathrm{op}}) = \int_a^b K(s,s)\,ds;$$

see Grothendieck [1956a, p. 375].

6.5.1.4 Continuous traces can live only on "small" quasi-Banach operator ideals $\mathfrak{A}$. More precisely, $\mathfrak{A}$ must be contained in the ideal of integral operators; see 5.7.3.9. Hence $\mathfrak{A}$ has eigenvalue type l_2. It seems to be an open problem whether the last result is optimal. All "bad" candidates such as $(\mathfrak{P}_2)^2 + (\mathfrak{P}_2^{\mathrm{dual}})^2$ have eigenvalue type not worse than $l_{4/3}$; see 6.5.1.9.

6.5.1.5 A classical theorem asserts that the trace of a matrix can be obtained as the sum of its eigenvalues. Extending this result to the infinite-dimensional setting is the main achievement of modern trace theory. One may even take the desired formula as a starting point.

Let $\mathfrak{A}$ be an ideal of eigenvalue type l_1, and define the **spectral sum** by

$$\lambda_{\mathrm{sum}}(T) := \sum_{n=1}^{\infty} \lambda_n(T) \quad \text{for all } T \in \mathfrak{A}(X).$$

Obviously, the functional $T \mapsto \lambda_{\mathrm{sum}}(T)$ has the properties $(\mathbf{T}_0)$, $(\mathbf{T}_2)$, and $(\mathbf{T}_3)$. The crucial point is additivity. If $(\mathbf{T}_1)$ holds, then we refer to $\operatorname{trace}(T) := \lambda_{\mathrm{sum}}(T)$ as the **spectral trace**. Maybe this is always true. Fortunately, White [1996, p. 672] gave an affirmative answer in the most important case; namely for quasi-Banach ideals. The proof of this striking result is based on techniques borrowed from the theory of analytic multi-valued functions.

6.5.1.6 White's theorem is the culmination of a long story, which will be told now.

First of all, Mercer [1909, p. 446] proved the trace formula

$$\int_a^b K(s,s)\,ds = \sum_{n=1}^{\infty} \lambda_n(K)$$

for positive definite continuous kernels K. An abstract version of this result was established by von Neumann [vNEU, pp. 97–99].

Next, I list the most important ideals on which the spectral sum defines a trace. The order is chronological:

$\mathfrak{F}_{2/3}$	6.3.2.11 :	Grothendieck [GRO$_1$, Chap. II, pp. 18–19]
$\mathfrak{S}_1(H)$	6.3.1.6 :	Lidskiĭ [1959b]; a remark in [GRO$_1$, Chap. II, p. 13] indicates that by 1955, Grothendieck was aware of this fact
$\mathfrak{L}_1^{app}$	6.3.3.2 :	König [1980a, p. 164]; the slightly smaller ideal of operators T such that $\sum_{n=1}^{\infty} a_n(T) \log\left(1+\frac{\|T\|}{a_n(T)}\right) < \infty$ was earlier treated by Markus/Matsaev [1971, стр. 307]
$\mathfrak{P}_2 \circ \mathfrak{P}_2$	6.3.2.9 :	König [1980b, pp. 259–260]; in a somewhat hidden form this result was proved by Ha [1975, p. 134]
$(\mathfrak{P}_2)_{2,1}^{app}$	6.3.3.11 :	Pietsch [1981b, p. 82]
$\mathfrak{L}_1^{gel}$, $\mathfrak{L}_1^{kol}$	6.3.3.2 :	Pietsch [1981b, p. 84]
$\mathfrak{L}_1^{weyl}$, $\mathfrak{L}_1^{ent}$	6.3.3.12 :	follows from White's theorem.

6.5.1.7 Grothendieck and Lidskiĭ obtained their trace formulas with the help of Fredholm denominators; see 6.5.5.7. Erdős [1974] and Power [1983] used the fact that every compact operator has a maximal nest of invariant subspaces; see 5.2.4.6. However, the latter method is restricted to Hilbert spaces. The most elementary approach in the setting of Banach spaces is due to Leiterer/Pietsch [1982], who borrowed the main idea of their proof from König [1980a].

6.5.1.8 Retherford [1980•, pp. 54–55] observed that there exist quasi-Banach ideals of eigenvalue type l_1 that contain non-compact operators. Take, for example, the class of operators factoring through $Id : l_1 \to c_0$.

6.5.1.9 The property of having eigenvalue type l_1 need not carry over from quasi-Banach ideals $\mathfrak{A}$ and $\mathfrak{B}$ to their sum $\mathfrak{A}+\mathfrak{B}$. Nevertheless, one may define a continuous trace on $\mathfrak{A}+\mathfrak{B}$. The most prominent example is $(\mathfrak{P}_2)^2+(\mathfrak{P}_2^{dual})^2$, which has optimal eigenvalue type $l_{4/3}$; see Pietsch [1981b, pp. 78–79].

6.5.1.10 Obviously, the ideal of finite rank operators supports precisely one trace. Hence, if $\mathfrak{F}$ is dense in a quasi-Banach ideal, then at most one continuous extension exists. This is not so in the general case. In the setting of Hilbert spaces, the first example of an “unusual” trace is due to Kalton [1987]. Subsequently, Figiel (unpublished) discovered a method for constructing different traces for operators on Banach spaces; see the survey of Pietsch [1990].

6.5.1.11 It is still unknown whether the finite rank operators are dense with respect to the quasi-norms of $\mathfrak{L}_1^{\text{gel}}$, $\mathfrak{L}_1^{\text{kol}}$, $\mathfrak{L}_1^{\text{weyl}}$, or $\mathfrak{L}_1^{\text{ent}}$. Thus it may well happen that these quasi-Banach ideals support continuous traces different from the spectral trace.

6.5.2 Fredholm denominators and determinants

The reader should consult Subsections 2.6.5 and 5.2.3.

6.5.2.1 The **Fredholm resolvent** of an operator $T \in \mathfrak{L}(X)$ is defined by

$$F(\zeta,T) := T(I-\zeta T)^{-1},$$

and we have $(I-\zeta T)^{-1} = I+\zeta F(\zeta,T)$.

In the special case of a Riesz operator, the $\mathfrak{L}(X)$-valued function $\zeta \mapsto F(\zeta,T)$ is meromorphic on the complex plane. Thus, by the **Weierstrass factorization theorem** [1876], there exist entire functions

$$d(\zeta,T) = \sum_{n=0}^{\infty} d_n \zeta^n \qquad \text{and} \qquad D(\zeta,T) = \sum_{n=0}^{\infty} D_n \zeta^n$$

such that

$$F(\zeta,T) = \frac{D(\zeta,T)}{d(\zeta,T)} \quad \text{and} \quad d_0 = 1;$$

see 5.2.3.3. Hence $d(\zeta,T)T = D(\zeta,T)(I-\zeta T)$. Substituting the power series and equating coefficients yields $D_0 = T$ and $D_n = d_n T + D_{n-1}T$, which in turn implies that

$$D_n = \sum_{h=0}^{n} d_h T^{n-h+1}. \tag{6.5.2.1.a}$$

These elementary but useful considerations are due to Ruston [1953, p. 369].

An entire function $d(\zeta,T)$ is called a **Fredholm denominator** or **Fredholm divisor** if its zeros are just the poles of the Fredholm resolvent. Furthermore, the multiplicity of every zero ζ_0 is supposed to coincide with the algebraic multiplicity of the eigenvalue $\lambda_0 = 1/\zeta_0$. In this way, $d(\zeta,T)$ is uniquely determined up to a factor $e^{f(\zeta)}$, where f denotes an entire function.

The corresponding function $D(\zeta,T)$ is said to be a **Fredholm numerator** or a **Fredholm minor** (of first order).

6.5.2.2 The main problem consists in finding explicit methods that yield the required Fredholm denominators. The basic tools are traces and determinants.

Let ζ_0 be any pole of the Fredholm resolvent. Then there is a Laurent expansion

$$-F(\zeta,T) = \frac{A_d}{(\zeta-\zeta_0)^d} + \cdots + \frac{A_1}{\zeta-\zeta_0} + \sum_{n=0}^{\infty} B_n(\zeta-\zeta_0)^n$$

in which the coefficients of the singular part have finite rank; compare with formula (5.2.3.2.a). In addition, we know that $A_d,\dots,A_2$ are nilpotent, whereas A_1 is just the spectral projection associated with the eigenvalue $\lambda_0 = 1/\zeta_0$; see 5.2.1.4.

Assume that $T \in \mathfrak{A}(X)$, where $\mathfrak{A}$ is any quasi-Banach ideal with a continuous trace τ. Then $\tau(A_d) = \cdots = \tau(A_2) = 0$, and $\tau(A_1)$ is equal to the algebraic multiplicity of λ_0; say m_0. This implies that

$$-\tau\big(F(\zeta,T)\big) = \frac{m_0}{\zeta-\zeta_0} + \sum_{n=0}^{\infty} \tau(B_n)(\zeta-\zeta_0)^n.$$

Hence a Fredholm denominator can be obtained by solving the differential equation

$$\big[\log d(\zeta,T)\big]' = \frac{d'(\zeta,T)}{d(\zeta,T)} = -\tau\big[F(\zeta,T)\big] \qquad (6.5.2.2.a)$$

under the initial condition $d(0,T) = 1$.

6.5.2.3 From (6.5.2.2.a) we get $d'(\zeta,T) = -\tau\big[D(\zeta,T)\big]$. Substituting the corresponding power series and equating coefficients yields

$$d_0 = 1 \quad \text{and} \quad d_n = -\tfrac{1}{n}\tau(D_{n-1}). \qquad (6.5.2.3.a)$$

In a next step, the numbers $d_0, d_1, \ldots \in \mathbb{C}$ can be determined recursively by (6.5.2.1.a) and (6.5.2.3.a):

$$-nd_n = \tau(D_{n-1}) = \sum_{h=0}^{n-1} d_h \tau(T^{n-h}). \qquad (6.5.2.3.b)$$

However, there are also direct formulas. With the abbreviation $\tau_k = \tau(T^k)$, we rewrite (6.5.2.3.b) in the form

$$\sum_{h=1}^{n-1} d_h \tau_{n-h} + nd_n = -\tau_n.$$

More explicitly,

$$\begin{pmatrix} 1 & 0 & \cdots\cdots & 0 & 0 \\ \tau_1 & 2 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{n-2} & \tau_{n-3} & \cdots\cdots & n-1 & 0 \\ \tau_{n-1} & \tau_{n-2} & \cdots\cdots & \tau_1 & n \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix} = - \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{n-1} \\ \tau_n \end{pmatrix}.$$

Now it follows from Cramer's rule that

$$d_n = \frac{(-1)^n}{n!} \det \begin{pmatrix} \tau_1 & 1 & \cdots\cdots & 0 & 0 \\ \tau_2 & \tau_1 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{n-1} & \tau_{n-2} & \cdots\cdots & \tau_1 & n-1 \\ \tau_n & \tau_{n-1} & \cdots\cdots & \tau_2 & \tau_1 \end{pmatrix}. \tag{6.5.2.3.c}$$

We also have

$$D_n = \frac{(-1)^n}{n!} \det \begin{pmatrix} \tau_1 & 1 & \cdots\cdots & 0 & 0 \\ \tau_2 & \tau_1 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_n & \tau_{n-1} & \cdots\cdots & \tau_1 & n \\ T^{n+1} & T^n & \cdots\cdots & T^2 & T^1 \end{pmatrix}. \tag{6.5.2.3.d}$$

Indeed, expanding the right-hand determinant by its last row yields (6.5.2.1.a).

The aesthetic formulas (6.5.2.3.c) and (6.5.2.3.d) were discovered by a mathematician from Slovenia and bear his name: **Plemelj's formulas**; see [1904, p. 122].

6.5.2.4 For sufficiently small ζ, the Fredholm resolvent in (6.5.2.2.a) admits a Neumann expansion, and term-by-term integration gives

$$\log d(\zeta,T) = -\sum_{n=1}^{\infty} \tfrac{1}{n}\tau(T^n)\zeta^n = \tau\big[\log(I-\zeta T)\big]. \tag{6.5.2.4.a}$$

This formula can be used as the starting point of determinant theory; see 6.5.2.7.

6.5.2.5 A **determinant** on an ideal $\mathfrak{A}$ is a scalar-valued function $\delta : T \mapsto \delta(I-T)$, defined on all components $\mathfrak{A}(X)$, such that the following conditions are satisfied:

(D$_0$**)** $\delta(I-x^*\otimes x) = 1-\langle x^*,x\rangle$ for $x^*\in X^*$ and $x\in X$.
(D$_1$**)** $\delta\big((I-S)(I-T)\big) = \delta(I-S)\delta(I-T)$ for $S,T\in\mathfrak{A}(X)$.
(D$_2$**)** $\delta(I-\zeta T)$ is an entire function of $\zeta\in\mathbb{C}$ for fixed $T\in\mathfrak{A}(X)$.
(D$_3$**)** $\delta(I_X-AT) = \delta(I_Y-TA)$ for $T\in\mathfrak{A}(X,Y)$ and $A\in\mathfrak{L}(Y,X)$.

6.5.2.6 The ideal of finite rank operators

$$T = \sum_{k=1}^{n} x_k^*\otimes x_k$$

supports one and only one determinant,

$$\det(I-T) := \det\Big(\delta_{hk} - \langle x_h^*, x_k\rangle\Big).$$

Of course, it must be shown that the right-hand quantity does not depend on the special choice of the finite representation.

6.5.2.7 Given any quasi-Banach ideal $\mathfrak{A}$ with a continuous trace τ, formula (6.5.2.4.a) suggests the following definition:

$$\delta(I-T):=\exp\big(\tau\big[\log(I-T)\big]\big) \text{ if } I-T \text{ is invertible, and } \delta(I-T):=0 \text{ otherwise.}$$

The logarithm above needs some explanation. Indeed,

$$\log(I-T):=-\int_{\mathscr{C}} F(\zeta,T)\,d\zeta$$

depends on the path $\mathscr{C}$ that, avoiding the poles of the Fredholm resolvent, is supposed to connect 0 and 1. However, different values deviate from each other only by finitely many summands of the form $2\pi\mathrm{i}A$, where A is a projection of finite rank. Passing to $\tau\big[\log(I-T)\big]$ reduces the indeterminateness down to a summand $2\pi\mathrm{i}m$ with some integer m. Therefore $\exp\big(\tau\big[\log(I-T)\big]\big)$ is well-defined. In this way, every continuous trace generates a continuous determinant. Just for completeness, I state that the Fredholm denominator associated with τ is now given by $d(\zeta,T)=\delta(I-\zeta T)$.

Conversely, starting with a continuous determinant δ, we get a continuous trace by letting

$$\tau(T):=\lim_{\zeta\to 0}\frac{\delta(I+\zeta T)-1}{\zeta} \quad \text{for } T\in\mathfrak{A}(X).$$

Hence the correspondence $\delta\leftrightarrow\tau$ is one-to-one.

6.5.2.8 So far, I have described the present state of the art. Now the reader should turn back his mind to the end of the nineteenth century.

The American astronomer and mathematician Hill [1877, see Acta Math., p. 26] was the first to define the determinant of an infinite matrix $A=(\alpha_{hk})$:

The question of the convergence, so to speak, of a determinant, consisting of an infinite number of constituents, has nowhere, so far as I am aware, been discussed. All such determinants [indexed by $\mathbb{Z}\times\mathbb{Z}$] *must be regarded as having a central constituent; when, in computing in succession the determinants formed from the* $3^2, 5^2, 7^2$ *& c., constituents symmetrically situated with respect to the central constituent, we approach, without limit, a determinate magnitude, the determinant may be called convergent, and the determinate magnitude is its value.*

Hill's approach was justified by Poincaré [1886, p. 84] in the case that $a_{hh}=1$ and

$$\sum_{\substack{h,k\in\mathbb{Z}\\ h\neq k}}|\alpha_{hk}|<\infty.$$

Next, von Koch [1892, pp. 220–221] had the clever idea to consider matrices of the form $I{+}A$ such that

$$\sum_{h,k\in\mathbb{Z}}|\alpha_{hk}|<\infty, \tag{6.5.2.8.a}$$

which he called *de la forme normale*. In this way, the artificial condition about the entries on the principal diagonal was removed.

Another observation of von Koch [1892, p. 228] says:

On peut démontrer d'une manière absolument analogue à celle employée pour le cas des déterminants finis, que l'égalité suivante a lieu:

$$D = 1 + \sum a_{pp} + \sum \begin{vmatrix} a_{pp} & a_{pq} \\ a_{qp} & a_{qq} \end{vmatrix} + \sum \begin{vmatrix} a_{pp} & a_{pq} & a_{pr} \\ a_{qp} & a_{qq} & a_{qr} \\ a_{rp} & a_{rq} & a_{rr} \end{vmatrix} + \cdots ,$$

les indices $p, q, r, \ldots$ *parcourant tous les nombres entiers positifs et négatifs qui satisfont aux conditions* $p < q < r < \ldots$; see also [1901, p. 93].

In [1901, pp. 107–108], von Koch passed to matrices indexed by $\mathbb{N} \times \mathbb{N}$, and more importantly, he replaced (6.5.2.8.a) by the weaker condition

$$\sum_{h=1}^{\infty} \sup_{k} |\alpha_{hk}| < \infty. \tag{6.5.2.8.b}$$

Ironically, the associated determinants were called *hypernormal*. The significance of this modification became transparent only later. Indeed, (6.5.2.8.b) characterizes just those matrices that generate nuclear operators on l_1; see 5.7.3.4.

6.5.2.9 Fredholm announced his celebrated determinant theory of integral equations in [1900]. The final version was published in Acta Mathematica [1903]. It seems that Fredholm was mainly inspired by a visit to Paris in spring 1899, where he attended lectures of Poincaré, Picard, and Hadamard. Only in [1909, p. 95] did he acknowledge:

... les travaux de mon collègue M. V. KOCH *sur les déterminants infinis ont beaucoup facilité mes recherches ... .*

The following starting point of Fredholm, [1900, p. 40], [1903, p. 367], should be compared with von Koch's formula in the preceding paragraph:

Pour définir D_f *j'introduis la notation abrégée*

$$f\begin{pmatrix} x_1, & x_2, & \ldots, x_n \\ y_1, & y_2, & \ldots, y_n \end{pmatrix} = \begin{vmatrix} f(x_1,y_1)\ , & f(x_1,y_2)\ , & \ldots, & f(x_1,y_n) \\ f(x_2,y_1)\ , & f(x_2,y_2)\ , & \ldots, & f(x_2,y_n) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ f(x_n,y_1)\ , & f(x_n,y_2)\ , & \ldots, & f(x_n,y_n) \end{vmatrix}$$

et je pose

$$D_f = 1 + \int_0^1 f(x,x)\,dx + \frac{1}{2!}\int_0^1\int_0^1 f\begin{pmatrix} x_1, & x_2 \\ x_1, & x_2 \end{pmatrix} dx_1\,dx_2 + \cdots .$$

Replacing f by ζf yields an entire function $\zeta \mapsto D_{\zeta f}$ on the complex plane. Fredholm [1903, p. 284] also discovered the formula

$$\log D_{\zeta f} = -\sum_{n=1}^{\infty} \frac{(-\zeta)^n}{n} \int_0^1 \cdots \int_0^1 f(x_1,x_2) f(x_2,x_3) \cdots f(x_{n-1},x_n) f(x_n,x_1) dx_1 \cdots dx_n;$$

see (6.5.2.4.a).

The following quotation from [1903, p. 372] shows that Fredholm already thought in terms of operators:

En considérant l'équation

$$\varphi(x) + \int_0^1 f(x,s)\varphi(s)ds = \psi(x)$$

comme transformant la fonction $\varphi(x)$ en une nouvelle fonction $\psi(x)$ j'écris cette même équation $S_f \varphi(x) = \psi(x)$, et je dis que la transformation S_f appartient à la fonction $f(x,y)$.

An important supplement to Fredholm's theory was given by Plemelj [1904, p. 122], who expressed the Taylor coefficients of $D_{\zeta f}$ by means of traces:

$$\tau_n := \int_0^1 \cdots \int_0^1 f(x_1,x_2) f(x_2,x_3) \cdots f(x_{n-1},x_n) f(x_n,x_1) dx_1 \cdots dx_n;$$

see (6.5.2.3.c).

6.5.2.10 As a decisive tool for proving the convergence of his power series, Fredholm used **Hadamard's inequality**, [1893a]:

$$|\det(A)| \leq \prod_{h=1}^{n} \left(\sum_{k=1}^{n} |\alpha_{hk}|^2 \right)^{1/2} \quad \text{for every } (n,n)\text{-matrix } A = (\alpha_{hk}).$$

This is another example of a happy coincidence: a powerful result was discovered, just in time, for the purpose of functional analysis; see also 1.1.3 and Subsection 6.5.5.

6.5.2.11 Michal/Martin [1934] made a first attempt of developing an abstract version of Fredholm's determinant theory. However, their approach was far too general.

These authors [1934, pp. 69–70] invented, even before Nagumo [1936] and the Russian school, the concept of a complete normed algebra $\mathscr{A}$ (Banach algebra with a unit element). In addition, they fixed some $\tau \in \mathscr{A}^*$ such that $\tau(ST) = \tau(TS)$ for $S,T \in \mathscr{A}$. This continuous linear functional was supposed to play the role of a trace. Fredholm denominators and numerators were defined via Plemelj's formulas, (6.5.2.3.c) and (6.5.2.3.d). Moreover, on p. 88 of the Michal/Martin paper we find the equation

$$\log d(\zeta,T) = -\sum_{n=1}^{\infty} \tfrac{1}{n} \tau(T^n)\zeta^n,$$

which holds for sufficiently small ζ's; see (6.5.2.4.a). Unfortunately, the power series defining $d(\zeta,T)$ and $D(\zeta,T)$ need not converge everywhere. As a negative example one may take the complex field with the "trace" $\tau(t) := \frac{1}{2}t$ for $t \in \mathbb{C}$. Then it follows that

$$\log d(\zeta,T) = -\tfrac{1}{2}\sum_{n=1}^{\infty}\tfrac{1}{n}t^n\zeta^n = \log\sqrt{1-\zeta t}.$$

Nevertheless, the Michal–Martin approach pointed in the right direction.

6.5.2.12 We owe the most important contributions to the determinant theory of abstract operators to Ruston [1951a], [1951b] and Grothendieck [1956a], who discovered, independently of each other, almost the same approach, which is based on the concept of a nuclear (trace class) operator. A similar theory for integral operators (in the sense of Grothendieck) was developed by Leżański and Sikorski; see 6.5.3.4.

6.5.2.13 Given $T \in \mathfrak{N}(X)$, we consider any nuclear representation

$$T = \sum_{k=1}^{\infty} x_k^* \otimes x_k$$

such that

$$c := \sum_{k=1}^{\infty}\|x_k^*\| < \infty \quad \text{and} \quad \|x_k\| \le 1 \quad \text{for } k = 1,2,\dots\,.$$

Then a Fredholm denominator

$$d(\zeta,T) = 1 + \sum_{n=1}^{\infty} d_n\zeta^n$$

is obtained by letting

$$d_n := \frac{(-1)^n}{n!}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\det\begin{pmatrix} \langle x_{k_1}^*, x_{k_1}\rangle, & \dots, & \langle x_{k_1}^*, x_{k_n}\rangle \\ \dots & ,\dots, & \dots \\ \langle x_{k_n}^*, x_{k_1}\rangle, & \dots, & \langle x_{k_n}^*, x_{k_n}\rangle \end{pmatrix}.$$

It follows from Hadamard's inequality that

$$|d_n| \le \frac{n^{n/2}}{n!}c^n.$$

Hence $d(\zeta,T)$ is indeed an entire function. However, since no approximation property is assumed, the preceding definition may yield different Fredholm denominators. But the indeterminateness turns out to be only a factor $\mathrm{e}^{\alpha\zeta}$, in which the constant α depends on the underlying representation.

6.5.2.14 Using the nuclear representation from the previous paragraph, we define operators $A \in \mathfrak{N}(X, l_1)$ and $B \in \mathfrak{L}(l_1, X)$ by

$$A : x \mapsto \langle x, x_h^*\rangle \quad \text{and} \quad B : (\xi_k) \mapsto \sum_{k=1}^{\infty}\xi_k x_k.$$

Therefore every nuclear operator $T=BA$ on an arbitrary Banach space X is related to a nuclear operator $S=AB$ on the Banach space l_1, and every Fredholm denominator of S can be used as a Fredholm denominator of T. Note that S is generated by the matrix $\big(\langle x_h^*, x_k\rangle\big)$, which satisfies (6.5.2.8.b). Thus Koch's determinant theory applies, and we are back at the beginning of our historical tour.

6.5.3 Regularized Fredholm denominators

6.5.3.1 Let $\mathfrak{A}$ be any quasi-Banach ideal that supports a continuous trace τ and hence a continuous determinant δ.

We now consider an operator $T\in\mathfrak{L}(X)$ such that $T^m\in\mathfrak{A}(X)$ for a sufficiently large exponent m. Then it is possible to construct a **regularized Fredholm denominator** $d_m(\zeta,T)$ by deleting in the relevant formulas all undesired powers of the operator T, namely $T,\dots,T^{m-1}$, as well as their (possibly) non-existent traces.

In detail, we let

$$F_m(\zeta,T):=\zeta^{m-1}T^m(I-\zeta T)^{-1}$$

and look for entire functions

$$d_m(\zeta,T)=\sum_{n=0}^{\infty}d_n^{(m)}\zeta^n \qquad\text{and}\qquad D_m(\zeta,T)=\sum_{n=0}^{\infty}D_n^{(m)}\zeta^n$$

such that

$$F_m(\zeta,T)=\frac{D_m(\zeta,T)}{d_m(\zeta,T)} \quad\text{and}\quad d_0^{(m)}=1.$$

It follows from

$$F(\zeta,T)=T+\zeta T^2+\dots+\zeta^{m-2}T^{m-1}+F_m(\zeta,T)$$

that the meromorphic functions $F(\zeta,T)$ and $F_m(\zeta,T)$ have the same poles with the same singular parts. Hence a Fredholm denominator can be obtained by solving the differential equation

$$\big[\log d_m(\zeta,T)\big]'=\frac{d_m'(\zeta,T)}{d_m(\zeta,T)}=-\tau\big[F_m(\zeta,T)\big]$$

under the initial condition $d_m(0,T)=1$. Then the coefficients $d_n^{(m)}$ and $D_n^{(m)}$ are given by modified Plemelj formulas.

Following the general strategy, one must replace the operators $T,\dots,T^{m-1}$ and their traces by zeros. Note that $d_1^{(m)}=\dots=d_{m-1}^{(m)}=0$ and $d_m^{(m)}=-\frac{1}{m}\tau(T^m)$.

6.5.3.2 The idea of regularization goes back to Plemelj [1904, p. 125], who conjectured:

Die Verhältnisse scheinen so zu stehen, daß man zur Lösung der Funktionalgleichung auch im Falle des Unendlichwerdens des Kernes die Reihen $\sum_{n=0}^{\infty}d_n\zeta^n$ und $\sum_{n=0}^{\infty}D_n\zeta^n$ verwenden kann, wenn nur in den Determinanten jene Koeffizienten $\tau(T),\tau(T^2),\dots$ weggelassen werden, die nicht endlich sind.

6.5.3.3 For kernels K with singularities on the diagonal, Hilbert [1904b, p. 82] proposed another regularization process:

Die Methode besteht darin, daß wir die Linie $s = t$ durch Gebietsstreifen der st-Ebene von beliebig geringer Breite ε ausschließen und alsdann eine Funktion $K_\varepsilon(s,t)$ construiren, die innerhalb jener Gebietsstreifen Null ist, während sie außerhalb derselben mit $K(s,t)$ übereinstimmt.

Passing to the limit as $\varepsilon \to 0$, he arrived at the conclusion to replace the diagonal entries $K(s_k, s_k)$ in Fredholm's formulas by zeros; see 6.5.2.9.

In [1910, p. 73], Poincaré generalized Hilbert's regularization method to $m \geq 2$. For every (n,n)-matrix $A = (\alpha_{hk})$, he defined a modified determinant

$$\det_m(A) := \sum_{\pi} \operatorname{sgn}(\pi) \alpha_{1\pi(1)} \cdots \alpha_{n\pi(n)},$$

where the right-hand sum ranges only over such permutations π for which all invariant subsets have at least m points. For example, if $m = 2$, then all terms containing a factor α_{kk} are canceled.

Carleman [1921] applied Hilbert's idea to arbitrary Hilbert–Schmidt kernels, and much later, Smithies [1941] realized that for $m = 2$, both approaches yield just the same Fredholm denominators. To the best of my knowledge, Poincaré's idea has not been pursued for $m > 2$.

Grobler/Raubenheimer/Eldik [1982] and Engelbrecht/Grobler [1983] carried over the regularization process to abstract operators; see also [PIE$_4$, pp. 200–206].

6.5.3.4 A determinant theory slightly different from that of Ruston–Grothendieck was created by Leżański [1953] and further developed by Sikorski [1961]. Based on our present knowledge, their approach can be described as follows.

Recall from 5.7.3.9 that an operator $T \in \mathfrak{L}(X)$ is integral if there exists a constant $c \geq 0$ such that

$$|\operatorname{trace}(ST)| \leq c\|S\| \quad \text{for all } S \in \mathfrak{F}(X).$$

In other words, the functional $\boldsymbol{t}_0\colon S \mapsto \operatorname{trace}(ST)$ is continuous on $\mathfrak{F}(X)$ with respect to the operator norm. Therefore, by the Hahn–Banach theorem, it admits continuous extensions $\boldsymbol{t} \in \mathfrak{L}(X)^*$, which are not assumed to be norm-preserving. Then a Fredholm denominator is obtained via Plemelj's formula (6.5.2.3.c) by letting $\tau_k := \boldsymbol{t}(T^{k-1})$ for $k = 1, 2, \dots$. Since $\boldsymbol{t}$ is uniquely determined on $\overline{\mathfrak{F}}(X)$, which contains the powers $T^2, T^3, \dots$, we have at most two degrees of freedom, namely τ_1 and τ_2. If X is infinite-dimensional, then $I \notin \overline{\mathfrak{F}}(X)$ and τ_1 may take any prescribed value. The same happens with τ_2, provided that $T \notin \overline{\mathfrak{F}}(X)$. More precisely, the Leżański–Sikorski denominator is given by

$$d_{\mathrm{LS}}(\zeta, T) = \exp\big(-\tau_1\zeta - \tfrac{1}{2}\tau_2\zeta^2\big) d_3(\zeta, T),$$

where $d_3(\zeta, T)$ does not depend on the special choice of the extension $\boldsymbol{t}$.

6.5.3.5 The Grothendieck–Ruston denominators are defined only for the smaller class of nuclear operators. Note that given $T \in \mathfrak{N}(X)$, the functional $\boldsymbol{t}_0 : S \mapsto \text{trace}(ST)$ is continuous even with respect to the $\mathscr{K}$-topology. Hence there exist $\mathscr{K}$-continuous extensions to all of $\mathfrak{L}(X)$. For spaces with the approximation property, $\mathfrak{F}(X)$ is $\mathscr{K}$-dense in $\mathfrak{L}(X)$. Consequently, assuming $\mathscr{K}$-continuity, the required extension is unique; see 5.7.4.1. Without the approximation property, we have one degree of freedom, namely τ_1:

$$d_{\text{GR}}(\zeta, T) = \exp(-\tau_1 \zeta) d_2(\zeta, T);$$

see 6.5.2.13.

6.5.4 The Gohberg–Goldberg–Krupnik approach

6.5.4.1 Gohberg/Goldberg/Krupnik, [1996] and[1997], proposed another approach to the theory of traces and determinants, which is also presented in [GOH$_2^+$].

As the domain of definition, they use a so-called **embedded subalgebra** $\mathscr{D}$ of $\mathfrak{L}(X)$, where X denotes a fixed Banach space; see [GOH$_2^+$, p. 26]. The algebra $\mathscr{D}$ is supposed to have its own topology, which comes from a norm such that

$$\|ST|\mathscr{D}\| \le \|S|\mathscr{D}\| \|T|\mathscr{D}\| \quad \text{and} \quad \|T\| \le c\|T|\mathscr{D}\| \quad \text{for} \quad S, T \in \mathscr{D}.$$

If the elementary trace is $\mathscr{D}$-continuous on $\mathscr{D} \cap \mathfrak{F}(X)$ and if $\mathscr{D} \cap \mathfrak{F}(X)$ is $\mathscr{D}$-dense in $\mathscr{D}$, then there exists a unique $\mathscr{D}$-continuous extension: $\text{trace}_{\mathscr{D}}$. From now on, the theory is–more or less–developed in the same way as in the ideal case. The insufficient proof of Theorem 6.1 in [GOH$_2^+$, p. 34] shows that working with subalgebras instead of ideals requires more care.

6.5.4.2 Next, I give two classical examples of embedded subalgebras that satisfy the preceding requirements.

Poincaré algebra [1886]: Let $\mathscr{P}$ denote the collection of all infinite matrices $A = (\alpha_{hk})$ such that

$$\|A|\mathscr{P}\| := \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{hk}|$$

is finite. Then $\mathscr{P}$ can be viewed as an embedded subalgebra of $\mathfrak{L}(l_p)$ with $1 \le p \le \infty$; see also [GOH$_2^+$, pp. 39–40].

Fredholm algebra [1903]: Let $\mathscr{F}$ denote the collection of all continuous kernels K on $[0,1] \times [0,1]$ and put

$$\|K|\mathscr{F}\| := \max_{0 \le s,t \le 1} |K(s,t)|.$$

Then $\mathscr{F}$ can be viewed as an embedded subalgebra of $\mathfrak{L}\big(C[0,1]\big)$ and $\mathfrak{L}\big(L_p[0,1]\big)$ with $1 \le p \le \infty$; see also [GOH$_2^+$, pp. 111–114].

Of course, these matrices and kernels induce nuclear operators on l_1 and $C[0,1]$, respectively. Hence the theory of ideals applies as well.

6.5.4.3 Strangely enough, Gohberg/Goldberg/Krupnik never considered the case in which $\mathscr{D}$ is complete, though this requirement, together with the density of $\mathscr{D} \cap \mathfrak{F}(X)$ in $\mathscr{D}$, implies a decisive property:

Let $I+T$ with $T \in \mathscr{D}$ be invertible. Then the inverse operator admits a representation $(I+T)^{-1} = I+S$ with $S \in \mathscr{D}$. In other words, T is even invertible in the algebra obtained from $\mathscr{D}$ by adjunction of the unit element I.

For the Fredholm algebra, this means that the inverse of $I+K_{\mathrm{op}}$ has the form $I+R_{\mathrm{op}}$, where R is a continuous kernel too. Thus one gets better information than provided by the ideal approach.

6.5.5 Eigenvalues and zeros of entire functions

Let $d(\zeta, T)$ be a Fredholm denominator of the operator T. Then ζ_0 is a zero of $d(\zeta, T)$ if and only if $\lambda_0 = 1/\zeta_0$ is an eigenvalue of T. In this case, the multiplicities of ζ_0 and λ_0 coincide. Hence theorems on the distribution of zeros of entire functions may be used to get theorems on the distribution of eigenvalues. Luckily enough, the theory of entire functions had been developed around 1900, just in time for the purposes of functional analysis; see [BOR$_2$].

The next paragraphs are devoted to this subject. We tacitly assume the entire functions $d(\zeta) = \sum_{n=0}^{\infty} d_n \zeta^n$ to be normalized such that $d_0 = 1$; formulas and estimates hold for all $\zeta \in \mathbb{C}$. The standard references on entire functions are [BOAS] and [LEVIN].

6.5.5.1 The famous **Weierstrass factorization theorem** [1876] says that every entire function d can be written in the form

$$d(\zeta) = \mathrm{e}^{f(\zeta)} \prod_{n=1}^{\infty} E(\zeta/\zeta_n, p_n),$$

where f is an entire function and the right-hand factors are **primary**:

$$E(\zeta/\zeta_0, p_0) := (1 - \zeta/\zeta_0) \exp\Big[(\zeta/\zeta_0) + \tfrac{1}{2}(\zeta/\zeta_0)^2 + \cdots + \tfrac{1}{p_0}(\zeta/\zeta_0)^{p_0}\Big].$$

Note that the zeros $\zeta_1, \zeta_2, \ldots$ are counted according to their multiplicities.

In the case that d has finitely many zeros, the above formula becomes trivial:

$$d(\zeta) = \mathrm{e}^{f(\zeta)} \prod_{n=1}^{N} (1 - \zeta/\zeta_n).$$

Otherwise, it is possible to choose non-negative integers $p_1, p_2, \ldots$ such that the infinite product converges.

6.5.5.2 Let $\rho > 0$, and assume that we can find constants $\gamma > 0$ and $C \geq 1$ such that

$$|d(\zeta)| \leq C \exp(\gamma |\zeta|^{\rho}). \tag{6.5.5.2.a}$$

Subject to this growth condition, d admits a **Hadamard factorization**

$$d(\zeta) = \mathrm{e}^{P(\zeta)} \prod_{n=1}^{\infty} E(\zeta/\zeta_n, p)$$

in which P is a polynomial of degree less than or equal to ρ and all p_n's are the same. More precisely, Hadamard [1893b, pp. 204, 209] showed that

$$\frac{1}{|\zeta_n|} = O\Big(\frac{1}{n^{1/\rho}}\Big).$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{p+1}} < \infty$$

for every integer p sufficiently large: $p+1 > \rho$.

Nowadays, the main tool for proving results about the distribution of zeros is a famous inequality of Jensen [1899, p. 362]:

$$\frac{r^n}{|\zeta_1 \cdots \zeta_n|} \leq \sup_{|\zeta|=r} |d(\zeta)| \quad \text{for } r > 0 \text{ and } n = 1, 2, \dots .$$

6.5.5.3 According to Laguerre, the **genus** of d is defined to be the smallest possible value of $\max\{\deg(P), p\}$. The genus does not exceed the exponent ρ in (6.5.5.2.a).

Most important for the purposes of determinant theory are the cases

$$\overbrace{d(\zeta) = \prod_{n=1}^{\infty} (1 - \zeta/\zeta_n)}^{\text{genus } 0} \quad \text{and} \quad \overbrace{d(\zeta) = \mathrm{e}^{\alpha\zeta} \prod_{n=1}^{\infty} (1 - \zeta/\zeta_n) \mathrm{e}^{\zeta/\zeta_n}}^{\text{genus } 1} .$$

6.5.5.4 Lindelöf [1905, p. 375] proved that d is of genus 0 if and only if

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|} < \infty \quad \text{and} \quad |d(\zeta)| \leq C_\varepsilon \exp(\varepsilon|\zeta|) \quad \text{for every } \varepsilon > 0 \text{ and suitable } C_\varepsilon \geq 1.$$

6.5.5.5 For applications to determinant theory it is helpful to express the estimate (6.5.5.2.a) in terms of the Taylor coefficients d_n of $d(\zeta) = \sum_{n=0}^{\infty} d_n \zeta^n$. Lindelöf [1902, pp. 34, 39], [1903, pp. 215–216] and Pringsheim [1904, pp. 273, 337–339] proved that the following condition is necessary and sufficient:

There exist constants $A > 0$ and $\alpha > 0$ such that

$$|d_n| \leq A \Big(\frac{\alpha}{n}\Big)^{n/\rho}.$$

6.5.5.6 Fredholm [1903, p. 368] used the estimate

$$|d_n| \leq \frac{n^{n/2}}{n!} c^n$$

in order to show that for every bounded and integrable kernel, the power series $d(\zeta,K) = \sum_{n=0}^{\infty} d_n \zeta^n$ defines an entire function. Probably, he did not know the preceding result of Lindelöf and Pringsheim. A first conclusion about the genus of $d(\zeta,K)$ was drawn by Plemelj [1904, p. 101]:

Das Laguerresche Geschlecht dieser ganzen Funktion kann den Wert 2 *nicht übersteigen, vermutlich ist es niemals* 2*, sondern im allgemeinen geradezu* 1*, bei durchweg stetigem* $K(s,t)$ *vielleicht sogar stets* 0.

Carleman verified the first part of Plemelj's conjecture and disproved the second one; see [1917, p. 5] and [1918, p. 381].

In a next step, Gheorghiu [1927, p. 1309] sketched a proof of the fact that the Fredholm denominator of the product of two continuous kernels has genus 0. The main idea was to employ Lindelöf's theorem 6.5.5.4.

6.5.5.7 After a long break, Grothendieck [GRO_1, Chap. II, pp. 18–19] obtained the first result in the abstract setting. He showed that the denominators of *opérateurs de Fredholm de puissance* $2/3$ *sommable* have genus 0. Unfortunately, the ideal $\mathfrak{F}_{2/3}$ is rather small; see 6.3.2.11.

Trace class operators in Hilbert spaces were treated by Lidskiĭ [1959b], who used the factorization

$$d(\zeta,T) = \prod_{n=1}^{\infty} (1-\zeta/\zeta_n)$$

in the proof of his famous trace formula. Obviously, Lidskiĭ was not aware of Gheorghiu's approach, because he quoted only [GRO_1].

6.5.5.8 For every continuous trace on a quasi-Banach ideal $\mathfrak{A}$, the associated Fredholm denominators of operators $T \in \mathfrak{A}(X)$ admit the factorization

$$d(\zeta,T) = \exp\Big(-\tau(T)\zeta - \tfrac{1}{2}\tau(T^2)\zeta^2 + \tfrac{1}{2}\sum_{n=1}^{\infty}(\zeta/\zeta_n)^2\Big)\prod_{n=1}^{\infty}(1-\zeta/\zeta_n)\exp\big(\zeta/\zeta_n\big).$$

If the restriction of τ to $\mathfrak{A}^2$ is spectral (which may always be true, [PIE_4, p. 184]), then we get a significant simplification:

$$d(\zeta,T) = \exp\big(-\tau(T)\zeta\big)\prod_{n=1}^{\infty}(1-\zeta/\zeta_n)\exp\big(\zeta/\zeta_n\big).$$

Finally, for the spectral trace it follows that

$$d(\zeta,T) = \prod_{n=1}^{\infty}(1-\zeta/\zeta_n).$$

In other words, the associated determinant is **spectral**:

$$\det(I-T) = \prod_{n=1}^{\infty}\big(1-\lambda_n(T)\big).$$

6.5.5.9 The Ruston–Grothendieck denominators have the form

$$d_{\mathrm{RG}}(\zeta,T) = \exp\big(-\tau_1\zeta\big)\prod_{n=1}^{\infty}(1-\zeta/\zeta_n)\exp\big(\zeta/\zeta_n\big),$$

where the constant τ_1 may depend on the underlying nuclear representation. In the Leżański–Sikorski theory, two degrees of freedom occur:

$$d_{\mathrm{LS}}(\zeta,T) = \exp\Big(-\tau_1\zeta - \tfrac{1}{2}\tau_2\zeta^2 + \tfrac{1}{2}\sum_{n=1}^{\infty}(\zeta/\zeta_n)^2\Big)\prod_{n=1}^{\infty}(1-\zeta/\zeta_n)\exp\big(\zeta/\zeta_n\big).$$

Compare with 6.5.3.4 and 6.5.3.5.

6.5.5.10 Finally, I mention a famous result of Carleman [1921, p. 208]. Using the symbol $D^*(\zeta)$ instead of $d_2(\zeta,K)$, he proved that the regularized Fredholm denominator of a Hilbert–Schmidt kernel satisfies the estimate

$$|D^*(\zeta)| \le C\exp\big(\tfrac{1}{2}\Omega|\zeta|^2\big) \quad \text{with} \quad \Omega = \int_a^b\int_a^b |K(s,t)|^2 ds\,dt.$$

His paper concludes as follows:

$D^*(\zeta)$ ist höchstens vom Geschecht 1 und besitzt die Produktdarstellung

$$D^*(\zeta) = \prod_{n=1}^{\infty}(1-\zeta/\zeta_n)\mathrm{e}^{\zeta/\zeta_n}.$$

6.5.6 Completeness of root vectors

6.5.6.1 Let T be a linear mapping on a finite-dimensional linear space X. Then there exists a basis of X such that the representing matrix of T has **Jordan form**:

$$\begin{pmatrix} \boxed{J_1} & \boxed{O} & \dots & \boxed{O} \\ \boxed{O} & \boxed{J_2} & \dots & \boxed{O} \\ \dots & \dots & \dots & \dots \\ \boxed{O} & \boxed{O} & \dots & \boxed{J_m} \end{pmatrix}.$$

The building blocks, **Jordan cells**, look as follows:

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda_k & 1 \\ 0 & \dots & \dots & 0 & \lambda_k \end{pmatrix}.$$

The required basis consists of **root vectors** (корневые векторы), say x, associated with the eigenvalues:

$$(\lambda I - T)^h x = \mathrm{o} \quad \text{for some sufficiently large exponent } h.$$

The terms **principal vector** (Hauptvektor) and **generalized eigenvector** are also common.

6.5.6.2 Concerning generalizations of the preceding result to the infinite-dimensional setting, there is good news and bad news.

Let us begin with the good ones, which go back to Hilbert [1904b, pp. 71–78] and Schmidt [1907a, p. 452]:

If T is a compact Hermitian operator on a separable infinite-dimensional Hilbert space, then there exists an orthonormal basis (u_n) such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x|u_n) u_n \quad \text{for all } x \in H.$$

The u_n's are eigenvectors associated with the real eigenvalues λ_n, and we have $\lambda_n \to 0$ as $n \to \infty$. In other words, the representing matrix of T with respect to the basis (u_n) has *diagonal form*.

6.5.6.3 Next, I comment on some bad news:

First of all, we know that there are separable Banach spaces without a basis. In this case, the search for nice representations of operators does not make sense. However, this is not the crucial point, since the most frustrating phenomenon occurs even in Hilbert spaces.

Everybody will agree that the rule

$$S : f(t) \mapsto \int_0^s f(t)\, dt,$$

which defines the **integration operator** on $L_2[0,1]$, looks quite nice and simple. Unfortunately, S has no eigenvalues at all. Hence there is no way to reconstruct this operator from its (non-existent) root vectors.

6.5.6.4 Let $T \in \mathfrak{L}(X)$ be an operator such that $T^{m+1} = O$. Then all members of X are root vectors associated with the eigenvalue $\lambda = 0$. Of course, this fact does not yield any useful information. Thus we realize that the eigenvalue $\lambda = 0$ plays an unpleasant role. Since the north pole of the Riemann sphere looks like an appropriate place, 0 is exiled by the transformation $\lambda \mapsto \zeta = 1/\lambda$; see also 2.6.5.1.

The preceding observation motivates the following notation:

Given $T \in \mathfrak{L}(X)$, the linear span of all root vectors associated with *non-zero* eigenvalues will be denoted by $R_{\text{oot}}(T)$.

Obviously, $R_{\text{oot}}(T)$ is contained in $M(T^m)$ (the range of T^m) for $m = 0, 1, 2, \ldots$.

6.5.6.5 Next, we consider the diagonal operator $D_t : (\xi_k) \mapsto (\tau_k \xi_k)$ generated by a null sequence $t = (\tau_k)$. If D_t is viewed as an operator on l_p with $1 \le p < \infty$, then everything goes fine: $\overline{R_{\text{oot}}(D_t)} = l_p$. But taking the Banach space of all convergent sequence gives $\overline{R_{\text{oot}}(D_t)} = c_0$. Hence the closed linear span of the root vectors turns out to be a proper subspace, even if D_t is one-to-one. This simple example shows that looking only for operators $T \in \mathfrak{L}(X)$ with $\overline{R_{\mathrm{oot}}(D_t)} = X$ would be too restrictive. Consequently, the main problem should be phrased as follows:

Find conditions that ensure that $M(T^m) \subseteq \overline{R_{\mathrm{oot}}(T)}$ for *fixed* $m = 0, 1, 2, \ldots$. If this is the case, then $\overline{M(T^m)} = \overline{R_{\mathrm{oot}}(T)}$, and the system of root vectors of the operator T is called m**-complete**.

6.5.6.6 The formulation of the final answer requires an additional definition. We say that an ideal $\mathfrak{A}$ is of **eigenvalue type** $o\big(\frac{1}{n^{1/p}}\big)$ if for every Banach space X, all operators $T \in \mathfrak{A}(X)$ are Riesz and

$$\lim_{n\to\infty} n^{1/p}\lambda_n(T) = 0.$$

This property is a little bit stronger than $\sup_n n^{1/p}|\lambda_n(T)| < \infty$; see 6.4.1.2.

6.5.6.7 Now we are able to state the **root vector completeness theorem**, which is due to Markus [1966, p. 63]. Its prehistory in the Hilbert space setting will be described in 6.5.6.11. The following version goes back to Reuter [1981, p. 393], [1985, p. 397].

Let $T \in \mathfrak{L}(X)$ belong to any quasi-Banach ideal of eigenvalue type $o\big(\frac{1}{n^{1/p}}\big)$. Suppose that the complex plane can be covered by a finite number of closed sectors

$$\Sigma(\zeta, \alpha, \beta) := \{\zeta + r\mathrm{e}^{\mathrm{i}\varphi} : r \ge 0,\ \alpha \le \varphi \le \beta\}$$

with opening $\beta - \alpha \le \min\{\pi/p, 2\pi\}$ such that

$$\limsup_{\zeta\to\infty} \frac{\|(I-\zeta T)^{-1}\|}{|\zeta|^m} < \infty \tag{6.5.6.7.a}$$

along their boundary rays. Moreover, it should be known that

$$\lim_{n\to\infty} \frac{\|(I-\zeta_n T)^{-1}\|}{|\zeta_n|^m} = 0 \tag{6.5.6.7.b}$$

for some sequence (ζ_n) tending to ∞.

Then the range of T^m is contained in the closed linear span of the root vectors associated with non-zero eigenvalues (m-completeness).

6.5.6.8 The main tool in proving the preceding assertion is a refined version of the classical **Phragmén–Lindelöf theorem** [1908]; see [HOL, p. 126]:

Let d be an analytic function that is defined in some open neighborhood of the closed sector $\Sigma_p := \{r\mathrm{e}^{\mathrm{i}\varphi} : r \ge 0,\ |\varphi| \le \pi/2p\}$, where $p > 1/2$. Assume that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon \ge 1$ such that

$$|d(\zeta)| \le C_\varepsilon \exp(\varepsilon|\zeta|^p) \quad \text{whenever } \zeta \in \Sigma_p. \tag{6.5.6.8.a}$$

If $|d(\zeta)| \le M$ on the boundary rays $r\mathrm{e}^{\pm\mathrm{i}\pi/2p}$, then $|d(\zeta)| \le M$ in the whole of Σ_p.

In fact, it suffices to know that the estimate (6.5.6.8.a) holds on a sequence of arcs $\mathscr{A}_r := \{r\mathrm{e}^{\mathrm{i}\varphi} : |\varphi| \le \pi/2p\}$ with arbitrarily large radii r.

6.5.6.9 Let $T \in \mathfrak{L}(X)$ belong to any quasi-Banach ideal $\mathfrak{A}$ that has eigenvalue type $o\big(\frac{1}{n^{1/p}}\big)$. Then there exist radii $r_k \to \infty$ and constants $C_\varepsilon \geq 1$ such that

$$\|(I-\zeta T)^{-1}\| \leq C_\varepsilon \exp(\varepsilon|\zeta|^p) \quad \text{whenever } |\zeta| = r_k \text{ and } \varepsilon > 0.$$

This estimate has a long history. Its up-to-date version is due to Reuter [1985, pp. 392–393]. However, the decisive idea of the proof was discovered by Matsaev [1964a] and Markus [1966, p. 59]; see also [GOH$_3^+$, pp. 242–245].

Passing from $\mathfrak{A}$ to $\mathfrak{A}^m$, one may arrange that the spectral determinant is available on the underlying ideal. Easy manipulations show that

$$I-\zeta(T+x^*\otimes x) = \big(I-\zeta T\big)\big(I-\zeta x^*\otimes(I-\zeta T)^{-1}x\big) \quad \text{for } x\in X \text{ and } x^*\in X^*.$$

Hence, in view of properties $(\mathbf{D}_0)$ and $(\mathbf{D}_1)$ from 6.5.2.5, we have

$$\zeta\langle x^*,(I-\zeta T)^{-1}x\rangle = 1 - \frac{\det\big(I-\zeta(T+x^*\otimes x)\big)}{\det\big(I-\zeta T\big)}.$$

The assumption about the eigenvalue type of $\mathfrak{A}$ yields information on the growth of $\det\big(I-\zeta(T+x^*\otimes x)\big)$ and $\det(I-\zeta T)$. These estimates carry over to the quotient, at least on a suitable sequence of circles; see [LEVIN, Chap. 1, § 9]. The final result follows by applying the principle of uniform boundedness.

6.5.6.10 Suppose that $T \in \mathfrak{L}(X)$ satisfies the conditions stated in the root vector completeness theorem 6.5.6.7, and put $N := \overline{R_{\text{oot}}(T)}$. Then T induces an operator V on X/N, and all estimates of $\|(I-\zeta T)^{-1}\|$ remain true for $\|(I-\zeta V)^{-1}\|$. Since we have killed the root vectors, $(I-\zeta V)^{-1}$ becomes analytic in the whole plane. In view of (6.5.6.7.a), the Phragmén–Lindelöf theorem implies that

$$V^m(I-\zeta V)^{-1} = \frac{(I-\zeta V)^{-1}-I-\zeta V-\cdots-\zeta^{m-1}V^{m-1}}{\zeta^m} = V^m+\zeta V^{m+1}+\cdots$$

is bounded. By Liouville's theorem, this function must be constant, $V^{m+1}=O$. Finally, it follows from $(I-\zeta V)^{-1} = I+\zeta V+\cdots+\zeta^m V^m$ and (6.5.6.7.b) that $V^m = O$. Hence $M(T^m) \subseteq N = \overline{R_{\text{oot}}(T)}$.

6.5.6.11 First results on completeness of root vectors of non-self-adjoint operators in Hilbert spaces were announced by Keldysh [1951], who considered operators of the form $(I+K)T$. Here $K \in \mathfrak{K}(H)$ is arbitrary, while $T \in \mathfrak{S}_p(H)$ is supposed to be Hermitian. The missing proofs were published with a delay of 20 years; see [1971]. In the meantime, Lidskiĭ [1959a] had written a remarkable thesis on the same subject. Among other results, he showed that an operator T on a Hilbert spaces is 1-complete if both Hermitian parts are positive, which means that all values $(Tx|x)$ belong to the sector $\{re^{i\varphi} : r \geq 0,\ 0 \leq \varphi \leq \pi/2\}$. In a joint lecture, Keldysh/Lidskiĭ [1961] presented their results at the Fourth Soviet Congress of Mathematics, Leningrad. Further progress was achieved by Matsaev [1964b]. This theory has been treated in two books: [DUN$_2^+$, pp. 1042, 1089, 1115] and [GOH$_3^+$, Chap. V, §§ 6 and 8].

6.5.6.12 In general, it is not easy to verify the growth conditions for the resolvent required in the root vector completeness theorem; see (6.5.6.7.a) and (6.5.6.7.b). Letting $T \in \mathfrak{L}(H)$, one may use the following observation; see [STONE, p. 147]:

If λ is a point of positive distance d from the numerical range

$$W(T) := \{\, (Tx|x) : \|x\| = 1 \,\},$$

then $(\lambda I - T)^{-1}$ exists and satisfies the inequality $\|(\lambda I - T)^{-1}\| \le 1/d$.

Of particular interest are the cases in which $W(T)$ belongs to the real axis (Hermitian operators) or to a half-plane (dissipative operators).

6.5.6.13 An operator $A \in \mathfrak{L}(H)$ is called **dissipative** if $\operatorname{Re}(Ax|x) \le 0$ for all $x \in H$. An equivalent property says that the semi-group generated by A consists of contractions, $\|e^{At}\| \le 1$ for $t \ge 0$.

The following explanation of Phillips [1959, p. 193] refers to this fact:

The term "dissipative" is used to emphasize the basic assumption imposed on the system, namely, that the energy is nonincreasing in time.

The definition above can be extended to the setting of Banach spaces. Lumer/Phillips [1961] require that

$$\operatorname{Re}\langle Ax, x^* \rangle \le 0 \quad \text{for every pair } x \in X \text{ and } x^* \in X^* \text{ such that } \langle x, x^* \rangle = \|x\| \|x^*\|.$$

Every dissipative operator $A \in \mathfrak{L}(X)$ that belongs to a quasi-Banach ideal of eigenvalue type $o\left(\frac{1}{n}\right)$ is 1-complete.

Indeed, we have

$$\frac{\|(I - \zeta A)^{-1}\|}{|\zeta|} \le \frac{1}{\operatorname{Re}\zeta} \quad \text{whenever } \operatorname{Re}\zeta > 0.$$

In particular,

$$\frac{\|(I - \zeta A)^{-1}\|}{|\zeta|} \le 1 \quad \text{if } \operatorname{Re}\zeta = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\|(I - nA)^{-1}\|}{n} = 0.$$

Thus, for $m = 1$, the root vector completeness theorem can be applied by dividing the complex plane into the half-planes $\operatorname{Re}\zeta \le 1$ and $\operatorname{Re}\zeta \ge 1$.

6.5.6.14 The Volterra operator

$$S : f(t) \mapsto \int_0^s f(t)\, dt,$$

viewed as a mapping on $L_2[0,1]$, yields an interesting example. Writing

$$S = A + \mathrm{i}B \quad \text{with} \quad A := \frac{S + S^*}{2} \quad \text{and} \quad B := \frac{S - S^*}{2\mathrm{i}},$$

we obtain

$$A : f(t) \mapsto \frac{1}{2}\int_0^1 f(t)\,dt \quad \text{and} \quad B : f(t) \mapsto \frac{1}{2\mathrm{i}}\int_0^1 \operatorname{sgn}(s-t)f(t)\,dt.$$

This means that the unpleasant operator S is obtained as a perturbation of the nice operator $\mathrm{i}B$ by a very small operator, $\operatorname{rank}(A)=1$. It also follows that $-S$ is dissipative.

On the other hand, we know from [GOH$_3^+$, p. 120] that $s_n(S) = O\big(\frac{1}{n}\big)$. Hence the *"little oh"* condition in the root vector completeness theorem cannot be replaced by the corresponding *"big Oh"* condition.

6.5.6.15 Following Markus [1966, p. 69], we refer to $A \in \mathfrak{L}(X)$ as an ***H*-operator** if there exists a constant $c \ge 1$ such that

$$\|(\lambda I - A)^{-1}\| \le \frac{c}{|\mathrm{Im}(\lambda)|} \quad \text{for } \lambda \notin \mathbb{R}. \tag{6.5.6.15.a}$$

This condition includes the requirement that the spectrum of A lie on the real axis.

In the special case when $c=1$, the property above characterizes Hermitian operators on a Hilbert space. Hence it can be expected that some results about Hermitian operators remain true for H-operators. An affirmative answer was given by Markus [1966, pp. 69–75]:

- root vectors are eigenvectors,
- Riesz H-operators are compact,
- $|\lambda_n(A)| \asymp a_n(A)$.

However, most important for the present considerations is the following result of Markus [1966, p. 72]:

Let $A \in \mathfrak{L}(X)$ be a compact H-operator. Then the linear span of all eigenvectors associated with non-zero eigenvalues is dense in $M(T)$.

Property (6.5.6.15.a) can be reformulated in terms of the Fredholm resolvent: There exists a constant $c \ge 1$ such that

$$\frac{\|(I-\zeta A)^{-1}\|}{|\zeta|} \le \frac{c}{|\mathrm{Im}(\zeta)|} \quad \text{for } \zeta \notin \mathbb{R}.$$

6.5.6.16 The following example is due to Markus [1966, pp. 70–71]:

Let (x_k) be any basis of a Banach space X, and let (x_k^*) denote the sequence of coordinate functionals. Then

$$\sum_{k=1}^{\infty} \tau_k x_k^* \otimes x_k$$

is a nuclear H-operator whenever $\sum_{k=1}^{\infty} |\tau_k| \|x_k^*\| \|x_k\| < \infty$.

Every operator similar to a Hermitian operator is an H-operator. If we take a conditional basis of l_2, then the preceding example shows that the converse implication fails.

6.5.6.17 Finally, I discuss a straightforward approach to a root vector completeness theorem that, unfortunately, is rather unsatisfactory.

Let $T\in\mathfrak{L}(X)$ be a Riesz operator, and choose some radius $r>0$ such that the circle $\mathscr{C}_r := \{\lambda\in\mathbb{C} : |\lambda| = r\}$ is contained in the Fredholm resolvent set of T. Then X splits into a direct sum $X = M_r \oplus N_r$. The finite-dimensional subspace N_r is generated by all root vectors associated with eigenvalues λ of modulus less than $1/r$, whereas M_r contains the rest. The spectral projection P_r onto N_r along M_r is given by

$$P_r = -\frac{1}{2\pi\mathrm{i}}\oint_{|\zeta|=r} F(\zeta,T)\,d\zeta = I - \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta|=r} \frac{(I-\zeta T)^{-1}}{\zeta}\,d\zeta.$$

Usually, the operational calculus is based on the resolvent $(\lambda I - T)^{-1}$, and passing to $(I-\zeta T)^{-1}$ makes the formulas less transparent. However, in order to be consistent with the previous considerations, I have used the parameter ζ.

Very strong assumptions are needed in order to guarantee that the finite rank operators $T^m P_r$ converge to T^m, in the operator norm and for some appropriate sequence of radii $r_k \to \infty$. In view of

$$T^m P_r = T^m - \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta|=r} \frac{(I-\zeta T)^{-1}}{\zeta^{m+1}}\,d\zeta,$$

the following growth condition suffices:

$$\liminf_{r\to\infty}\ \max_{|\zeta|=r} \frac{\|(I-\zeta T)^{-1}\|}{|\zeta|^m} = 0. \tag{6.5.6.17.a}$$

In the case $m=0$, this can happen only when X is finite-dimensional.

The reader is recommended to compare (6.5.6.7.a) and (6.5.6.17.a).

6.5.7 Determinants: pros and cons

pros: At the beginning of functional analysis there were determinants.
Indeed, one may say without exaggeration that the ideas of von Koch and Fredholm gave the decisive impetus for creating the theory of linear operators.

cons: The most important outcome of Fredholm's work, however, was a determinant-free statement, his famous alternative; see 2.6.4.4. Therefore it seemed natural to look for a determinant-free approach. This goal was achieved by Hilbert [1906a], Schmidt [1907a], [1907b], [1908], Riesz [1918], and Schauder [1930c]. Schur's proof [1909] of his famous inequality

$$\sum_{n=1}^{\infty} |\lambda_n(K)|^2 \le \int_a^b\int_a^b |K(s,t)|^2 ds\,dt$$

is determinant-free as well.

pros: Many authors employed the growth of the Fredholm denominator for obtaining results about the distribution of eigenvalues. The landmark paper in this field was written by Hille/Tamarkin [1931].

cons: As a consequence of Weyl's paper on *Inequalities between the two kinds of eigenvalues of a linear transformation* [1949], the philosophy of eigenvalue distribution changed completely: s-numbers became the main tool. This new method yields results that are much sharper than those obtained via determinants.

pros: The determinant-free Riesz–Schauder theory of compact operators in Banach spaces called for a counterpart in terms of determinants. Thus it came as no surprise that even three theories of this kind were independently developed by Ruston [1951a], [1951b], Leżański [1953], and Grothendieck [1956a]. These determinants became the decisive tool in the first proofs of trace formulas.

cons: An important byproduct of the abstract determinant theory was the concept of a nuclear operator. Grothendieck himself showed by determinant-free methods that the eigenvalue sequences of those operators are square summable. Hence, once again, determinants became superfluous.

pros: I know of only one situation in which determinants are indispensable. At least presently, estimates of the growth of resolvents as described in 6.5.6.9 cannot be obtained by another tool.

Final conclusion: In my opinion, the question, *Are determinants important or not?* should be prohibited. Every good mathematical concept has its intrinsic beauty, and this is certainly true for determinants. Thus I do not agree with Axler [1995], who published a (didactic) paper under the title

Down with determinants!

6.6 Interpolation theory

If not otherwise stated, numbers p, p_0, and p_1 are always related by the formula

$$1/p = (1-\theta)/p_0 + \theta/p_1,$$

where θ is a parameter with $0<\theta<1$. The same convention will be applied to $q, r, \dots$.

Standard references are [BERG$^+$], [KREIN$^+$], [TRI$_1$], [BENN$^+$], and [BRUD$^+$].

6.6.1 Classical convexity theorems

6.6.1.1 Interpolation theory has its origin in the **convexity theorem** of Marcel Riesz, which was proved in the finite-dimensional setting of bilinear forms. However, in his seminal paper [1926, p. 481] one also finds a generalization to operators between infinite-dimensional spaces:

The function $(1/p, 1/q) \mapsto \log \|T : L_p \to L_q\|$ is convex on the triangle defined by $1 \le p \le q \le \infty$.

Riesz simultaneously dealt with real and complex scalars. Furthermore, he showed that in the real case, the restriction $1 \le p \le q \le \infty$ is indeed necessary. Subsequently, one of his pupils settled the complex case. Using tools from the theory of analytic functions, Thorin [1938] found an elegant approach that works even for $0<p,q\leq\infty$.

6.6.1.2 The essential point of the convexity theorem is the following:

If a given rule defines both

an operator from L_{p_0} into L_{q_0} and an operator from L_{p_1} into L_{q_1},

then it also defines

an operator from L_p into L_q.

This means that having information about the behavior at the endpoints $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$ provides information about the behavior at all interior points $(1/p, 1/q)$ of the connecting line.

6.6.1.3 As a typical application of his convexity theorem, Riesz [1926, p. 482] presented a proof of the **famous Hausdorff–Young inequality**:

$$\Big(\sum_{k\in\mathbb{Z}}|\gamma_k(f)|^{p^*}\Big)^{1/p^*} \leq \Big(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p dt\Big)^{1/p} \tag{6.6.1.3.a}$$

for $f\in L_p(\mathbb{T})$ and $1<p<2$.

Indeed, the rule that assigns to every periodic function f the sequence of its Fourier coefficients $\big(\gamma_k(f)\big)$ defines an operator from $L_1(\mathbb{T})$ into $c_0(\mathbb{Z})$ (Riemann–Lebesgue lemma) and an operator from $L_2(\mathbb{T})$ onto $l_2(\mathbb{Z})$ (Fischer–Riesz theorem). Therefore $f\in L_p(\mathbb{T})$ implies $\big(\gamma_k(f)\big)\in l_{p^*}(\mathbb{Z})$.

6.6.2 Interpolation methods

6.6.2.1 For many years, research in *interpolation* was exclusively concerned with the classical spaces L_p and the Marcinkiewicz spaces $L_{p,\infty}$. The big bang, when this theory developed to a separate branch of functional analysis, happened only in the first half of the 1960s. Among the pioneers of the new era were Aronszajn, Gagliardo, Calderón, Selim Kreĭn, Lions, and Peetre. Starting from couples of arbitrary Banach spaces X_0 and X_1, they constructed new Banach spaces X that play the same role as L_p with respect to L_{p_0} and L_{p_1}.

6.6.2.2 First of all, one needs information about the mutual positions of X_0 and X_1. To this end, both spaces should be algebraically embedded into a topological linear space $\mathfrak{X}$ such that the identity maps $X_0\to\mathfrak{X}$ and $X_1\to\mathfrak{X}$ are continuous; the topology must be Hausdorff. Then $X_0\cap X_1$ and X_0+X_1 become Banach spaces under the norms

$$\|x|X_0\cap X_1\| := \max\big\{\|x|X_0\|, \|x|X_1\|\big\}$$

and

$$\|x|X_0+X_1\| := \inf\big\{\|x_0|X_0\|+\|x_1|X_1\| : x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\big\},$$

respectively. From now on, we may use X_0+X_1 instead of $\mathfrak{X}$. This fact justifies an **interpolation couple** being denoted simply by $\{X_0, X_1\}$.

By an **intermediate space** we mean a Banach space X that is located between $X_0 \cap X_1$ and X_0+X_1 such that the identity maps are continuous:

$$X_0 \cap X_1 \xrightarrow{Id} X \xrightarrow{Id} X_0 + X_1.$$

The preceding definitions go back to Gagliardo [1961, p. 247].

6.6.2.3 In modern terminology, interpolation couples may be considered as the objects of a category. Following Aronszajn/Gagliardo [1965, p. 105], one defines the corresponding morphism as follows:

Given $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, we take all couples $\{T_0, T_1\}$ of operators $T_0 \in \mathfrak{L}(X_0, Y_0)$ and $T_1 \in \mathfrak{L}(X_1, Y_1)$ that coincide on $X_0 \cap X_1$. Let

$$\|\{T_0, T_1\}\| := \max\big\{\|T_0 : X_0 \to Y_0\|, \|T_1 : X_1 \to Y_1\|\big\}.$$

Obviously, $T_0+T_1 : x_0+x_1 \to T_0x_0+T_1x_1$ defines an operator from X_0+X_1 into Y_0+Y_1.

6.6.2.4 The fundamental concept of an **interpolation method** was invented by Aronszajn/Gagliardo [1965, p. 113]. Taking the categorical point of view, one also speaks of an **interpolation functor**. This is a rule Φ that assigns to every couple of spaces an intermediate space such that every couple of operators $T_0 : X_0 \to Y_0$ and $T_1 : X_1 \to Y_1$ induces an operator $T = \Phi\{T_0, T_1\}$ from $X = \Phi\{X_0, X_1\}$ into $Y = \Phi\{Y_0, Y_1\}$. Then, by the closed graph theorem, there exists a constant $c_\Phi > 0$ such that

$$\|T : X \to Y\| \le c_\Phi \max\big\{\|T_0 : X_0 \to Y_0\|, \|T_1 : X_1 \to Y_1\|\big\}.$$

In the case that

$$\|T : X \to Y\| \le c_\Phi \|T_0 : X_0 \to Y_0\|^{1-\theta} \|T_1 : X_1 \to Y_1\|^{\theta},$$

the interpolation method is said to be of **type** θ. If $c_\Phi = 1$, then we refer to Φ as **exact**.

6.6.2.5 Finally, I stress that from the historical point of view, this subsection should be interchanged with the next one. Indeed, Aronszajn/Gagliardo [1965, p. 51] motivated their abstract investigations by stating:

In the presence of so many different interpolation methods it seemed timely to study the general structure of all possible methods.

The preceding remark was relativized in [BRUD$^+$, p. vi, footnote [3]]:

This is how things appeared in 1965. Fifteen years later, it was found that the number of interpolation methods at our disposal is not large.

6.6.3 Complex and real interpolation methods

6.6.3.1 Complex interpolation methods were introduced by three authors who worked independently from one another. Lions [1960] and Calderón [1960] discovered the definition that is still used today, whereas a more general approach proposed by Selim Kreĭn [1960] did not survive. The standard reference for the complex method is Calderón's paper [1964].

6.6.3.2 Given any couple of complex Banach spaces X_0 and X_1 as well as a parameter $0 < \theta < 1$, the interpolation space $[X_0, X_1]_\theta$ is obtained as follows.

We denote by $\mathscr{H}\{X_0, X_1\}$ the collection of all functions $\boldsymbol{f}$ that satisfy the conditions:

- $\boldsymbol{f}$ is an analytic (X_0+X_1)-valued function on the open strip
$$S := \{\, \zeta = \xi + \mathrm{i}\eta \,:\, 0 < \xi < 1,\ \eta \in \mathbb{R}\}.$$
- $\boldsymbol{f}$ is a continuous (X_0+X_1)-valued function on the closed strip
$$\overline{S} := \{\, \zeta = \xi + \mathrm{i}\eta \,:\, 0 \le \xi \le 1,\ \eta \in \mathbb{R}\}.$$
- $\boldsymbol{f}$ is a bounded continuous X_0-valued function on the line $\xi = 0$.
- $\boldsymbol{f}$ is a bounded continuous X_1-valued function on the line $\xi = 1$.

It easily turns out that $\mathscr{H}\{X_0, X_1\}$ becomes a Banach space under the norm

$$\|\boldsymbol{f}|\mathscr{H}\| := \max\Big\{ \sup_{\eta\in\mathbb{R}} \|\boldsymbol{f}(0+\mathrm{i}\eta)|X_0\|, \sup_{\eta\in\mathbb{R}} \|\boldsymbol{f}(1+\mathrm{i}\eta)|X_1\| \Big\}.$$

Now $[X_0, X_1]_\theta$ can be defined to consist of all elements $x = \boldsymbol{f}(\theta)$ obtained from some $\boldsymbol{f} \in \mathscr{H}\{X_0, X_1\}$, and the norm is given by

$$\big\|x\big|[X_0, X_1]_\theta\big\| := \inf\big\{\|\boldsymbol{f}|\mathscr{H}\| \,:\, x = \boldsymbol{f}(\theta)\big\}.$$

The final result says that the map

$$\Phi_\theta : \{X_0, X_1\} \mapsto [X_0, X_1]_\theta$$

is an exact interpolation functor of type θ.

6.6.3.3 The classical example of complex interpolation, namely $[L_{p_0}, L_{p_1}]_\theta = L_p$, is due to Calderón [1964, p. 125], who treated even the case of vector-valued functions:

$$\big[[L_{p_0}, X_0], [L_{p_1}, X_1]\big]_\theta = \big[L_p, [X_0, X_1]_\theta\big].$$

This equation is an isometry.

6.6.3.4 In contrast to the **complex interpolation method** just described, there are also **real methods**; the latter are more flexible. Whereas complex methods require complex Banach spaces, the real ones work in the real as well as in the complex case.

Beginning in 1959, Lions published a series of papers in which he introduced and studied *"espaces de traces."* Subsequently, Lions/Peetre [1961] discovered the *"espaces de moyennes"* and showed that both concepts amount to the same thing. The theory of these spaces, denoted by $S(q_0, \xi_0, X_0, q_1, \xi_1, X_1)$, was developed in a seminal paper, Lions/Peetre [1964].

Seemingly, $S(q_0, \xi_0, X_0, q_1, \xi_1, X_1)$ depends on four real parameters: $\xi_0\xi_1 < 0$ and $1 \le q_0, q_1 \le \infty$. However, as already observed by Lions/Peetre [1964, p. 15], one has

$$S(q_0, \xi_0, X_0, q_1, \xi_1, X_1) = S(q_0, \lambda\xi_0, X_0, q_1, \lambda\xi_1, X_1) \quad \text{for all } \lambda \neq 0.$$

Hence it may be arranged that $\xi_0 = -\theta$ and $\xi_1 = 1-\theta$, where $0 < \theta < 1$. In a next step, Peetre [1963a], [1963b] showed that even two parameters suffice: $0 < \theta < 1$ and $1 \le q \le \infty$. More precisely, he replaced $1/q_0$ and $1/q_1$ by $1/q = (1-\theta)/q_0 + \theta/q_1$.

6.6.3.5 According to Lions/Peetre [1964, p. 9], the espaces de moyennes are defined as follows.

Consider the collection of all elements $x \in X_0 + X_1$ that can be written in the form

$$x = \int_{-\infty}^{+\infty} \boldsymbol{f}(s)\, ds, \tag{6.6.3.5.a}$$

where $\boldsymbol{f}$ is an $(X_0 \cap X_1)$-valued function such that

$$\mathrm{e}^{\xi_0 s} \boldsymbol{f}(s) \in [L_{q_0}(\mathbb{R}), X_0] \quad \text{and} \quad \mathrm{e}^{\xi_1 s} \boldsymbol{f}(s) \in [L_{q_1}(\mathbb{R}), X_1].$$

The preceding conditions ensure that $\boldsymbol{f} \in [L_1(\mathbb{R}), X_0 + X_1]$. Thus the right-hand integral in (6.6.3.5.a) exists. In this way, we get a Banach space $S(q_0, \xi_0, X_0, q_1, \xi_1, X_1)$ with the norm

$$\|x|S\| := \inf\left\{ \max\left[\left\|\mathrm{e}^{\xi_0 s} \boldsymbol{f}(s) \middle| L_{q_0}\right\|, \left\|\mathrm{e}^{\xi_1 s} \boldsymbol{f}(s) \middle| L_{q_1}\right\| \right] : x = \int_{-\infty}^{+\infty} \boldsymbol{f}(s) ds \right\}.$$

6.6.3.6 Lions/Peetre [1964, p. 9] observed that it is sometimes advantageous to replace $s \in \mathbb{R}$ by $t = \mathrm{e}^s \in \mathbb{R}_+$. Then $L_q\big(\mathbb{R}_+, \frac{dt}{t}\big)$ must be used instead of $L_q(\mathbb{R})$. Nowadays, the variable t has become standard.

6.6.3.7 The most elegant constructions of the real interpolation spaces are based on two concepts introduced by Peetre in 1963. I fully agree with a quotation from [BENN$^+$, p. 436]:

The J- and K-methods are due to Peetre, whose manifold contributions have made his name almost synonymous with interpolation theory.

The ***J*-functional**

$$J(t,x) := \max\left\{ \|x|X_0\|, t\|x|X_1\| \right\}$$

and the ***K*-functional**

$$K(t,x) := \inf\left\{ \|x_0|X_0\| + t\|x_1|X_1\| : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1 \right\}$$

are defined for all $t > 0$ on $X_0 \cap X_1$ and $X_0 + X_1$, respectively. Sometimes the more precise symbols $J(t,x|X_0,X_1)$ and $K(t,x|X_0,X_1)$ are used. According to [PEE$_1$, p. 8], *the choice of the letters J and K is quite arbitrary.*

The space $(X_0, X_1)^J_{\theta,q}$ consists of all elements that can be written in the form

$$x = \int_0^\infty \boldsymbol{f}(t) \frac{dt}{t}, \tag{6.6.3.7.a}$$

where $\boldsymbol{f}$ is a Bochner measurable $(X_0 \cap X_1)$-valued function such that

$$t^{-\theta} J\big(t, \boldsymbol{f}(t)\big) \in L_q\big(\mathbb{R}_+, \tfrac{dt}{t}\big).$$

The preceding condition ensures that $\boldsymbol{f} \in \big[L_1\big(\mathbb{R}_+, \frac{dt}{t}\big), X_0{+}X_1\big]$. Hence the right-hand integral in (6.6.3.7.a) exists. By the way, it suffices to consider such $\boldsymbol{f}$'s that are either constant on the intervals $[2^n, 2^{n+1})$ with $n = \dots, -2, -1, 0, +1, +2, \dots$ or infinitely differentiable. The norm is given by

$$\big\|x\big|(X_0,X_1)^J_{\theta,q}\big\| := \inf\left\{ \left(\int_0^\infty \big[t^{-\theta} J\big(t, \boldsymbol{f}(t)\big)\big]^q \frac{dt}{t}\right)^{1/q} : x = \int_0^\infty \boldsymbol{f}(t)\frac{dt}{t}\right\}.$$

The definition of $(X_0,X_1)^K_{\theta,q}$ turns out to be even simpler. One just takes all elements $x \in X_0{+}X_1$ such that

$$\big\|x\big|(X_0,X_1)^K_{\theta,q}\big\| := \left(\int_0^\infty \big[t^{-\theta} K\big(t,x\big)\big]^q \frac{dt}{t}\right)^{1/q}$$

is finite.

The maps

$$\Phi^J_{\theta,q} : \{X_0,X_1\} \mapsto (X_0,X_1)^J_{\theta,q} \quad \text{and} \quad \Phi^K_{\theta,q} : \{X_0,X_1\} \mapsto (X_0,X_1)^K_{\theta,q}$$

are exact interpolation functors of type θ.

6.6.3.8 A fundamental result of Peetre [1963b, p. 251] says that

$$(X_0,X_1)^K_{\theta,q} = (X_0,X_1)^J_{\theta,q} = S(q_0, -\theta, X_0, q_1, 1-\theta, X_1),$$

the norms being equivalent. This observation justifies that the real interpolation functors are simply denoted by $\Phi_{\theta,q} : \{X_0,X_1\} \mapsto (X_0,X_1)_{\theta,q}$.

6.6.3.9 The space $X_0 \cap X_1$ is dense in $[X_0,X_1]_\theta$ and $(X_0,X_1)_{\theta,q}$ whenever $1 \le q < \infty$.

Replacing X_0 and X_1 by the closed hulls of $X_0 \cap X_1$ in X_0 and X_1, respectively, does not change the final results: $[X_0,X_1]_\theta$ and $(X_0,X_1)_{\theta,q}$; see [TRI$_1$, pp. 39, 59]. Thus, without loss of generality, one may assume that $X_0 \cap X_1$ is dense in both X_0 and X_1. This property is required to get the *duality theorem* (equivalent norms):

$$(X_0,X_1)^*_{\theta,q} = (X_0^*,X_1^*)_{\theta,q^*} \quad \text{if } 1 \le q < \infty.$$

For the complex method, $[X_0,X_1]^*_\theta = [X_0^*,X_1^*]_\theta$ holds only under additional assumptions. For example, it suffices to know that one of the spaces X_0 or X_1 is reflexive.

6.6.3.10 Let $0 < \theta_0 < \theta_1 < 1$, $0 < \eta < 1$, and $\theta = (1-\eta)\theta_0 + \eta\theta_1$. Then

$$\big((X_0,X_1)_{\theta_0,u_0}, (X_0,X_1)_{\theta_1,u_1}\big)_{\eta,q} = (X_0,X_1)_{\theta,q},$$

the norms being equivalent. This **reiteration theorem** goes back to Lions/Peetre [1964, p. 33]. The above version was proved by Holmstedt [1970, pp. 183–184]. Its complex counterpart is due to Calderón [1964, pp. 121, 159].

6.6.3.11 Until now, complex interpolation methods were applied only to Banach spaces. On the other hand, real methods easily extend to the more general setting of quasi-Banach spaces or even normed abelian groups. Then it is natural to define $(X_0,X_1)_{\theta,q}$ also for parameters q with $0<q<1$. These generalizations were carried out by Krée [1967], Holmstedt [1970], Peetre/Sparr [1972], Sagher [1972], and Kradzhov [1973].

6.6.4 Lorentz spaces

6.6.4.1 Let $(M,\mathscr{M},\mu)$ be a measure space. The **non-increasing rearrangement** f^* of a measurable scalar-valued function f on M is defined to be the inverse

$$f^*(t) := \sup\{s \geq 0 : F(s) > t\} \quad \text{for } t \geq 0$$

of the *distribution function*

$$F(s) := \mu\{\xi \in M : |f(\xi)| > s\} \quad \text{for } s \geq 0.$$

Given $0 < p < \infty$ and $0 < q < \infty$, the **Lorentz space** $L_{p,q}(M,\mathscr{M},\mu)$, in shorthand $L_{p,q}$, consists of all μ-measurable functions f for which

$$\|f|L_{p,q}\| := \left(\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t}\right)^{1/q} \tag{6.6.4.1.a}$$

is finite. In the limiting case $q = \infty$, we let

$$\|f|L_{p,\infty}\| := \sup_{t\geq 0} t^{1/p} f^*(t). \tag{6.6.4.1.b}$$

Frequently, $L_{p,\infty}$ is called a **Marcinkiewicz space** or **weak L_p-space**.

The case $p = \infty$ and $0 < q < \infty$ can be excluded, since $L_{\infty,q}$ consists only of the zero element. Note that $L_{p,p}$ is just the classical space L_p.

Taking the counting measure on $\mathbb{N}$ yields the sequence spaces $l_{p,q}$; see 6.3.1.5.

6.6.4.2 Apart from the case $1 \leq q \leq p < \infty$, the expression $\|f|L_{p,q}\|$ is only a quasi-norm. Hunt [1966, p. 259] identified all parameters for which $L_{p,q}$ can be made a Banach space by a suitable renorming. This is indeed possible for $1 < p < \infty$ and $1 \leq q \leq \infty$. The required norms are obtained by using the average

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\,du$$

instead of $f^*(t)$ in (6.6.4.1.a) and (6.6.4.1.b). At first glance, it is surprising that $L_{1,q}$ with $1 < q \leq \infty$ never becomes a Banach space.

6.6.4.3 Given $\alpha > 0$ and $p \geq 1$, Lorentz [1950, p. 41] introduced the collection of all Lebesgue measurable functions f on $[0,1]$ for which

$$\|f|L(\alpha,q)\| := \left(\alpha \int_0^1 t^{\alpha-1} f^*(t)^q\,dt\right)^{1/q}$$

is finite. In modern terminology: $\alpha = q/p$.

The name **Lorentz space** was coined by Calderón at a conference in Warsaw; see Calderón [1960]. He also proposed the symbol $\Lambda_{p,q}$, which was later changed into $L_{p,q}$; see Calderón [1964, p. 123]. Important contributions are due to O'Neil [1963], Oklander [1966], Hunt [1966], and Sagher [1972], members of the Zygmund school in Chicago.

At the beginning, the considerations were restricted to parameters $1 < p < \infty$ and $1 \le q \le \infty$. Later on, Hunt and Sagher dealt with the full scale, where $0 < p < \infty$ and $0 < q \le \infty$.

A parallel development was initiated by Lions/Peetre [1964, pp. 49–51], who proved that $S(\infty, -\theta, L_1, \infty, 1-\theta, L_\infty) = L_{p,\infty}$, where $1/p = 1-\theta$. Further contributions are due to Holmstedt [1970] and Peetre/Sparr [1972]; see also [PEE$_1$, Chap. X].

6.6.4.4 Lorentz spaces can be obtained by real interpolation:

$$\big(L_1, L_\infty\big)_{\theta,q} = L_{p,q} \quad \text{if } 1/p = 1-\theta \text{ and } 1 < p < \infty. \tag{6.6.4.4.a}$$

This result is an immediate consequence of **Hardy's inequality** [1920, p. 316], [HARD$^+$, pp. 239–243], Kufner/Maligranda/Persson [2006$^\bullet$],

$$\left\{ \int_0^\infty \left[t^{1/p} \Big(\frac{1}{t} \int_0^t f^*(u)du \Big) \right]^q \frac{dt}{t} \right\}^{1/q} \le p^* \left\{ \int_0^\infty \left[t^{1/p} f^*(t)dt \right]^q \frac{dt}{t} \right\}^{1/q}$$

and the formula

$$K(t, f|L_1, L_\infty) = \int_0^t f^*(u)du = t f^{**}(t) \quad \text{for } f \in L_1 + L_\infty, \tag{6.6.4.4.b}$$

which has a somehow confused history. The case of non-atomic measure spaces was treated in [PEE$_1$, pp. 54–56] via

$$K(t, f|L_1, L_\infty) = \sup_{\mu(A) \le t} \int_A |f(\xi)| d\mu(\xi).$$

In Peetre's Comptes Rendus note [1963a, p. 1425] the above supremum ranges only over all sets with $\mu(A) = t$. Thus it may be badly defined. Oklander settled the general case in his unpublished thesis (Chicago, 1964); this proof was reproduced in [BUT$_1^+$, pp. 184–186, 220]. Meanwhile, the matter has become quite simple: [BERG$^+$, p. 109].

The interpolation formula (6.6.4.4.a) extends to all parameters for which the Lorentz spaces make sense. As usual for real interpolation, the norms or quasi-norms are equivalent. The most general result, obtained by reiteration, says that

$$\big(L_{p_0,u_0}, L_{p_1,u_1}\big)_{\theta,q} = L_{p,q}.$$

6.6.4.5 In the case of vector-valued functions, we have

$$\big([L_{p_0}, X], [L_{p_1}, X]\big)_{\theta,q} = \big[L_{p,q}, X\big];$$

see [TRI$_1$, p. 134]. Therefore it seemed reasonable to hope that

$$\big([L_{p_0}, X_0], [L_{p_1}, X_1]\big)_{\theta,q} = \big[L_{p,q}, (X_0, X_1)_{\theta,q}\big].$$

According to Lions/Peetre [1964, p. 47], this formula is indeed true for $p = q$. On the other hand, Cwikel [1974, p. 287] observed that for $p \ne q$ and $p_0 \ne p_1$, there exist spaces X_0 and X_1 for which the members of $\big([L_{p_0}, X_0], [L_{p_1}, X_1]\big)_{\theta,q}$ admit no satisfactory description. This defect is the reason why the 3-parameter scale of Besov spaces behaves badly with respect to real interpolation; see 6.7.5.3 and 6.7.7.4.

6.6.4.6 The starting point of the theory of Lorentz spaces was the **Marcinkiewicz interpolation theorem** [1939], whose modern version reads as follows ($1 \le w \le \infty$):

If a given rule defines both

$$\text{an operator from } L_{p_0,1} \text{ into } L_{q_0,\infty} \quad \text{and} \quad \text{an operator from } L_{p_1,1} \text{ into } L_{q_1,\infty},$$

then it also defines

$$\text{an operator from } L_{p,w} \text{ into } L_{q,w}.$$

This means that having "weak" information about the behavior at the endpoints $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$ provides "strong" information about the behavior at all interior points $(1/p, 1/q)$ of the connecting line; see 6.6.1.2.

This phenomenon was first observed by Kolmogoroff [1925] in the case of the Hilbert transform.

6.6.5 Applications of interpolation theory

6.6.5.1 Given an interpolation scheme

$$\begin{tikzcd}
 & X_0 \arrow[r, "T_0"] & Y_0 \arrow[dr, "Id"] & \\
X_0 \cap X_1 \arrow[ur, "Id"] \arrow[r, "Id"] \arrow[dr, "Id"'] & X \arrow[r, "T"] & Y \arrow[r, "Id"] & Y_0 + Y_1\,, \\
 & X_1 \arrow[r, "T_1"] & Y_1 \arrow[ur, "Id"'] &
\end{tikzcd}$$

one may ask whether specific properties of T_0 and/or T_1 carry over to T. Letting $X = (X_0, X_1)_{\theta,p}$ and $Y = (Y_0, Y_1)_{\theta,p}$ with $1 \le p < \infty$, Hayakawa [1969, p. 189] proved that $T_0 \in \mathfrak{K}$ and $T_1 \in \mathfrak{K}$ imply $T \in \mathfrak{K}$. Subject to a specific approximation property, Persson [1964] had earlier shown that the compactness of one boundary operator T_0 or T_1 suffices.

For $X = (X_0, X_1)_{\theta,p}$ and $Y = (Y_0, Y_1)_{\theta,p}$ with $1 < p < \infty$, Beauzamy [BEAU$_1$, p. 40] established a far-reaching result along the same lines. Here is a slight generalization:

If T, viewed as an operator from $X_0 \cap X_1$ into $Y_0 + Y_1$, is weakly compact, then it is also weakly compact as an operator from X into Y. This is, in particular, the case when $T_0 \in \mathfrak{W}$ or $T_1 \in \mathfrak{W}$.

A corollary says that reflexivity of X_0 or X_1 implies reflexivity of $(X_0, X_1)_{\theta,p}$; see Morimoto [1967].

6.6.5.2 A famous result of Davis/Figiel/Johnson/Pełczyński [1974] states that every weakly compact operator factors through a reflexive space. Obviously, the four authors did not realize that their construction was borrowed from interpolation theory. We owe this observation to Beauzamy [BEAU$_1$, p. 32]. A modern proof reads as follows.

Let $Y_0 := M(T)$ be the range of the operator $T \in \mathfrak{W}(X,Y)$ equipped with the quotient norm $\|y|Y_0\| := \inf\{\|x\| : y = Tx\}$, and define $T_0 : X \to Y_0$ by $T_0 x := Tx$. Then $Y_0 \xrightarrow{Id} Y$ is weakly compact and

$$T : X \xrightarrow{T_0} Y_0 \xrightarrow{Id} (Y_0, Y)_{\frac{1}{2},2} \xrightarrow{Id} Y$$

yields the required factorization.

6.6.5.3 Sometimes we must restrict our considerations to simplified interpolation schemes,

$$\begin{tikzcd} & Y_0 \arrow[dr, "Id"] & \\ X \arrow[ur, "T_0"] \arrow[r, "T"] \arrow[dr, "T_1"'] & Y \arrow[r, "Id"] & Y_0+Y_1 \\ & Y_1 \arrow[ur, "Id"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & X_0 \arrow[dr, "T_0"] & \\ X_0 \cap X_1 \arrow[ur, "Id"] \arrow[r, "Id"] \arrow[dr, "Id"'] & X \arrow[r, "T"] & Y \\ & X_1 \arrow[ur, "T_1"'] & \end{tikzcd} .$$

An intermediate space Y is said to be of **class** $J(\theta, Y_0, Y_1)$ if there exists a constant $c > 0$ such that for arbitrary X, every compatible couple $T_0 \in \mathfrak{L}(X, Y_0)$ and $T_1 \in \mathfrak{L}(X, Y_1)$ induces an operator $T \in \mathfrak{L}(X,Y)$ with

An intermediate space X is said to be of **class** $K(\theta, X_0, X_1)$ if there exists a constant $c > 0$ such that for arbitrary Y, every compatible couple $T_0 \in \mathfrak{L}(X_0, Y)$ and $T_1 \in \mathfrak{L}(X_1, Y)$ induces an operator $T \in \mathfrak{L}(X,Y)$ with

$$\|T\| \le c\|T_0\|^{1-\theta}\|T_1\|^{\theta}.$$

Equivalently,

$$(Y_0, Y_1)^J_{\theta,1} \subseteq Y \qquad \text{and} \qquad X \subseteq (X_0, X_1)^K_{\theta,\infty},$$

where the norms of the embeddings are less than or equal to c.

Subject to the conditions above, Heinrich [1980b, pp. 401–403] proved the following interpolation theorems that are, in some sense, dual to each other:

Suppose that the ideal $\mathfrak{A}$ is injective and closed. Then

$$T_0 \in \mathfrak{A}(X, Y_0) \;\Rightarrow\; T \in \mathfrak{A}(X,Y).$$

Suppose that the ideal $\mathfrak{B}$ is surjective and closed. Then

$$T_0 \in \mathfrak{B}(X_0, Y) \;\Rightarrow\; T \in \mathfrak{B}(X,Y).$$

Lions/Peetre [1964, pp. 36–37] had earlier obtained both conclusions for the ideal of compact operators, which, by the way, is injective, surjective, and closed. In view of 6.2.3.10, the same result follows from Triebel's inequalities [1970, pp. 99–100]:

$$c_{2n-1}(T : X \to Y) \le c\, c_n(T_0 : X \to Y_0)^{1-\theta} c_n(T_1 : X \to Y_1)^{\theta} \qquad \text{(left-hand diagram)}$$

and

$$d_{2n-1}(T : X \to Y) \le c\, d_n(T_0 : X_0 \to Y)^{1-\theta} d_n(T_1 : X_1 \to Y)^{\theta} \qquad \text{(right-hand diagram).}$$

6.6.5.4 Let $\mathfrak{A}$ and $\mathfrak{B}$ be closed ideals such that $\mathfrak{A}$ is injective and $\mathfrak{B}$ is surjective. Combining his preceding interpolation theorems, Heinrich [1980b, p. 401] showed that $\mathfrak{B} \circ \mathfrak{A} = \mathfrak{A} \cap \mathfrak{B}$.

The diagram below proves, without words, the non-trivial inclusion $\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{B} \circ \mathfrak{A}$:

$$\begin{tikzcd}
X \arrow[r, "T\in\mathfrak{A}"] \arrow[dr, "A\in\mathfrak{A}"] \arrow[ddr, "T_0"'] & Y \arrow[r, "Id"] & Y \\
& (Y_0,Y)_{\frac{1}{2},2} \arrow[u, "B:=Id\in\mathfrak{B}"'] \arrow[ur] & \\
& Y_0 \arrow[u] \arrow[uur, "Id\in\mathfrak{B}"'] &
\end{tikzcd} \quad .$$

Note that Y_0 and T_0 are defined as in 6.6.5.2.

6.6.5.5 Most ideals behave very badly with respect to interpolation.

We know that every continuous kernel induces nuclear operators on $L_1(\mathbb{T})$ and $L_\infty(\mathbb{T})$. However, the theory of trigonometric series provides continuous functions for which the associated convolution operators fail to be nuclear on $L_p(\mathbb{T})$ with $1 < p < \infty$; see 6.4.4.2.

6.6.5.6 Pisier/Xu [1987, p. 189] used the real interpolation method to construct a space with many interesting properties.

Recall that the summation operator $\Sigma : l_1 \to l_\infty$ is defined by

$$\Sigma : (\xi_k) \mapsto \Big(\sum_{k=1}^{h} \xi_k \Big).$$

Then $v_1 := M(\Sigma)$ becomes a Banach space under the norm

$$\|y|v_1\| := |\eta_1| + |\eta_2 - \eta_1| + |\eta_3 - \eta_2| + \cdots \quad \text{for } y = (\eta_h).$$

Since $\{v_1, l_\infty\}$ is an interpolation couple, one may form $\big(v_1, l_\infty\big)_{\frac{1}{2},2}$. This space is of Rademacher type p and cotype q whenever $1 < p < 2 < q < \infty$. Moreover, there exists a constant $c > 0$ such that

$$d(E_n, l_2^n) \le c(1 + \log n) \quad \text{for every } n\text{-dimensional subspace, } n = 1, 2, \dots .$$

Hence $\big(v_1, l_\infty\big)_{\frac{1}{2},2}$ is “almost” Hilbertizable. On the other hand, it has codimension 1 in its bidual.

6.6.6 Interpolation of operator ideals

6.6.6.1 Components $\mathfrak{A}_0(X,Y)$ and $\mathfrak{A}_1(X,Y)$ of Banach ideals always constitute an interpolation couple, since they are continuously embedded into $\mathfrak{L}(X,Y)$. Therefore, given any exact interpolation functor Φ, one may define

$$\mathfrak{A}_\Phi(X,Y) := \Phi\{\mathfrak{A}_0(X,Y), \mathfrak{A}_1(X,Y)\}.$$

Obviously, these components form a Banach ideal. Although this definition looks very promising, it turned out that, very seldom, the members of $\mathfrak{A}_\Phi$ admit a handy characterization. For example, almost nothing is known about $[\mathfrak{N}, \mathfrak{L}]_\theta$ and $(\mathfrak{N}, \mathfrak{L})_{\theta,q}$.

6.6.6.2 However, there are rays of hope as well.

For $1 \le p_0 < p < p_1 \le \infty$ and $1 \le q \le \infty$, Triebel [1967, pp. 277–279] invented the Lorentz ideals $\mathfrak{S}_{p,q}(H)$ and proved the real interpolation formula

$$\big(\mathfrak{S}_{p_0}(H), \mathfrak{S}_{p_1}(H)\big)_{\theta,q} = \mathfrak{S}_{p,q}(H). \tag{6.6.6.2.a}$$

Subsequently, interpolating between quasi-Banach ideals, Oloff [1970, p. 216] extended this result to $0 < p_0 < p < p_1 \le \infty$. In a next step, Peetre/Sparr [1972, p. 257] passed from Hilbert to Banach spaces:

$$\big(\mathfrak{L}^{\text{app}}_{p_0,u_0}, \mathfrak{L}^{\text{app}}_{p_1,u_1}\big)_{\theta,q} = \mathfrak{L}^{\text{app}}_{p,q}.$$

see also König [1978, p. 37].

Using a three-lines theorem from [GOH$_3^+$, pp. 136–137], Pietsch/Triebel [1968, p. 100] established the complex counterpart of (6.6.6.2.a), which requires the assumption $1 \le p_0 < p < p_1 \le \infty$:

$$\big[\mathfrak{S}_{p_0}(H), \mathfrak{S}_{p_1}(H)\big]_{\theta} = \mathfrak{S}_{p}(H).$$

6.6.6.3 Apart from $\mathfrak{L}^{\text{app}}_p$, I do not know any other 1-parameter scale of ideals $\mathfrak{A}_p$ that could be reproduced by real or complex interpolation: $\big(\mathfrak{A}_{p_0}, \mathfrak{A}_{p_1}\big)_{\theta,p} = \mathfrak{A}_p$ or $\big[\mathfrak{A}_{p_0}, \mathfrak{A}_{p_1}\big]_{\theta} = \mathfrak{A}_p$. However, sometimes we have at least one-sided inclusions. For example, König [1978, p. 42] and Pietsch/Triebel [1968, p. 101] showed that

$$\big(\mathfrak{P}_{p_0,q}, \mathfrak{P}_{p_1,q}\big)_{\theta,p} \subset \mathfrak{P}_{p,q} \quad \text{and} \quad \big[\mathfrak{P}_{p_0,q}, \mathfrak{P}_{p_1,q}\big]_{\theta} \subset \mathfrak{P}_{p,q}.$$

Both inclusions are strict; see [KÖN, pp. 126–127].

6.6.7 New trends in interpolation theory

6.6.7.1 In a survey article, Peetre [1985, p. 178] expressed a very critical opinion:

… the theory of interpolation spaces has developed in a far too abstract direction, getting more and more divorced from the rest of Analysis. This is no good sign. Perhaps it is still time to stem the tide. In particular, it should be of major interest to see what the new developments associated with the "new" generation really mean for the down to earth analysis problems from which interpolation theory once arose.

Let me mention that according to Peetre [1984, p. 3], [1985, p. 178], the "new" generation includes Ovchinnikov, Brudnyĭ, Kruglyak, Cwikel, Janson, Nilsson, and many others.

6.6.7.2 Obviously, Peetre has described a general trend that in no way is restricted to interpolation theory or functional analysis.

As commonly accepted, the significance of modern theories should be evaluated by looking at the extent to which they can be used in solving classical problems. I am not so happy about this "mathematical" criterion. Here is an example:

Special cases of the interpolation formula

$$\big(L_{p_0},L_{p_1}\big)_{\theta,q}=L_{p,q} \tag{6.6.7.2.a}$$

say that

$$\big(L_1(\mathbb{T}),L_2(\mathbb{T})\big)_{\theta,p}=L_{p,p}(\mathbb{T}) \quad\text{and}\quad \big(c_0(\mathbb{Z}),l_2(\mathbb{Z})\big)_{\theta,p}=l_{p*,p}(\mathbb{Z})$$

with $p_0=1$, $p_1=2$, and $1/p=(1-\theta)/p_0+\theta/p_1=1-\theta/2$. If $\big(\gamma_n^*(f)\big)$ denotes the non-decreasing rearrangement of $\big(\gamma_k(f)\big)$, then we obtain

$$\Big(\sum_{n=1}^{\infty}n^{p-2}\gamma_n^*(f)^p\Big)^{1/p}\le c_p\Big(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p dt\Big)^{1/p} \tag{6.6.7.2.b}$$

for $f\in L_p(\mathbb{T})$ and $1<p<2$. This refinement of the Hausdorff–Young inequality was discovered by Hardy/Littlewood [1931, p. 4]; see also (6.6.1.3.a). For arbitrary orthonormal systems of uniformly bounded functions on $[0,1]$, a proof is due to Paley [1931, p. 230]. In my opinion, both (6.6.7.2.a) and (6.6.7.2.b) are beautiful results. However, I cannot see why the former needs the latter in its defense.

6.6.7.3 Recall that the real interpolation space $(X_0,X_1)_{\theta,q}$ consists of all elements $x\in X_0+X_1$ such that $K(t,x)$, viewed as a function of t, belongs to $L_{p,q}\big(\mathbb{R}_+,\frac{dt}{t}\big)$ with $1/p=1-\theta$.

In *modern* interpolation theory, the *classical* Lorentz spaces are replaced by much more general Banach lattices L_ρ whose members are measurable functions on $(\mathbb{R}_+,\frac{dt}{t})$. Suppose that $\min\{t,1\}\in L_\rho$. Then

$$\Phi_\rho:\{X_0,X_1\}\mapsto(X_0,X_1)_\rho:=\big\{\,x\in X_0+X_1\,:\,K(\cdot,x)\in L_\rho\big\}$$

is an interpolation functor. This concept goes back to the Brazilian lecture notes of Peetre [PEE$_1$, pp. 10–15], who generated the underlying Banach lattice from a *function norm* ρ, which explains the symbol L_ρ.

6.6.7.4 For many years, modern interpolation theory remained unsatisfactory; see Cwikel/Peetre [1981]. The decisive breakthrough was achieved only in [1981, стр. 16], when Brudnyĭ/Kruglyak announced their famous ***K*-divisibility theorem**:

There exists a universal constant $c\ge1$ with the following property:

Let (ψ_k) be a sequence of positive concave functions on $\mathbb{R}_+$, and let $\{X_0,X_1\}$ be any interpolation couple. Then every $x\in X_0+X_1$ such that

$$K(t,x)\le\sum_{k=1}^{\infty}\psi_k(t)<\infty\quad\text{for } t>0$$

admits a decomposition

$$x=\sum_{k=1}^{\infty}x_k,\quad\text{converging in } X_0+X_1,$$

such that

$$K(t,x_k)\le c\psi_k(t)\quad\text{for } t>0 \text{ and } k=1,2,\dots\,.$$

Most important is the case that $\psi_k(t):=K(t,u_k)$ with $u_k\in X_0+X_1$.

Though this "lemma" looks quite technical, according to [CWI$^{\cup}$, pp. 11–12],
it is something of a revolution in the real interpolation method.
...
The K-divisibility theorem makes it possible to establish all sorts of results for the spaces $(X_0,X_1)_\rho$, reiteration, duality, equivalence of J-functional and K-functional constructions, which had earlier been known only for the space $(X_0,X_1)_{\theta,q}$ and various rather specialized variants of them.

With a large delay, the original proof of Brudnyĭ/Kruglyak was published only in [BRUD$^+$, pp. 315–337]. Meanwhile, Cwikel [1984] had presented an alternative approach; see also [OVCH, pp. 414–415]. Additionally, I refer to papers of Cwikel/Peetre [1981] and Cwikel/Jawerth/Milman [1990], which are devoted to a strengthening of the *"fundamental lemma of interpolation theory."*

6.6.7.5 Concerning its significance, the K-divisibility theorem can be compared with Grothendieck's fundamental theorem. There is, however, another analogy: we have a **K-divisibility constant** whose exact value is unknown as well.

Let c_D be the infimum of all constants c for which the K-divisibility theorem holds, regardless how we choose the majorizing functions ψ_k and the underlying interpolation couple. Then

$$1.609\ldots = \frac{2+2\sqrt{2}}{3} \le c_D \le 3+2\sqrt{2} = 5.828\ldots\ .$$

Kruglyak [1981] discovered the lower bound by using the couple $\{C[0,1],C^1[0,1]\}$; see also [BRUD$^+$, p. 335]. The upper estimate is due to Cwikel/Jawerth/Milman [1990, p. 77]. The original proof of Brudnyĭ/Kruglyak [1981, стр. 16] gave $1<c_D<14$.

6.6.7.6 An intermediate space X of $\{X_0,X_1\}$ is said to be an **interpolation space** if every couple of operators $T_0\in\mathfrak{L}(X_0)$ and $T_1\in\mathfrak{L}(X_1)$ that coincide on $X_0\cap X_1$ induces an operator $T\in\mathfrak{L}(X)$. In this case, there exists a constant $c_X>0$, not depending on T_0 and T_1, such that

$$\|T:X\to X\| \le c_X \max\big\{\|T_0:X_0\to X_0\|,\|T_1:X_1\to X_1\|\big\}.$$

If $c_X=1$, then the interpolation space X is called **exact**; compare with 6.6.2.4.

According to [BRUD$^+$, p. 250], the *"basic problem"* of interpolation theory consists in describing *all* interpolation spaces of a given couple.

6.6.7.7 An intermediate space X is called **K-monotone** if

$$x\in X_0{+}X_1,\ \ u\in X,\ \text{ and }\ K(t,x)\le K(t,u)\ \text{ for } t>0\ \text{ imply }\ x\in X.$$

Clever applications of the closed graph theorem show the existence of a constant $c>0$, not depending on u and x, such that $\|x|X\|\le c\|u|X\|$; see [BRUD$^+$, pp. 501–502].

6.6.7.8 Clearly, every real interpolation space $(X_0,X_1)_\rho$ is K-monotone. A deep theorem of Brudnyĭi/Kruglyak [1981, стр. 16], [BRUD$^+$, p. 504] states the converse:

Every K-monotone intermediate space X can be represented in the form $X=(X_0,X_1)_\rho$, the required function norm being given by

$$\rho(f):=\inf\left\{\sum_{k=1}^{\infty}\|u_k|X\| : u_k\in X,\quad |f(t)|\leq\sum_{k=1}^{\infty}K(t,u_k)\ \text{ for } t>0\right\}.$$

6.6.7.9 We refer to $\{X_0,X_1\}$ as a **Calderón couple** if all of its interpolation spaces are K-monotone. This terminology is justified by the fact that Calderón [1966, p. 280] discovered the first couple with this property: $\{L_1,L_\infty\}$. More precisely, he proved that an intermediate space X of $\{L_1,L_\infty\}$ is an exact interpolation space if and only if

$$f\in L_1+L_\infty,\ g\in X\ \text{ and }\ f^{**}(t)\leq g^{**}(t)\ \text{ for } t>0\ \text{ imply }\ f\in X \text{ and } \|f|X\|\leq\|g|X\|,$$

a condition that anticipated the concept of K-monotonicity modulo (6.6.4.4.b); see also 6.3.1.10.

Mityagin [1965] obtained a slightly weaker result. Further authors and examples are listed in [KREIN$^+$, p. 350]. The most general theorem about couples of weighted L_p spaces was obtained by Sparr [1978].

An elegant counterexample is due to Maligranda/Ovchinnikov [1992], who showed that $L_p+L_{p^*}$ and $L_p\cap L_{p^*}$ with $1<p<p^*<\infty$ are interpolation spaces of the couple $\{L_1\cap L_\infty, L_1+L_\infty\}$, which, however, fail to be K-monotone.

6.6.7.10 Finally, I present a classical result that is concerned with "weak" interpolation. To simplify matters, only function spaces over the real half-line $\mathbb{R}_+$ are considered.

Let X be an interpolation space of the couple $\{L_1(\mathbb{R}_+),L_\infty(\mathbb{R}_+)\}$. Then the *dilation operator* $D_a : f(t)\mapsto f(t/a)$ with $0<a<\infty$ takes X into itself.

The **Boyd indices** are defined by

$$p_0(X):=\lim_{a\to\infty}\frac{\log a}{\log\|D_a:X\to X\|}=\sup\left\{p_0 : \|D_a:X\to X\|\leq a^{1/p_0}\quad\text{for large } a\right\}$$

and

$$p_1(X):=\lim_{a\to 0}\frac{\log a}{\log\|D_a:X\to X\|}=\inf\left\{p_1 : \|D_a:X\to X\|\leq a^{1/p_1}\quad\text{for small } a\right\}.$$

We conclude from $1\leq\|D_a:X\to X\|\,\|D_{1/a}:X\to X\|$ that $1\leq p_0(X)\leq p_1(X)\leq\infty$.

In his original work, Boyd [1969, p. 1251] used the reciprocals of $p_0(X)$ and $p_1(X)$. The present modification is adopted from Lindenstrauss/Tzafriri, [LIND$_2^+$, pp. 130–131], who motivated their decision by the formula $p_0(L_p)=p_1(L_p)=p$.

Boyd's theorem [1969, p. 1252] says that for every space X as above and exponents $1\leq p_0,p_1\leq\infty$, the conditions $p_0<p_0(X)$ and $p_1(X)<p_1$ are necessary and sufficient to ensure the following interpolation property:

Every rule that defines both an operator from $L_{p_0,1}$ into $L_{p_0,\infty}$ and an operator from $L_{p_1,1}$ into $L_{p_1,\infty}$ also defines an operator from X into X.

6.7 Function spaces

The theory of function spaces has developed into an autonomous branch of functional analysis. Since the structure of the usual spaces H_p, W_p^m, $B_{p,q}^\sigma$, ... is relatively simple, they are only of little interest for Banach space geometers. Some prominent exceptions are discussed in 6.7.6.9 and 6.7.12.25. On the other hand, function spaces provide the connecting link to all kinds of applications in the theory of partial differential equations, harmonic analysis and approximation theory.

The Euclidean norm of $x \in \mathbb{R}^N$ will be denoted by $|x|$.

6.7.1 Hölder–Lipschitz spaces

6.7.1.1 Spaces of differentiable functions were considered already in Banach's thesis [1922, pp. 134, 167]:

$\mathscr{C}^p\mathscr{C}$ is defined to be *l'ensemble de fonctions ayant la* $p^{\text{ième}}$ *dérivée continue,* equipped with the norm

$$\|f\| = \max_{a \le x \le b} |f(x)| + \max_{a \le x \le b} \left| \frac{d^p f(x)}{dx^p} \right|.$$

6.7.1.2 Let Ω be a domain in $\mathbb{R}^N$ and $0 < \sigma \le 1$. Conditions of the form

$$|f(x) - f(y)| \le c|x-y|^\sigma \quad \text{for all } x, y \in \Omega \text{ and some } c \ge 0 \qquad (6.7.1.2.a)$$

occurred for the first time in the work of Lipschitz on trigonometric series [1864, p. 301] and on ordinary differential equations [1876] as well as in Hölder's *Beiträgen zur Potential Theorie* [1882]. Nowadays, whenever $0 < \sigma < 1$, the collection of all (bounded) functions f defined on Ω and satisfying (6.7.1.2.a) is denoted by $\mathrm{Lip}^\sigma(\Omega)$. The symbol $C^\sigma(\Omega)$ is common as well, and one speaks of **Lipschitz spaces** or **Hölder spaces**. To the best of my knowledge, the definition of the norm

$$\|f|\mathrm{Lip}^\sigma\| := \sup_{x \in \Omega} |f(x)| + \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|f(x) - f(y)|}{|x-y|^\sigma}$$

goes back to Schauder [1934, p. 259]. From the historical point of view, the periodic case is of special interest.

6.7.1.3 For every function $f \in C(\mathbb{T})$, we let

$$E_n(f|C) := \inf \|f - T|C\|,$$

where the infimum ranges over all trigonometric polynomials T of degree less than or equal to n.

In its final form, the famous **Jackson–Bernstein theorem** says that for $0 < \sigma < 1$,

$$f \in \mathrm{Lip}^\sigma(\mathbb{T}) \quad \text{if and only if} \quad E_n(f|C) = O\left(\tfrac{1}{n^\sigma}\right).$$

This result became a landmark of modern analysis. Jackson [1911, 1912] was awarded a prize by the University of Göttingen, and Bernstein [1912] by the Belgian Academy.

6.7.1.4 The original Lipschitz space $\mathrm{Lip}(\mathbb{T})$, which is obtained for $\sigma = 1$, plays the role of an outsider, since the Jackson–Bernstein theorem fails. In order to circumvent this defect, Zygmund [1945] used differences of second order. More precisely, for $0 < \sigma < 2$, he considered the class $\mathrm{Lip}^{\sigma,2}(\mathbb{T})$ consisting of all $f \in C(\mathbb{T})$ such that

$$|f(x+h) - 2f(x) + f(x-h)| \le ch^\sigma \quad \text{for every } x \in \mathbb{T}, \text{ every } h > 0 \text{ and some } c \ge 0.$$

Then

$$f \in \mathrm{Lip}^{\sigma,2}(\mathbb{T}) \quad \text{if and only if} \quad E_n(f|C) = O\big(\tfrac{1}{n^\sigma}\big).$$

Hence $\mathrm{Lip}^\sigma(\mathbb{T}) = \mathrm{Lip}^{\sigma,2}(\mathbb{T})$ whenever $0 < \sigma < 1$. On the other hand, $\mathrm{Lip}(\mathbb{T})$ is a proper subset of $\mathrm{Lip}^{1,2}(\mathbb{T})$.

This process can be extended to get $\mathrm{Lip}^\sigma(\mathbb{T}) := \mathrm{Lip}^{\sigma,m}(\mathbb{T})$ for any $\sigma > 0$. Let $\Delta_h f(x) := f(x+h) - f(x)$. Using differences Δ_h^m of order $m > \sigma$, one defines $\mathrm{Lip}^{\sigma,m}(\mathbb{T})$ to be the collection of all $f \in C(\mathbb{T})$ such that

$$|\Delta_h^m f(x)| \le ch^\sigma \quad \text{for every } x \in \mathbb{T}, \text{ every } h > 0, \text{ and some } c \ge 0.$$

Of course, it must be shown that the space $\mathrm{Lip}^{\sigma,m}(\mathbb{T})$ does not depend on the special choice of m. For a long time, this was done via the criterion

$$f \in \mathrm{Lip}^{\sigma,m}(\mathbb{T}) \quad \text{if and only if} \quad E_n(f|C) = O\big(\tfrac{1}{n^\sigma}\big).$$

Now we have a direct proof, which will be discussed in the next paragraph.

The classical way of defining $\mathrm{Lip}^\sigma(\mathbb{T})$ is the following. Non-integral exponents are written in the form $\sigma = [\sigma] + \sigma_0$ with $[\sigma] \in \mathbb{N}_0$ (integral part) and $0 < \sigma_0 < 1$. Then $\mathrm{Lip}^\sigma(\mathbb{T})$ consists of all functions f with derivatives up to the order $[\sigma]$ such that $f^{[\sigma]} \in \mathrm{Lip}^{\sigma_0}(\mathbb{T})$. For an integer $\sigma = m$, the functions $f \in \mathrm{Lip}^\sigma(\mathbb{T})$ are supposed to have derivatives of order $m-1$ that belong to $\mathrm{Lip}^{1,2}(\mathbb{T})$.

6.7.1.5 One refers to

$$\omega_m(t, f|C) := \sup_{|h| \le t} \|\Delta_h^m f|C\| \quad \text{with } t > 0$$

as the **modulus of smoothness** of order m of the function $f \in C(\mathbb{T})$. For $m = 1$, this expression was defined by de la Vallée Poussin, who also coined the term *module de continuité*; see [VAL, p. 7]. Moduli of higher order $m = 2, 3, \ldots$ were introduced by Marchaud [1927, p. 338]. We have

$$f \in \mathrm{Lip}^{\sigma,m}(\mathbb{T}) \quad \text{if and only if} \quad \omega_m(t, f|C) = O\big(t^\sigma\big).$$

Besides the trivial fact that $\omega_{m+1} \le 2\omega_m$, Marchaud [1927, pp. 371–372] proved an estimate of ω_m by ω_{m+1} that was overlooked for many years; [LORZ, pp. 48, 53]. **Marchaud's inequalities** show that $\mathrm{Lip}^\sigma(\mathbb{T}) := \mathrm{Lip}^{\sigma,m}(\mathbb{T})$ is independent of $m > \sigma$; see also [ZYG, Vol. I, footnote on p. 44].

Stechkin [1951, стр. 226] generalized a classical result of Jackson:

$$E_n(f|C) \le a_m\, \omega_m(\tfrac{1}{n}, f|C).$$

More importantly, he discovered a converse estimate, [1951, стр. 234]:

$$\omega_m(\tfrac{1}{n}, f|C) \le b_m \frac{1}{n^m} \sum_{k=1}^{n} k^{m-1} E_k(f|C).$$

Here a_m and b_m depend only on m. In other words, these inequalities hold uniformly for $n=1,2,\ldots$. The Jackson–Bernstein theorem is an immediate consequence.

6.7.1.6 From the standpoint of Banach space theory, Lipschitz spaces are not quite interesting. Indeed, Ciesielski [1960] showed that $\mathrm{Lip}^\sigma(\mathbb{T})$ is isomorphic to l_∞.

In view of this observation, the formula $l_\infty = c_0^{**}$ suggested to look for a space $\mathrm{lip}^\sigma(\mathbb{T})$ such that $\mathrm{Lip}^\sigma(\mathbb{T}) = \mathrm{lip}^\sigma(\mathbb{T})^{**}$. This goal was achieved by using a uniform *"little oh"* condition $|\Delta_h^m f(x)| = o(h^\sigma)$ instead of $|\Delta_h^m f(x)| = O(h^\sigma)$; see Glaeser [1958, p. 96] and de Leeuw [1961, p. 56].

6.7.1.7 For $m=1,2,\ldots$, the Banach space $C^m(\mathbb{T})$ consists of all periodic functions that have continuous derivatives up to order m. A suitable norm is given by

$$\|f|C^m\| := \sup_{x\in\mathbb{T}} |f(x)| + \sup_{x\in\mathbb{T}} |f^{(m)}(x)|.$$

Then

$$\big(C(\mathbb{T}), C^m(\mathbb{T})\big)_{\theta,\infty} = \mathrm{Lip}^{\sigma,m}(\mathbb{T}), \quad \text{where } \sigma = \theta m. \tag{6.7.1.7.a}$$

This formula seems to be some kind of the folklore. Though not stated explicitly by Peetre, it was certainly known to him around 1963; see [PEE$_1$, pp. 63–73]. Nowadays, (6.7.1.7.a) is considered as the limiting case of a general result to be discussed in 6.7.5.2, namely

$$\big(L_p(\mathbb{T}), W_p^m(\mathbb{T})\big)_{\theta,q} = B_{p,q}^{\theta m}(\mathbb{T}).$$

As just observed, the spaces $C(\mathbb{T}), C^1(\mathbb{T}), C^2(\mathbb{T}), \ldots$ can be used as endpoints of interpolation. Unfortunately, they do not fit into the scale $\big\{\mathrm{Lip}^\sigma(\mathbb{T})\big\}_{\sigma>0}$ so obtained. Indeed, $C^1(\mathbb{T})$ is separable, whereas $\mathrm{Lip}^{1,2}(\mathbb{T}) = \big(C(\mathbb{T}), C^2(\mathbb{T})\big)_{1/2,\infty}$ is not. Hence the inclusion $C^1(\mathbb{T}) \subset \big(C(\mathbb{T}), C^2(\mathbb{T})\big)_{1/2,\infty}$ turns out to be "very" strict. The situation is even worse: Mityagin/Semenov [1977, стр. 1290] showed that there cannot exist any interpolation functor Φ such that $C^1(\mathbb{T}) = \Phi\{C(\mathbb{T}), C^2(\mathbb{T})\}$.

6.7.2 Sobolev spaces

6.7.2.1 In the 1-dimensional case, Sobolev spaces $\mathscr{C}^p\mathscr{S}^r$ were invented already by Banach [1922, pp. 134, 167], who considered

l'ensemble de fonctions ayant $(p-1)^{\text{ième}}$ dérivée absolument continue et la $p^{\text{ième}}$ dérivée intégrable (L) avec la $r^{\text{ième}}$ puissance,

the norm being given by

$$\|f(x)\| = \max_{a\le x\le b} |f(x)| + \sqrt[r]{\int_a^b \left|\frac{d^p f(x)}{dx^p}\right|^r dx}.$$

6.7.2.2 The definition of Sobolev spaces over domains in $\mathbb{R}^N$ was delayed by a decisive obstacle: one had to find a suitable concept of differentiation. Sobolev [1938, Résumé, стр. 496] succeeded in solving this problem:

Appelons espace $L_p^{(\nu)}$ [modern symbol: W_p^m] *l'espace fonctionnel linéaire qui est formé de toutes les fonctions de n variables réelles* $\varphi(x_1,\dots,x_n)$ *dont les dérivées partielles jusq'à l'ordre* ν *existent et sont sommables à la puissance* $p>1$ *dans chaque partie bornée de l'espace* $x_1,\dots,x_n$. *La dérivée*

$$\frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

est définie comme une fonction qui satisfait à l'équation

$$\int\limits_\infty\!\cdots\!\int \psi\frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,dx_1\cdots dx_n = \int\limits_\infty\!\cdots\!\int (-1)^\alpha \varphi\frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,dx_1\cdots dx_n,$$

quelle que soit la fonction ψ *continue ayant des dérivées jusq'à l'ordre* ν *et s'annulant en dehors d'un domaine borné.*

In his early work, Sobolev did not use the concept of a Banach space. In particular, norms occur only implicitly; see Sobolev [1938, стр. 486]. However, his famous book [SOB] is based on a modern point of view. In § 7 he identified W_p^m as a normed space, and completeness was proved in § 10.

Nowadays, for $m=1,2\dots,$ $1\le p<\infty$, and a domain Ω in $\mathbb{R}^N$, the **Sobolev space** $W_p^m(\Omega)$ is defined to be the collection of all functions $f\in L_p(\Omega)$ whose partial derivatives (in the above sense),

$$D^\alpha f := \frac{\partial^\alpha}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}} f \quad\text{and}\quad \alpha=(\alpha_1,\dots,\alpha_N),$$

are p-integrable whenever $|\alpha| := \alpha_1+\cdots+\alpha_N\le m$. The norm is given by

$$\|f|W_p^m\| := \Big\{\sum_{|\alpha|\le m}\int_\Omega \big|D^\alpha f(x)\big|^p dx\Big\}^{1/p}.$$

6.7.2.3 Nikodym [1933, p. 131] treated a particular space that plays a decisive role in connection with the Dirichlet principle. A member of $W_2^1(\Omega)$, called a Beppo Levi function, was supposed to have the following properties:

1° *elle est définie presque partout dans* Ω,

2° *elle est intérieurement absolument continue sur presque toute droite relative à* Ω,

3° *les dérivées* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ *sont à carré sommable dans* Ω.

Nikodym showed completeness with respect to the semi-metric

$$\left\{\iiint\limits_\Omega \left(\left|\frac{\partial(f-g)}{\partial x}\right|^2+\left|\frac{\partial(f-g)}{\partial y}\right|^2+\left|\frac{\partial(f-g)}{\partial z}\right|^2\right)dxdydz\right\}^{1/2}.$$

Passing to the quotient space modulo constants, he gave the first example of a homogeneous function space, namely $\dot{W}_2^1(\Omega)$; see 6.7.6.3.

6.7.2.4 In what follows, $S(\mathbb{R}^N)$ denotes the locally convex linear space formed by all rapidly decreasing functions on $\mathbb{R}^N$. Members of the dual $S(\mathbb{R}^N)^*$ are called **tempered distributions**. The **Fourier transform**

$$F_{\text{our}} : f(x) \mapsto \widehat{f}(y) := \int_{\mathbb{R}} f(x) \mathrm{e}^{-\mathrm{i}(x|y)} dx$$

yields an isomorphism on $S(\mathbb{R}^N)$ as well as on $S(\mathbb{R}^N)^*$. These fundamental results are due to (Laurent) Schwartz; see [SCHW$_1$, tome II, Chap. VII].

6.7.2.5 Calderón [1961, p. 33] observed that the rule

$$J_\sigma : f \mapsto F_{\text{our}}^{-1}\big[(1+|y|^2)^{-\sigma/2} F_{\text{our}} f\big]$$

yields a 1-parameter group of continuous linear mappings on $S(\mathbb{R}^N)$ and $S(\mathbb{R}^N)^*$. Obviously, $J_{-2} = Id - \Delta$.

For every real σ, Calderón [1961, p. 35] defined the **Lebesgue space** $L_p^\sigma(\mathbb{R}^N)$ to be the image of $L_p(\mathbb{R}^N)$ under J_σ. Letting $\|f|L_p^\sigma\| := \|J_{-\sigma} f|L_p\|$, the Banach spaces $L_p^\sigma(\mathbb{R}^N)$ and $L_p(\mathbb{R}^N)$ are isometric, by definition. The special case $L_2^\sigma(\mathbb{R}^N)$ and $\sigma > 0$ was treated by Aronszajn/Smith [1961]. These authors also coined the term **Bessel potential of order** σ, which is justified by the fact that $(1+|y|^2)^{-\sigma/2}$ with $\sigma > 0$ turns out to be the Fourier transform of

$$G_\sigma(x) := \frac{1}{2^{(N+\sigma-2)/2} \pi^{N/2} \Gamma\big(\frac{\sigma}{2}\big)} \frac{K_{(N-\sigma)/2}\big(|x|\big)}{|x|^{(N-\sigma)/2}},$$

where $K_{(N-\sigma)/2}$ is a modified Bessel function of the third kind. We have $G_\sigma(x) > 0$ everywhere and $\int_{\mathbb{R}^N} G_\sigma(x) dx = 1$. Hence the convolution operator $J_\sigma : f \mapsto G_\sigma * f$ is a contraction on $L_p(\mathbb{R}^N)$ whenever $1 \le p \le \infty$.

Stein [STEIN, p. 133] pointed out:

We shall, however, not need any of the properties of the Bessel functions and so the terminology of "Bessel potential" has for us only a vestigial significance.

Despite this remark, most authors refer to $L_p^\sigma(\mathbb{R}^N)$ as a **Bessel potential space**.

6.7.2.6 Lizorkin [1963, стр. 332, 334, 340] defined $L_p^\sigma(\mathbb{R}^N)$ via a sophisticated extension of the classical operator of fractional integration, $\sigma > 0$:

$$f(x) \mapsto f_\sigma(x) := \frac{1}{\Gamma(\sigma)} \int_{-\infty}^{x} (x-t)^{\sigma-1} f(t) dt.$$

This forgotten approach seems to be the reason why the Russian school [NIK, Chap. 9] prefers the name **Liouville space**; see Liouville [1832].

6.7.2.7 For natural exponents m and $1 < p < \infty$, the Bessel potential spaces $L_p^m(\mathbb{R}^N)$ coincide with the classical Sobolev spaces $W_p^m(\mathbb{R}^N)$; see Calderón [1961, p. 36].

6.7.3 Besov spaces

6.7.3.1 In the definition of $\mathrm{Lip}^\sigma(\mathbb{T})$ the sup-norm can be replaced by the L_p-norm. In this way, Hardy/Littlewood [1928a, p. 612] obtained a useful generalization: $\mathrm{Lip}_p^\sigma(\mathbb{T})$ for $0<\sigma<1$ and $1\le p<\infty$. At the end of their paper, they stated without proof that the periodic Jackson–Bernstein theorem 6.7.1.3 remains true in this modified setting. This assertion was verified by Quade [1937].

6.7.3.2 Functions on the real line were treated in a book of Akhieser that appeared in 1947; see [AKH, German transl.: pp. 162–165]. Since Akhieser did not give any reference, it seems reasonable to assume that the following contributions are due to him.

The **Lipschitz space** $\mathrm{Lip}_p^\sigma(\mathbb{R})$ with $0<\sigma<1$ and $1\le p<\infty$ consists of all functions $f\in L_p(\mathbb{R})$ such that

$$\left\{\int_{\mathbb{R}}|f(x+h)-f(x)|^p dx\right\}^{1/p}\le c\,h^\sigma \quad \text{for all } h>0 \text{ and some } c\ge 0.$$

In the case $\sigma=1$, according to Zygmund's advice, the estimate

$$\left\{\int_{\mathbb{R}}|f(x+h)-2f(x)+f(x-h)|^p dx\right\}^{1/p}\le c\,h$$

is required. Using the same ideas as in 6.7.1.4, we can define Lipschitz spaces $\mathrm{Lip}_p^\sigma(\mathbb{R})$ for any $\sigma>0$.

6.7.3.3 Akhieser's investigations were inspired by Bernstein's work on best approximation. In 1946, the latter published a series of Doklady Notes in which he considered непрерывные функции на всей вещественной оси. The underlying space, equipped with the sup-norm, was supposed to consist of all bounded and uniformly continuous functions on $\mathbb{R}$. Instead, by *trigonometric sums*

$$\sum_{k=-n}^{n} c_k e^{ikx},$$

Bernstein [1946] approximated the given functions by *trigonometric integrals*

$$T(z)=\int_{-\tau}^{+\tau} c(y)e^{iyz}dy \quad \text{for } z\in\mathbb{C}. \tag{6.7.3.3.a}$$

The famous Paley–Wiener–Schwartz theorem, [PAL$^+$, pp. 12–13] and [SCHW$_1$, Vol. II, p. 128], characterizes entire functions obtained via (6.7.3.3.a):

For every $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|T(z)|\le C_\varepsilon e^{(\tau+\varepsilon)|z|} \quad \text{whenever } z\in\mathbb{C}. \tag{6.7.3.3.b}$$

Such functions are said to be of **exponential type**.

6.7.3.4 For every function $f \in L_p(\mathbb{R})$, let

$$E_n(f|L_p) := \inf \|f - T|L_p\|,$$

where the infimum ranges over all entire functions T satisfying (6.7.3.3.b) with $\tau = n$. Additionally, the restriction of T to $\mathbb{R}$ is assumed to be p-integrable.

Here is Akhieser's version of the Jackson–Bernstein theorem, [AKH, German transl.: p. 211]:

$$f \in \mathrm{Lip}_p^\sigma(\mathbb{R}) \quad \text{if and only if} \quad E_n(f|L_p) = O\big(\tfrac{1}{n^\sigma}\big).$$

6.7.3.5 In a next step, Nikolskiĭ [1951, p. 14] extended the preceding theory to functions of N variables. These results and much more were presented in a remarkable monograph, [NIK]. As a tribute to him, $\mathrm{Lip}_p^\sigma(\mathbb{R}^N)$ is often called a **Nikolskiĭ space**. For $m > \sigma$, the space $\mathrm{Lip}_p^\sigma(\mathbb{R}^N)$ may be equipped with the equivalent norms

$$\|f|\mathrm{Lip}_p^{\sigma,m}\| := \|f|L_p\| + \sup_{h \neq 0} \left\{ \int_{\mathbb{R}^N} \left| \frac{\Delta_h^m f(x)}{|h|^\sigma} \right|^p dx \right\}^{1/p}.$$

6.7.3.6 We have just seen that approximation theory gave a decisive impetus for creating new function spaces. However, there is a second and even more important source. Indeed, the following theorem of Slobodetskiĭ [1958, p. 264] shows that Sobolev spaces with *fractional* orders of differentiation necessarily occur in relation to partial differential equations.

The Dirichlet problem

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega \quad \text{and} \quad u(x) = \varphi(x) \quad \text{for } x \in \partial\Omega$$

has a solution $u \in W_2^1(\Omega)$ if and only if $\varphi \in W_2^{1/2}(\partial\Omega)$.

More precise information can be found in Subsection 6.7.11.

6.7.3.7 **Slobodetskiĭ spaces** on a domain Ω in $\mathbb{R}^N$ are defined only for non-integral exponents $\sigma > 0$. To this end, we write $\sigma = [\sigma] + \sigma_0$ with $[\sigma] \in \mathbb{N}_0$ (integral part) and $0 < \sigma_0 < 1$. Let $W_p^\sigma(\Omega)$ denote the collection of all functions $f \in L_p(\Omega)$ having derivatives up to the order $[\sigma]$ such that

$$\|f|W_p^\sigma\| := \|f|L_p\| + \left\{ \iint\limits_{\Omega\times\Omega} \sum_{|\alpha|=[\sigma]} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{N+\sigma_0 p}} dx dy \right\}^{1/p}$$

is finite. Similar expressions had appeared earlier in a paper of Gagliardo [1957, p. 289].

In the case $\Omega = \mathbb{R}^N$, the preceding double integral can be transformed as follows:

$$\|f|W_p^\sigma\| := \|f|L_p\| + \left\{ \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \sum_{|\alpha|=[\sigma]} \left| \frac{\Delta_h D^\alpha f(x)}{|h|^{\sigma_0}} \right|^p dx \frac{dh}{|h|^N} \right\}^{1/p}.$$

Choosing $m > \sigma$, we obtain further equivalent norms:

$$\|f|W_p^\sigma\|_{\Delta_h^m} := \|f|L_p\| + \left\{ \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \left| \frac{\Delta_h^m f(x)}{|h|^\sigma} \right|^p dx \frac{dh}{|h|^N} \right\}^{1/p}.$$

The symbol W_p^σ suggests that the Slobodetskiĭ spaces together with the Sobolev spaces W_p^m for $m = 1, 2, \dots$ form a nice scale. Unfortunately, this is the case only for $p = 2$. Otherwise we have a similar situation as described in 6.7.1.7 for $\mathrm{Lip}^\sigma(\mathbb{T})$.

6.7.3.8 The most decisive discovery was made by Besov [1961], who observed that the generalized Lipschitz spaces $\mathrm{Lip}_p^\sigma(\mathbb{R}^N)$ as well as the Slobodetskiĭ spaces $W_p^\sigma(\mathbb{R}^N)$ are contained in a scale of spaces that depend on a third parameter $1 \le q \le \infty$. The members of $B_{p,q}^\sigma(\mathbb{R}^N)$ may be specified, for example, by the property that

$$\|f|B_{p,q}^\sigma\|_{\Delta_h^m} := \|f|L_p\| + \left\{ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \frac{\Delta_h^m f(x)}{|h|^\sigma} \right|^p dx \right)^{q/p} \frac{dh}{|h|^N} \right\}^{1/q}$$

is finite, $m > \sigma$. Obviously, $\mathrm{Lip}_p^\sigma(\mathbb{R}^N) = B_{p,\infty}^\sigma(\mathbb{R}^N)$ and $W_p^\sigma(\mathbb{R}^N) = B_{p,p}^\sigma(\mathbb{R}^N)$. The latter formula holds only for non-integral σ.

The definition above differs from Besov's original one. Roughly speaking, he had used $\Delta_h D^\alpha f$ and $\Delta_h^2 D^\alpha f$ instead of $\Delta_h^m f$; see [1961, pp. 90–91].

6.7.3.9 The historical starting point of the following result was the fact that members of the Hardy space $H_p(\mathbb{T})$ can be viewed as "boundary values" of functions on the open unit disk $\mathbb{D}$; see 6.7.12.3. In his thesis [1964/66, Part I, pp. 421–426], Taibleson presented an analogous approach to Besov spaces.

The **Poisson integral** associates with every $f \in L_p(\mathbb{R}^N)$ the harmonic function

$$u(x,t) := \frac{\Gamma\big((N+1)/2\big)}{\pi^{(N+1)/2}} \int_{\mathbb{R}^N} \frac{t}{\big(t^2 + |x-y|^2\big)^{(N+1)/2}} f(y)dy$$

for $x \in \mathbb{R}^N$ and $t > 0$. Roughly speaking, we have $u(x,0) = f(x)$.

Fix any natural number $m > \sigma$. Then f belongs to the **Besov space** $B_{p,q}^\sigma(\mathbb{R}^N)$ if and only if

$$\|f|B_{p,q}^\sigma\|_m^{\mathrm{harm}} := \|f|L_p\| + \left\{ \int_0^\infty \left(\int_{\mathbb{R}^N} \left| t^{m-\sigma} \frac{\partial^m}{\partial t^m} u(x,t) \right|^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q}$$

is finite.

A similar criterion holds when $u(x,t)$ is replaced by a temperature $v(x,t)$, that is, a solution of the heat equation $\Delta v(x,t) = \frac{\partial}{\partial t} v(x,t)$ that satisfies the initial condition $v(x,0) = f(x)$. This approach is due to Taibleson [1964/66, Part I, pp. 431–435] and Flett [1971].

6.7.3.10 The quantity $E_n(f|L_p)$, defined in 6.7.3.4, also makes sense for functions of N variables, and we have the following Jackson–Bernstein theorem, which is due to Besov [1961, pp. 90–91]:

$$f \in B^\sigma_{p,q}(\mathbb{R}^N) \quad \text{if and only if} \quad \sum_{k=0}^{\infty} E_{2^k}\big(2^{k\sigma} f\big|L_p)^q < \infty.$$

The right-hand condition means that $\big(E_n(f|L_p)\big)$ is a member of the Lorentz space $l_{1/\sigma,q}$. In the limiting case $q=\infty$, one gets Akhieser's result 6.7.3.4.

6.7.3.11 A function $f \in L_p(\mathbb{R}^N)$ belongs to $B^\sigma_{p,q}(\mathbb{R}^N)$ if and only if it can be written in the form

$$f = \sum_{k=0}^{\infty} a_k \tag{6.7.3.11.a}$$

such that the support of the Fourier transform of $a_k \in L_p(\mathbb{R}^N)$ is contained in the ball $\{y \in \mathbb{R}^N : |y| \le 2^k\}$ and $\big(\|2^{k\sigma} a_k|L_p\|\big) \in l_q$. An equivalent norm on $B^\sigma_{p,q}(\mathbb{R}^N)$ is obtained by letting

$$\|f|B^\sigma_{p,q}\|_\Sigma := \inf\left\{ \sum_{k=0}^{\infty} \big\|2^{k\sigma} a_k\big|L_p\big\|^q \right\}^{1/q},$$

where the infimum ranges over all representations described above.

An analogous criterion holds in the periodic case. If $N=1$, then a_k is a trigonometric polynomial of degree less than or equal to 2^k.

For $B^\sigma_{p,\infty}(\mathbb{R}^N)$, the preceding characterization was implicitly used by Nikolskiĭ [1951, pp. 28–29]; see also [NIK, Section 5.5]. Later on, representations of the form (6.7.3.11.a) occurred in the abstract setting of approximation spaces; see [PEE_1, pp. 45–53], [BUT_2^+, p. 54], and Pietsch [1981a, p. 120]. A particular case is treated in 6.3.3.6.

6.7.3.12 Finally, I describe the Fourier analytic approach to Besov spaces. This technique yields representations $f = \sum_{k=0}^\infty a_k$ in which the summands a_k depend on f *linearly*. More precisely, it can be arranged that $a_k = \varphi_k * f$; see (6.7.3.13.a).

According to Besov/Kudryavtsev/Lizorkin/Nikolskiĭ [1988],

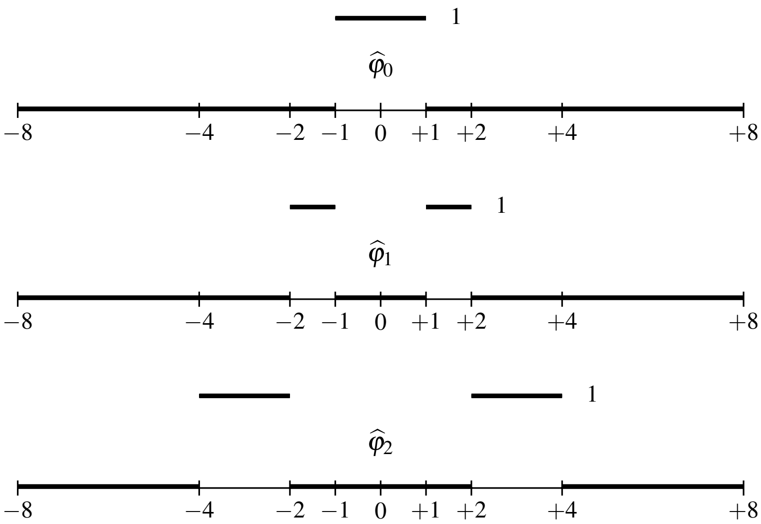
the decomposition method was conceived and essentially developed within the walls of the Steklov Institute.

Indeed, the first result along these lines is due to Lizorkin [1965, стр. 1319], who stated that for $1 < p < \infty$, $1 \le q < \infty$, a distribution $f \in S(\mathbb{R})^*$ belongs to $B^\sigma_{p,q}(\mathbb{R})$ if and only if

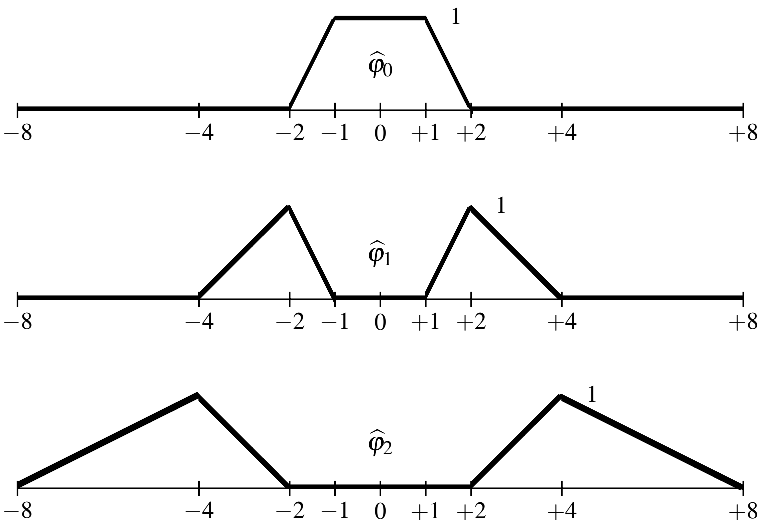
$$\|f|B^\sigma_{p,q}\|_\Phi := \left\{ \sum_{k=0}^{\infty} \big\|2^{k\sigma} \varphi_k * f\big|L_p\big\|^q \right\}^{1/q}$$

is finite; see also [NIK, стр. 374]. Here Φ denotes a sequence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ that are defined via their Fourier transforms.

In the 1-dimensional case, the simplest choice is dyadic blocks; see also (6.7.4.5.a):



Lizorkin's characterizations fails for $p = 1$. In order to overcome this defect, Nikolskiĭ [1966, стр. 514] used a modified decomposition related to the well-known **de la Vallée Poussin sums**; see also [NIK, стр. 369]. The crucial point consists in improving the quality of the φ_k's, which are now chosen to be piecewise linear:



Not aware of the results obtained in the Soviet Union, Peetre [1967, p. 281] also defined Besov spaces via the norm $\|f|B^{\sigma}_{p,q}\|_{\Phi}$. His main idea was to use infinitely differentiable functions $\varphi_0, \varphi_1, \varphi_2, \dots$. This modification has the advantage that it works also for negative σ's as well as for exponents p and q strictly less than 1; see Peetre [1973] and [PEE_3, Chapter 11].

According to Wallin [$\mathrm{CWI}^{\cup}$, p. ix], Peetre's book
New Thoughts on Besov Spaces is written in such a good way that, when you read it, you can even forgive Besov spaces for having three indices.

6.7.3.13 Of course, Peetre's approach should yield the same space (with equivalent norms) regardless of the choice of $\Phi = (\varphi_k)$. This is, in particular, the case for systems obtained as follows.

Fix any function $\varphi \in S(\mathbb{R})$ such that $\widehat{\varphi}(y) = 1$ if $|y| \le 1$ and $\widehat{\varphi}(y) = 0$ if $|y| > 2$. Let $\widehat{\varphi}_0 := \widehat{\varphi}$ and

$$\widehat{\varphi}_k(y) := \widehat{\varphi}(y/2^k) - \widehat{\varphi}(y/2^{k-1}) \quad \text{for } k = 1, 2, \dots .$$

Then

$$\sum_{k=0}^{\infty} \widehat{\varphi}_k(y) = 1 \quad \text{for all } y \in \mathbb{R}.$$

Because of this equation, $(\widehat{\varphi}_k)$ is referred to as a **partition** or **resolution of unity**. As a consequence, one has

$$f = \sum_{k=0}^{\infty} \varphi_k * f \quad \text{for all } f \in S(\mathbb{R}^N)^*. \tag{6.7.3.13.a}$$

Our notation is adopted from Peetre [PEE_3, p. 48] and [TRI_1, p. 172]. In his other books, Triebel writes φ_k instead of $\widehat{\varphi}_k$, and as a consequence, the convolution $\varphi_k * f$ is replaced by $\varphi_k(D)f := F_{\mathrm{our}}^{-1}[\varphi_k F_{\mathrm{our}} f]$; see [$\mathrm{TRI}_4$, p. 45] and [$\mathrm{TRI}_5$, pp. 92–93]. These changes are a matter of taste.

6.7.4 Lizorkin–Triebel spaces

6.7.4.1 Let $0 < p < q < \infty$ and suppose that the scalar-valued function f is measurable on the product of the measure spaces $(M, \mathscr{M}, \mu)$ and $(N, \mathscr{N}, \nu)$. Then, substituting the $L_{q/p}$-valued function $\boldsymbol{f} : s \mapsto |f(s,t)|^p$ into

$$\left\| \int_M \boldsymbol{f}(s) d\mu(s) \right\| \le \int_M \|\boldsymbol{f}(s)\| d\mu(s)$$

yields **Minkowski's inequality** for integrals, which is due to Jessen [1933, p. 9]:

$$\left(\int_N \left[\int_M |f(s,t)|^p d\mu(s) \right]^{q/p} d\nu(t) \right)^{1/q} \le \left(\int_M \left[\int_N |f(s,t)|^q d\nu(t) \right]^{p/q} d\mu(s) \right)^{1/p}.$$

6.7.4.2 Several norms of $B^\sigma_{p,q}(\mathbb{R}^N)$ are defined by expressions that contain two integrals. For example,

$$\|f|B^\sigma_{p,q}\|_{\Delta^m_h} := \|f|L_p\| + \left\{ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \frac{\Delta^m_h f(x)}{|h|^\sigma} \right|^p dx \right)^{q/p} \frac{dh}{|h|^N} \right\}^{1/q},$$

$$\|f|B^\sigma_{p,q}\|^{\mathrm{harm}}_m := \|f|L_p\| + \left\{ \int_0^\infty \left(\int_{\mathbb{R}^N} \left| t^{m-\sigma} \frac{\partial^m}{\partial t^m} u(x,t) \right|^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q},$$

$$\|f|B^\sigma_{p,q}\|_\Phi := \left\{ \sum_{k=0}^\infty \left(\int_{\mathbb{R}^N} |2^{k\sigma} \varphi_k * f(x)|^p dx \right)^{q/p} \right\}^{1/q}.$$

Independently of each other, Lizorkin [1972, p. 255] and Triebel [1973a, p. 29] observed that another 3-parameter scale of function spaces $F^\sigma_{p,q}(\mathbb{R}^N)$ can be obtained by interchanging the order of the integrals above:

$$\|f|F^\sigma_{p,q}\|_{\Delta^m_h} := \|f|L_p\| + \left\{ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \frac{\Delta^m_h f(x)}{|h|^\sigma} \right|^q \frac{dh}{|h|^N} \right)^{p/q} dx \right\}^{1/p},$$

$$\|f|F^\sigma_{p,q}\|^{\mathrm{harm}}_m := \|f|L_p\| + \left\{ \int_{\mathbb{R}^N} \left(\int_0^\infty \left| t^{m-\sigma} \frac{\partial^m}{\partial t^m} u(x,t) \right|^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p},$$

$$\|f|F^\sigma_{p,q}\|_\Phi := \left\{ \int_{\mathbb{R}^N} \left(\sum_{k=0}^\infty |2^{k\sigma} \varphi_k * f(x)|^q \right)^{p/q} dx \right\}^{1/p}.$$

Triebel [1980, pp. 252–253], [1982, p. 286] showed that these norms are equivalent, but only subject to the condition $\sigma > N/\min(p,q)$. Hence, in order to define $F^\sigma_{p,q}(\mathbb{R}^N)$ for any $\sigma > 0$, one had to select the "best" candidate: $\|f|F^\sigma_{p,q}\|_\Phi$. By the way, this choice works for all real exponents σ.

Nowadays, following Peetre [1975, p. 124] and assuming the commutative law, one refers to $F^\sigma_{p,q}$ as a **Lizorkin–Triebel** or **Triebel–Lizorkin space**; see also Besov/Kalyabin [2003, pp. 6–7]. Recent developments have shown that Peetre [PEE$_3$, p. 261] was wrong when he presented the spaces $F^\sigma_{p,q}$ in a chapter entitled *"Some strange new spaces."*

6.7.4.3 We know from [TRI$_4$, p. 108] that replacing the second term in $\|f|F^\sigma_{p,q}\|_{\Delta^m_h}$ by

$$\left\{ \int_{\mathbb{R}^N} \left(\int_0^\infty \left[\int_{t<|h|<2t} \frac{|\Delta^m_h f(x)|}{|h|^\sigma} \frac{dh}{|h|^N} \right]^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p}$$

or

$$\left\{ \int_{\mathbb{R}^N} \left(\int_0^\infty \left[\int_{t<|h|<2t} |\Delta^m_h f(x)| dh \right]^q \frac{dt}{t^{(\sigma+N)q+1}} \right)^{p/q} dx \right\}^{1/p}$$

yields norms that are equivalent to $\|f|F^\sigma_{p,q}\|_\Phi$ for every $\sigma > 0$. These are almost the original definitions of Lizorkin [1972, pp. 255, 276].

6.7.4.4 In view of 6.7.4.1, we have

$$B^{\sigma}_{p,p}(\mathbb{R}^N) = F^{\sigma}_{p,p}(\mathbb{R}^N) \subseteq F^{\sigma}_{p,q}(\mathbb{R}^N) \subseteq B^{\sigma}_{p,q}(\mathbb{R}^N) \quad \text{if } p \le q$$

and

$$B^{\sigma}_{p,q}(\mathbb{R}^N) \subseteq F^{\sigma}_{p,q}(\mathbb{R}^N) \subseteq F^{\sigma}_{p,p}(\mathbb{R}^N) = B^{\sigma}_{p,p}(\mathbb{R}^N) \quad \text{if } q \le p,$$

with strict inclusions for $p \neq q$. Hence the spaces $B^{\sigma}_{p,q}(\mathbb{R}^N)$ and $F^{\sigma}_{p,q}(\mathbb{R}^N)$ are different from, but very close to, each other.

6.7.4.5 Using the dyadic Fourier decomposition

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nx} = c_0 + \sum_{k=1}^{\infty} \Bigg[\sum_{2^{k-1} \le |n| < 2^k} c_n \mathrm{e}^{\mathrm{i}nx} \Bigg], \tag{6.7.4.5.a}$$

Littlewood/Paley [1931/37, Part I, p. 231] proved that

$$\|f|L_p\|_{LP} := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left(|c_0|^2 + \sum_{k=1}^{\infty} \Bigg| \sum_{2^{k-1} \le |n| < 2^k} c_n \mathrm{e}^{\mathrm{i}nx} \Bigg|^2 \right)^{p/2} dx \right\}^{1/p}$$

defines an equivalent norm on $L_p(\mathbb{T})$ whenever $1 < p < \infty$. This is the famous **Littlewood–Paley theorem**, which was the starting point of the Fourier theoretic approach to the theory of function spaces.

The first result along these lines is due to Hirschman [1953, pp. 536–541], who implicitly showed that

$$F^{\sigma}_{p,2}(\mathbb{T}) = L^{\sigma}_{p}(\mathbb{T}) \quad \text{for } 0 < \sigma < 1 \text{ and } 1 < p < \infty.$$

In modern terminology, the following counterpart was established by Triebel [1973a, p. 33]:

$$F^{\sigma}_{p,2}(\mathbb{R}^N) = L^{\sigma}_{p}(\mathbb{R}^N) \quad \text{for } -\infty < \sigma < +\infty \text{ and } 1 < p < \infty,$$

the norms being equivalent. Because of the preceding formulas, the Russian school uses the symbol $L^{\sigma}_{p,q}$ instead of $F^{\sigma}_{p,q}$.

6.7.4.6 Subject to the restriction $0 < \sigma < 1$, Hirschman [1953, p. 545] also proved (modulo misprints) that

$$B^{\sigma}_{p,p}(\mathbb{T}) \subseteq L^{\sigma}_{p}(\mathbb{T}) \subseteq B^{\sigma}_{p,2}(\mathbb{T}) \quad \text{if } 1 < p \le 2$$

and

$$B^{\sigma}_{p,2}(\mathbb{T}) \subseteq L^{\sigma}_{p}(\mathbb{T}) \subseteq B^{\sigma}_{p,p}(\mathbb{T}) \quad \text{if } 2 \le p < \infty.$$

This result was extended to $\sigma > 0$ and function spaces over $\mathbb{R}^N$ by Lizorkin [1964, стр. 534] as well as by Taibleson [1964/66, Part I, p. 452].

6.7.5 Interpolation of function spaces

The following interpolation formulas hold for function spaces over $\mathbb{R}^N$ as well as over any domain with a sufficiently smooth boundary; see 6.7.6.6. All equations should be understood in the sense that the corresponding norms are equivalent.

Let $0 < \theta < 1$ and $-\infty < \sigma, \sigma_0, \sigma_1 < +\infty$. To be on the safe side, I assume that $1 < p, p_0, p_1, q, q_0, q_1, u, u_0, u_1 < \infty$. In the case of the real method, most of the results also remain true for $1 \le p, p_0, p_1, q, q_0, q_1, u, u_0, u_1 \le \infty$ or even for $0 < p, p_0, p_1, q, q_0, q_1, u, u_0, u_1 \le \infty$.

Exponents σ, σ_0, and σ_1 appearing simultaneously are supposed to satisfy the relations $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$. The same convention is applied to the parameters $1/p$, $1/q$, and $1/u$.

6.7.5.1 As shown by Calderón [1964, pp. 126–128], Schechter [1967, p. 128], and others, the 2-parameter scale of the Bessel potential spaces $\{L_p^\sigma\}$ is stable under complex interpolation:

$$\big[L_{p_0}^{\sigma_0}, L_{p_1}^{\sigma_1}\big]_\theta = L_p^\sigma.$$

Because of $\big[L_p, W_p^m\big]_\theta = L_p^{\theta m}$, this scale can be generated by the classical spaces L_p and the Sobolev spaces W_p^m.

Since $L_p^\sigma = F_{p,2}^\sigma$, the 2-parameter scale $\{L_p^\sigma\}$ is contained in the 3-parameter scale $\{F_{p,q}^\sigma\}$. The latter is stable as well:

$$\big[F_{p_0,q_0}^{\sigma_0}, F_{p_1,q_1}^{\sigma_1}\big]_\theta = F_{p,q}^\sigma;$$

see Triebel [1973a, p. 62], [TRI_4, p. 69].

Taibleson [1964/66, Part II, p. 837] and Grisvard [1966, Part I, p. 177] proved that the 3-parameter scale of Besov spaces has the same stability property:

$$\big[B_{p_0,q_0}^{\sigma_0}, B_{p_1,q_1}^{\sigma_1}\big]_\theta = B_{p,q}^\sigma.$$

6.7.5.2 The basic formula concerned with the real method goes back to Lions/Peetre [1964, p. 60] and Peetre [1966, p. 285]:

$$\big(L_p, W_p^m\big)_{\theta,q} = B_{p,q}^{\theta m}.$$

Reiteration yields

$$\big(B_{p,u_0}^{\sigma_0}, B_{p,u_1}^{\sigma_1}\big)_{\theta,q} = B_{p,q}^\sigma \quad \text{whenever } \sigma_0 \neq \sigma_1.$$

I stress that the parameter p is fixed. Without this assumption, Grisvard [1966, Part I, p. 177] showed that

$$\big(B_{p_0,p_0}^{\sigma_0}, B_{p_1,p_1}^{\sigma_1}\big)_{\theta,p} = B_{p,p}^\sigma.$$

But this is again a particular case.

6.7.5.3 Peetre [1967, p. 281] discovered the fact that the 3-parameter scale $\{B^{\sigma}_{p,q}\}$ behaves badly with respect to real interpolation; see 6.6.4.5. Indeed, $(B^{\sigma_0}_{p_0,u_0}, B^{\sigma_1}_{p_1,u_1})_{\theta,q}$ with $p_0 \neq p_1$ need not be a Besov space. The same holds for $(B^{\sigma}_{p,u_0}, B^{\sigma}_{p,u_1})_{\theta,q}$ whenever $u \neq q$. On the other hand, we know from [PEE$_3$, p. 107] that

$$\big(B^{\sigma}_{p,q_0}, B^{\sigma}_{p,q_1}\big)_{\theta,q} = B^{\sigma}_{p,q}.$$

6.7.5.4 In general, real interpolation between Lizorkin–Triebel spaces yields Besov spaces. This is a consequence of the reiteration theorem as well as of the inclusions stated in 6.7.4.4.

The exception $\big(F^{\sigma}_{p_0,q}, F^{\sigma}_{p_1,q}\big)_{\theta,p} = F^{\sigma}_{p,q}$ was proved by Triebel [1973a, p. 52].

6.7.6 Spaces of smooth functions: supplements

The standard references about Sobolev and Besov spaces are, in chronological order, [SOB], [NIK], [STEIN], [ADAMS], [PEE$_3$], [TRI$_1$], [MAZ], as well as further books of Triebel; see also the surveys of Besov/Kalyabin [2003] and Triebel [1991].

6.7.6.1 According to Browder [1975$^\bullet$, p. 579]:

The conceptual change that brought about the transformation to the functional analytic viewpoint in the theory of ordinary and partial differential equations, the calculus of variations, Fourier analysis, and many other areas was the realization that their substantial concern was not the single function and its internal structure but the whole family of related functions.

Indeed, spaces of smooth functions play a central role in real and complex analysis. However, in contrast to the quotation above, one of the main goals of these theories is the problem to characterize the members of a given space in various ways. Since each characterization gives rise to an equivalent norm, such undertakings may be considered as the search for clever renormings.

6.7.6.2 Triebel's books are based on the philosophy that every "good" function space is contained in (at least) one of the scales $\{B^{\sigma}_{p,q}\}$ and $\{F^{\sigma}_{p,q}\}$. Thus the undesirable members of the society must be excluded by a suitable criterion; [TRI$_4$, pp. 38–40]. The following quotation is from [TRI$_5$, p. 3]:

We are looking at function spaces from the point of view of possible applications to (pseudo) partial differential equations. Then $L_p(\Omega)$ with $1 < p < \infty$ are good spaces; whereas $L_1(\Omega)$, $L_\infty(\Omega)$ and $C^k(\overline{\Omega})$ with $k = 0, 1, 2, \ldots$ have well-known shortcomings; and the spaces $L_p(\Omega)$ with $0 < p < 1$ are really nasty.

The new favorites are $B^{\sigma}_{1,1}$ and $B^{\sigma}_{\infty,\infty}$ as well as H^{real}_p with $0 < p \leq 1$ and *BMO*.

Some bad properties of C^k are described in 6.7.1.7 and 6.7.11.6, while the spaces $L_p[0,1]$ with $0 < p < 1$ were disqualified a long time ago when Day [1940] showed that $L_p[0,1]^*$ is trivial. Frankly, some colleagues like the "nasty" L_p's very much.

Taken as a general rule, the above restriction would be a Procrustean bed. Of course, $C(K)$ and $L_1(M,\mathscr{M},\mu)$ are the most important non-reflexive spaces, and no analyst is willing to throw them away.

6.7.6.3 Many functions spaces possess a *homogeneous* double ganger. This phenomenon will be explained by means of the Lipschitz space $\mathrm{Lip}^\sigma(\mathbb{R}^N)$ equipped with the norm

$$\|f|\mathrm{Lip}^\sigma\| := \sup_{x\in\mathbb{R}^N}|f(x)| + \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\sigma}.$$

Clearly,

$$\|f|\mathrm{Lip}^\sigma\|^{\mathrm{hom}} := \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\sigma}$$

is the decisive term. However, this expression yields only a semi-norm. Thus we need to add $\sup_{x\in\mathbb{R}^N}|f(x)|$ in order to get a norm. However, there is another possibility: we may pass to the quotient space modulo those functions for which $\|f|\mathrm{Lip}^\sigma\|^{\mathrm{hom}}$ vanishes. These are just the constants.

The superscript "**hom**"ogeneous is justified by the formula

$$\|f(\lambda x)|\mathrm{Lip}^\sigma\|^{\mathrm{hom}} = \lambda^\sigma\|f(x)|\mathrm{Lip}^\sigma\|^{\mathrm{hom}},$$

which holds for all $f\in\mathrm{Lip}^\sigma(\mathbb{R}^N)$ and $\lambda>0$.

More generally, a semi-norm on a function space $F(\mathbb{R}^N)$ may be called **σ-homogeneous** if

$$\|f(\lambda x)|F\| = \lambda^\sigma\|f(x)|F\| \quad \text{for all } f\in F(\mathbb{R}^N) \text{ and } \lambda>0, \tag{6.7.6.3.a}$$

where $\sigma\in\mathbb{R}$ is a fixed exponent; see [TRI$_5$, p. 26]. In the 1-dimensional setting, it follows that $\|x^n|F\|=0$ whenever $x^n\in F(\mathbb{R})$ and $n\neq\sigma$.

To the best of my knowledge, there is no general procedure that associates with every function space its homogeneous counterpart (if it exists at all). Each case must be treated by individual inspiration. For example, on the Besov space $B^\sigma_{p,q}(\mathbb{R}^N)$ a $(\sigma-N/p)$-homogeneous semi-norm is given by

$$\|f|B^\sigma_{p,q}\|^{\mathrm{hom}}_{\Delta^m_h} := \left\{\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left|\frac{\Delta^m_h f(x)}{|h|^\sigma}\right|^p dx\right)^{q/p}\frac{dh}{|h|^N}\right\}^{1/q},$$

where $m>\sigma$. Another $(\sigma-N/p)$-homogeneous semi-norm

$$\|f|B^\sigma_{p,q}\|^{\mathrm{hom}}_{\Phi} := \left\{\sum_{k\in\mathbb{Z}}\|2^{k\sigma}\varphi_k * f|L_p\|^q\right\}^{1/q}$$

can be defined via the decomposition method. To this end, the construction presented in 6.7.3.13 must be modified as follows. Given φ as before, one defines $\Phi=(\varphi_k)$ by $\widehat{\varphi}_k(y) := \widehat{\varphi}(y/2^k) - \widehat{\varphi}(y/2^{k-1})$. But now the index k ranges over all integers. Note, however, that the homogeneity relation (6.7.6.3.a) holds only for factors $\lambda = 2^k$ with $k \in \mathbb{Z}$.

The homogeneous Besov space, mostly denoted by $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^N)$, consists of equivalence classes of functions for which the above semi-norms are finite. The null space of $\|f|B^{\sigma}_{p,q}\|^{\text{hom}}_{\Delta^m_h}$ is formed by all polynomials of degree less than m, while $\|f|B^{\sigma}_{p,q}\|^{\text{hom}}_{\Phi}$ vanishes for polynomials of arbitrary degree. Nevertheless, the quotient spaces are isomorphic.

Homogeneous function spaces were systematically studied for the first time by Peetre [1966]; see also [PEE$_3$, pp. 50–57]. However, $\dot{W}^1_2(\Omega)$ was already considered by Nikodym [1933] in connection with the Dirichlet principle; see 6.7.2.3.

6.7.6.4 The map J_σ defined in 6.7.2.5 yields an isomorphism between $B^{\tau}_{p,q}(\mathbb{R}^N)$ and $B^{\sigma+\tau}_{p,q}(\mathbb{R}^N)$. For $-\infty < \sigma, \tau < +\infty$, $1 < p < \infty$, and $1 \le q < \infty$, this result was proved by Taibleson [1964/66, Part I, p. 429] and Lizorkin [1965, стр. 1319]. The remaining cases were settled in [TRI$_3$, p. 68] and [TRI$_4$, pp. 58–59].

The same assertion holds for $F^{\sigma}_{p,q}(\mathbb{R}^N)$; see Triebel [1973a, p. 35] and [TRI$_4$, pp. 58–59]. This isomorphism is obvious in the case of $L^{\sigma}_p(\mathbb{R}^N) = F^{\sigma}_{p,2}(\mathbb{R}^N)$, since it follows immediately from the very definition of the Bessel potential spaces.

6.7.6.5 Besov spaces $B^{\sigma}_{p,q}(\mathbb{R}^N)$ with a negative exponent of smoothness, $\sigma < 0$, were invented by Taibleson [1964/66, Part I, pp. 436–437]. Then $B^{\sigma}_{p,q}(\mathbb{R}^N)$ not only contains functions, but also tempered distributions. For example, $\delta \in B^{1/p-1}_{p,\infty}(\mathbb{R})$; see [PEE$_3$, pp. 50, 241].

Real σ's are needed in order to state the following duality relations:

$$B^{\sigma}_{p,q}(\mathbb{R}^N)^* = B^{-\sigma}_{p^*,q^*}(\mathbb{R}^N) \quad \text{and} \quad F^{\sigma}_{p,q}(\mathbb{R}^N)^* = F^{-\sigma}_{p^*,q^*}(\mathbb{R}^N),$$

the norms being equivalent.

For $1 < p, q < \infty$, this result was proved by Taibleson [1964/66, Part I, p. 833]. Subject to suitable modifications, it remains true in some limiting cases.

6.7.6.6 Most spaces treated so far are built from functions on $\mathbb{R}^N$. However, quite often, the underlying definitions make sense for functions on a domain Ω in $\mathbb{R}^N$.

There is a general procedure that assigns to every function space $F(\mathbb{R}^N)$ a function space $F(\Omega)$. The latter consists, by definition, of all restrictions to Ω of functions $f_0 \in F(\mathbb{R}^N)$. The associated norm is obtained by letting

$$\|f|F(\Omega)\| := \inf\Big\{ \|f_0|F(\mathbb{R}^N)\| \,:\, f_0 \in F(\mathbb{R}^N), \;\; \text{rest}_\Omega f_0 = f \Big\}.$$

Since $F(\mathbb{R}^N)$ may contain distributions, the restriction $\text{rest}_\Omega f_0$ must be interpreted in this general sense.

If the boundary of Ω is sufficiently smooth, then for all standard spaces there exists a (bounded and linear) extension operator ext_Ω from $F(\Omega)$ into $F(\mathbb{R}^N)$. Hence $F(\Omega)$ is isomorphic to a complemented subspace of $F(\mathbb{R}^N)$.

6.7.6.7 In order to define $B^{\sigma}_{p,q}$ and $F^{\sigma}_{p,q}$ over manifolds, it is necessary to show that the spaces $B^{\sigma}_{p,q}(\mathbb{R}^N)$ and $F^{\sigma}_{p,q}(\mathbb{R}^N)$ are invariant under changes of the underlying coordinates. More precisely, if φ is any infinitely differentiable mapping from $\mathbb{R}^N$ onto itself, then the composition $f(y) \mapsto f\big(\varphi(x)\big)$ should be an isomorphism.

A final and affirmative answer was given in [TRI$_4$, p. 174]. However, Peetre [PEE$_3$, pp. 5, 65] had earlier observed that the desired invariance is trivial for $L_p(\mathbb{R}^N)$ and $W^m_p(\mathbb{R}^N)$. Hence it carries over to $B^{\theta m}_{p,q}(\mathbb{R}^N) = \big(L_p(\mathbb{R}^N), W^m_p(\mathbb{R}^N)\big)_{\theta,q}$.

6.7.6.8 Many efforts have been made to extend the results about $B^{\sigma}_{p,q}$ and $F^{\sigma}_{p,q}$ to parameters p and q strictly less than 1; see [TRI$_4$] and [TRI$_5$]. However, this theory is outside of the range of the present text, since it concerns quasi-Banach spaces.

6.7.6.9 The classical Sobolev spaces $W^m_p(\Omega)$ consist of all functions $f \in L_p(\Omega)$ such that $D^{\alpha} f \in L_p(\Omega)$ for every $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_1 + \dots + \alpha_N \le m$.

Anisotropic Sobolev spaces are obtained if the index of differentiation ranges over a prescribed **smoothness**; this is a finite set $\mathbb{S}$ of N-tuples $(\alpha_1, \dots, \alpha_N)$ with $\alpha_k = 0, 1, 2, \dots$ such that

$$(\alpha_1, \dots, \alpha_N) \in \mathbb{S} \text{ and } \beta_1 \le \alpha_1, \dots, \beta_N \le \alpha_N \quad \text{imply} \quad (\beta_1, \dots, \beta_N) \in \mathbb{S}.$$

The structure of anisotropic Sobolev spaces strongly depends on the shape of the underlying smoothness $\mathbb{S}$. More information can be found in a survey of Pełczyński/Wojciechowski [2003, pp. 1408–1419].

Several books on function spaces, for example [NIK], treat the special case

$$\mathbb{S} := \big\{(\alpha_1, \dots, \alpha_N) : \alpha_1 \le m_1, \dots, \alpha_N \le m_N\big\}.$$

6.7.6.10 **Weighted spaces** have been introduced in order to get compact embeddings for functions defined on unbounded domains, and in particular, on $\mathbb{R}^N$. There exists an abundant literature on this subject.

6.7.6.11 The newest fashion (dernier cri) are function spaces over fractals; see [TRI$_6$].

6.7.6.12 Finally, I summarize the basic tools that are used to define Banach spaces of smooth functions:

- smoothness and differentiability properties,
- best approximation by entire functions of exponential type
  or by trigonometric polynomials,
- boundary values of solutions of partial differential equations,
- interpolation methods,
- Fourier analytic decompositions,
- maximal functions, ...

The order above is not supposed to indicate any priority.

6.7.7 Bases of Besov spaces

6.7.7.1 Triebel [1973b, pp. 89–93] was the first to construct an unconditional basis of $B^{\sigma}_{p,q}(\mathbb{R}^N)$. According to 6.7.6.4, all spaces $B^{\sigma}_{p,q}(\mathbb{R}^N)$ and $B^{\tau}_{p,q}(\mathbb{R}^N)$ are isomorphic. Hence it suffices to consider the case $0<\sigma<1/p$, in which the Haar basis can be used.

Triebel did not identify the associated sequence spaces; see 5.6.1.4. Nevertheless, he showed that $B^{\sigma}_{p,q}(\mathbb{R}^N)$ is isomorphic to $[l_q, l_p]$, the direct l_q-sum of countably many copies of l_p; see also Lemarié/Meyer [1986, p. 13].

6.7.7.2 Let $-\infty<\sigma<+\infty$ and $1\le p,q\le\infty$. The **Besov sequence space** $b^{\sigma}_{p,q}$ consists of all $x=(\xi_{ik})$ with $i=1,\dots,2^k$ and $k=0,1,\dots$ such that

$$\|x|b^{\sigma}_{p,q}\| := \left\{ \sum_{k=0}^{\infty} \left[2^{k\sigma} \Big(\sum_{i=1}^{2^k} |\xi_{ik}|^p \Big)^{1/p} \right]^q \right\}^{1/q}$$

is finite. In other words, we define $b^{\sigma}_{p,q} := [l_q, 2^{k\sigma} l_p^{2^k}]$ as the direct l_q-sum of spaces $l_p^{2^k}$ whose norms are multiplied by $2^{k\sigma}$.

6.7.7.3 Lizorkin [1976, pp. 102–103] discovered a simple construction that yields a common basis for all Besov spaces $B^{\sigma}_{p,q}(\mathbb{T})$ with $\sigma>0$ and $1<p,q<\infty$. Moreover, he was able to describe explicitly the associated sequence spaces. Unfortunately, the limiting cases $p=1$ and $p=\infty$ were excluded.

First of all, $B^{\sigma}_{p,q}(\mathbb{T})$ is represented as the weighted l_q-sum of the 2^k-dimensional spaces

$$B_0 := \mathbb{K} \quad \text{and} \quad B_k := \left\{ \sum_{2^{k-1}\le|n|<2^k} c_n \mathrm{e}^{\mathrm{i}nx} \, : \, c_n\in\mathbb{K} \right\} \quad \text{for } k=1,2,\dots$$

equipped with the $L_p(\mathbb{T})$-norm. Thus we get an isomorphism between $B^{\sigma}_{p,q}(\mathbb{T})$ and $\big[l_q, 2^{k\sigma}B_k\big]$. Secondly, a classical theorem of Marcinkiewicz/Zygmund [1937, p. 132] provides us with isomorphisms $U_k : B_k \to 2^{-k/p} l_p^{2^k}$ (not depending on p) such that

$$\|U_k\|\|U_k^{-1}\| \le c\,\frac{p^2}{p-1};$$

see also [WOJ_1, p. 72]. Hence one and the same rule defines an isomorphism

$$B^{\sigma}_{p,q}(\mathbb{T}) \quad \rightleftarrows \quad b^{\sigma-1/p}_{p,q} \tag{6.7.7.3.a}$$

for any choice of the parameters $\sigma>0$ and $1<p,q<\infty$.

6.7.7.4 The Polish school found another approach, which has a great advantage: it also works in the limiting cases. The main idea was to use splines. At the very beginning, Ciesielski [1960], [1963/66, Part II, p. 317] considered the Franklin system in the spaces $\mathrm{Lip}^\sigma[0,1]$ and $B^\sigma_{p,\infty}(0,1)$ with $0<\sigma<1$ and $1\le p<\infty$. In a next step, Ropela [1976] constructed an isomorphism

$$B^\sigma_{p,q}(0,1) \quad \rightleftarrows \quad b^{\sigma-1/p+1/2}_{p,q}. \tag{6.7.7.4.a}$$

Given $\sigma_0>0$, one and the same rule can be used for $0<\sigma<\sigma_0$ and $1\le p,q\le\infty$.

The different exponents $\sigma-1/p$ and $\sigma-1/p+1/2$ in (6.7.7.3.a) and (6.7.7.4.a), respectively, are obtained because of different normalizations.

6.7.7.5 Finally, Ciesielski/Figiel [1983, Part I, p. 2] extended the previous results to Besov spaces over compact N-dimensional manifolds. In the particular case of the N-dimensional torus, they provided an isomorphism

$$B^\sigma_{p,q}(\mathbb{T}^N) \quad \rightleftarrows \quad b^{\sigma/N-1/p+1/2}_{p,q}. \tag{6.7.7.5.a}$$

6.7.7.6 Formula (6.7.7.5.a) remains true if $\mathbb{T}^N$ is replaced by the N-dimensional cube; see Ciesielski/Figiel [1983, Part II, p. 127]. Nowadays, an isomorphism

$$B^\sigma_{p,q}\big((0,1)^N\big) \quad \rightleftarrows \quad b^{\sigma/N-1/p+1/2}_{p,q} \tag{6.7.7.6.a}$$

can also be defined with the help of wavelet bases; see Cohen/Dahmen/DeVore [2000, pp. 3669–3672] and [FRA$^+$, pp. 57, 103].

6.7.7.7 We have $F^\sigma_{p,2}(\mathbb{T})\cong L_p(\mathbb{T})$ if $1<p<\infty$ and $F^\sigma_{p,p}(\mathbb{T})=B^\sigma_{p,p}(\mathbb{T})\cong l_p$ if $1\le p\le\infty$. For all other choices of the parameters, the structure of the Lizorkin–Triebel spaces $F^\sigma_{p,q}(\mathbb{T})$ seems to be unknown.

6.7.8 Embedding operators

Let Ω denote a *bounded* domain in $\mathbb{R}^N$ that for simplicity is supposed to possess a sufficiently smooth boundary.

6.7.8.1 Embedding theorems tell us that a function spaces $F(\Omega)$ is contained in a function space $G(\Omega)$. Then we may consider the **embedding map** $Id: F(\Omega)\to G(\Omega)$. In all relevant cases, the closed graph theorem implies that this map is continuous. Quite often, one even gets compactness.

6.7.8.2 Of course, some obvious inclusions follow from Hölder's inequality:

$$C(\overline{\Omega})\subset L_\infty(\Omega)\subset L_p(\Omega)\subset L_q(\Omega)\subset L_1(\Omega) \quad \text{whenever } 1<q<p<\infty.$$

The attentive reader will observe that $C(\overline{\Omega})\subset L_\infty(\Omega)$ needs some explanation.

For $0<\tau\le\sigma<\infty$, $1\le q\le p\le\infty$ and $1\le u\le v\le\infty$, we have

$$B^\sigma_{p,u}(\Omega)\subseteq B^\tau_{q,v}(\Omega) \quad \text{and} \quad F^\sigma_{p,u}(\Omega)\subseteq F^\tau_{q,v}(\Omega).$$

6.7.8.3 Hardy/Littlewood [1928b, p. 605], [1932, pp. 435–437] proved the first non-trivial result, *by the use of complex function theory*:

$$\mathrm{Lip}^\sigma_p(\mathbb{T})\subseteq \mathrm{Lip}^\tau_q(\mathbb{T}) \quad \text{if } 0<\tau<\sigma<1,\ 1\le p<q\le\infty,\ \sigma-1/p=\tau-1/q.$$

6.7.8.4 Sobolev [1938, стр. 486] achieved the decisive breakthrough. Using his famous integral representation, he established the inclusion

$$W_p^m(\Omega) \subset L_q(\Omega) \quad \text{if } 1/q = 1/p - m/N > 0 \text{ and } 1 < p < N/m.$$

The case $p > N/m$ was earlier treated by Sobolev [1936, $p = 2$] and Kondrashov [1938]:

$$W_p^m(\Omega) \subset C(\overline{\Omega}).$$

Subsequently, Kondrashov [1945, p. 536] observed that the embedding maps above are compact. The compactness of $Id : W_2^1(\Omega) \to L_2(\Omega)$ was proved earlier by Rellich [1930, p. 30].

The classical embedding theorems are summarized in [SOB, § 8].

6.7.8.5 In a next step, Nikolskiĭ [1951, p. 27] extended the Hardy/Littlewood result to smoothness exponents that may be larger than 1:

$$B_{p,\infty}^{\sigma}(\mathbb{T}) \subseteq B_{q,\infty}^{\tau}(\mathbb{T}) \quad \text{if } 0 < \tau < \sigma < \infty,\ 1 \le p < q \le \infty,\ \sigma - 1/p = \tau - 1/q.$$

His proof was based on a dyadic decomposition as described in 6.7.3.11 and on **Nikolskiĭ's inequality** that asserts that with a universal constant $c > 1$,

$$\|T|L_q\| \le cn^{1/p-1/q}\|T|L_p\| \quad \text{for } 1 \le p < q \le \infty \tag{6.7.8.5.a}$$

and every trigonometric polynomial T of degree less than or equal to n; Nikolskiĭ [1951, p. 13], [NIK, стр. 159], and [ZYG, Vol. I, pp. 154, 378].

6.7.8.6 The final result was obtained by Besov [1961, p. 96]:
For $0 < \tau < \sigma < \infty$, $1 \le p < q \le \infty$, $1 \le u < v \le \infty$ and $\sigma - N/p = \tau - N/q$, we have

$$B_{p,u}^{\sigma}(\Omega) \subseteq B_{q,v}^{\tau}(\Omega).$$

6.7.8.7 Using a common spline or wavelet basis of $B_{p,u}^{\sigma}(\mathbb{T}^N)$ and $B_{q,v}^{\tau}(\mathbb{T}^N)$ yields the diagram

$$\begin{array}{ccc}
B_{p,u}^{\sigma}(\mathbb{T}^N) & \xrightarrow{\quad Id \quad} & B_{q,v}^{\tau}(\mathbb{T}^N) \\
U \big\downarrow \big\uparrow U^{-1} & & V \big\downarrow \big\uparrow V^{-1} \\
[l_u, 2^{k\alpha} l_p^{2^k}] & \xrightarrow[\quad Id \quad]{} & [l_v, 2^{k\beta} l_q^{2^k}]\ ,
\end{array}$$

where U and V are isomorphisms; see (6.7.7.5.a). The exponents of smoothness of the Besov sequence spaces are given by $\alpha := \sigma/N - 1/p + 1/2$ and $\beta := \tau/N - 1/q + 1/2$.

6.7.8.8 In the special case that $u = p$ and $v = q$, we get

$$\begin{array}{ccc} B^\sigma_{p,p}(\mathbb{T}^N) & \xrightarrow{\quad Id \quad} & B^\tau_{q,q}(\mathbb{T}^N) \\ U_0 \big\downarrow \big\uparrow U_0^{-1} & & V_0 \big\downarrow \big\uparrow V_0^{-1} \\ l_p & \xrightarrow[\quad D_\lambda \quad]{} & l_q \end{array} . \tag{6.7.8.8.a}$$

The diagonal operator $D_\lambda : (\xi_n) \mapsto \left(\frac{1}{n^\lambda}\xi_n\right)$ has the exponent

$$\lambda := \alpha - \beta = (\sigma/N - 1/p) - (\tau/N - 1/q).$$

Hence the above embedding exists whenever $\sigma - \tau > N(1/p - 1/q)_+$.

6.7.8.9 The **Sobolev limit order** of an operator ideal $\mathfrak{A}$ is defined by

$$\lambda^\Omega_{\text{emb}}(p,q|\mathfrak{A}) := \inf\left\{\, \sigma > 0 : Id \in \mathfrak{A}\big(W^\sigma_p(\Omega), L_q(\Omega)\big) \,\right\}.$$

Note that instead of $Id : W^\sigma_p(\Omega) \to L_q(\Omega)$, the embeddings $Id : B^\sigma_{p,u}(\Omega) \to L_q(\Omega)$ or $Id : B^{\sigma+\tau}_{p,u}(\Omega) \to B^\tau_{q,v}(\Omega)$ may be used.

6.7.8.10 There is a very close connection between the limit orders $\lambda^\Omega_{\text{emb}}(p,q|\mathfrak{A})$ and $\lambda_{\text{diag}}(p,q|\mathfrak{A})$; see 6.3.10.1. This phenomenon was first observed in [PIE$_2$, p. 135] by an inspection of all known examples. Later on, König [1974, Part I, p. 54] showed that

$$\lambda^\Omega_{\text{emb}}(p,q|\mathfrak{A})/N = \lambda_{\text{diag}}(p,q|\mathfrak{A}) + 1/p - 1/q,$$

where $N = \dim(\Omega)$. His original proof worked only for quasi-Banach ideals. However, in view of the diagram (6.7.8.8.a), the same result holds for all ideals.

Let $\lambda_{\text{emb}}(p,q|\mathfrak{A})$ denote the Sobolev limit order associated with the unit interval $(0,1)$ or $\mathbb{T}$. Then

$$\lambda^\Omega_{\text{emb}}(p,q|\mathfrak{A}) = N\lambda_{\text{emb}}(p,q|\mathfrak{A}).$$

In other words, $\lambda^\Omega_{\text{emb}}(p,q|\mathfrak{A})$ depends only on the dimension of Ω and not on its shape.

6.7.8.11 Finally, I present some results about s-numbers and entropy numbers of embedding operators. To simplify matters, the function spaces W^σ_p, $B^\sigma_{p,u}$, and L_q are defined over the unit interval, and we always suppose that $\sigma > (1/p - 1/q)_+$, which implies that $Id : W^\sigma_p \to L_q$ and $Id : B^\sigma_{p,u} \to L_q$ exist.

6.7.8.12 The relationship between s-numbers of Sobolev embedding operators and diagonal operators was discovered by Maĭorov only in [1975]. Using a discretization technique, he proved that

$$d_n(Id : l^{2n}_p \to l^{2n}_q) \le c_{pq} n^{\sigma - 1/p + 1/q} d_n\big(Id : W^\sigma_p \to L_q\big).$$

A more complicated estimate in the converse direction was obtained as well. Nowadays, these observations have become “trivial” because of

$$\begin{array}{ccc} B^{\sigma}_{p,u} & \xrightarrow{\quad Id \quad} & B^{\tau}_{q,v} \\ U\downarrow\ \uparrow U^{-1} & & V\downarrow\ \uparrow V^{-1} \\ b^{\alpha}_{p,u} & \xrightarrow[\quad Id \quad]{} & b^{\beta}_{q,v} \end{array},$$

where $\alpha := \sigma - 1/p + 1/2$ and $\beta := \tau - 1/q + 1/2$. Unfortunately, it seems to be unknown whether the asymptotic behavior of $s_n(Id : b^{\alpha}_{p,u} \to b^{\beta}_{q,v})$ may depend on the fine indices u and v.

The diagram above implies that for any choice of s-numbers $s : T \mapsto \big(s_n(T)\big)$, the **Sobolev limit order**

$$\rho_{\text{emb}}(\sigma, p, q|s) := \sup\Big\{\rho \geq 0 : s_n(Id : W^{\sigma}_p \to L_q) \preceq \tfrac{1}{n^{\rho}}\Big\}$$

coincides with the diagonal limit order $\rho_{\text{diag}}(\sigma - 1/p + 1/q, p, q|s)$ defined in 6.2.5.3.

6.7.8.13 What follows is a list of some typical results, arranged chronologically:

Kolmogorov [1936, p. 108]:

$$d_n(Id : W^s_2 \to L_2) \asymp n^{-s} \quad \text{for } s = 1, 2, \ldots,$$

Tikhomirov [1960, стр. 89]:

$$d_n(Id : W^s_{\infty} \to L_{\infty}) \asymp n^{-s} \quad \text{for } s = 1, 2, \ldots,$$

Birman/Solomyak [1967, стр. 343–344]:

$$a_n(Id : W^{\sigma}_p \to L_q) \asymp \begin{cases} n^{-\sigma} & \text{if } 1 \leq q \leq p \leq \infty, \\ n^{-(\sigma - 1/p + 1/q)} & \text{if } 1 \leq p \leq q \leq \infty, \end{cases}$$

Makovoz [1972, стр. 142]:

$$d_n(Id : W^{\sigma}_p \to L_q) \asymp n^{-\sigma} \quad \text{if } 1 \leq q \leq p \leq \infty,$$

Ismagilov [1974, стр. 175]:

$$d_n(Id : W^{\sigma}_p \to L_q) \asymp n^{-(\sigma - 1/p + 1/q)} \quad \text{if } 1 \leq p \leq q \leq 2,$$

Gluskin [1974, p. 1594], based on Kashin [1974, p. 304]:

$$d_n(Id : W^{\sigma}_1 \to C) \asymp n^{-(\sigma - 1/2)} \quad \text{if } \sigma \geq 2,$$

Scholz [1976, p. 261]:

$$a_n(Id : W^{\sigma}_p \to L_q) \asymp n^{-(\sigma - 1/p + 1/q)} \quad \text{if } 2 \leq p \leq q \leq \infty,$$

Kashin [1977, p. 318]:

$$\left.\begin{array}{ll} d_n(Id : W^{\sigma}_p \to L_q) \asymp n^{-\sigma} & \text{if } 2 \leq p \leq q \leq \infty, \\ d_n(Id : W^{\sigma}_p \to L_q) \asymp n^{-(\sigma - 1/p + 1/2)} & \text{if } 1 \leq p \leq 2 \leq q \leq \infty, \end{array}\right. \quad \sigma \geq 1,$$

Höllig [1979, pp. 280–281], Maĭorov [1980, стр. 450]:

$$d_n(Id : W_p^\sigma \to L_q) \asymp n^{-(\sigma-1/p+1/2)} \quad \text{if } 1 \le p \le 2 \le q \le \infty \text{ and } \sigma \ge 1,$$

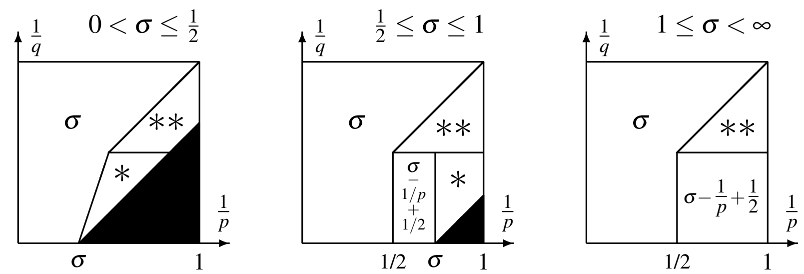
Kashin [1981, стр. 50]:

$$d_n(Id : W_1^\sigma \to L_q) \asymp n^{-\frac{q}{2}(\sigma-1+1/q)} \quad \text{if } 2 < q < \infty \text{ and } \sigma < 1,$$

Lubitz [1982, pp. 74–81], based on Gluskin [1981a, p. 164]:

$$d_n(Id : W_p^\sigma \to L_q) \asymp n^{-\frac{q}{2}(\sigma-1/p+1/q)} \quad \begin{array}{l} \text{if } 1 \le p \le 2 \le q < \infty \text{ and } \sigma < 1/p, \\ \text{if } 2 \le p \le q < \infty \text{ and } 2\sigma < \frac{1/p-1/q}{1/2-1/q}. \end{array}$$

6.7.8.14 The diagrams below provide complete information about the Sobolev limit order of the Kolmogorov numbers, $\rho_{\text{emb}}(\sigma, p, q|\boldsymbol{d}) = \rho_{\text{diag}}(\sigma - 1/p + 1/q, q^*, p^*|\boldsymbol{c})$; compare with 6.2.5.4.



Here $*$ stands for $\frac{q}{2}(\sigma - 1/p + 1/q)$ and $**$ stands for $\sigma - 1/p + 1/q$.

6.7.8.15 At first, Soviet mathematicians investigated the ε-entropy of the closed unit ball of W_p^σ viewed as a compact subset of L_q. Their results can easily be translated into the language of entropy numbers. First of all, Kolmogorov [1956] showed that

$$e_n(Id : C^s \to C) \asymp n^{-s} \quad \text{for } s = 1, 2, \dots .$$

Kolmogorov/Tikhomirov [1959, p. 31, German translation] extended this formula to arbitrary exponents $\sigma > 0$,

$$e_n(Id : \text{Lip}^\sigma \to C) \asymp n^{-\sigma}.$$

Next, Birman/Solomyak [1967, стр. 349] obtained the upper estimate

$$e_n(Id : W_p^\sigma \to L_q) \preceq n^{-\sigma}$$

for any choice of $1 \le p, q \le \infty$. Based on Mostefai's thesis (Univ. d'Alger, 1970, unpublished), the relation

$$e_n(Id : W_p^\sigma \to L_q) \asymp e_n(Id : B_{p,u}^\sigma \to L_q) \asymp n^{-\sigma}$$

was obtained by Triebel [1975, p. 27]. Finally, Carl [1981c, p. 68] proved the same result using the diagram from 6.7.8.12.

6.7.8.16 The concept of ε-entropy has played an important role in connection with Hilbert's 13th problem; see Tikhomirov [1963•, стр. 75–90] and Lorentz [1990•, pp. 65–67], [LORZ, pp. 174–177].

Hilbert conjectured that there exists a continuous function F of three variables that cannot be represented as a superposition of functions of two variables: This means that $F \neq a_0^{(1)}$ for any dyadic tree $\{a_k^{(i)}\}$ whose members are connected by the formula

$$a_k^{(i)}(x,y,z) = f_k^{(i)}\big(a_{k+1}^{(2i-1)}(x,y,z), a_{k+1}^{(2i)}(x,y,z)\big);$$

see 6.1.9.2. The starting functions $a_n^{(1)}, \dots, a_n^{(2^n)}$ are supposed to depend on at most two of the variables x,y,z.

In the 1950s, a negative answer was given by Kolmogorov and his pupil Arnold, who showed that every continuous function on the cube $0 \le x,y,z \le 1$ is the sum of 7 functions of the form $f\big(a(x)+b(y)+c(z)\big)$ with $a,b,c \in C[0,1]$ and $f \in C(\mathbb{R})$.

On the other hand, Vitushkin [1954] verified the following version of Hilbert's conjecture; see also Vitushkin [2003] and Kolmogorov/Tikhomirov [1959, Anhang I]: If $\frac{r}{2} > \frac{s}{3}$, then there exists a C^s-function of three variables that cannot be represented as superposition of C^r-functions of two variables.

Roughly speaking, this results follows from

$$e_n\big(Id : C^r([0,1]^2) \to C([0,1]^2)\big) \asymp n^{-r/2} \text{ and } e_n\big(Id : C^s([0,1]^3) \to C([0,1]^3)\big) \asymp n^{-s/3}.$$

6.7.9 Spaces of vector-valued functions

6.7.9.1 Recall that the Bochner space $[L_p(M,\mathscr{M},\mu),X]$ is defined just as in the case of scalar-valued functions, the only change being that absolute values are replaced by the norms of the elements from the underlying Banach space X. The same procedure works for spaces of X-valued smooth functions. Therefore the symbols $[W_p^m(\Omega),X]$, $[L_p^\sigma(\Omega),X]$, $\dots$, $[B_{p,q}^\sigma(\Omega),X]$, and $[F_{p,q}^\sigma(\Omega),X]$ make sense. As usual, Ω denotes a "nice" domain of $\mathbb{R}^N$.

Sobolev and Besov spaces of vector-valued functions were considered for the first time by Grisvard [1966, pp. 171–176]; his approach was based on interpolation theory. Almost simultaneously, Wloka [1967, pp. 305, 308] treated the case of Slobodetskiĭ spaces. I stress the fact that Birman/Solomyak [1967/70, Part II, стр. 22; Part III, стр. 38–39] used the condition $K \in [W_p^\sigma(\Omega), W_q^\tau(\Omega)]$ in order to characterize the smoothness of a kernel K; see Subsection 6.7.10.

There is an unpleasant handicap. Since most proofs can easily be adapted from the scalar-valued setting, no author liked to waste his time writing up a detailed introduction to this subject. On the other hand, it will be shown below that some results do not carry over to the vector-valued setting. Hence care is required.

For more information the reader may consult a survey of Schmeißer [1987] and a paper of Amann [1997].

6.7.9.2 Let $m \in L_1(\mathbb{R}^N)$ and $1 \le p \le \infty$. Then

$$m_{\mathrm{op}} : f(t) \mapsto m * f(x) = \int_{\mathbb{R}^N} m(x-t) f(t)\, dt$$

defines an operator on $L_p(\mathbb{R}^N)$ with $\|m_{\mathrm{op}}|L_p \to L_p\| \le \|m|L_1\|$. Plainly, this conclusion remains true when $L_p(\mathbb{R}^N)$ is replaced by $[L_p(\mathbb{R}^N), X]$. As a consequence, it follows that the Fourier theoretic definition of Besov spaces described in 6.7.3.12 also works in the vector-valued case. Indeed, norms

$$\|\boldsymbol{f}|B^{\sigma}_{p,q}\|_{\Phi} := \left\{ \sum_{k=0}^{\infty} \left\|2^{k\sigma}\varphi_k * \boldsymbol{f}\,\middle|L_p\right\|^p \right\}^{1/q}$$

obtained from different systems $\Phi = (\varphi_k)$ are equivalent.

6.7.9.3 If $p=2$, then m_{op} can be written in the form $F^{-1}_{\mathrm{our}}\widehat{m}F_{\mathrm{our}}$, where the middle term just means multiplication by $\widehat{m}$. However, the new expression $F^{-1}_{\mathrm{our}}\widehat{m}F_{\mathrm{our}}$ is even meaningful for all distributions m with a bounded and measurable Fourier transform. Hence the map $f \mapsto F^{-1}_{\mathrm{our}}[\widehat{m}F_{\mathrm{our}}f]$ is well-defined on $f \in L_p(\mathbb{R}^N) \cap L_2(\mathbb{R}^N)$, and we may ask whether it extends to an operator on $L_p(\mathbb{R}^N)$. In the affirmative case, m is called an L_p-**multiplier**.

The Marcel Riesz theorem asserts that $h(x) := \dfrac{1}{\pi x}$ is an L_p-**multiplier**, $1<p<\infty$.

6.7.9.4 Multiplier theorems are an important tool in the theory of function spaces. Most useful is a version that goes back to Mikhlin [1956] and Hörmander [1960, pp. 120–121]:

Let m be a distribution on $\mathbb{R}^N$ whose Fourier transform is infinitely differentiable on $\mathbb{R}^N \setminus \{o\}$. Assume that there exists a constant $c>0$ such that

$$\left|D^{\alpha}\widehat{m}(y)\right| \le c|y|^{-|\alpha|} \quad \text{whenever } y \ne o \text{ and } |\alpha| \le N+1.$$

Then m is an L_p-multiplier for $1<p<\infty$.

Since $\widehat{h}(y) = -\mathrm{i}\,\mathrm{sgn}\, y$, the previous result applies to the Hilbert transform $H_{\mathrm{ilb}} = h_{\mathrm{op}}$. This basic example helps to explain why it is useful to allow a singularity of $\widehat{m}$ at zero.

6.7.9.5 Based on the fundamental work of Burkholder and Bourgain, McConnell [1984, p. 741] and Zimmermann [1989, p. 204] were able to show that the Mikhlin–Hörmander multiplier theorem extends to $[L_p(\mathbb{R}^N), X]$ if and only if X is a *UMD* space.

This result implies that

$$\left[W^m_p(\mathbb{R}^N), X\right] = \left[L^m_p(\mathbb{R}^N), X\right] \quad \text{whenever } X \text{ is } \mathit{UMD};$$

see 6.7.2.7 for the scalar-valued case. According to Bukhvalov [1991b, p. 2178], the *UMD* property of X is not only sufficient, but also necessary.

6.7.9.6 Let $1<p<\infty$, and recall from 6.7.4.5 that in the scalar-valued setting, the identity $F^{\sigma}_{p,2}(\mathbb{R}^N) = L^{\sigma}_p(\mathbb{R}^N)$ is deduced from the $\mathbb{R}$-version of the Littlewood–Paley theorem 6.7.4.5. This famous result was extended by (Jacob) Schwartz [1961, pp. 795, 797] to functions with values in an arbitrary Hilbert space:

There exist finite constants a_p and b_p such that

$$a_p\bigg(\int_{\mathbb{R}}|\boldsymbol{f}(x)|^p dx\bigg)^{1/p} \le \bigg(\int_{\mathbb{R}}\bigg(\sum_{n\in\mathbb{Z}}|\boldsymbol{f}_n(x)|^2\bigg)^{p/2} dx\bigg)^{1/p} \le b_p\bigg(\int_{\mathbb{R}}|\boldsymbol{f}(x)|^p dx\bigg)^{1/p}.$$

Here $\boldsymbol{f}$ stands for an arbitrary function in $[L_p(\mathbb{R}),H]$, and $\boldsymbol{f}_n$ denotes the function whose Fourier transform is identical to that of $\boldsymbol{f}$ in the range $2^n < |y| < 2^{n+1}$ and vanishes outside this range.

Conversely, using Kwapień's type-cotype theorem, McConnell [1984, p. 742] and Rubio de Francia/Torrea [1987, p. 283] proved, independently of each other, that X must be isomorphic to a Hilbert space if the two-sided inequality above holds for all $\boldsymbol{f}\in[L_p(\mathbb{R}),X]$.

Hence we have $[F^\sigma_{p,2}(\mathbb{R}^N),X] = [L^\sigma_p(\mathbb{R}^N),X]$ precisely when X is Hilbertian.

6.7.10 Integral operators

6.7.10.1 In this subsection, I discuss a number of results that give answers to the following question of Hille/Tamarkin [1931, p. 1]:

What can be said about the distribution of the characteristic values of the Fredholm integral equation

$$y(x) = \zeta\int_a^b K(x,\xi)y(\xi)d\xi$$

on the basis of the general analytic properties of the kernel $K(x,\xi)$ such as integrability, continuity, differentiability, analyticity and the like?

The answer can be formulated as a rule of thumb:

The better the kernel, the faster the decay of the eigenvalues.

In order to make this assertion meaningful, we need a classification of kernels. The basic idea consists in regarding kernels as vector-valued functions. This point of view already occurred in 6.4.3.8 when the concept of a Hille–Tamarkin kernel was invented.

6.7.10.2 Let $F(\Omega)$ be a Banach space of scalar-valued functions on a domain Ω in $\mathbb{R}^N$, and let X be any Banach space. Suppose that we have defined a Banach space $[F(\Omega),X]$ that consists of X-valued functions $\boldsymbol{k}$ on Ω such that all scalar-valued functions $s\mapsto\langle\boldsymbol{k}(s),x^*\rangle$ with $x^*\in X^*$ are contained in $F(\Omega)$.

In addition, we need an embedding $f\mapsto x^*$ of $F(\Omega)$ into X^*. Then, under appropriate conditions, every member $\boldsymbol{k}$ of $[F(\Omega),X]$ induces a Riesz operator on $F(\Omega)$,

$$\boldsymbol{k}_{\mathrm{op}} : f\mapsto x^*\mapsto\langle\boldsymbol{k}(s),x^*\rangle.$$

The associated sequence of eigenvalues will be denoted by $\big(\lambda_n(\boldsymbol{k})\big)$. In the case that $\boldsymbol{k}$ is induced by a kernel K (see 5.1.2.1), we write $\big(\lambda_n(K)\big)$.

6.7.10.3 First of all, I present some results that were obtained with the help of Fredholm's determinant theory. Indeed, the master himself [1903, p. 368] dealt with continuous kernels K on $[0,1]\times[0,1]$ satisfying a Hölder condition

$$|K(s_1,t)-K(s_2,t)|\leq c|s_1-s_2|^{\sigma},$$

where $0<\sigma\leq 1$. For the associated determinant $d(\zeta)=\sum_{n=0}^{\infty}d_n\zeta^n$, he proved that

$$|d_n|\leq\frac{n^{1/2-\sigma}}{n!}c^n.$$

Subsequently, Lalesco [LAL, pp. 86–89] inferred from this estimate that

$$K\in\big[\mathrm{Lip}^{\sigma}[0,1],C[0,1]\big]\quad\text{implies}\quad\lambda_n(K)=O\big(n^{-(\sigma+1/2-\varepsilon)}\big).$$

Had he looked carefully at the theorems about zeros of entire functions, he would have been able to avoid the superfluous $\varepsilon>0$; see Subsection 6.5.5.

Differentiable kernels were considered by Mazurkiewicz [1915, Part II, p. 810] and Gelfond [1931, p. 831]. The latter showed that

$$K\in\big[C^{m_1}[0,1],C^{m_2}[0,1]\big]\quad\text{implies}\quad\lambda_n(K)=O\big(n^{-(m_1+m_2+1/2)}\big).$$

The most important result was obtained by Hille/Tamarkin [1931, p. 46], who proved that

$$K\in\big[L_{q^*}(\mathbb{T}),L_q^{\sigma}(\mathbb{T})\big]\quad\text{implies}\quad\lambda_n(K)=O\big(n^{-(\sigma+1/q^*)}\big)$$

whenever $\sigma>0$ and $1<q\leq 2$. In addition, these authors posited the technical assumption that K be a Hilbert–Schmidt kernel.

6.7.10.4 In the special case of *symmetric* kernels, Weyl [1912, p. 449] was able to show that

$$K\in\big[C^1[0,1],C[0,1]\big]\quad\text{implies}\quad\lambda_n(K)=o\big(n^{-3/2}\big).$$

The following extended and improved version is due to Kreĭn [1937, стр. 725]:

$$K\in\big[C^m[0,1],C^m[0,1]\big]\quad\text{implies}\quad\big(\lambda_n(K)\big)\in l_{r,2},$$

where $1/r=2m+1/2$. All proofs were based on Hilbert space techniques.

6.7.10.5 Without the assumption of symmetry, the previous theorem was significantly generalized in [GOH$_3^+$, pp. 121–122]:

$$K\in\big[L_2(0,1),W_2^m(0,1)\big]\quad\text{implies}\quad\big(\lambda_n(K)\big)\in l_{r,2},$$

where $1/r=m+1/2$. Chang [1952, pp. 22–23] had earlier studied the s-numbers of the associated operator: $s_n(K_{\mathrm{op}})=o\big(n^{-1/r}\big)$.

Gohberg/Kreĭn and Paraska [1965] were the first authors who estimated s-numbers in order to get results about eigenvalue distributions. In the following years, this approach has been successfully applied to various classes of kernels. Of particular interest is a series of papers as well as a survey [1977] written by Birman/Solomyak. In [1967/70, Part I, стр. 47–48], they obtained the following theorem:

Let $\tau/N > (1/q-1/2)_+$ and $1/r=\tau/N+\begin{cases} 1/q^* & \text{if } 1\le q\le 2,\\ 1/2 & \text{if } 2\le q\le\infty.\end{cases}$
Then

$$K\in\big[L_2(\Omega),W_q^\tau(\Omega)\big] \quad\text{implies}\quad \big(s_n(K_{\text{op}}|L_2\to L_2)\big)=O\big(n^{-1/r}\big).$$

In order to ensure compactness of the involved embedding operators, the underlying domain Ω is supposed to be bounded.

Next, in [Part II, стр. 23–25] and [Part III, стр. 39], these authors studied s-numbers of operators generated by kernels $K\in\big[W_p^\sigma(\Omega),W_q^\tau(\Omega)\big]$. However, most of their estimates turned out to yield non-optimal results about eigenvalues. Indeed, before 1979, Weyl's inequality was known only in the Hilbert space setting. Thus the associated integral operators K_{op} were forced to live in L_2, which is sometimes quite unnatural. This convincing example shows that we really need Banach spaces!

6.7.10.6 The final result about eigenvalue distributions of smooth kernels is due to Pietsch [1980c, p. 176], [1981a, p. 133]:
Let $(\sigma+\tau)/N>(1/p+1/q-1)_+$ and $1/r=(\sigma+\tau)/N+\begin{cases} 1/q^* & \text{if } 1\le q\le 2,\\ 1/2 & \text{if } 2\le q\le\infty.\end{cases}$
Then

$$K\in\big[B_{p,u}^\sigma(\Omega),B_{q,v}^\tau(\Omega)\big] \quad\text{implies}\quad \big(\lambda_n(K)\big)\in l_{r,u}.$$

6.7.10.7 One of the main problems of the theory of trigonometric series is the question of how the asymptotic behavior of the Fourier coefficients

$$\gamma_k(f):=\frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\,dt$$

depends on the properties of the function $f\in L_1(\mathbb{T})$.

Here are some classical results:

Riemann–Lebesgue lemma: $\big(\gamma_k(f)\big)\in c_0(\mathbb{Z})$ for $f\in L_1(\mathbb{T})$,
Fischer–Riesz theorem: $\big(\gamma_k(f)\big)\in l_2(\mathbb{Z})$ for $f\in L_2(\mathbb{T})$,
Hausdorff–Young theorem: $\big(\gamma_k(f)\big)\in l_{p^*}(\mathbb{Z})$ for $f\in L_p(\mathbb{T})$ and $1<p<2$.

The connection between Fourier coefficients and eigenvalues was realized very late. The first result along these lines is due to Carleman [1918, pp. 380–381]. Nowadays, it looks like a triviality that the convolution operator

$$\varphi(t)\mapsto\frac{1}{2\pi}\int_0^{2\pi} f(s-t)\varphi(t)\,dt$$

has the eigenfunctions e^{ikt} with the Fourier coefficients $\gamma_k(f)$ as eigenvalues.

Hence every result about distributions of eigenvalues implies a corresponding result about distributions of Fourier coefficients. This observation was, in particular, used in the converse direction; see Hille/Tamarkin [1931, p. 8]:

The main interest of these [read: convolution] *kernels lies in the fact that they are very well fitted for construction of examples and "Gegenbeispiele" in order to illustrate various situations of the general theory.*

6.7.10.8 Finally, I report about a historical curiosity.

Unaware of the Fredholm–Lalesco theorem 6.7.10.3 and using direct methods, Bernstein [1914] "rediscovered" that every function $f \in \mathrm{Lip}^{\sigma}(\mathbb{T})$ with $\sigma > 1/2$ has an absolutely convergent Fourier series. In [1934], he improved this result by showing that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} E_n(f|L_\infty) < \infty \quad \text{implies} \quad \sum_{k\in\mathbb{Z}} |\gamma_k(f)| < \infty.$$

Later on, Stechkin [1947, стр. 178], [1955, стр. 38] replaced $E_n(f|L_\infty)$ by $E_n(f|L_2)$. In view of 6.7.3.10, this improvement means that $f \in B^{1/2}_{2,1}(\mathbb{T})$. Thus one may say that in a hidden form, Besov spaces were used before their official invention.

6.7.11 Differential operators

This subsection is supposed to give, among others things, an answer to a "serious" question of Peetre, [PEE$_2$, p. 67]:
Isn't this ironical. The functions being as nice as possible in the interior [that is: harmonic]*, why then bother about the boundary where they get really nasty?*

6.7.11.1 Throughout, Ω denotes a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary.

I stress that the following considerations require function spaces on $\partial\Omega$. Since it would be beyond the scope of this text to treat functions and distributions on manifolds, the reader is advised to take a naive point of view. Precise information can be found in [TRI$_5$, Chap. 7]; see also 6.7.6.7.

6.7.11.2 In order to treat boundary value problems, one has to know the answer to the question, *What are the boundary values of a function given on* Ω?

The simplest method is to look for a limit of $f(x_k)$ if $x_k \in \Omega$ approaches the point $x \in \partial\Omega$ in a prescribed way. However, this pointwise definition does not work for functions that may be changed in a null set. Hence a functional analytic approach seems to be more appropriate.

As described in 6.7.6.6, the space $F(\Omega)$ consists of all restrictions to Ω of functions $f_0 \in F(\mathbb{R}^N)$. Clearly, $\mathrm{rest}_{\partial\Omega} f_0$ is well-defined whenever $f_0 \in S(\mathbb{R}^N)$. Thus, supposing that $S(\mathbb{R}^N)$ is dense in $F(\mathbb{R}^N)$, one may hope that $f_0 \mapsto \mathrm{rest}_{\partial\Omega} f_0$ extends continuously to a map from $F(\mathbb{R}^N)$ into some space $G(\partial\Omega)$. This is quite often the case. Subject to the condition $\sigma > 1/p$, there exist operators

$$R : B^{\sigma}_{p,q}(\Omega) \longrightarrow B^{\sigma-1/p}_{p,q}(\partial\Omega) \quad \text{and} \quad R : F^{\sigma}_{p,q}(\Omega) \longrightarrow F^{\sigma-1/p}_{p,p}(\partial\Omega)$$

that yield the classical boundary values when applied to "nice" functions.

Теоремы вложения разных измерений go back to Nikolskiĭ [1953], Aronszajn [1955], Gagliardo [1957], and Slobodetskiĭ [1958]. The final version of the left-hand case is due to Besov [1961, p. 96], whereas the right-hand one was established by Triebel [1980, p. 277], [TRI$_3$, p. 105].

I stress the astonishing fact that functions from Lizorkin–Triebel spaces $F^{\sigma}_{p,q}$ have boundary values in Besov spaces $B^{\sigma-1/p}_{p,p} = F^{\sigma-1/p}_{p,p}$, which do not depend on the fine index q.

For Besov as well as for Lizorkin–Triebel spaces, the boundary operator R admits a right inverse S. This means that every function φ on $\partial\Omega$ can be extended to a function $S\varphi$ on Ω such that $RS\varphi = \varphi$; see Besov [1961, p. 98] and Triebel [1980, p. 278].

A summary of the results above can be found in [TRI$_4$, p. 200].

6.7.11.3 The assumption $\sigma > 1/p$ required in the previous paragraph is almost necessary. Indeed, if $\sigma < 1/p$, then the infinitely differentiable functions with compact support are dense in $B^{\sigma}_{p,q}(\Omega)$ and $F^{\sigma}_{p,q}(\Omega)$; see [TRI$_1$, p. 318] and [TRI$_4$, p. 210]. Hence, in this case, looking for boundary values does not make sense.

6.7.11.4 Now we are prepared to present one of the most beautiful applications of functional analysis. In order to make the presentation more transparent, I have treated only the simple case of the Laplace operator, though the following results hold as well for regular elliptic differential equations; see Carleman [1936, p. 132] for a classical approach.

We consider the **Dirichlet problem**

$$\Delta u(x) = f(x) \quad \text{for } x \in \Omega \quad \text{and} \quad u(x) = \varphi(x) \quad \text{for } x \in \partial\Omega.$$

Then the correspondence $u \leftrightarrow f \oplus \varphi$ defines an isomorphism between

$$B^{\sigma}_{p,q}(\Omega) \quad \text{and} \quad B^{\sigma-2}_{p,q}(\Omega) \oplus B^{\sigma-1/p}_{p,q}(\partial\Omega) \tag{6.7.11.4.a}$$

as well as between

$$F^{\sigma}_{p,q}(\Omega) \quad \text{and} \quad F^{\sigma-2}_{p,q}(\Omega) \oplus F^{\sigma-1/p}_{p,p}(\partial\Omega); \tag{6.7.11.4.b}$$

see [TRI$_1$, pp. 389–391] and [TRI$_4$, pp. 233–234].

The model case

$$W^2_2(\Omega) \longleftrightarrow L_2(\Omega) \oplus W^{3/2}_2(\partial\Omega)$$

is due to Slobodetskiĭ [1958, p. 266].

According to [TRI$_4$, p. 212]:

... the milestone paper by Agmon/Douglis/Nirenberg marks the beginning of the up-to-date part of this theory.

These authors [1959, pp. 703–706] treated the case $m \geq 2$ and $1 < p < \infty$. Then

$$W_p^m(\Omega) \longleftrightarrow W_p^{m-2}(\Omega) \oplus W_p^{m-1/p}(\partial\Omega).$$

However, their approach was based partly on a "proof by definition," [1959, p. 699]:

$A_p^{m-1/p}(\partial\Omega)$ *is to be the class of functions* φ *which are the boundary values of functions u belonging to* $W_p^m(\Omega)$. *In this class we introduce the norm*

$$\|\varphi|A_p^{m-1/p}\| := \text{g.l.b.}\,\|u|W_p^m\| \qquad \text{[slightly modernized version].}$$

Note that the exponent of smoothness of $A_p^{m-1/p}(\partial\Omega)$, namely $m-1/p$, was properly chosen. Ironically, the three authors were stimulated by a wrong, but not too wrong, conjecture [1959, p. 649]. They expected that $A_p^{m-1/p}(\partial\Omega)$ would be the Bessel potential space $L_p^{m-1/p}(\partial\Omega)$. However, this is the case only for $p=2$. Otherwise, one gets the Slobodetskiĭ space $W_p^{m-1/p}(\partial\Omega)$.

6.7.11.5 If $p=q=\infty$, then (6.7.11.4.a) becomes an isomorphism between

$$\mathrm{Lip}^\sigma(\Omega) \quad \text{and} \quad \mathrm{Lip}^{\sigma-2}(\Omega) \oplus \mathrm{Lip}^\sigma(\partial\Omega); \qquad (6.7.11.5.a)$$

see [TRI$_4$, p. 235]. For non-integral σ's, this result is due to Schauder [1934], who created the technique of *a priori estimates*. His famous contributions to the theory of elliptic differential equations were the real starting point of all developments described above.

6.7.11.6 Note that $C^m(\overline{\Omega})$ is the collection of all functions f on Ω whose derivatives $D^\alpha f$ with $|\alpha| \leq m$ exist and admit continuous extensions to $\overline{\Omega}$.

One may wonder whether an analogue of (6.7.11.5.a) also holds in this classical setting. Unfortunately, the answer is frustrating: $u \leftrightarrow f \oplus \varphi$ does *not* define an isomorphism between

$$C^m(\overline{\Omega}) \quad \text{and} \quad C^{m-2}(\overline{\Omega}) \oplus C^m(\partial\Omega).$$

The reason for this defect was discovered only many years after Schauder's work.

Mityagin [1959] succeeded in constructing an example (quite involved) of a function u of two variables that possesses continuous second derivatives $\frac{\partial^2}{\partial x^2}u$ and $\frac{\partial^2}{\partial y^2}u$ everywhere, while $\frac{\partial^2}{\partial x \partial y}u$ fails to exist at $(0,0)$. The following simplification goes back to Boman [1972, p. 206].

Let

$$u(x,y) := \begin{cases} 0 & \text{if } x = y = 0, \\ xy \log_2 \log_2 \frac{2}{x^2+y^2} & \text{if } 0 < x^2+y^2 \le 1, \\ 0 & \text{if } x^2+y^2 = 1. \end{cases}$$

Then $f := \Delta u \in C(\overline{\mathbb{D}})$, and u is the unique solution of the equation $\Delta u = f$ in the open unit disk that satisfies the boundary condition $u(x,y)=0$ if $x^2+y^2=1$. Hence $f \oplus o \in C(\overline{\mathbb{D}}) \oplus C^2(\mathbb{T})$ has no preimage in $C^2(\overline{\mathbb{D}})$.

6.7.11.7 Next, I treat the **eigenvalue problem** under the boundary condition $\varphi = 0$. To this end, let $W_2^2(\Omega)_0$ denote that subspace of $W_2^2(\Omega)$ whose members vanish on $\partial\Omega$. Then $u \mapsto \Delta u$ defines an isomorphism from $W_2^2(\Omega)_0$ onto $L_2(\Omega)$. On the other hand, the approximation numbers of $Id : W_2^2(\Omega)_0 \to L_2(\Omega)$ behave like $n^{-2/N}$; see 6.7.8.13. Hence the diagram

$$\begin{tikzcd} L_2(\Omega) \arrow[r, shift left, "\Delta^{-1}"] & W_2^2(\Omega)_0 \arrow[l, shift left, "\Delta"] \arrow[r, "Id"] & L_2(\Omega) \end{tikzcd}$$

tells us that Δ^{-1} defines an approximable operator on $L_2(\Omega)$ with

$$a_n\big(\Delta^{-1} : L_2 \to L_2\big) \asymp n^{-2/N}.$$

Passing to $-\Delta^{-1}$ ensures positivity, which implies that the eigenvalues coincide with the s-numbers. Therefore we arrive at

$$\lambda_n\big(-\Delta\big) \asymp n^{2/N}.$$

For $N=2$ and $N=3$, this formula follows from a classical result of Weyl [1912, pp. 457, 469]. The extension to arbitrary dimensions is straightforward:

$$\lim_{n\to\infty} \frac{\lambda_n\big(-\Delta\big)^{N/2}}{n} = \frac{(4\pi)^{N/2}\Gamma\big(\frac{N+2}{2}\big)}{|\Omega|}.$$

Note that the asymptotic relations proved by "hard" analysis are more precise than those obtained by Banach space techniques.

6.7.11.8 The original way of treating the equation $\Delta u = f$ under homogeneous boundary conditions was to represent Δ^{-1} as an integral operator:

$$\Delta^{-1} : f(y) \mapsto u(x) = \int_\Omega G(x,y) f(y)\,dy.$$

Then the problem consists in exploring the differentiability properties of the **Green function** G.

6.7.12 Hardy spaces

The theory of **Hardy spaces** is a child of complex analysis; conformal mappings, inner and outer functions, and Blaschke products are among the most useful tools. However, Fefferman/Stein [1972, p. 138] developed *a new viewpoint about H^p which pushes these ideas into the background, but which brings to light the real variable meaning of H^p.*

It is my intention to convince the reader that mathematics should be as colorful as possible and that propaganda for one-sided standpoints should remain an established right of politicians.

6.7.12.1 The story began with a classical result of Hardy [1914, p. 269], who proved that for any analytic function f on the open unit disk and for $\delta > 0$,

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^\delta d\theta$$

is a *steadily increasing and logarithmically convex* function of $r \in (0,1)$.

6.7.12.2 The next step was done by the brothers Riesz [1916, p. 42]:

Damit es zu einer innerhalb des Einheitskreises regulär analytischen Funktion $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ eine längs des Einheitskreises erklärte im Lebesgue'schen Sinne integrierbare Funktion $\varphi(z)$ gebe derart, dass

$$\frac{1}{2\mathrm{i}\pi}\int\limits_{|z|=1} \frac{\varphi(z)}{z^{n+1}} dz = a_n \quad \textit{und} \quad \frac{1}{2\mathrm{i}\pi}\int\limits_{|z|=1} \varphi(z) z^n dz = 0 \qquad (n = 0, 1, 2, \dots)$$

seien, ist es notwendig und hinreichend, dass das Integral

$$I(r) = \int\limits_{|z|=r} |f(z)||dz| = r\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})| d\theta$$

für alle $r < 1$ unterhalb einer endlichen, von r unabhängigen Schranke bleibe.

Riesz [1923, p. 87] coined the symbol H_δ (nowadays H_p):

Es sei $f(z)$ eine innerhalb des Einheitskreises regulär analytische Funktion; δ sei eine positive Zahl. Der Mittelwert

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^\delta d\theta \qquad (r < 1)$$

ist, laut einem Satz von Herrn Hardy, eine wachsende Funktion von r und es bestehen daher die folgenden beiden Möglichkeiten: entweder wächst $\mu_\delta(r)$ für $r \to 1$ ins Unendliche, oder aber ist $\mu_\delta(r)$ beschränkt und strebt für $r \to 1$ gegen einen Grenzwert. Im letzteren Falle wollen wir sagen, die Funktion $f(z)$ gehöre zur Klasse H_δ.

6.7.12.3 The preceding remarks show that Hardy classes were invented in order to study *Randwerte analytischer Funktionen*. Indeed, if $f \in H_p$, then $f(z)$ behaves nicely as $z = re^{i\theta}$ approaches the boundary: θ fixed, $r \nearrow 1$. Consequently, for more than 30 years, the interest was directed to the properties of the members of H_p and not to H_p as a whole; see [PRI]. It was only at the end of the 1940s that Taylor [1950, p. 121], [1950/51, Part II, p. 26] treated H_p as a Banach space with the norm

$$\|f|H_p\| := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \qquad \text{where } p \geq 1.$$

The first monograph on *Banach Spaces of Analytic Functions* (1962) was published by Hoffman, [HOF]. Further references are [DUR], [GARN], [KOO], and the lecture notes [PEŁ]. The reader may also consult [MÜLL], a recent book on H_1.

6.7.12.4 In what follows, $\mathbb{D}$ denotes the open unit disk in the complex as well as in the real plane,

$$\{z \in \mathbb{C} : |z| < 1\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

Similarly, $\mathbb{T}$ stands for

$$\{z \in \mathbb{C} : z = e^{i\theta},\ 0 \leq \theta < 2\pi\} = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 : 0 \leq \theta < 2\pi\}.$$

Quite often, $\mathbb{T}$ is also identified with $\mathbb{R}/2\pi\mathbb{Z}$. The last convention causes some ambiguity, since several authors use the symbols $f(e^{i\theta})$ and $f(\theta)$ synonymously. However, there is no serious danger of confusion.

6.7.12.5 Members of Hardy spaces, $1 \leq p < \infty$, behave like chameleons. They may appear as

$\mathscr{H}_p(\mathbb{D})$: analytic (complex-valued) functions f on the open unit disk of the complex plane for which

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

is finite.

$\mathscr{H}_p^{\text{real}}(\mathbb{D})$: harmonic (real-valued) functions u on the open unit disk of the real plane such that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \sup_{(x,y) \in \Omega_r(\theta)} |u(x,y)|^p d\theta \right)^{1/p}$$

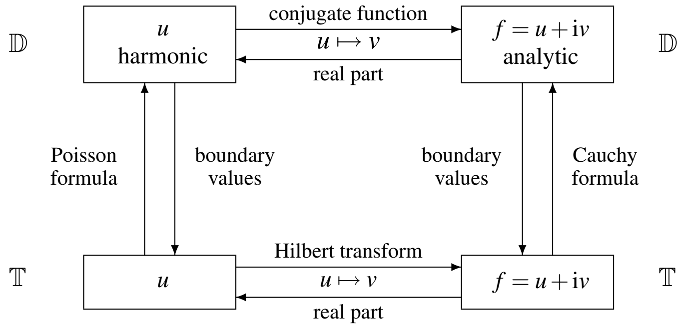
is finite for some/all $0 < r < 1$. Here $\Omega_r(\theta)$ denotes the interior of the convex hull of the disk $r\mathbb{D}$ and the point $e^{i\theta}$.

$H_p(\mathbb{T})$: p-integrable (complex-valued) functions f on the unit circle for which all Fourier coefficients with negative index vanish:

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0 \quad \text{for } n = 1, 2, \ldots .$$

$H_p^{\rm real}(\mathbb{T})$: p-integrable (real-valued) functions u on the unit circle whose Hilbert transform
$$v(s) := \frac{1}{2\pi}\int_0^{2\pi} \cot\Big(\frac{s-t}{2}\Big) u(t)\,dt$$
is p-integrable as well; see 6.7.12.15.

These spaces can be identified according to the following scheme:



The crucial point is the characterization of $\mathscr{H}_p^{\rm real}(\mathbb{D})$ via the *non-tangential maximal function*
$$u^*(r,\theta) := \sup_{(x,y)\in\Omega_r(\theta)} |u(x,y)|,$$
which was prepared by Hardy/Littlewood [1930, p. 112] and finally established by Burkholder/Gundy/Silverstein [1971, pp. 137–138].

The underlying setting is indicated as follows: spaces of analytic or harmonic functions are denoted by $\mathscr{H}_p$, whereas H_p stands for spaces of boundary values.

Note that the definition of $H_p^{\rm real}(\mathbb{T})$ also makes sense if its members are supposed to be complex-valued functions.

Using Walsh–Paley martingales on $[0,1]$, Maurey [1980, p. 82] defined the space $H_1(\delta)$ in which the Haar functions form an unconditional basis. Further information about the use of martingale techniques may be found in [MÜLL, Chap. 4].

6.7.12.6 In the limiting case $p=\infty$, the Hardy space $\mathscr{H}_\infty(\mathbb{D})$, equipped with the sup-norm, consists of all bounded analytic functions on $\mathbb{D}$. The corresponding space of boundary values is denoted by $H_\infty(\mathbb{T})$.

The **disk algebra** $\mathscr{A}(\mathbb{D})$ is the collection of all continuous functions on the closed unit disk $\overline{\mathbb{D}}$ that are analytic in the interior. This algebra appeared for the first time as an example of an anti-symmetric subalgebra of $C(\overline{\mathbb{D}})$, where anti-symmetry means that $f\in\mathscr{A}(\mathbb{D})$ and $\overline{f}\in\mathscr{A}(\mathbb{D})$ imply $f=\text{const}$; Shilov [1951]. However, as an example of a Banach space, it can already be found in [BAN, p. 12].

Let $A(\mathbb{T})$ denote the closed linear span of $1, z, z^2, \ldots$ in $C(\mathbb{T})$. Since every function in $\mathscr{A}(\mathbb{D})$ is uniquely determined by its boundary values, $\mathscr{A}(\mathbb{D})$ and $A(\mathbb{T})$ can be identified.

6.7.12.7 As observed by Krylov [1939], it is often useful to replace the open unit disk by the upper half-plane $\mathbb{R}^2_+$,

$$\big\{z\in\mathbb{C} : \mathrm{Im}(z)>0\big\} = \big\{(x,y)\in\mathbb{R}^2 : y>0\big\}.$$

Then $\mathbb{T}$ passes into the real axis $\mathbb{R}$,

$$\big\{z\in\mathbb{C} : \mathrm{Im}(z)=0\big\} = \big\{(x,0)\in\mathbb{R}^2 : x\in\mathbb{R}\big\}.$$

The corresponding spaces can be identified in the same way as indicated in the scheme above.

$\mathscr{H}_p(\mathbb{R}^2_+)$: analytic (complex-valued) functions f on the complex upper half-plane for which

$$\sup_{y>0}\left(\int_{-\infty}^{+\infty}|f(x+\mathrm{i}y)|^p dx\right)^{1/p}$$

is finite.

$\mathscr{H}_p^{\mathrm{real}}(\mathbb{R}^2_+)$: harmonic (real-valued) functions u on the real upper half-plane such that

$$\left(\int_{-\infty}^{+\infty}\sup_{(x,y)\in C_\alpha(t)}|u(x,y)|^p dt\right)^{1/p}$$

is finite for some/all $\alpha>0$. Here $C_\alpha(t) := \{(x,y) : |x-t| < \alpha y,\ y>0\}$ denotes a vertical cone.

$H_2(\mathbb{R})$: 2-integrable (complex-valued) functions f on the real line whose Fourier transforms vanish for $y<0$; see [KOO, p. 131]. For $p\neq 2$, no appropriate characterization seems to be available.

$H_p^{\mathrm{real}}(\mathbb{R})$: p-integrable (real-valued) functions u on the real line whose Hilbert transform

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(t)}{s-t}\,dt$$

is p-integrable as well.

6.7.12.8 Taking into account that $z\mapsto w=\frac{\mathrm{i}-z}{\mathrm{i}+z}$ defines a conformal mapping from $\mathbb{R}^2_+$ onto $\mathbb{D}$, Hoffman [HOF] found an isometry between $\mathscr{H}_p(\mathbb{D})$ and $\mathscr{H}_p(\mathbb{R}^2_+)$:

$$f(w)\mapsto\frac{c_p}{(\mathrm{i}+z)^{2/p}}f\left(\frac{\mathrm{i}-z}{\mathrm{i}+z}\right),$$

c_p being a suitable norming factor.

6.7.12.9 Let $1/p=(1-\theta)/p_0+\theta/p_1$. Then, in the case $1\le p_0,p_1\le\infty$, Rivière/Sagher [1973] proved the interpolation formula

$$\big(H_{p_0}^{\mathrm{real}}(\mathbb{R}),H_{p_1}^{\mathrm{real}}(\mathbb{R})\big)_{\theta,p}=H_p^{\mathrm{real}}(\mathbb{R}).$$

This result was extended to $0<p_0,p_1\le\infty$ by Fefferman/Rivière/Sagher [1974].

For the complex versions of the real Hardy spaces, Fefferman/Stein [1972, p. 156] stated without proof that

$$\big[H^{\mathrm{real}}_{p_0}(\mathbb{R}), H^{\mathrm{real}}_{p_1}(\mathbb{R})\big]_\theta = H^{\mathrm{real}}_p(\mathbb{R}).$$

First results about interpolation of $\mathscr{H}_p(\mathbb{D})$ were earlier obtained by Thorin [1948] and Salem/Zygmund [1948]. However, the interest of these authors was limited to convexity theorems. Then the main period followed. Concerning recent developments, the reader may consult a survey of Kislyakov [1999].

6.7.12.10 A real-valued function a on $\mathbb{R}$, vanishing outside an interval I, is said to be an **atom** if

$$\int_I a(x)\,dx = 0 \quad \text{and} \quad |I| \sup_{x\in I} |a(x)| \le 1. \tag{6.7.12.10.a}$$

I stress that the term "atom" has nothing to do with indecomposability. It means only that atoms are the building blocks of larger functions. According to Coifman [1974, p. 269], the following criterion was discovered by Fefferman (unpublished):

A function $f \in L_1(\mathbb{R})$ belongs to $H^{\mathrm{real}}_1(\mathbb{R})$ if and only if there exists a sequence of atoms a_k and $(\lambda_k) \in l_1$ such that

$$f = \sum_{k=1}^\infty \lambda_k a_k.$$

An equivalent norm is obtained by letting

$$\|f|H_1\|^{\mathrm{atom}} := \inf \sum_{k=1}^\infty |\lambda_k|,$$

where the infimum ranges over all representations described above; see also [BENN$^+$, pp. 362–373].

This characterization turned out to be quite flexible. For example, using atomic decompositions, one may define $H^{\mathrm{real}}_1[0,1]$; see Wojtaszczyk [1980, p. 294].

6.7.12.11 A function g on $\mathbb{R}$ is said to be **locally integrable** if it is integrable over every interval I. In this case, the *averages*

$$\frac{1}{|I|}\int_I g(x)dx$$

exist. Following John/Nirenberg [1961, p. 416], one says that g has **bounded mean oscillation** if

$$\|g|BMO\| := \sup_I \frac{1}{|I|}\int_I \Big| g(x) - \frac{1}{|I|}\int_I g(y)dy \Big| dx$$

is finite. The collection of these functions will be denoted by $BMO(\mathbb{R})$. Obviously, $\|\cdot|BMO\|$ is a semi-norm that vanishes on the constant functions. Therefore, in order to get a Banach space, we must identify functions whose difference is constant.

Bounded measurable functions have bounded mean oscillation. However, $BMO(\mathbb{R})$ also contains unbounded members. A typical example is $\log|x|$.

6.7.12.12 A celebrated duality theorem says that $H_1^{\text{real}}(\mathbb{R})^* = BMO(\mathbb{R})$; see Fefferman [1971, p. 587] and Fefferman/Stein [1972, p. 145]. Indeed, let

$$\langle a, g\rangle := \int_{\mathbb{R}} a(x)g(x)dx$$

for every atom $a \in H_1^{\text{real}}(\mathbb{R})$ and $g \in BMO(\mathbb{R})$. Then it follows from (6.7.12.10.a) that

$$|\langle a, g\rangle| = \left| \int_I a(x)\Big(g(x) - \frac{1}{|I|}\int_I g(y)dy\Big)dx \right| \leq \|g|BMO\|.$$

Hence continuous extension yields a functional on $H_1^{\text{real}}(\mathbb{R})$.

6.7.12.13 According to Sarason [1975, p. 392], a function g with the property that

$$\frac{1}{|I|}\int_I \left| g(x) - \frac{1}{|I|}\int_I g(y)dy \right| dx \to 0 \quad \text{as} \quad |I| \to 0$$

is said to have **vanishing mean oscillation**. These functions form a closed subspace of $BMO(\mathbb{R})$ that is just the closure of all uniformly continuous functions. Nowadays, one prefers to work with a slightly smaller space: $VMO(\mathbb{R})$ is defined to be the closure of the continuous functions with compact support. This modification goes back to Coifman/Weiss [1977, p. 638], who proved a remarkable supplement to the duality relation $H_1^{\text{real}}(\mathbb{R})^* = BMO(\mathbb{R})$, namely $VMO(\mathbb{R})^* = H_1^{\text{real}}(\mathbb{R})$.

6.7.12.14 The researchers interested in Hardy spaces split into two almost disjoint groups; more than half of those people are interested in properties of the members of H_p, while the other ones like structure theory. The rest of this subsection is devoted to the latter aspect.

The modern H_p-theory was decisively stimulated by *Pełczyński's ten-lecture marathon* delivered at a conference on *Banach Spaces and Analytic Functions* held at Kent State University in 1976; see [PEŁ].

Further surveys were given by Wojtaszczyk [1983], [2003], Kislyakov [1991] and Gamelin/Kislyakov [2001].

6.7.12.15 Let $1 < p < \infty$. Then, by the Marcel Riesz theorem, $H_p(\mathbb{T})$ is a complemented subspace of $L_p(\mathbb{T})$. On the other hand, Boas [1955] showed that $H_p(\mathbb{T})$ and $L_p(\mathbb{T})$ are isomorphic. Taylor [1950/51, Part II, pp. 46–47] had earlier identified $H_p(\mathbb{T})^*$ and $H_{p^*}(\mathbb{T})$. In the real setting, we have $H_p^{\text{real}}(\mathbb{T}) = L_p(\mathbb{T})$. Hence the representation of $H_p^{\text{real}}(\mathbb{T})^*$ becomes trivial.

The preceding facts show that for $1 < p < \infty$, the structure of H_p does not yield anything new. On the other hand, these spaces play an important role in operator theory; see Subsections 6.9.9 and 6.9.10.

6.7.12.16 Next, I treat the limiting cases $p=1$ and $p=\infty$ in which the situation is completely different. As a general rule, one may state that $H_1(\mathbb{T})$ is quite different from any space $L_1(M,\mathscr{M},\mu)$, while $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ are rather far from $C(K)$.

Newman [1961] proved that $H_1(\mathbb{T})$ fails to be complemented in $L_1(\mathbb{T})$. Similarly, according to Rudin [1962, p. 432] and Pełczyński [1964a, p. 289], the disk algebra $A(\mathbb{T})$ is not complemented in $C(\mathbb{T})$. Later on, it was proved in [PEŁ, p. 24] that neither $A(\mathbb{T})$ nor $H_\infty(\mathbb{T})$ can be isomorphic to quotients of any $C(K)$. Consequently,

- $H_1(\mathbb{T})$ is not an $\mathscr{L}_1$-space; see also 6.7.12.21.
- $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ are not $\mathscr{L}_\infty$-spaces.

6.7.12.17 The brothers Riesz [1916] proved a famous theorem that in modern terminology says that every $\mu\in C(\mathbb{T})^*$ with the property

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\theta}\,d\mu(\theta)=0 \quad \text{for } k=1,2,\dots$$

is absolutely continuous with respect to the Lebesgue measure; see also 6.7.12.2. Hence there exists $g\in L_1(\mathbb{T})$ such that

$$\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,d\mu(\theta)=\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})g(\mathrm{e}^{\mathrm{i}\theta})\,d\theta \quad \text{for } f\in C(\mathbb{T}).$$

The **Riesz–Riesz theorem** has multifarious consequences. As complex counterparts of the duality relations $\mathit{VMO}(\mathbb{R})^*=H_1^{\mathrm{real}}(\mathbb{R})$ and $H_1^{\mathrm{real}}(\mathbb{R})^*=\mathit{BMO}(\mathbb{R})$, we get

$$\big(C(\mathbb{T})/zA(\mathbb{T})\big)^*=H_1(\mathbb{T}) \quad \text{and} \quad H_1(\mathbb{T})^*=L_\infty(\mathbb{T})/zH_\infty(\mathbb{T}).$$

The left-hand formula yields further evidence that $H_1(\mathbb{T})$ and $L_1(\mathbb{T})$ behave quite differently. The first space has a predual, while the second one does not.

The extreme points of the closed unit ball of $H_1(\mathbb{T})$ were described by de Leeuw/ Rudin [1958]; see also [DUR, p. 124]. On the other hand, Kreĭn/Milman [1940, p. 137] knew that the closed unit ball of $L_1(\mathbb{T})$ does not have any extreme point; see also 5.4.1.6.

6.7.12.18 The basis problem for H_1 was open for a long time. First of all, Billard [1971, p. 317] discovered that the Haar functions on $\mathbb{T}$ form a basis of $H_1^{\mathrm{real}}(\mathbb{T})$. However, as observed by Kwapień/Pełczyński [1976, pp. 285–286], this basis fails to be unconditional. The final goal was achieved by Maurey [1980], Carleson [1980], and Wojtaszczyk. The latter proved in [1980, p. 294] that the Franklin system is an unconditional basis of $H_1^{\mathrm{real}}[0,1]$. Note that the Franklin system had already been used by Bochkarev [1974] for constructing a basis of the disk algebra. I also stress the remarkable interplay with the theory of wavelets; see 5.6.6.12.

With the help of real techniques, Jones [1985] showed (by a proof of 40 pages) that the non-separable space $\mathit{BMO}(\mathbb{R})$ has the bounded approximation property. Frustratingly, the approximation problem is still open for $H_\infty(\mathbb{R})$.

6.7.12.19 Obviously, $A(\mathbb{T})$ is a subspace of $H_\infty(\mathbb{T})$. On the other hand, Pełczyński [PEŁ, p. 11] inferred from the Riesz–Riesz theorem that $H_\infty(\mathbb{T})$ can be identified with a complemented subspace of $A(\mathbb{T})^{**}$. Thus, by the principle of local reflexivity, $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ have the same local properties. In particular, Pełczyński [1974] showed that $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ do not possess local unconditional structure. He even proved that the 1-summing embedding map $Id : A(\mathbb{T}) \to H_1(\mathbb{T})$ does not factor through any space $L_1(M,\mathscr{M},\mu)$. In other words, $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ fail to have the Gordon–Lewis property; see 6.3.13.8.

6.7.12.20 Further properties of the Banach spaces $A(\mathbb{T})$ and $H_\infty(\mathbb{T})$ were established in a seminal paper of Bourgain [1984a]. He discovered that $A(\mathbb{T})^*$ and $H_\infty(\mathbb{T})^*$ have Rademacher cotype 2.

6.7.12.21 Let

$$J_1 : (\xi_k) \mapsto \sum_{k=0}^{\infty} \xi_k e^{i2^k t} \quad \text{and} \quad Q_1 : f(t) \mapsto \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-i2^k t} dt \right).$$

Extending earlier results on lacunary coefficients of trigonometric series, Paley [1933] showed that Q_1 is a surjection from $H_1(\mathbb{T})$ onto l_2. A corollary of this fact says that $H_1(\mathbb{T})$ contains a complemented subspace isomorphic to l_2: the range of the **Paley projection** J_1Q_1; see Kwapień/Pełczyński [1976, p. 269]. This remarkable property implies that $H_1(\mathbb{T})$ fails to be an $\mathscr{L}_1$-space; see Lindenstrauss/Pełczyński [1968, p. 295].

6.7.12.22 Now I am able to present a clever proof of Grothendieck's theorem 6.3.9.1, which is based on the following diagram:

$$\begin{tikzcd}
A(\mathbb{T}) \arrow[r, "Id"] \arrow[dr, "Q_0"] & H_1(\mathbb{T}) \arrow[d, "Q_1"] \\
l_1 \arrow[u, "T_0"] \arrow[r, "T"] & l_2
\end{tikzcd}.$$

Indeed, Vinogradov [1968, стр. 512–513] improved Paley's result by showing that Q_0, the restriction of Q_1 to the disk algebra $A(\mathbb{T})$, remains a surjection onto l_2; see Fournier [1974, pp. 403–405] for a constructive approach. Hence every $T \in \mathfrak{L}(l_1, l_2)$ admits a lifting T_0. Finally, it follows that $T = Q_0T_0 = Q_1\, Id\, T_0$ is 1-summing, since it contains the 1-summing factor $Id : A(\mathbb{T}) \to H_1(\mathbb{T})$.

Usually, people are fighting for priority. In this case, however, we have the opposite situation. According to Pełczyński [PEŁ, p. 23], he discovered the proof in collaboration with his pupil Wojtaszczyk. On the other hand, Wojtaszczyk claims that, being unaware of Grothendieck's theorem, he did nothing other than communicating to Pełczyński the theorem of Vinogradov mentioned above.

6.7.12.23 Most of the preceding considerations were extended to functions of N complex variables. However, this was a hard task. The main achievements are due to Stein and his school; see Stein/Weiss [1960], Fefferman/Stein [1972], [STEIN], and [UCH]. A first summary can be found in Peetre's seminar notes, [PEE$_2$].

6.7.12.24 In several variables, there are two substitutes for the open unit disk:

the N-ball $\qquad \mathbb{B}^N := \Big\{ (z_1,\dots,z_N) : \sum_{k=1}^{N} |z_k|^2 < 1 \Big\}$

and

the N-disk $\qquad \mathbb{D}^N := \Big\{ (z_1,\dots,z_N) : \max_{1\le k\le N} |z_k| < 1 \Big\}.$

Properties that characterize the members of $\mathscr{H}_p(\mathbb{B}^N)$ and $\mathscr{H}_p(\mathbb{D}^N)$ were considered by Bochner [1960]. Subsequently, the case of the N-ball was treated in [RUD$_2$, Section 5.6]. Concerning the N-disk the reader is referred to [STEIN$^+$, pp. 114–131]. In fact, these authors dealt with the N-fold upper half-plane. Finally, considering boundary values led to the real Hardy spaces on $\mathbb{R}^N$, which have become the favorite choice.

6.7.12.25 The isomorphic classification of spaces of analytic functions depends on the dimension as well as on the shape of the underlying domain of definition.

- The question whether $\mathscr{A}(\mathbb{B}^M)$ and $\mathscr{A}(\mathbb{B}^N)$ are isomorphic for $M > N > 1$ is open. However, Mityagin/Pełczyński [1976] showed that $\mathscr{A}(\mathbb{B}^M) \ncong \mathscr{A}(\mathbb{B}^1)$ for $M > 1$.
- $\mathscr{A}(\mathbb{D}^M)$ and $\mathscr{A}(\mathbb{D}^N)$ are non-isomorphic for $M \neq N$, Mityagin/Pełczyński [1976], Bourgain [1984b].
- $\mathscr{A}(\mathbb{B}^N)$ and $\mathscr{A}(\mathbb{D}^N)$ are non-isomorphic for $N \geq 2$, Khenkin [1968].
- all spaces $\mathscr{H}_1(\mathbb{B}^N)$ with $N = 1,2,\dots$ are isomorphic, Wolniewicz [1989].
- $\mathscr{H}_1(\mathbb{D}^M)$ and $\mathscr{H}_1(\mathbb{D}^N)$ are non-isomorphic for $M \neq N$, Bourgain [1982b], [1983a].
- $\mathscr{H}_1(\mathbb{B}^N)$ and $\mathscr{H}_1(\mathbb{D}^N)$ are non-isomorphic for $N \geq 2$, since $\mathscr{H}_1(\mathbb{B}^N) \cong \mathscr{H}_1(\mathbb{D}) \ncong \mathscr{H}_1(\mathbb{D}^N)$.
- all spaces $\mathscr{H}_1^{\mathrm{real}}(\mathbb{R}^N)$ with $N = 1,2,\dots$ are isomorphic, Maurey [1980, p. 117].

6.7.13 Bergman spaces

6.7.13.1 Let $1 \leq p < \infty$. The **Bergman space** $\mathscr{L}\mathscr{A}_p(\mathbb{D})$ consists of all analytic functions f on the open unit disk $\mathbb{D}$ for which

$$\|f|\mathscr{L}\mathscr{A}_p\| := \left(\frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \, r\, dr\, d\theta \right)^{1/p}$$

is finite. In the case $p = 2$, this definition goes back to Bergman(n) [1922], who, however, dealt with harmonic functions. A closely related paper was written by Bochner [1922].

The classical reference is [BERG]. Modern presentations are give in [HED$^+$], [ZHU], and in Wojtaszczyk's survey [2003].

6.7.13.2 Obviously, $\mathscr{L}\mathscr{A}_p(\mathbb{D})$ can be viewed as a closed subspace of $L_p(\mathbb{D})$, where $\mathbb{D}$ is equipped with the normalized Lebesgue measure: $dA(z) = \frac{1}{\pi} r\,dr\,d\theta$. Analyticity provides us with an important tool. We have the representation

$$f(w) = \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^2}\,dA(z) \quad \text{for } f \in \mathscr{L}\mathscr{A}_p(\mathbb{D}).$$

One refers to

$$K(w,z) := \frac{1}{(1-w\bar{z})^2} = \sum_{k=1}^{\infty} (k+1)\, w^k \bar{z}^k$$

as the **reproducing kernel** or **Bergman kernel**.

Zakharyuta/Yudovich [1964, стр. 141] were the first to show that for $1<p<\infty$,

$$P : f(z) \mapsto g(w) := \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^2}\,dA(z)$$

defines a map from $L_p(\mathbb{D})$ onto $\mathscr{L}\mathscr{A}_p(\mathbb{D})$, which is called the **Bergman projection**. According to Shields/Williams [1971, p. 294], the same holds for $p=1$. Using this fact, Lindenstrauss/Pełczyński [1971, p. 247] proved that $\mathscr{L}\mathscr{A}_p(\mathbb{D})$ is isomorphic to the sequence space l_p.

6.7.13.3 As another consequence, Zakharyuta/Yudovich [1964, стр. 139] obtained the formula $\mathscr{L}\mathscr{A}_p(\mathbb{D})^* = \mathscr{L}\mathscr{A}_{p^*}(\mathbb{D})$ for $1<p<\infty$, the norms being equivalent. This identification is based on the pairing

$$\langle f,g\rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)g(\bar{z})\,dA(z) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta})g(re^{-i\theta})\,r\,dr\,d\theta = \sum_{k=0}^{\infty} \tfrac{1}{k+1} a_k b_k$$

whenever

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{L}\mathscr{A}_p(\mathbb{D}) \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathscr{L}\mathscr{A}_{p^*}(\mathbb{D}).$$

6.7.13.4 The **Bloch space** $\mathscr{B}(\mathbb{D})$ consists of all analytic functions f on the open unit disk for which

$$\|f|\mathscr{B}\| := |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)|$$

is finite. The functions $f \in \mathscr{B}(\mathbb{D})$ such that

$$\lim_{|z|\nearrow 1} (1-|z|^2)|f'(z)| = 0$$

form the **little Bloch space** $\mathscr{B}_0(\mathbb{D})$, which can also be obtained as the closure of the set of all polynomials; see Anderson/Clunie/Pommerenke [1974, pp. 13–14].

The Bergman projection P maps $L_\infty(\mathbb{D})$ onto $\mathscr{B}(\mathbb{D})$ and $C(\overline{\mathbb{D}})$ onto $\mathscr{B}_0(\mathbb{D})$; see Coifman/Rochberg/Weiss [1976, pp. 631–632] and Axler [1988, pp. 17, 21].

I stress that in this case, the name *Bergman projection* is misleading: P fails to be a projection, since $\mathscr{B}(\mathbb{D}) \not\subseteq L_\infty(\mathbb{D})$ and $\mathscr{B}_0(\mathbb{D}) \not\subseteq C(\overline{\mathbb{D}})$.

Axler [1988, pp. 14–15, 19] proved that with respect to the pairing

$$\langle f,g\rangle_{\mathbb{D}} := \lim_{r\nearrow 1}\int_{|z|\le r} f(z)g(\overline{z})\,dA(z) = \lim_{r\nearrow 1}\sum_{k=0}^{\infty}\tfrac{1}{k+1}a_k b_k r^{2k+1},$$

the following formulas hold:

$$\mathscr{B}_0(\mathbb{D})^* = \mathscr{L}\mathscr{A}_1(\mathbb{D}) \quad\text{and}\quad \mathscr{B}(\mathbb{D}) = \mathscr{L}\mathscr{A}_1(\mathbb{D})^*,$$

the norms being equivalent. Earlier results of Duren/Romberg/Shields [1969, p. 35], Shields/Williams [1971, pp. 292, 296], and Anderson/Clunie/Pommerenke [1974, pp. 16–17] were obtained using slightly different pairings such as

$$\langle f,g\rangle_{\mathbb{T}} := \lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta})g(r\mathrm{e}^{-\mathrm{i}\theta})d\theta = \lim_{r\nearrow 1}\sum_{k=0}^{\infty}a_k b_k r^{2k}.$$

Note that the sum $\sum_{k=0}^\infty a_k b_k$ need not converge.

Since $\mathscr{L}\mathscr{A}_1(\mathbb{D})^*$ is isomorphic to l_1, we may conclude from $\mathscr{B}(\mathbb{D}) = \mathscr{L}\mathscr{A}_1(\mathbb{D})^*$ that $\mathscr{B}(\mathbb{D})$ is isomorphic to l_∞. Moreover, the formula $\mathscr{B}(\mathbb{D}) = \mathscr{B}_0(\mathbb{D})^{**}$ suggests that $\mathscr{B}_0(\mathbb{D})$ should be isomorphic to c_0. A proof of this fact can be obtained by carefully reading a paper of Shields/Williams [1978]. On pp. 275–279 they establish this result for the "harmonic" little Bloch space that contains its "analytic" counterpart as a complemented subspace; see p. 263. Another approach via M-ideals is sketched in 6.9.4.12.

6.7.13.5 For $\alpha > -1$, the **weighted Bergman space** $\mathscr{L}\mathscr{A}_p^\alpha(\mathbb{D})$ is defined to be the collection of all analytic functions f on the open unit disk $\mathbb{D}$ for which

$$\|f|\mathscr{L}\mathscr{A}_p^\alpha\| := \left(\frac{1}{\pi}\int_0^1\int_0^{2\pi}|f(r\mathrm{e}^{\mathrm{i}\theta})|^p\,(1-r)^\alpha r\,dr\,d\theta\right)^{1/p}$$

is finite; see Horowitz [1974, p. 702].

Some authors prefer to consider Bergman spaces over the upper complex half-plane $\mathbb{C}_+ := \{\, z = x+\mathrm{i}t : t>0\,\}$:

$$\|f|\mathscr{L}\mathscr{A}_p^\alpha\| := \left(\int_0^\infty\int_{-\infty}^{+\infty}|f(x+\mathrm{i}t)|^p dx\, t^\alpha dt\right)^{1/p};$$

see Rochberg [1982, p. 914], [1985, p. 226].

6.7.13.6 Without any reference to the results just described, Oswald [1983, p. 421] invented the **analytic Besov space** $\mathscr{B}\mathscr{A}_{p,q}^\sigma(\mathbb{D})$, where $-\infty<\sigma<+\infty$ and $0<p,q<\infty$. The underlying quasi-norms

$$\|f|\mathscr{B}\mathscr{A}_{p,q}^\sigma\|_m := \sum_{0\le k<m}|f^{(k)}(0)| + \left\{\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}\big|f^{(m)}(r\mathrm{e}^{\mathrm{i}\theta})\big|^p d\theta\right)^{q/p}(1-r)^{(m-\sigma)q-1}dr\right\}^{1/q}$$

are equivalent for all $m = 0, 1, 2, \ldots$ such that $m > \sigma$. A special case was earlier considered by Peller [1980, стр. 542–547]. Quite likely, this is the reason why Peetre [1983, p. 305] refers to the approach above as well known.

The term *"Besov space"* is justified in view the following criterion; see Oswald [1983, p. 418]:

A function f belongs to $\mathscr{BA}^{\sigma}_{p,q}(\mathbb{D})$ if and only if the "boundary" distribution defined by

$$\langle f, \varphi \rangle := \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta}) \varphi(\mathrm{e}^{-\mathrm{i}\theta})\, d\theta \quad \text{for} \quad \varphi \in C^{\infty}(\mathbb{T})$$

is a member of the "ordinary" Besov space $B^{\sigma}_{p,q}(\mathbb{T})$ and all of its Fourier coefficients with negative indices vanish. In other words, one may identify $\mathscr{BA}^{\sigma}_{p,q}(\mathbb{D})$ with the "analytic part" of $B^{\sigma}_{p,q}(\mathbb{T})$; see also Peetre [1983, pp. 304–305].

If one allows *harmonic* functions instead of *analytic* ones, then the considerations above yield all of $B^{\sigma}_{p,q}(\mathbb{T})$; see Flett [1972, pp. 130, 133–135].

6.7.13.7 Since passing from $r\,dr$ to dr is a harmless step, it follows that

$$\mathscr{LA}^{\alpha}_{p}(\mathbb{D}) = \mathscr{BA}^{-(\alpha+1)/p}_{p,p}(\mathbb{D}), \quad \text{and in particular,} \quad \mathscr{LA}_{p}(\mathbb{D}) = \mathscr{BA}^{-1/p}_{p,p}(\mathbb{D}).$$

6.7.13.8 For smoothness exponents $\sigma < 0$, Wojtaszczyk [1997, p. 13] constructed unconditional bases of $\mathscr{BA}^{\sigma}_{p,q}(\mathbb{D})$, and he was able to identify the associated sequence space as $b^{\sigma}_{p,q}$. This result should be compared with that about bases in $B^{\sigma}_{p,q}(\mathbb{T})$; see 6.7.7.4. As a special case, one gets a concrete isomorphism between $\mathscr{LA}_p(\mathbb{D})$ and l_p, which improves the Lindenstrauss–Pełczyński theorem 6.7.13.2.

6.7.13.9 Dzhrbashyan [1948] was among the first to consider weighted Bergman spaces $\mathscr{LA}^{\alpha}_{p}(\mathbb{D})$, usually denoted by A^{p}_{α}. He published his paper in a journal of the Armenian Academy of Sciences (Yerevan). Hence, apart from the Soviet Union, his results were not available to the mathematical community. Now an English translation can be found on the Internet; see the bibliography.

The authors of the Teubner Text [DJR$^+$, p. 8] vehemently fight for the Armenian priority:

In view of what has been said it is natural to call the spaces A^{p}_{α} M.M. Djrbashian spaces. Of course, we have no objection against the term "Bergman kernel" as a substitute for the term "representing or reproducing kernel". However, we want to stress that S. Bergman NEVER considered the class A^{p}_{α} in his works and hence there is no scientific reason to call them "Bergman spaces".

Unfortunately, this pleading is pointless inasmuch as the condition that defines $\mathscr{BA}^{\sigma}_{p,q}(\mathbb{D})$ for $\sigma < m := 0$ was already invented by Hardy/Littlewood [1932, p. 412] when they proved that $\mathscr{H}_p(\mathbb{D}) \subset \mathscr{BA}^{\sigma}_{p_0,p}(\mathbb{D})$ if $\sigma = 1/p_0 - 1/p < 0$.

6.7.14 Orlicz spaces

6.7.14.1 A convex, non-decreasing, and continuous function Φ is called a **Young function**, or **Orlicz function**, if

$$\Phi(0)=0, \quad \Phi(u)\geq 0 \quad \text{for } u>0 \quad \text{and} \quad \lim_{u\to\infty}\Phi(u)=\infty.$$

Some authors require additional properties such as $\Phi(u)>0$ for $u>0$,

$$\lim_{u\to 0}\frac{\Phi(u)}{u}=0 \quad \text{and} \quad \lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty. \tag{6.7.14.1.a}$$

6.7.14.2 Birnbaum/Orlicz [1931, p. 13] proved that Young functions can be written in the form

$$\Phi(u)=\int_0^u p(s)\,ds,$$

where the *right* continuous density p is obtained as the *right* derivative of Φ. There is also a "left wing"; see [LUX, p. 37], [BENN$^+$, p. 265], and [RAO$^+$, p. 7].

6.7.14.3 The *right* inverse of p is defined by $q(t):=\inf\{s\geq 0 : p(s)>t\}$. Letting

$$\Psi(v):=\sup_{u\geq 0}\big[uv-\Phi(u)\big]=\int_0^v q(t)\,dt$$

yields the **complementary Young function**, and we have **Young's inequality**:

$$uv\leq \Phi(u)+\Psi(v) \quad \text{for } u,v\geq 0;$$

see Young [1912, p. 226] and Birnbaum/Orlicz [1931, pp. 15–16].

6.7.14.4 The classical examples are

$$\Phi(u)=\frac{u^p}{p} \quad \text{and} \quad \Psi(v)=\frac{v^q}{q} \quad \text{with } 1<p,q<\infty \text{ and } 1/p+1/q=1.$$

In order to include the limiting case $p=1$ and $q=\infty$,

$$\Phi(u)=u \quad \text{and} \quad \Psi(v)=\begin{cases} 0 & \text{if } 0\leq v\leq 1,\\ \infty & \text{if } 1<v<\infty,\end{cases}$$

one frequently allows that Young functions may take the value ∞.

6.7.14.5 According to Birnbaum/Orlicz [1931, p. 20], a Young function Φ is said to satisfy the Δ_2**-condition** if there exists a constant $c\geq 2$ such that

$$\Phi(2u)\leq c\,\Phi(u) \quad \text{for } u>0. \tag{6.7.14.5.a}$$

This condition splits into two parts: it may be required *"for small values of u"* and/or *"for large values of u,"* which means that $0<u<a$ and/or $b<u<\infty$.

The Young functions

$$\Phi(u):=(1+u)\log(1+u)-u \quad \text{and} \quad \Psi(v):=\exp v-v-1$$

are complementary. While Φ satisfies the Δ_2-condition, Ψ does not; see Orlicz [1936, p. 96].

6.7.14.6 Fix any Young function Φ, and let $(M,\mathscr{M},\mu)$ be a measure space. Then, by analogy with the definition of $L_p(M,\mathscr{M},\mu)$, one may consider the collection of all measurable functions f for which

$$\rho_\Phi(f) := \int_M \Phi\big(|f(t)|\big)\,d\mu(t) \tag{6.7.14.6.a}$$

is finite. Unfortunately, we do not always obtain a linear space. However, already Orlicz [1932, p. 209] discovered that everything becomes fine if the underlying Young function satisfies (6.7.14.1.a) and (6.7.14.5.a). Subject to these conditions, one even gets a Banach space $L_\Phi(M,\mathscr{M},\mu)$, in shorthand L_Φ, equipped with the **Orlicz norm**

$$\|f|L_\Phi\|_{\mathrm{Orl}} := \sup\left\{\left|\int_M f(t)g(t)\,d\mu(t)\right| : \rho_\Psi(g)\le 1\right\},$$

where Ψ denotes the complementary Young function.

Only much later, Luxemburg observed in his thesis [LUX, pp. 46–47] that the situation improves if L_Φ is defined to consist of all measurable functions f for which $\|f|L_\Phi\|_{\mathrm{Orl}} < \infty$. Then, without additional assumptions, L_Φ becomes a Banach space.

Another approach is more elementary. According to Orlicz [1936, pp. 93–94] and Luxemburg [LUX, pp. 43, 47], the **Orlicz space** L_Φ coincides with the linear hull of the absolutely convex set $B_\Phi := \big\{f : \rho_\Phi(f)\le 1\big\}$; and the **Luxemburg norm**

$$\|f|L_\Phi\|_{\mathrm{Lux}} := \inf\Big\{\rho > 0 : f\in\rho B_\Phi\Big\}$$

is nothing but a Minkowski functional 3.3.1.2.

6.7.14.7 Orlicz spaces can be defined over arbitrary measure spaces. However, to be on the safe side, one should assume σ-finiteness. The most typical examples are $[0,1]$, the unit interval equipped with the Lebesgue measure, and $\mathbb{N}$, the set of natural numbers equipped with the counting measure. These two cases lead to the two parts of the Δ_2-condition: $[0,1]$ requires large arguments, while $\mathbb{N}$ requires small arguments; see Orlicz [1936, Bedingungen (8) and (20)]. The corresponding Orlicz spaces are denoted by $L_\Phi[0,1]$ and l_Φ. It is also quite common to work on $\mathbb{R}_+$.

6.7.14.8 Let Φ and Ψ be complementary Young functions. Then, in view of Young's inequality, L_Φ and L_Ψ form a dual system with respect to the bilinear form

$$\langle f,g\rangle := \int_M f(t)g(t)\,d\mu(t).$$

It follows that every $g\in L_\Psi$ defines a functional $\ell : f\mapsto\langle f,g\rangle$ on L_Φ. Moreover, assuming (6.7.14.1.a) and (6.7.14.5.a), Orlicz [1932, p. 216] showed that every functional ℓ on L_Φ can be obtained in this way: $L_\Phi^* = L_\Psi$.

The general situation was treated by Krasnoselskiĭ/Rutitskiĭ [1954] and Luxemburg [LUX, p. 56]. Since L_Φ is sometimes too big, one must pass to a smaller subspace E_Φ, which consists of all functions f such that $\rho_\Phi(\lambda f) < \infty$ for every $\lambda > 0$. The subspace E_Φ can also be obtained as the closed linear span of all characteristic functions of sets with finite measure; see [LUX, p. 55]. Now it follows that $E_\Phi^* = L_\Psi$.

6.7.14.9 Orlicz [1932, p. 218], [1936, pp. 98, 104] proved that for complementary Young functions Φ and Ψ, the following are equivalent:

$L_\Phi[0,1]$ is reflexive $\Leftrightarrow$ Φ and Ψ satisfy the Δ_2-condition for large arguments.

l_Φ is reflexive $\Leftrightarrow$ Φ and Ψ satisfy the Δ_2-condition for small arguments.

To ensure the reflexivity of $L_\Phi(M,\mathscr{M},\mu)$ for arbitrary σ-finite measure spaces, Φ and Ψ must satisfy the Δ_2-condition for all arguments; see [LUX, p. 60].

6.7.14.10 According to [MAL, p. 125], every Orlicz space $L_\Phi(M,\mathscr{M},\mu)$ is an interpolation space of the couple $\big\{L_1(M,\mathscr{M},\mu), L_\infty(M,\mathscr{M},\mu)\big\}$. In the classical case of a finite interval, this result is due to Orlicz [1935, p. 133]. First investigations about interpolation between Orlicz spaces L_{Φ_0} and L_{Φ_1} were made by Gustavsson/Peetre [1977].

6.7.14.11 In the case of Orlicz spaces over $\mathbb{R}_+$, the Boyd indices (see 6.6.7.10)

$$p_0(L_\Phi) := \lim_{a\to\infty} \frac{\log a}{\log \|D_a : L_\Phi \to L_\Phi\|} = \sup\big\{\, p_0 : \|D_a : L_\Phi \to L_\Phi\| \le a^{1/p_0} \quad \text{for large } a\big\}$$

and

$$p_1(L_\Phi) := \lim_{a\to 0} \frac{\log a}{\log \|D_a : L_\Phi \to L_\Phi\|} = \inf\big\{\, p_1 : \|D_a : L_\Phi \to L_\Phi\| \le a^{1/p_1} \quad \text{for small } a\big\}$$

are obtained by using

$$d_{\Phi^{-1}}(a) := \sup\left\{ \frac{\Phi^{-1}(t)}{\Phi^{-1}(t/a)} : 0 < t < \infty \right\}$$

instead of $\|D_a : L_\Phi \to L_\Phi\|$. Letting

$$d_\Phi(b) := \sup\left\{ \frac{\Phi(t)}{\Phi(t/b)} : 0 < t < \infty \right\},$$

we formally have

$$\frac{1}{a} = d_\Phi(b) \qquad \Leftrightarrow \qquad \frac{1}{b} = d_{\Phi^{-1}}(a);$$

see Boyd [1967, p. 612] or [MAL, Chap. 11]. Hence

$$p_0(L_\Phi) := \lim_{b\to 0} \frac{\log d_\Phi(b)}{\log b} = \sup\big\{\, p_0 : d_\Phi(b) \le b^{p_0} \quad \text{for small } b\big\}$$

and

$$p_1(L_\Phi) := \lim_{b\to\infty} \frac{\log d_\Phi(b)}{\log b} = \inf\big\{\, p_1 : d_\Phi(b) \le b^{p_1} \quad \text{for large } b\big\}.$$

Matuszewska/Orlicz [1960, p. 439] were the first to introduce various indices of Φ-functions. The results above are due to Boyd [1971, p. 316]; see also [BENN$^+$, p. 277] and [LIND$_2^+$, p. 139].

6.7.14.12 In [1951], Lorentz considered function spaces whose members are defined by properties like

$$\left(\int_0^\infty \big(w(t)f^*(t)\big)^q \frac{dt}{t}\right)^{1/q} < \infty,$$

where w is a certain weight, say $w(t) = t^{1/p}(1+|\log t|)$. Later on, he looked for conditions under which such spaces can be viewed as Orlicz spaces and vice versa. It turned out that this happens only in exceptional cases; see Lorentz [1961, p. 132].

6.7.14.13 Taking (6.7.14.6.a) as a pattern, Nakano [1951] introduced the concept of a **modular**. This is a non-negative function ρ on a linear space X such that

$$\rho(x)=0 \text{ if and only if } x=\mathrm{o}, \quad \rho(\lambda x) = \rho(x) \text{ for } |\lambda|=1,\ x\in X,$$

and

$$\rho\big((1-\theta)x_0+\theta x_1\big) \le (1-\theta)\rho(x_0)+\theta\rho(x_1) \quad \text{for } 0<\theta<1,\ x_0,x_1\in X.$$

The value ∞ is admitted.

Obviously, norms are modulars. An illuminating example is given by

$$\rho(x) := \sum_{k=1}^{\infty} |\xi_k|^k \quad \text{for } \textit{every} \text{ sequence } x=(\xi_k).$$

Nakano's theory was further developed by Musielak/Orlicz [1959, p. 49], who, instead of convexity, used the weaker condition

$$\rho\big(\theta x_0+(1-\theta)x_1\big) \le \rho(x_0)+\rho(x_1) \quad \text{for } 0<\theta<1,\ x_0,x_1\in X.$$

6.7.14.14 To simplify matters, the following considerations are concerned only with **convex** modulars in the sense of Nakano. Then $B_\rho := \{x \in X : \rho(x) \le 1\}$ is an absolutely convex subset of the linear space

$$X_\rho := \{x\in X : \rho(\lambda x) \text{ is finite for some } \lambda>0\}.$$

The Minkowski functional associated with B_ρ coincides with the *second norm* in Nakano's paper [1951, p. 113]. In the case of Orlicz spaces, one gets the Luxemburg norm. Concerning Nakano's *first norm*, which corresponds to the Orlicz norm, the reader should consult [1951, p. 95].

6.7.14.15 There exists an abundant literature on Orlicz spaces and modular spaces. The book [NAK] can be viewed as a forerunner of the classical monograph [KRA$^+$]. Further references are [KOZ], [MUS], [MAL], and [RAO$^+$]; see also [BENN$^+$, pp. 265–280]. Geometric properties of Orlicz spaces are treated in [LIND$_1^+$, Chap. 4] and [CHEN].

6.8 Probability theory on Banach spaces

Having in mind applications to probability theory, we deal only with measures of *total mass* 1. The underlying linear spaces are supposed to be *real*.

6.8.1 Baire, Borel, and Radon measures

Locally convex linear spaces are completely regular Hausdorff spaces. Hence all relevant results from Section 4.6 apply. This is, in particular, true in the case of Banach spaces equipped with the norm topology or the weak topology.

6.8.1.1 Measure theory on Banach spaces began when Löwner [1939, pp. 816–817] proved that there exists no translation invariant measure on the infinite-dimensional Hilbert space:

Sei $\mathscr{K}$ eine Kugel vom Radius $a > 0$. Man denke sich nun ein cartesisches Achsenkreuz, dessen Mittelpunkt mit dem Mittelpunkt von $\mathscr{K}$ zusammenfällt. Um die Mittelpunkte der auf den positiven und negativen Achsen liegenden Radien von $\mathscr{K}$ schlage man je eine Kugel vom Radius $b < a/(2\sqrt{2})$. So erhält man eine Folge von Kugeln von gleichem Radius, welche paarweise punktfremd sind und alle in $\mathscr{K}$ liegen.

6.8.1.2 On Banach spaces X, several canonical σ-algebras have been considered:

$\mathscr{B}_{\text{aire}}(X, \text{norm})$: Baire subsets in the norm topology,
$\mathscr{B}_{\text{orel}}(X, \text{norm})$: Borel subsets in the norm topology,
$\mathscr{B}_{\text{aire}}(X, \text{weak})$: Baire subsets in the weak topology,
$\mathscr{B}_{\text{orel}}(X, \text{weak})$: Borel subsets in the weak topology.

These σ-algebras coincide on separable spaces. In the general case, we know only that

$$\mathscr{B}_{\text{aire}}(X, \text{norm}) = \mathscr{B}_{\text{orel}}(X, \text{norm})$$

and

$$\mathscr{B}_{\text{aire}}(X, \text{weak}) \subseteq \mathscr{B}_{\text{orel}}(X, \text{weak}) \subseteq \mathscr{B}_{\text{orel}}(X, \text{norm}).$$

A criterion of Edgar implies that $\{o\} \notin \mathscr{B}_{\text{aire}}(l_2(\mathbb{I}), \text{weak})$ if $\mathbb{I}$ is uncountable; see 6.8.2.1 and [VAKH$^+$, стр. 26]. Therefore $\mathscr{B}_{\text{aire}}(l_2(\mathbb{I}), \text{weak}) \neq \mathscr{B}_{\text{orel}}(l_2(\mathbb{I}), \text{weak})$. According to Talagrand [1978, p. 1002], the right-hand inclusion may be strict as well: $\mathscr{B}_{\text{orel}}(l_\infty, \text{weak}) \neq \mathscr{B}_{\text{orel}}(l_\infty, \text{norm})$.

On the other hand, Fremlin [1980] observed that there exist non-separable Banach spaces for which $\mathscr{B}_{\text{aire}}(X, \text{weak}) = \mathscr{B}_{\text{orel}}(X, \text{norm})$. Take, for example, $l_1(\mathbb{I})$ with $\text{card}(\mathbb{I}) = \aleph_1$. A result of Edgar [1977, p. 664], [1979, p. 561] says that

$$\mathscr{B}_{\text{orel}}(X, \text{weak}) = \mathscr{B}_{\text{orel}}(X, \text{norm})$$

for all Banach spaces that admit an equivalent norm with the Kadets–Klee property. This is, in particular, true for weakly compactly generated spaces.

6.8.1.3 On dual spaces, there exist two more canonical σ-algebras, namely

$\mathscr{B}_{\text{aire}}(X^*, \text{weak}^*)$: Baire subsets in the weak* topology,
$\mathscr{B}_{\text{orel}}(X^*, \text{weak}^*)$: Borel subsets in the weak* topology.

As above, we have $\mathscr{B}_{\text{aire}}(l_2(\mathbb{I}), \text{weak}^*) \neq \mathscr{B}_{\text{orel}}(l_2(\mathbb{I}), \text{weak}^*)$ for every uncountable index set $\mathbb{I}$. Moreover, $\mathscr{B}_{\text{aire}}(l_\infty, \text{weak}^*) \neq \mathscr{B}_{\text{aire}}(l_\infty, \text{weak})$; see [VAKH$^+$, стр. 29]. Finally, Talagrand [1978, p. 1002] showed that $l_\infty \notin \mathscr{B}_{\text{orel}}(l_\infty^{**}, \text{weak}^*)$, which implies that $\mathscr{B}_{\text{orel}}(l_\infty^{**}, \text{weak}^*) \neq \mathscr{B}_{\text{orel}}(l_\infty^{**}, \text{weak})$, since l_∞ is a weakly closed subset of l_∞^{**}.

6.8.1.4 By a **Radon measure** on a locally convex linear space X we mean a measure μ on $\mathscr{B}_{\text{orel}}(X)$ such that

$$\mu(B) = \sup\{\mu(K) : K \subseteq B, \ K \text{ compact}\} \quad \text{for } B \in \mathscr{B}_{\text{orel}}(X);$$

see [SCHW$_2$, p. 13]. Many authors use the name **tight measure**; see 4.6.10.

Obviously, every Radon measure on a Banach space is concentrated on a separable closed subspace.

6.8.1.5 A classical result of Ulam implies that for separable Banach spaces X, all measures on $\mathscr{B}_{\text{aire}}(X, \text{norm})$ are not only inner (and outer) regular but even Radon measures. According to Oxtoby/Ulam [1939, footnote ^{3}) on p. 561], a full proof was supposed to be published in a note *Sur la mesure dans un espace séparable et complet* submitted to Comptes Rendus. This paper of Ulam never did appear (quite likely because of the German invasion of France in 1940).

The following example shows that the conclusion above may fail in the non-separable case:

Consider the canonical map from $[0,1]$ into $l_2([0,1])$ that assigns to every point $t \in [0,1]$ its characteristic function e_t. If we assume the existence of a measurable cardinal, then the Lebesgue measure on $[0,1]$ admits an extension to $\mathscr{P}([0,1])$, the power set of $[0,1]$; see Solovay [1970, 1971] and [TAL, p. 189]. The image of this extension is defined for all subsets of $l_2([0,1])$. Obviously, compact subsets have measure zero, since they contain only a finite number of e_t's.

6.8.1.6 Roughly speaking, we get the same Radon measures for the norm topology and the weak topology of every Banach space X. Indeed, using the Dunford–Pettis–Phillips theorem 5.1.3.3, Grothendieck [GRO$_1$, pp. 100–104] showed that every Radon measure on $\mathscr{B}_{\text{aire}}(X, \text{weak})$ can be (uniquely) extended to a Radon measure on $\mathscr{B}_{\text{aire}}(X, \text{norm}) = \mathscr{B}_{\text{orel}}(X, \text{norm})$.

A measure-theoretic proof of this fact is due to Tortrat [1977, p. 135]; see also Schwartz [1976, p. 5] and [TAL, pp. 19–20, 142, 209].

6.8.1.7 According to Alexandroff [1940/43, Part III, p. 169], a sequence of measures μ_k is said to converge **weakly** to μ if

$$\lim_{k\to\infty} \int_X f(x)\, d\mu_k(x) = \int_X f(x)\, d\mu(x)$$

for every bounded continuous real function f on X. In view of the representation theorem 4.6.6, the adverb *"weakly"* should be replaced by **weakly***. Maybe, this ambiguity was the reason why Bourbaki [BOU$_{6d}$, p. 59] preferred the term **narrow topology** (topologie étroite); see also [SCHW$_2$, p. 249].

The concept above generalizes the classical convergence in distribution. In the case of the real line, Alexandroff [1940/43, Part III, p. 225] showed that measures μ_k converge weakly to μ if and only if their distribution functions $F_k(x) := \mu_k(-\infty, x)$ converge to $F(x) := \mu(-\infty, x)$ at every point of continuity of F.

6.8.1.8 For separable Banach spaces, the narrow topology is metrizable. Prokhorov [1956, p. 166] defined a metric $d_{\text{prokh}}(\mu, \nu)$ as the infimum of all $\varepsilon > 0$ such that

$$\mu(F) \le \nu(F_\varepsilon) + \varepsilon \quad \text{and} \quad \nu(F) \le \mu(F_\varepsilon) + \varepsilon$$

for every closed subset F of X and its ε-neighborhood $F_\varepsilon := \bigcup\limits_{x_0 \in F} \{x \in X : \|x - x_0\| < \varepsilon\}$.

6.8.2 Cylindrical measures

6.8.2.1 Let X be any locally convex linear space. Given $x_1^*, \dots, x_n^* \in X^*$, the rule

$$C_{x_1^*, \dots, x_n^*} : x \mapsto \big(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle\big)$$

yields an operator from X into $\mathbb{R}^n$. The preimage of $\mathscr{B}_{\text{orel}}(\mathbb{R}^n)$ is a σ-algebra, denoted by $\mathscr{C}_{x_1^*, \dots, x_n^*}(X)$, and the members of the algebra

$$\mathscr{C}_{\text{yl}}(X) := \bigcup_{x_1^*, \dots, x_n^* \in X^*} \mathscr{C}_{x_1^*, \dots, x_n^*}(X)$$

are called **cylindrical sets**. A result of Edgar [1977, p. 668] implies that $\mathscr{C}_{\text{yl}}(X)$ generates the σ-algebra $\mathscr{B}_{\text{aire}}(X, \text{weak})$.

Several authors prefer to use the surjections $Q_N^X : X \to X/N$ instead of $C_{x_1^*, \dots, x_n^*}$. Here N ranges over all finite-codimensional closed subspaces of X.

6.8.2.2 By a **cylindrical measure**, we mean a set function ζ on $\mathscr{C}_{\text{yl}}(X)$ whose restrictions to the σ-algebras $\mathscr{C}_{x_1^*, \dots, x_n^*}(X)$ are σ-additive. As stated at the beginning of this subsection, we assume that $\zeta(X) = 1$. Hence the term **cylindrical probability** is common as well.

6.8.2.3 A fundamental result of Prokhorov [1956, p. 161] implies that a cylindrical measure ζ admits a unique extension to a Radon measure (denoted by ζ too) if and only if given $\varepsilon > 0$, there exists a compact subset K_ε such that $\zeta(K_\varepsilon + N) \ge 1 - \varepsilon$ for all finite-codimensional closed subspaces N of X; see also Prokhorov [1961, p. 411], [BOU$_{6d}$, pp. 52, 71], and [SCHW$_2$, pp. 74–75, 174]. In this case,

$$\zeta(K) = \inf\{\zeta(K + N) : \text{cod}(N) < \infty\}$$

for every compact subset K.

6.8.2.4 Let $L_0(M,\mathscr{M},\mu)$ denote the collection of all measurable scalar functions on a measure space $(M,\mathscr{M},\mu)$. Then $L_0(M,\mathscr{M},\mu)$ is a complete topological linear space under the equivalent metrics

$$\int_M \frac{|f(\xi)-g(\xi)|}{1+|f(\xi)-g(\xi)|}\,d\mu(\xi) \quad \text{and} \quad \int_M \min\{|f(\xi)-g(\xi)|,1\}\,d\mu(\xi);$$

see [BAN, p. 9]. As usual, functions f and g are identified if they coincide almost everywhere. The corresponding convergence is **convergence in measure**:

$$\lim_{k\to\infty} \mu\big\{\xi \in M : |f_k(\xi)-f(\xi)| \le \varepsilon\big\} = 1 \quad \text{whenever } \varepsilon > 0.$$

This terminology goes back to Riesz [1909b, p. 1303].

6.8.2.5 A **linear random process** on a locally convex linear space X is a linear mapping $Z : X \to L_0(M,\mathscr{M},\mu)$. Under the name **weak distribution**, this concept was introduced by Segal [1954, p. 730]; see also 6.8.4.6.

6.8.2.6 With every linear random process $Z : X^* \to L_0(M,\mathscr{M},\mu)$ we associate the cylindrical measure ζ on the locally convex linear space X defined by

$$\zeta\big(C^{-1}_{x_1^*,\dots,x_n^*}(B)\big) := \mu\big\{\xi \in M : \big(Z(x_1^*,\xi),\dots,Z(x_n^*,\xi)\big) \in B\big\} \quad \text{for } B \in \mathscr{B}\mathrm{orel}(\mathbb{R}^n).$$

Here $Z(x^*,\xi)$ denotes the value of the function $Z(x^*) \in L_0(M,\mathscr{M},\mu)$ at the point $\xi \in M$.

Most importantly, every cylindrical measure is obtained in this way. In view of Kolmogoroff's *Erweiterungssatz* [KOL, p. 15], the required measure space $(M,\mathscr{M},\mu)$ can be built over the Cartesian product $\mathbb{R}^{X^*}$:

$$\begin{tikzcd} X \arrow[r, "J"] \arrow[dr, "C_{x_1^*,\dots,x_n^*}"'] & \mathbb{R}^{X^*} \arrow[d, "P_{x_1^*,\dots,x_n^*}"] \\ & \mathbb{R}^n \end{tikzcd},$$

where $J : x \mapsto (\langle x,x^*\rangle)$ and $P_{x_1^*,\dots,x_n^*} : (\xi_{x^*}) \mapsto (\xi_{x_1^*},\dots,\xi_{x_n^*})$. The measure μ is given by

$$\mu\big(P^{-1}_{x_1^*,\dots,x_n^*}(B)\big) := \zeta\big(C^{-1}_{x_1^*,\dots,x_n^*}(B)\big) \quad \text{for } B \in \mathscr{B}\mathrm{orel}(\mathbb{R}^n).$$

Since Schwartz wanted to obtain a Radon measure μ, he replaced $\mathbb{R}$ by its Stone–Čech compactification $\beta\mathbb{R}$; see [SCHW$_2$, pp. 256–258].

6.8.2.7 Let $0 \le p < \infty$. In this text, a cylindrical measures ζ induced by a *continuous* linear map $Z : X^* \to L_p(M,\mathscr{M},\mu)$ is called **L_p-continuous**. Schwartz [1969b, p. 1410] used the term *p-typique*. If X is a Banach space and $p > 0$, then the condition above means that the **weak p^{th} moment**

$$\|\zeta|W_p\| := \sup_{\|x^*\|\le 1} \left(\int_X |\langle x,x^*\rangle|^p\, d\zeta(x)\right)^{1/p} = \|Z : X^* \to L_p\|$$

is finite; see [PAR$^{\Sigma}_{69}$, exposé 6, p. 8].

6.8.3 Characteristic functionals

6.8.3.1 The **characteristic functional** of a cylindrical measure ζ is defined by

$$\widehat{\zeta}(x^*) := \int_X e^{i\langle x, x^* \rangle} d\zeta(x) \quad \text{for all } x^* \in X^*.$$

In the case of separable Banach spaces, this definition was proposed by Kolmogoroff [1935] in a short note whose misleading title is *La transformation de Laplace dans les espaces linéraires*. Kolmogoroff also observed that ζ is uniquely determined by $\widehat{\zeta}$.

Obviously, for the standard Gaussian measure

$$\gamma(B) := \frac{1}{\sqrt{2\pi}} \int_B \exp(-\tfrac{1}{2}t^2)\, dt,$$

we have

$$\widehat{\gamma}(s) = \exp(-\tfrac{1}{2}s^2).$$

6.8.3.2 A complex-valued function φ on X^* is called **positive definite** if

$$\sum_{h=1}^{n} \sum_{k=1}^{n} \zeta_h \varphi(x_h^* - x_k^*) \overline{\zeta_k} \geq 0$$

for any choice of $x_1^*, \ldots, x_n^* \in X^*$, $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $n = 1, 2, \ldots$.

Then the following generalization of the classical **Bochner theorem** 5.4.2.7 holds; see Getoor [1956, p. 887]:
A positive definite function φ on X^* with $\varphi(o) = 1$ is the characteristic functional of a cylindrical measure ζ if and only if all restrictions of φ to finite-dimensional subspaces are continuous.

Moreover, from Getoor [1956, p. 888] or [PAR$^{\Sigma}_{72}$, exposé 9, p. 3] we learn that norm continuity of φ is equivalent to L_0-continuity of ζ.

6.8.3.3 The most important problem is to find conditions on φ that guarantee that ζ is a Radon measure. To the best of my knowledge, a satisfactory solution is known only for Hilbert spaces H.

Minlos–Sazonov theorem:
A cylindrical measure ζ on H is Radon if and only if its characteristic functional

$$\widehat{\zeta}(x^*) := \int_H e^{i(x|x^*)} d\zeta(x)$$

is continuous with respect to the locally convex topology induced by the semi-norms $p_S(x) := \|Sx\|$, where S ranges over all Hilbert–Schmidt operators on H.

Based on preliminary work of Prokhorov [1956, pp. 171–177], this fundamental result was independently obtained by Sazonov [1958, p. 204], Minlos [1959, pp. 310–311], and Gross [GROSS, pp. 20–21]; see also Prokhorov's excellent survey [1961]. The standard reference about measures on the separable Hilbert space is [SKO].

6.8.4 Radonifying operators

6.8.4.1 An operator $T \in \mathfrak{L}(X,Y)$ is called ***p*-radonifying** if every L_p-continuous cylindrical measure ζ on X is taken to a Radon measure $T\zeta$ on Y:

$$T\zeta(B) := \zeta\big(T^{-1}(B)\big) \quad \text{for } B \in \mathscr{B}_{\mathrm{orel}}(Y, \mathrm{norm}).$$

In his original definition, Schwartz [1969b, p. 1410] additionally assumed that the **absolute** p^{th} **moment**

$$\|T\zeta|L_p\| := \left(\int_Y \|y\|^p dT\zeta(y)\right)^{1/p} = \left(\int_X \|Tx\|^p d\zeta(x)\right)^{1/p}$$

is finite. Later on, Kwapień [1969] observed that this requirement is automatically satisfied; see also Schwartz [1971, p. 225].

6.8.4.2 According to Schwartz [$\mathrm{PAR}^{\Sigma}_{69}$, exposé 11, p. 2]:

L'idée de comparer les applications p-sommantes et p-radonifiantes revient à Kwapień [1969].

The main theorem states that for $1 < p < \infty$, the p-summing operators and the p-radonifying operators coincide.

One direction of the proof is based on the fact that given any sequence (x_k),

$$\zeta := \sum_{k=1}^{\infty} \tfrac{1}{2^k} \delta_{\{2^{k/p} x_k\}}$$

defines a cylindrical Radon measure on X such that

$$\|\zeta|W_p\| = \sup_{\|x^*\| \le 1} \left(\sum_{k=1}^{\infty} |\langle x_k, x^*\rangle|^p\right)^{1/p} = \|(x_k)|w_p\|$$

and

$$\|T\zeta|L_p\| = \left(\sum_{k=1}^{\infty} \|Tx_k\|^p\right)^{1/p} = \|(Tx_k)|l_p\|.$$

Hence $\|T\zeta|L_p\| \le \pi_p(T)\|\zeta|W_p\|$.

6.8.4.3 If $p=1$, then the situation turns out to be more complicated:

An operator $T \in \mathfrak{L}(X,Y)$ is 1-summing if and only if $K_Y T$ maps every L_1-continuous cylindrical measure ζ on X to a Radon measure $K_Y T\zeta$ on Y^{**} equipped with the weak* topology.

The transition from Y to Y^{**} is unavoidable, since Linde [1975] proved that a Banach space Y has the Radon–Nikodym property if and only if the 1-summing operators from any Banach space X into Y are 1-radonifying.

6.8.4.4 For $0 \le p < 1$, almost nothing is known about p-radonifying operators. In order to get satisfactory results, one deals only with L_p-continuous cylindrical measures that are supposed to be somehow “approximable”; Schwartz [1969b, p. 1410], [1971, p. 179]. For details the reader is referred to [PAR$^{\Sigma}_{69}$, exposés 16 and 17], [PAR$^{\Sigma}_{72}$, exposé 9] and Schwartz [1971, § 4].

A striking example of Pisier [1985] shows that $\mathfrak{R}_p$, the ideal of p-radonifying operators, is considerably smaller than $\mathfrak{P}_p$. While all $\mathfrak{P}_p$'s coincide for $0 \le p < 1$, it seems to be open whether the same holds for the $\mathfrak{R}_p$'s.

6.8.4.5 Let ζ be an L_0-continuous cylindrical measure on a Hilbert space H. Then, in view of 6.8.3.2, its characteristic functional $\widehat{\zeta}$ is norm continuous. Hence the Minlos–Sazonov theorem 6.8.3.3 implies that $S\zeta$ is a Radon measure for every Hilbert–Schmidt operator S on H. In other words, Hilbert–Schmidt operators are 0-radonifying. A direct proof of this fact was given by Kwapień [1968b, p. 699], who showed that $Id : l_1 \to l_2$ is 0-radonifying.

On the other hand, we know from Schwartz [1971, p. 240], [SCHW$_2$, p. 301] that 0-radonifying operators between Hilbert spaces are Hilbert–Schmidt. Thus the two classes of operators coincide. This observation was the starting point of the theory of radonifying operators.

6.8.4.6 Inspired by preliminary work of Wiener and Kolmogorov, the concept of a *generalized random process* was introduced by Gelfand [1955, pp. 553–554]. Subsequently, in his article *On some problems of functional analysis*, Gelfand [1956, pp. 323–324] wrote:

Apart from quantum theory of fields, integration and analysis in functional spaces are interesting for the theory of stochastic processes, the theory of information, statistical physics. The development of this branch of analysis may accordingly turn out to be very appropriate for future progress.

The realization of this proposal has led to Chapter IV of [GEL$^{+}_{4}$]. An account of the Russian contributions to *“Probability measures in infinite-dimensional spaces”* was also given by Vershik/Sudakov [1969]. Most important is the

Minlos theorem, [1959, p. 303]:
Every L_0-continuous cylindrical measure on the dual of a countably Hilbertian nuclear space (счетно-гильбертово ядерное пространство) is σ-additive.

Almost simultaneously, related studies were carried out in the USA by Bochner [BOCH$_2$], Segal [1954], and their followers; see [GROSS] and the readable survey of Dudley [1969]. I also mention the contributions of Itô [1953] from Japan.

The decisive breakthrough was achieved by Schwartz [1967, 1969a, 1969b, 1969c] and Kwapień [1968b, 1969, 1970b]. The most important reference, however, is the

Séminaire Laurent Schwartz 1969–1970.

A modernized account (with few proofs) can be found in [SCHW$_3$, Chap. II].

6.8.5 Gaussian measures

6.8.5.1 A cylindrical measure ζ on a Banach space X is called **Gaussian** if

$$\widehat{\zeta}(x^*) := \int_X \mathrm{e}^{\mathrm{i}\langle x,x^*\rangle} d\zeta(x) = \exp\big(-\tfrac{1}{2}v(x^*)\big) \qquad (6.8.5.1.a)$$

with some non-negative function v such that $v(\lambda x^*) = \lambda^2 v(x^*)$ for $x^* \in X^*$ and $\lambda \in \mathbb{R}$. Instead of (6.8.5.1.a) one may require that

$$\zeta\big\{x \in X : \langle x,x^*\rangle \in \sqrt{v(x^*)}B\big\} = \gamma(B) \quad \text{whenever } B \in \mathscr{B}\mathrm{orel}(\mathbb{R}),$$

where γ denotes the standard Gaussian measure on $\mathbb{R}$; see 6.1.7.14.

Elementary manipulations show that the **covariance functional** v is given by

$$v(x^*) = \int_X \langle x,x^*\rangle^2 \, d\zeta(x).$$

Conversely, every non-negative quadratic form v on X^* induces a Gaussian cylindrical measure on X.

To simplify matters, the presentation is restricted to measures *centered* at o. Note that the Dirac measure $\delta_\circ$ is Gaussian.

6.8.5.2 The concept of a Gaussian measure on the Borel sets of a separable Banach space was invented by Kolmogoroff [1935], who proposed the term *normal distribution*. For a while, the French school used the adjective *laplacien*; see Fréchet [1951].

6.8.5.3 Cylindrical sets of a Hilbert space H can be represented in the form

$$C = \big\{x \in H : \big((x|e_1),\dots,(x|e_n)\big) \in B\big\}$$

with an orthonormal system $(e_1,\dots,e_n)$ and $B \in \mathscr{B}\mathrm{orel}(\mathbb{R}^n)$. If γ^n denotes the standard Gaussian measure on $\mathbb{R}^n$, then $\gamma^H(C) := \gamma^n(B)$ yields a well-defined cylindrical measure on H that is often called **white noise**. We have $\widehat{\gamma}^H(x^*) = \exp\big(-\frac{1}{2}\|x^*\|^2\big)$.

6.8.5.4 Unfortunately, γ^H fails to be σ-additive on infinite-dimensional Hilbert spaces. This fact, which was certainly known to Lévy in the early 1920s, can be proved as follows:

Fix any orthonormal sequence (e_k), and let

$$C_{n,k} := \big\{x \in H : \big|(x|e_1)\big| \le k,\dots,\big|(x|e_n)\big| \le k\big\}.$$

Obviously,

$$\gamma^H(C_{n,k}) = (c_k)^n \quad \text{with} \quad c_k := \frac{1}{\sqrt{2\pi}} \int_{-k}^{+k} \mathrm{e}^{-t^2/2} dt < 1.$$

Choose n_k so large that $(c_k)^{n_k} \le \varepsilon/2^k$. Then the σ-additivity of γ^H would imply the contradiction

$$1 = \gamma^H(H) = \gamma^H\Big(\bigcup_{k=1}^{\infty} C_{n_k,k}\Big) \le \sum_{k=1}^{\infty} \gamma^H(C_{n_k,k}) \le \varepsilon.$$

In view of Kolmogoroff's *Hauptsatz* [KOL, p. 27], we may say that $H = l_2(\mathbb{I})$ is a null set in $\mathbb{R}^{\mathbb{I}}$ endowed with the σ-additive $\mathbb{I}$-fold power $\gamma^{\mathbb{I}}$ of the standard Gaussian measure γ on $\mathbb{R}$.

6.8.5.5 As implicitly shown by Mourier [1953, pp. 233–243], a Gaussian cylindrical measure ζ on a Hilbert space is Radon if and only if there exists a Hilbert–Schmidt operator T on H such that $\widehat{\zeta}(x^*) = \exp\left(-\frac{1}{2}\|T^*x^*\|^2\right)$, or equivalently, $\zeta = T\left(\gamma^H\right)$.

This result, which is a predecessor of the famous Minlos–Sazonov theorem 6.8.3.3, was rediscovered by Varadhan [1962, p. 226].

6.8.5.6 Linde/Pietsch [1974] have generalized the criterion above to the setting of Banach spaces. Unfortunately, the situation is not as simple, since one must pass to the second dual:

A Gaussian cylindrical measure ζ on X admits a Radon extension to X^{**} equipped with the weak* topology if and only if there exists a γ-summing operator T from a Hilbert space H into X such that $\widehat{\zeta}(x^*) = \exp\left(-\frac{1}{2}\|T^*x^*\|^2\right)$.

Gaussian cylindrical measures associated to p-summing operators $T : H \to X$ are even Radon on X.

6.8.5.7 An account of the theory of Gaussian measures on Banach spaces is given in the lecture notes [KUO].

6.8.6 Wiener measure

6.8.6.1 The **Wiener space** $C_0[0,1]$ consists of all continuous real functions f on $[0,1]$ with $f(0) = 0$. Since the integration operator

$$S : f(t) \mapsto \int_0^s f(t)\,dt$$

acts from $L_2[0,1]$ into $C_0[0,1]$, the **Wiener measure** w on $C_0[0,1]$ can be defined as the image of the cylindrical Gaussian measure γ^{L_2}.

Even more is true. In view of 6.3.10.3 and 6.7.8.10, it follows from

$$S : L_2[0,1] \xrightarrow{\;S\;} W_2^1[0,1] \cap C_0[0,1] \xrightarrow{\;Id\;} \mathrm{Lip}^\sigma[0,1] \cap C_0[0,1]$$

that S is p-summing and hence p-radonifying whenever $0 < \sigma < 1/2 - 1/p$. Since the weak p^{th} moments of γ^{L_2} are finite for $0 < p < \infty$, the image $w := S\gamma^{L_2}$ must be a Radon measure on the space $\mathrm{Lip}^\sigma[0,1] \cap C_0[0,1]$ for $0 < \sigma < 1/2$. This is a modern proof of a classical result that will be described next.

6.8.6.2 Note that

$$\mathrm{Lip}^\sigma[0,1] \cap C_0[0,1] = \bigcup_{n=1}^{\infty} \overbrace{\{f \in C_0[0,1] : |f(s) - (t)| \le n|s-t|^\sigma \quad \text{for} \quad 0 \le s,t \le 1\}}^{\text{closed}}$$

is a Borel set in $C_0[0,1]$. Hence $w\big(\mathrm{Lip}^\sigma[0,1]\cap C_0[0,1]\big)$ makes sense. Paley/Wiener/Zygmund [1933, pp. 666–668] proved that

$$w\big(\mathrm{Lip}^\sigma[0,1]\cap C_0[0,1]\big)=\begin{cases}1 & \text{if } 0<\sigma<\frac{1}{2},\\ 0 & \text{if } \frac{1}{2}<\sigma\le 1;\end{cases}$$

see also [PAL$^+$, § 38]. Wiener [1930, pp. 214–221] had treated the case $\sigma=1/4$. Concerning $w\big(\mathrm{Lip}^{1/2}[0,1]\cap C_0[0,1]\big)=0$, the reader may consult [BAU, § 47].

The most striking consequence that arose from this circle of ideas says that almost all functions in $C_0[0,1]$ are nowhere differentiable; see p. 567. In order to make this statement meaningful, one has to use the completion of the Wiener measure.

6.8.6.3 Viewing $\mu\in C_0[0,1]^*$ as a bounded measure on $(0,1]$, the covariance functional of the Wiener measure is given by

$$v(\mu)=\|S^*\mu\|^2=\int_0^1\int_0^1 \min\{s,t\}\,d\mu(s)d\mu(t);$$

see [BOU$_{6d}$, p. 87] and 6.8.7.4.

6.8.6.4 The mathematical theory of Brownian motion was founded by Wiener. His main paper on *Differential spaces* appeared in 1923. According to Ito [WIE$^{\bowtie}$, p. 514],

It is written in a heuristic way so that we can see how he approached the rigorous definition of Wiener measure.

The opinion of Kac [1966$^\bullet$, p. 55] is less optimistic:

The [read: Wiener's] *early papers were very difficult to read (as I know from personal experience, when as a student in Lwów I tried to read them with a depressing lack of success) and they fell on deaf ears. Only Paul Lévy in France, who had himself been thinking along similar lines, fully appreciated their significance.*

6.8.6.5 A historical survey was given by Kahane [1998$^\bullet$].

6.8.7 Vector-valued random variables

In the rest of this section, I adopt the usual notation from probability theory that means that $(\Omega,\mathscr{F},\mathsf{P})$ consists of a set Ω, a σ-algebra $\mathscr{F}$ of subsets of Ω, and a σ-additive measure P defined on $\mathscr{F}$ with $\mathsf{P}(\Omega)=1$.

Instead of *"almost everywhere"* the term *"almost surely"* will be used, and *"convergence in measure"* becomes *"convergence in probability."*

The standard reference is [LED$^+$]. The reader may also consult the older books [GREN, Chap. 6] and [PARTH, Chap. 6 and 7] as well as the excellent surveys of Hoffmann-Jørgensen [1977], [1982]. The state of the art at the beginning of the modern era is described in addresses of Fréchet [1954] and Kolmogoroff/Prochorow [1954] delivered at a conference in Berlin; see [GNE$^\cup$].

6.8.7.1 In his introduction to the proceedings of the *First International Conference on Probability in Banach Spaces* Beck wrote; see [BECK$^{\cup}$]:

Probability in Banach spaces originated in 1953 in the thesis of Dr. Edith Mourier, who proved the first Strong Law of Large Numbers for random variables taking their values in a Banach space. Early work seemed largely to parallel the corresponding theorems for real or complex valued random variables, and for that reason did not attract much attention. Beginning in the early sixties, however, there began to develop a body of material tying the truth or falsity of probabilistic theorems in Banach spaces to the geometry of the spaces.

...

The search for probabilistic theorems has led us into considerations of new classes of Banach spaces, and the dissection of the theorems in Banach spaces has led us into new probabilistic conditions.

6.8.7.2 By an X**-valued Borel random variable** we mean a function $\boldsymbol{f}$ from Ω into a Banach space X such that $\boldsymbol{f}^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{B}_{\text{orel}}(X, \text{norm})$.

At first glance, this seems to be an appropriate concept, which, for example, was used in Kahane's monograph [KAH, p. 3]. Unfortunately, Nedoma [1957] observed that an unpleasant phenomenon may occur.

Call a Banach space X **measurable** if the sum of two X-valued Borel measurable functions is Borel measurable as well. A criterion of Talagrand [1979b, p. 256] shows that this property may fail for large spaces:

Fix any norm-dense subset D of minimal cardinality, $\text{card}(D) = \text{dense}(X)$; see 6.9.1.15. Then X is measurable if and only if the power set $\mathscr{P}(D \times D)$ is the smallest σ-algebra containing all rectangles $A \times B$ with $A, B \in \mathscr{P}(D)$.

Thus the question of measurability is reduced to a purely set-theoretic problem. The surprising result reads as follows; see Kunen [1968], Rao [1969], and Nedoma [1957]:

A set X is measurable if $\text{dense}(X) \leq \aleph_1$ and non-measurable if $2^{\aleph_0} < \text{dense}(X)$.

Refuting the continuum hypothesis, we get the additional case $\aleph_1 < \text{dense}(X) \leq 2^{\aleph_0}$ in which the answer depends on the underlying model. For example, Martin's axiom implies that X is measurable. On the other hand, the non-measurability of l_∞ is consistent with (ZF). For more information about the set-theoretic background, the reader is referred to Miller [1979, pp. 253–254].

6.8.7.3 In order to avoid the pathology above, one requires an additional condition:

X-valued random variables are supposed to have an (almost) separable range.

In other words, Borel measurability is replaced by measurability in the sense of Bochner; see 5.1.2.2.

The **expectation** of a Bochner integrable random variable $\boldsymbol{f}$ is defined by

$$\mathsf{E}(\boldsymbol{f}) := \int_\Omega \boldsymbol{f}(\omega)\, d\mathsf{P}(\omega).$$

6.8.7.4 Vector-valued random variables naturally arise in the theory of real-valued *stochastic processes* $f : T \times \Omega \to \mathbb{R}$, where $f_t(\omega) := f(t, \omega)$ is supposed to be a random variable on Ω for every fixed t in the parameter set T.

Fixing $\omega \in \Omega$, we get the **sample functions** $f_\omega(t) := f(t, \omega)$. Of particular interest is the case in which the f_ω's belong almost surely to a separable Banach space X such that $\boldsymbol{f} : \omega \mapsto f_\omega$ becomes Borel measurable.

The most prominent example is the *Wiener process*

$$w(t,\omega) := \sqrt{2} \sum_{k=1}^{\infty} g_k(\omega) \frac{\sin(k+\frac{1}{2})\pi t}{(k+\frac{1}{2})\pi},$$

where (g_k) denotes any sequence of independent standard Gaussian random variables on a suitable probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Then $w_\omega \in \mathrm{Lip}^\sigma[0,1] \cap C_0[0,1]$ almost surely whenever $0 < \sigma < 1/2$; see 6.8.6.2. Note that

$$\left(\sqrt{2} \frac{\sin(k+\frac{1}{2})\pi t}{(k+\frac{1}{2})\pi} \right)$$

is a complete orthonormal sequence of eigenfunctions of the integration operator with the eigenvalues $\big[(k+\frac{1}{2})\pi\big]^{-1}$. According to Mercer's theorem [1909, p. 444],

$$\min\{s,t\} = 2 \sum_{k=1}^{\infty} \frac{\sin(k+\frac{1}{2})\pi s}{(k+\frac{1}{2})\pi} \frac{\sin(k+\frac{1}{2})\pi t}{(k+\frac{1}{2})\pi}.$$

6.8.7.5 Every X-valued Borel random variable $\boldsymbol{f}$ induces a Borel measure $\mathrm{dist}(\boldsymbol{f})$ on X called the **distribution** of $\boldsymbol{f}$:

$$\mathrm{dist}(\boldsymbol{f})(B) := \mathsf{P}\big(\boldsymbol{f}^{-1}(B)\big) \quad \text{for } B \in \mathscr{B}_{\text{orel}}(X, \text{norm}).$$

X-valued (Bochner) random variables $\boldsymbol{f}$ are characterized by the property that $\mathrm{dist}(\boldsymbol{f})$ is a Radon measure.

6.8.7.6 A sequence of X-valued random variables $\boldsymbol{f}_k$ may converge to $\boldsymbol{f}$

(1) **almost surely**: $\mathsf{P}\{\omega \in \Omega : \lim_{k\to\infty} \boldsymbol{f}_k(\omega) = \boldsymbol{f}(\omega)\} = 1$,

(2) **in probability**: $\lim_{k\to\infty} \mathsf{P}\{\omega \in \Omega : \|\boldsymbol{f}_k(\omega) - \boldsymbol{f}(\omega)\| \le \varepsilon\} = 1$ whenever $\varepsilon > 0$,

(3) **in distribution**: $\lim_{k\to\infty} d_{\text{prokh}}\big(\mathrm{dist}(\boldsymbol{f}_k), \mathrm{dist}(\boldsymbol{f})\big) = 0$; see 6.8.1.8.

The following are obvious:

almost sure convergence $\Rightarrow$ convergence in probability $\Rightarrow$ convergence in distribution.

Even in the scalar-valued case, none of these implications can be reversed; see [STROM, pp. 93–94].

6.8.8 Rademacher series

6.8.8.1 Important examples of X-valued (Bochner) random variables are **Rademacher series**

$$\boldsymbol{f}(t) := \sum_{k=1}^{\infty} r_k(t)x_k \quad \text{with } x_1, x_2, \ldots \in X, \tag{6.8.8.1.a}$$

provided that the right-hand sum converges almost surely. The underlying probability P is just the Lebesgue measure on $[0,1]$.

In the scalar-valued setting, this concept goes back to Rademacher [1922], Khintchine/Kolmogoroff [1925], and in particular, to a series of papers by Paley/Zygmund [1930/32]. The first abstract result is due to Nordlander [1961, pp. 290, 295]:

In a Hilbert space, the series (6.8.8.1.a) converges almost surely if and only if we have $(\|x_k\|) \in l_2$.

For general Banach spaces, no satisfactory criterion seems to be known. Based on preliminary work of Kahane [1964], Kwapień [1975, p. 157] obtained a necessary condition:

$$\int_0^1 \exp\big(\alpha \|\boldsymbol{f}(t)\|^2\big)dt < \infty \quad \text{whenever } \alpha > 0.$$

Almost sure convergence of random series was a decisive source of the theory of type and cotype; see 6.1.7.26.

The standard reference is [KAH]. This monograph, whose first edition appeared in 1968, gave the impetus for developing probability theory in Banach spaces.

6.8.8.2 A theorem of Itô/Nisio [1968, p. 37] tells us that for the partial sums

$$\boldsymbol{f}_n(t) := \sum_{k=1}^{n} r_k(t)x_k,$$

almost sure convergence, convergence in probability, and convergence in distribution are equivalent.

Moreover, Hoffmann-Jørgensen [1974, pp. 161–162] observed that $(\boldsymbol{f}_n)$ is bounded almost surely if and only if it is bounded in probability. The latter property means that given $\varepsilon > 0$, we can find some $c > 0$ such that

$$\mathsf{P}\big\{t \in [0,1] : \|\boldsymbol{f}_n(t)\| \geq c\big\} \leq \varepsilon \quad \text{for } n = 1, 2, \ldots .$$

Obviously, almost sure convergence of $(\boldsymbol{f}_n)$ implies almost sure boundedness. It was conjectured by Hoffmann-Jørgensen [1974, p. 176] and proved by Kwapień [1974] that the converse implication holds if and only if the underlying Banach space does not contain an isomorphic copy of c_0.

6.8.9 The law of large numbers

6.8.9.1 In this subsection, I discuss one of the pearls of probability theory: the *law of large numbers*, which goes back to Jacob Bernoulli [BERN]; see also [SCHN$^{\cup}$, p. 124].

As in the scalar-valued case, a sequence of X-valued random variables f_k is called **independent** if

$$\mathsf{P}\{\omega \in \Omega : f_1(\omega) \in B_1, \dots, f_n(\omega) \in B_n\} = \prod_{k=1}^{n} \mathsf{P}\{\omega \in \Omega : f_k(\omega) \in B_k\}$$

for $B_1, \dots, B_n \in \mathscr{B}_{\mathrm{orel}}(X, \mathrm{norm})$ and $n = 2, 3, \dots$. It will be further assumed that the f_k's are Bochner integrable and $\mathsf{E}(f_k) = o$.

The problem consists in finding conditions that guarantee that

$$\frac{f_1 + \cdots + f_n}{n} \xrightarrow{\|\cdot\|} o \quad \text{almost surely.}$$

Any statement of this kind is said to be a **strong law of large numbers**. If we have only convergence in probability, then the name **weak law of large numbers** is used. Hence the attributes *"strong"* and *"weak"* have nothing to do with the strong or weak topology of the underlying Banach space X.

6.8.9.2 The first strong law of large numbers in separable Banach spaces was proved by Mourier [1953, pp. 190–196] subject to the condition that the random variables f_k are **identically distributed**:

$$\mathsf{P}\{\omega \in \Omega : f_k(\omega) \in B\} \quad \text{does not depend on } k.$$

In addition, she assumed that

$$\mathsf{E}\big(\|f_k\|\big) < \infty \quad \text{and} \quad \mathsf{E}(f_k) = o.$$

The latter conditions are necessary as well; see Hanš [1957, p. 100] and Woyczyński [1974, p. 222].

Since, by definition, the values of a random variable belong almost surely to a separable subspace, Mourier's law of large numbers holds in *every* Banach space; see Hoffmann-Jørgensen [1977, p. 131] for a simplified proof.

6.8.9.3 The real breakthrough was achieved with respect to sequences of random variables that are not assumed to be identically distributed.

First of all, Fortet/Mourier [1954, p. 25], [1955, pp. 65–66] obtained a strong law of large numbers in certain uniformly convex spaces whenever

$$\sup_k \mathsf{E}\big(\|f_k\|^2\big) < \infty \quad \text{and} \quad \mathsf{E}(f_k) = o. \tag{6.8.9.3.a}$$

Next, Beck [1958, p. 36] showed the same for every uniformly convex space. His final result [1962, pp. 330, 333] says that given any Banach space X, the strong law of large numbers holds for all sequences of X-valued random variables $\boldsymbol{f}_k$ satisfying (6.8.9.3.a) if and only if X is B-convex.

Let $0<\alpha\le 1$. According to Woyczyński [1973, стр. 362], we refer to X as a G_α space if there exists a (uniquely determined; see 5.5.3.2) map $G: X\to X^*$ such that

$$\|G(x)\|=\|x\|^\alpha, \quad \langle G(x),x\rangle=\|x\|^{\alpha+1}, \quad \text{and} \quad \|G(x+h)-G(x)\|\le A\|h\|^\alpha$$

for $x,h\in X$ and some constant $A\ge 1$. He showed in [1973, стр. 366] that for such spaces the strong law of large numbers holds whenever

$$\sum_{k=1}^\infty \frac{\mathsf{E}\big(\|\boldsymbol{f}_k\|^{\alpha+1}\big)}{k^{\alpha+1}}<\infty \quad \text{and} \quad \mathsf{E}(\boldsymbol{f}_k)=\mathsf{o}.$$

The case $\alpha=1$ was already treated by Fortet/Mourier in the paper [1955, pp. 65–66] mentioned above.

Hoffmann-Jørgensen/Pisier [1976, p. 590] observed that the G_α spaces are just the p-smooth spaces with $p=\alpha+1$, considered in 5.5.2.9. Therefore G_α spaces have Haar type p; see 6.1.9.9. This fact raised the natural question, *What happens in spaces of Rademacher type p?* The answer obtained by Hoffmann-Jørgensen/Pisier [1976, pp. 588–589] turned out to be highly satisfactory:
The strong law of large numbers holds for all sequences of X-valued random variables $\boldsymbol{f}_k$ with

$$\sum_{k=1}^\infty \frac{\mathsf{E}\big(\|\boldsymbol{f}_k\|^{p}\big)}{k^{p}}<\infty \quad \text{and} \quad \mathsf{E}(\boldsymbol{f}_k)=\mathsf{o}$$

if and only if the Banach space X is of Rademacher type p.

6.8.10 The central limit theorem

6.8.10.1 Let $(\boldsymbol{f}_k)$ be any sequence of independent copies of an X-valued random variable $\boldsymbol{f}$. Then $\boldsymbol{f}$ is said to satisfy the **central limit theorem** if

$$\operatorname{dist}\left(\frac{\boldsymbol{f}_1+\cdots+\boldsymbol{f}_n}{\sqrt{n}}\right)$$

converges in the narrow topology to a Radon probability ζ on X as $n\to\infty$. If so, then

$$\lim_{t\to\infty} t^2\mathsf{P}\big\{\omega\in\Omega : \|\boldsymbol{f}(\omega)\|>t\big\}=0;$$

see Jain [1975, p. 128]. Hence all absolute moments $\mathsf{E}\big(\|\boldsymbol{f}\|^p\big)$ with $0<p<2$ exist. Moreover, $\mathsf{E}(\boldsymbol{f})=\mathsf{o}$, and the limit probability ζ is Gaussian,

$$\widehat{\zeta}(x^*)=\exp\big(-\tfrac{1}{2}\nu(x^*)\big) \quad \text{with} \quad \nu(x^*):=\int_\Omega \langle \boldsymbol{f}(\omega),x^*\rangle^2 d\mathsf{P}(\omega) \quad \text{for } x^*\in X^*.$$

I stress that these results hold in arbitrary Banach spaces.

6.8.10.2 For any scalar-valued random variable f the following properties are equivalent:

$$f \text{ satisfies the central limit theorem} \iff \text{the moment } \mathsf{E}\big(|f|^2\big) \text{ is finite.}$$

In the vector-valued setting the situation becomes more complicated. Aldous [1976, p. 376], Hoffmann-Jørgensen/Pisier [1976, pp. 596–597], and Jain [1975, p. 126], [1977, p. 57] discovered necessary and sufficient conditions on a Banach space X under which we have one-sided implications, for all X-valued random variables $\boldsymbol{f}$:

$$\boldsymbol{f} \text{ satisfies the central limit theorem} \overset{\text{type}\,2}{\Longleftarrow} \text{the moment } \mathsf{E}\big(\|\boldsymbol{f}\|^2\big) \text{ is finite,}$$

$$\boldsymbol{f} \text{ satisfies the central limit theorem} \overset{\text{cotype}\,2}{\Longrightarrow} \text{the moment } \mathsf{E}\big(\|\boldsymbol{f}\|^2\big) \text{ is finite.}$$

Hence, in view of Kwapień's theorem 6.1.7.19, the equivalence

$$\boldsymbol{f} \text{ satisfies the central limit theorem} \iff \text{the moment } \mathsf{E}\big(\|\boldsymbol{f}\|^2\big) \text{ is finite}$$

characterizes Hilbertian spaces.

The central limit theorem in separable Hilbert spaces goes back to Mourier [1953, p. 243]; see also Fortet [1954, pp. 32–33] and Prokhorov [1956, p. 173].

6.8.10.3 The standard reference is [LED$^+$, Chap. 10]. In addition, the reader may consult [ARAU$^+$]. A historical survey concerned with the scalar-valued case was given by Le Cam [1986$^\bullet$].

6.8.11 Vector-valued martingales

6.8.11.1 For every random variable $f \in L_1(\Omega, \mathscr{F}, \mathsf{P})$ and every sub-σ-algebra $\mathscr{F}_0$ of $\mathscr{F}$ there exists $f_0 \in L_1(\Omega, \mathscr{F}_0, \mathsf{P})$ such that

$$\int_A f_0(\omega)\,d\mathsf{P}(\omega) = \int_A f(\omega)\,d\mathsf{P}(\omega) \quad \text{whenever } A \in \mathscr{F}_0.$$

The new random variable f_0, usually denoted by $\mathsf{E}(f|\mathscr{F}_0)$, is called the **conditional expectation** of f with respect to $\mathscr{F}_0$.

This concept was invented by Kolmogoroff [KOL, p. 47], who proved the existence of f_0 via the Radon–Nikodym theorem.

The conditional expectation operator $\mathsf{E}(\mathscr{F}_0) : f \mapsto f_0$ is a positive projection on every Banach space $L_p(\Omega, \mathscr{F}, \mathsf{P})$ with $1 \le p \le \infty$. In particular, $\mathsf{E}(\mathscr{F}_0)$ turns out to be the orthogonal projection from $L_2(\Omega, \mathscr{F}, \mathsf{P})$ onto the closed subspace $L_2(\Omega, \mathscr{F}_0, \mathsf{P})$. Note that $\|\mathsf{E}(\mathscr{F}_0) : L_p \to L_p\| = 1$.

6.8.11.2 It is not hard to check that the conditional expectation $\mathsf{E}(\boldsymbol{f}|\mathscr{F}_0)$ can also be defined for X-valued integrable random variables $\boldsymbol{f}$; see Chatterji [1968, p. 22] and [DIE$_2^+$, p. 123].

A functional analytic approach goes back to Dunford/Pettis [1940, p. 371]; see also [DUN$_1^+$, p. 297]:

Let π be any partition of Ω into subsets $A_1,\dots,A_n \in \mathscr{F}_0$, and define

$$P_\pi \boldsymbol{f}(\omega) := \frac{1}{\mathsf{P}(A_k)} \int_{A_k} \boldsymbol{f}(\xi)\,d\mathsf{P}(\xi) \quad \text{for } \omega \in A_k.$$

Then the π's form a directed system with respect to the canonical ordering, and we have $\mathsf{E}(\boldsymbol{f}|\mathscr{F}_0) = L_p\text{-}\lim_\pi P_\pi \boldsymbol{f}$ whenever $f \in L_p$.

6.8.11.3 Fix sub-σ-algebras $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots$ of the σ-algebra $\mathscr{F}$. A sequence of X-valued $\mathscr{F}_n$-measurable random variables $\boldsymbol{f}_n$ is called a **martingale** if

$$\boldsymbol{f}_n = \mathsf{E}(\boldsymbol{f}_{n+1}|\mathscr{F}_n) \quad \text{for } n = 0, 1, \dots .$$

6.8.11.4 The term *"martingale"* was coined by Ville [VIL, p. 89]. Bauer gives the following explanation, [BAU, p. 144]:

Vermutungen [see 3rd edition, footnote on p. 315] *auf die Ableitung des Wortes vom provenzalischen Namen der französischen Gemeinde Martigues im Departement Bouches-du-Rhone sind nicht ohne Widerspruch geblieben. Das Wort hat mehrere Bedeutungen, u.a.: Hilfszügel beim Zaumzeug des Reitpferdes, welches zu starke Kopfbewegungen des Pferdes verhindert, sowie ein die Takelage bei Segelschiffen absicherndes Seil. Vor allem aber bedeutet es eine Strategie beim Roulettespiel, im Provenzalischen genannt „jouga a la martegalo“ . Diese besteht in der jeweiligen Verdopplung des beim vorausgegangenen Spiel verlorenen Einsatzes.*

According to The American Heritage Dictionary a martingale is

(1) *the strap of a horse's harness that connects the girth to the nose band and is designed to prevent the horse from throwing back its head,*

(2) *any of several parts of standing rigging strengthening the bowsprit and jib boom against the force of the head stays,*

(3) *a method of gambling in which one doubles the stakes after each loss,*

(4) *a loose half belt or strap placed on the back of a garment, such as a coat or jacket.*

See Snell [1982$^\bullet$, p. 123] for nice pictures.

6.8.11.5 The theory of scalar-valued martigales was mainly developed by Doob [1940, p. 460], [DOOB, Chap. VII]. Standard references about the vector-valued setting are Chatterji's seminal paper [1968] as well as [NEV, Chap. V-2] and [DIE$_2^+$, Chap. V]; see also Woyczyński's survey [1975].

6.8.11.6 Let $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}$. For every $\boldsymbol{f} \in [L_1(\Omega, \mathscr{F}, \mathsf{P}), X]$, the X-valued random variables $\boldsymbol{f}_n := \mathsf{E}(\boldsymbol{f}|\mathscr{F}_n)$ form an L_1-bounded martingale, $\sup_n \|\boldsymbol{f}_n|L_1\| < \infty$. We even have a stronger property: the $\boldsymbol{f}_n$'s are **uniformly integrable**,

$$\lim_{t\to\infty} \int\limits_{\|\boldsymbol{f}_n(\omega)\|\geq t} \|\boldsymbol{f}_n(\omega)\| \, d\mathsf{P}(\omega) = 0.$$

Let $\boldsymbol{f}_\infty := \mathsf{E}(\boldsymbol{f}|\mathscr{F}_\infty)$, where $\mathscr{F}_\infty$ denotes the smallest sub-σ-algebra containing all $\mathscr{F}_n$'s. If $1 \leq p < \infty$ and $\boldsymbol{f} \in [L_p(\Omega, \mathscr{F}, \mathsf{P}), X]$, then

$$\|\boldsymbol{f}_n - \boldsymbol{f}_\infty|L_p\| \to 0 \quad \text{and} \quad \|\boldsymbol{f}_n(\omega) - \boldsymbol{f}_\infty(\omega)\| \to 0 \quad \text{almost surely;}$$

see Chatterji [1964, pp. 143, 146].

6.8.11.7 The **martingale convergence theorems** provide conditions that ensure that an X-valued martingale can be obtained in the way just described. Interestingly enough, such a characterization is possible only in a special kind of Banach space.

A Banach space X has the Radon–Nikodym property if and only if every L_1-bounded X-valued martingale $(\boldsymbol{f}_n)$ converges to a limit $\boldsymbol{f}_\infty \in [L_1(\Omega, \mathscr{F}_\infty, \mathsf{P}), X]$, that is $\|\boldsymbol{f}_n(\omega) - \boldsymbol{f}_\infty(\omega)\| \to 0$ almost surely.

Even in the scalar-valued case, the formula $\boldsymbol{f}_n := \mathsf{E}(\boldsymbol{f}_\infty|\mathscr{F}_n)$ need not hold; see [NEV, p. 38]. Uniform integrability turns out to be a necessary and sufficient condition, which, in particular, is satisfied for L_p-bounded martingales, $1 < p < \infty$.

For reflexive spaces or spaces with a separable dual, martingale convergence theorems were proved by Chatterji [1960, p. 397], [1964, p. 144], Ionescu Tulcea [1963, p. 121], Métivier [1967, p. 196], and Scalora [1961, p. 366]. Finally, Rønnow [1967, p. 50] and Chatterji [1968, p. 31] discovered the important role that is played by the Radon–Nikodym property.

6.8.11.8 Martingales can be considered over any upward directed index set $\mathbb{I}$. Important examples are $\{0, 1, 2, \dots\}$, $[0, 1)$, and $[0, \infty)$. In Subsection 6.1.9 we have seen that X-valued Walsh–Paley martingales of finite length are useful tools in the geometry of Banach spaces.

6.9 Further topics

This final section contains a few remarks about some, but **not all**, topics that have been omitted so far.

6.9.1 Topological properties of Banach spaces

Among the many problems concerned with the relationship between general topology and Banach space theory (see also 7.1.2), I have decided to present the following aspect.

Characterize properties of a compact Hausdorff space K by properties of the corresponding Banach space $C(K)$ and vice versa; see 4.8.4.5.

6.9.1.1 A classical result says that K is metrizable if and only if $C(K)$ is separable.

As already mentioned in 4.5.7, Borsuk has proved the "only if" part. The "if" part can be inferred from the fact that for a separable Banach space X, the weak* topology of the closed unit ball of X^* is metrizable; see Alaoglu [1940, p. 255] or Shmulyan [1940b, pp. 439–440] and the metrization theorem of Urysohn [1924a].

6.9.1.2 A Banach space X is called **weakly compactly generated** if there exists a weakly compact subset K whose linear span is dense in X. This concept goes back to Corson [1961, p. 1]. More information can be found in a survey of Lindenstrauss [1972] as well as in [DAY, 3rd edition, pp. 72–77], [DIE$_1$, Chap. 5], [FAB$^+$, Chap. 11], and Zizler [2003, pp. 1760–1765].

6.9.1.3 Reflexive spaces and separable spaces are weakly compactly generated. Further examples are the spaces $c_0(\mathbb{I})$. Moreover, we know from Lindenstrauss [1972, p. 240] that $L_1(M,\mathscr{M},\mu)$ is weakly compactly generated if and only if μ is σ-finite.

Rosenthal [1974b, p. 86] has constructed an example that shows that the property of being weakly compactly generated is not preserved in passing to closed subspaces.

6.9.1.4 A theorem of Amir/Lindenstrauss [1968, p. 35] says that a Banach space X is weakly compactly generated if and only if there exists a one-to-one operator from X into a suitable space $c_0(\mathbb{I})$.

Moreover, by Davis/Figiel/Johnson/Pełczyński [1974, p. 315], a Banach space X is weakly compactly generated if and only if there exists a one-to-one operator from a reflexive space (with an unconditional basis) into X that has dense range.

6.9.1.5 Namioka/Phelps [1975, p. 741] showed that weakly compactly generated duals have the Radon–Nikodym property.

6.9.1.6 According to Amir/Lindenstrauss [1968, p. 36], a compact Hausdorff space K is said to be **Eberlein compact** if it is homeomorphic to a weakly compact subset of a Banach space X. By a fundamental theorem of these authors, one may always arrange that $X = c_0(\mathbb{I})$ for some index set $\mathbb{I}$.

6.9.1.7 A criterion of Amir/Lindenstrauss [1968, p. 37] states that a Banach space $C(K)$ is weakly compactly generated if and only if the underlying Hausdorff space K is Eberlein compact.

6.9.1.8 Benyamini/Starbird [1976, p. 138] have invented the concept of **uniform Eberlein compactness**: K is homeomorphic to a weakly compact subset of a Hilbert space.

The term "uniform" was motivated by the original definition: K is homeomorphic to a weakly compact subset K_0 of some $c_0(\mathbb{I})$ such that for every $\varepsilon > 0$ and all $(\xi_i) \in K_0$, the number of indices i with $|\xi_i| \geq \varepsilon$ is uniformly bounded.

6.9.1.9 A compact Hausdorff space K is called **Talagrand compact** if $C(K)$ is $\mathscr{K}$-analytic under its weak topology. This property means that $C(K)$ can be represented as the continuous image of a countable intersection of countable unions of compact subsets of a completely regular Hausdorff space; see Choquet [1955, p. 139].

A topological characterization of Talagrand compact spaces in the spirit of 6.9.1.6 was given by Mercourakis [1987, p. 317].

6.9.1.10 A Banach space X is **weakly countably $\mathscr{K}$-determined** if we can find a sequence of weakly* compact subsets K_n of X^{**} with the following property:

Given $x \in X$ and $x^{**} \in X^{**} \backslash X$, there exists n_0 such that $x \in K_{n_0}$ and $x^{**} \notin K_{n_0}$.

In a purely topological context, a condition of this kind was earlier invented by Henriksen/Isbell/Johnson [1961, Lemma on p. 113]. The functional analytic version goes back to Vašák [1981, p. 12].

6.9.1.11 On the suggestion of Argyros/Mercourakis/Negrepontis [1982, p. 13], a compact Hausdorff space K is said to be **Gulko compact** if $C(K)$ is weakly countably $\mathscr{K}$-determined.

In his fundamental paper [1979, стр. 33], Gulko dealt with compact Hausdorff spaces K for which $C(K)$, equipped with the topology of pointwise convergence, is a Lindelöf Σ-space (финально компактное Σ-пространство). This means that $C(K)$ can be represented as the continuous image of a subspace of some Cartesian product $X \times Y$, where X admits a countable basis of open sets and Y is compact; see [ARK$^\cup$, pp. 41–42, 110–112] for more information. Talagrand [1979a, pp. 414–415] showed that Gulko's property is equivalent to Gulko compactness. His proof uses the following theorem of Grothendieck [1952, p. 182]:

A subset of $C(K)$ is weakly compact if and only if it is bounded and compact in the topology of pointwise convergence.

The results above were obtained by topologists (Arkhagelskiĭ, Nagami, Gulko) and functional analysts (Vašák, Talagrand, Argyros, Mercourakis, Negrepontis). These schools used different mathematical languages, and no attempt has been made to unify their work. Therefore a full understanding is possible only for insiders. For further information, the reader may consult the original literature as well as [FAB, Chap. 7], [HART$^\cup$], and a survey of Godefroy [2002].

6.9.1.12 A compact Hausdorff space K is said to be **Corson compact** if there exists an index set $\mathbb{I}$ such that K is homeomorphic to a subset K_0 of $\Sigma(\mathbb{R}^{\mathbb{I}})$. The Σ-product $\Sigma(\mathbb{R}^{\mathbb{I}})$ was defined by Corson [1959, p. 785]; it consists of all scalar families $(\xi_i) \in \mathbb{R}^{\mathbb{I}}$ that have at most a countable number of non-zero coordinates.

6.9.1.13 Deville/Godefroy [1993, p. 185] refer to a compact Hausdorff space K as **Valdivia compact** if there exists a homeomorphism to a subset K_0 of some Cartesian product $\mathbb{R}^{\mathbb{I}}$ such that $K_0 \cap \Sigma(\mathbb{R}^{\mathbb{I}})$ is dense in K_0.

6.9.1.14 A **Lindelöf space** is defined by the property that every open cover contains a countable subcover. As is well known, regular Lindelöf spaces are normal; see [KEL, p. 113]. Obviously, normal spaces need not be Lindelöf. However, based on preliminary work of Baturov [1987], Reznichenko [1990, p. 26] was able to prove that the two concepts coincide for the weak topology of any Banach space.

According to a lemma of Henriksen/Isbell/Johnson [1961, p. 113], weakly countably $\mathscr{K}$-determined Banach spaces are weakly Lindelöf. Not aware of this result from general topology, Vašák [1981, p. 13] found his own proof. The converse implication may fail; see Argyros/Mercourakis/Negrepontis [1988, pp. 216, 224].

Argyros/Mercourakis/Negrepontis [1988, pp. 215–216] observed that the "axiom"

$C(K)$ is weakly Lindelöf for every Corson compact set K

is independent of the Zermelo–Fraenkel system. For a detailed discussion of similar problems the reader may consult Subsection 7.5.

6.9.1.15 In every infinite-dimensional Banach X, we can find a norm-dense subset D having minimal cardinality. Then $\text{dense}(X) := \text{card}(D)$ is called the **density character** of X. Let μ denote the smallest ordinal with $\text{card}(\mu) = \text{dense}(X)$.

A **projectional resolution of the identity** of X is a family of projections P_α, indexed by ordinals $\alpha \in [\omega_0, \mu]$, such that the following conditions are satisfied:

(1) $\|P_\alpha\| = 1$ for $\alpha \in [\omega_0, \mu]$.

(2) $\text{dense}(P_\alpha(X)) \leq \text{card}(\alpha)$ for $\alpha \in [\omega_0, \mu]$.

(3) $P_\mu = I_X$.

(4) $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$ for $\alpha, \beta \in [\omega_0, \mu]$.

(5) $\bigcup_{\beta<\alpha} P_\beta(X)$ is norm-dense in $P_\alpha(X)$ for every limit ordinal $\alpha \in (\omega_0, \mu]$.

The reader should realize that the concept above is trivial for separable spaces.

According to [DAY, 3rd edition, p. 72], the existence of a projectional resolution of the identity *gives much structural information about such a space, nearly as much as would the existence of a generalized basis*.

Lindenstrauss [1965, p. 203] was the first to construct "long sequences of projections" in reflexive spaces. Later on, Amir/Lindenstrauss [1968, p. 44] observed that the same technique also works in weakly compactly generated Banach spaces. A further extension is due to Vašák [1981, p. 14], who invented the weakly countably $\mathscr{K}$-determined spaces just for this purpose. Finally, I mention that $C(K)$ has a projectional resolution of the identity if K is Valdivia compact; see Valdivia [1990].

6.9.1.16 The following table illustrates the hierarchy of the properties discussed above.

properties of K (and implications)		properties of $C(K)$ (and counterexamples)
metrizable	$\xleftrightarrow{\text{classical result}}$	separable
Urysohn [1924b]	$\downarrow\ \not\uparrow$	$B_{l_2(\mathbb{I})}$, $\mathbb{I}$ uncountable
uniform Eberlein		?
obvious	$\downarrow\ \not\uparrow$	Benyamini/Starbird [1976, p. 140]
Eberlein	$\underset{\text{[1968, p. 37]}}{\xleftrightarrow{\text{Amir/Lindenstrauss}}}$	weakly compactly generated
Talagrand [1979a, p. 413]	$\downarrow\ \not\uparrow$	Talagrand [1979a, p. 418]
Talagrand	$\xleftrightarrow{\text{by definition}}$	weakly $\mathscr{K}$-analytic
Talagrand [1979a, p. 409]	$\downarrow\ \not\uparrow$	Talagrand [1984, p. 79]
Gulko	$\xleftrightarrow{\text{by definition}}$	weakly countably $\mathscr{K}$-determined
Gulko [1979, стр. 33]	$\downarrow\ \not\uparrow$	Argyros/Mercourakis/Negrepontis [1982, p. 21]
Corson		?
obvious	$\downarrow\ \not\uparrow$	$[0,1]^{\mathbb{I}}$, $\mathbb{I}$ uncountable
Valdivia		?
obvious	$\downarrow\ \not\uparrow$	Deville/Godefroy [1993, p. 185]
arbitrary		arbitrary

The question marks indicate that no characteristic property of $C(K)$ is known.

6.9.2 Topological classification of Banach spaces

The results described in this subsection show the fascinating interplay between non-linear and linear phenomena in Banach space theory.

6.9.2.1 We know from 4.9.1.2 that in the real case, a map U from X onto Y such that $U\mathrm{o}=\mathrm{o}$ and $\|U(x+h)-Ux\|=\|h\|$ for $x,h\in X$ is linear. On the other hand, if U is merely a homeomorphism, then nothing can be said about its algebraic properties. Indeed, by the Kadets theorem 5.6.1.5, *all* separable infinite-dimensional Banach spaces are homeomorphic to each other. Toruńszyk [1981, p. 258] proved that every Banach space is homeomorphic to a Hilbert space. Hence infinite-dimensional Banach spaces are homeomorphic if and only if they have the same density character.

6.9.2.2 Between the two extremal cases just described, various other kinds of homeomorphisms have been considered.

Instead of isometries, we may use isomorphisms as defined in 4.9.1.1. For the present purpose, however, the name **linear homeomorphism** is more suggestive.

A one-to-one map U from X onto Y is called a **Lipschitz homeomorphism** if there exist constants $a, b > 0$ such that

$$a\|h\| \leq \|U(x+h) - Ux\| \leq b\|h\| \quad \text{for } x, h \in X.$$

A **uniform homeomorphism** U is defined by the property that both U and U^{-1} are uniformly continuous.

Banach spaces X and Y are referred to as **linearly homeomorphic**, **Lipschitz homeomorphic**, and **uniformly homeomorphic**, respectively, if we can find homeomorphisms $U : X \to Y$ of the specified kind. Some authors also use the attribute **Lipschitz equivalent** or **Lipschitz isomorphic**.

The implications below are obvious, and the subsequent examples show that none of them can be reversed.

$$\begin{array}{c}
\text{isometric} \\
\downarrow \;\; \not\uparrow \;\; 6.9.2.3 \\
\text{linearly homeomorphic} \\
\downarrow \;\; \not\uparrow \;\; 6.9.2.5 \\
\text{Lipschitz homeomorphic} \\
\downarrow \;\; \not\uparrow \;\; 6.9.2.6 \\
\text{uniformly homeomorphic} \\
\downarrow \;\; \not\uparrow \;\; 6.9.2.4 \\
\text{homeomorphic}
\end{array}$$

6.9.2.3 By the theorems of Banach–Stone 4.5.5 and Milyutin 4.9.3.6, for different n's the spaces $C\big([0,1]^n\big)$ are not isometric but linearly homeomorphic (isomorphic).

6.9.2.4 Lindenstrauss [1964, p. 283] gave the first example of two Banach spaces that fail to be uniformly homeomorphic: $L_p[0,1]$ and $L_q[0,1]$ with $p \neq q$ and $\max\{p,q\} > 2$. The case $\max\{p,q\} \leq 2$ was added by Enflo [1969, p. 105].

6.9.2.5 Aharoni/Lindenstrauss [1978] constructed a Banach space that is Lipschitz homeomorphic but not linearly homeomorphic to $c_0(\mathbb{I})$ for some uncountable index set $\mathbb{I}$. Therefore it is worthwhile to look for conditions that eliminate such counterexamples. According to [BENY$_1^+$, p. 169],

it is quite possible that the Lipschitz and linear classifications coincide for all separable (*or at least for separable and reflexive*) *Banach spaces.*

Here are two partial results of Heinrich/Mankiewicz [1982, p. 234] that support this conjecture.

(1) Let X and Y be separable dual Banach spaces that are linearly homeomorphic to their Cartesian squares. Then Lipschitz homeomorphy implies linear homeomorphy.

(2) Let $1<p<\infty$. If a Banach space X is Lipschitz homeomorphic to l_p, then it is also linearly homeomorphic to l_p. The same is true for $L_p[0,1]$.

Using renorming techniques, Godefroy/Kalton/Lancien [2001, p. 801] were able to verify (2) for c_0 as well. Until now, the case of l_1 was solved only subject to the condition that X be a dual space.

The standard method of obtaining a linear homeomorphism from a Lipschitz homeomorphism U is Gâteaux differentiation; see 5.1.8.3. If there is at least one point $x_0 \in X$ at which U has a Gâteaux derivative, then $\delta U(x_0)$ yields an injection from X into Y. Unfortunately, we do not know whether $\delta U(x_0)$ is surjective as well. The difficulties that may occur are described in [BENY$_1^+$, p. 170]; see also Preiss [2003, pp. 1539–1542].

6.9.2.6 A theorem of Enflo [1970, Part II, p. 266] says that every Banach space that is uniformly homeomorphic to a Hilbert space is also linearly homeomorphic to a Hilbert space.

On the other hand, Ribe [1984, p. 139] discovered a pair of spaces that are uniformly homeomorphic but not linearly homeomorphic. His construction was generalized by Aharoni/Lindenstrauss [1985, p. 59]. They considered the pair $[l_q, l_{p_n}]$ and $l_p \oplus [l_q, l_{p_n}]$, where all exponents $1 \le p, p_n, q < \infty$ are different and $p_n \to p$. It follows from (1) in 6.9.2.5 that these spaces even fail to be Lipschitz homeomorphic.

Letting $p=1$ shows that reflexive spaces can be uniformly homeomorphic to non-reflexive spaces. On the other hand, in view of the next theorem, superreflexivity is stable when we pass to uniformly homeomorphic copies.

6.9.2.7 The famous **Ribe theorem** [1976, p. 238] asserts that, roughly speaking, uniformly homeomorphic Banach spaces have the same finite-dimensional subspaces. More precisely, they are crudely finitely representable in each other; see 6.1.3.1.

An elegant approach to this result was presented in a seminal paper of Heinrich/Mankiewicz [1982, p. 240], who discovered that uniformly homeomorphic Banach spaces X and Y admit Lipschitz homeomorphic ultrapowers $X^{\mathscr{U}}$ and $Y^{\mathscr{U}}$, where $\mathscr{U}$ is any free (non-principal, 7.5.21) ultrafilter on $\mathbb{N}$.

6.9.2.8 The fact that for $1 \le p < \infty$ and $p \ne 2$, the spaces $L_p[0,1]$ and l_p fail to be uniformly homeomorphic was proved in several steps. Enflo (unpublished) treated the case $p=1$; see Benyamini [1984, pp. 30–32]. His proof was extended to $1 \le p < 2$ by Bourgain [1986a, p. 158], and the most complicated part, namely $2<p<\infty$, is due to Gorelik [1994, p. 2].

Since $L_p[0,1]$ and l_p are finitely representable in each other, the implication in Ribe's theorem cannot be reversed.

6.9.2.9 A supplement of Ribe's theorem [1978, p. 7] says that the property of being an $\mathscr{L}_p$-space, $1<p<\infty$, is stable when passing to uniformly homeomorphic copies.

6.9.2.10 Finally, I mention that the first result in the spirit of this subsection was obtained by Keller. In [1931, p. 756], he proved that all infinite-dimensional convex compact subsets of Hilbert spaces are homeomorphic to each other. In view of the Kadets theorem, the same is true in Banach spaces.

6.9.3 The analytic Radon–Nikodym property

In this subsection, all Banach spaces under consideration are *complex*. With regard to Hardy spaces, the reader is advised to consult Subsection 6.7.12.

6.9.3.1 For every Banach space X and $1\le p<\infty$, the Hardy space $[\mathscr{H}_p(\mathbb{D}),X]$ consists of all analytic X-valued functions $\boldsymbol{f}$ on the open unit disk of the complex plane for which

$$\|\boldsymbol{f}|\mathscr{H}_p\| := \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\|\boldsymbol{f}(r\mathrm{e}^{\mathrm{i}\theta})\|^p d\theta\right)^{1/p}$$

is finite. In the limiting case $p=\infty$, the functions $\boldsymbol{f}$ are supposed to be bounded.

6.9.3.2 In [1976, p. 1055], Bukhvalov wrote $X\in(RN)_a$ if whenever $1\le p\le\infty$, every function $\boldsymbol{f}\in[\mathscr{H}_p(\mathbb{D}),X]$ possesses well-defined boundary values:

$$\boldsymbol{f}(\mathrm{e}^{\mathrm{i}\theta}) := \lim_{r\nearrow 1}\boldsymbol{f}(r\mathrm{e}^{\mathrm{i}\theta}) \quad \text{exists for almost all } \theta\text{'s.}$$

The name **analytic Radon–Nikodym property**, which certainly gave rise to the symbol $(RN)_a$, appeared only in Bukhvalov/Danilevich [1982, стр. 205]. This paper also contains the observation that one does not have to verify the condition above for all p's; some fixed p suffices.

6.9.3.3 According to Bukhvalov [1991a, p. 212], the following criterion is already contained in his unpublished д-р. физ.-мат. наук thesis (Leningrad, 1984). Explicit statements are due to Dowling [1985, p. 143] and Hensgen [1986], [1987, p. 404].

A Banach space X has the analytic Radon–Nikodym property if and only if given any X-valued Borel measure $\boldsymbol{m}$ on $\mathbb{T}$ with bounded variation and

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}n\theta}d\boldsymbol{m}(\theta)=\mathrm{o} \quad \text{for } n=1,2,\dots,$$

there exists a Bochner integrable X-valued function $\boldsymbol{f}$ such that

$$\boldsymbol{m}(B)=\int_B\boldsymbol{f}(\mathrm{e}^{\mathrm{i}\theta})d\theta \quad \text{for } B\in\mathscr{B}_{\mathrm{orel}}(\mathbb{T}).$$

By the Riesz–Riesz theorem 6.7.12.17, the measure m is absolutely continuous with respect to the Lebesgue measure. Hence every Banach space with the "ordinary" Radon–Nikodym property also possesses the "analytic" Radon–Nikodym property. The reverse implication fails, since $L_1[0,1]$ has only the analytic Radon–Nikodym property; see Bukhvalov/Danilevich [1982, стр. 208].

6.9.3.4 A continuous real function u on a Banach space X is said to be **pluri-subharmonic** if

$$u(x) \leq \frac{1}{2\pi}\int_0^{2\pi} u(x+\mathrm{e}^{\mathrm{i}\theta}h)\,d\theta \quad \text{for } x,h\in X.$$

The collection of these functions is denoted by $PSH(X)$. Please, note that different authors have used different assumptions concerning continuity; the spectrum ranges from upper semi-continuity to Lipschitz continuity.

6.9.3.5 If $\{x\in A : u(x)>0\}$ is non-empty, then we refer to this subset as the ***PSH*-slice** of A generated by $u\in PSH(X)$.

A subset A is called ***PSH*-dentable** if it has PSH-slices with arbitrarily small diameters. By definition, a ***PSH*-denting point** of A is contained in PSH-slices with diameters as small as we please.

6.9.3.6 A point x_0 in a subset A of X is called ***PSH*-exposed** or a **barrier** if there exists a function $u\in PSH(X)$ such that $u(x)<u(x_0)$ for all $x\in A\setminus\{x_0\}$.

In the case that

$$\sup_{x\in A\setminus B_\varepsilon(x_0)} u(x) < u(x_0) \quad \text{for all } \varepsilon>0 \text{ and some } u\in PSH(X),$$

we speak of a **strongly *PSH*-exposed point** or a **strong barrier**.

6.9.3.7 Let A be a subset in a complex Banach space X. Then $x_0\in A$ is called a **complex extreme point** if there does not exist any $h\in A$ such that $h\neq o$ and

$$x_0+\mathrm{e}^{\mathrm{i}\theta}h\in A \quad \text{whenever } 0\leq\theta<2\pi.$$

Thorp/Whitley [1967, p. 644] proved that *every point on the surface of the unit ball of $L_1(M,\mathscr{M},\mu)$ is a complex extreme point*; see also 5.5.5.1.

6.9.3.8 Subject to the replacements

convex function	$\Longrightarrow$	pluri-subharmonic function,
extreme point	$\Longrightarrow$	complex extreme point,
denting point	$\Longrightarrow$	PSH-denting point,
exposed point	$\Longrightarrow$	PSH-exposed point,
strongly exposed point	$\Longrightarrow$	strongly PSH-exposed point,

the relationship between the different kinds of points is the same as that in 5.4.4.1.

Moreover, Ghoussoub/Lindenstrauss/Maurey [1989, pp. 122–123] and Bu [1988, p. 759] established a full analogue of the geometric characterization of Banach spaces having the Radon–Nikodym property. With the replacements above, the criterion 5.4.4.2 remains true for the *analytic* Radon–Nikodym property if we use, instead of the closed convex hull of a subset A, its *PSH*-**convex hull**, which consists of all elements $x \in X$ such that $u(x) \leq 0$ for every $u \in PSH(X)$ vanishing on A.

6.9.3.9 By analogy with 6.8.11.7, the analytic Radon–Nikodym property can be characterized in terms of martingales. This criterion is due to Edgar [1986, p. 277], who used so-called analytic martingales, a rather technical concept that goes back to Davis/Garling/Tomczak-Jaegermann [1984, p. 130]. The attribute *"analytic"* was suggested by Bourgain/Davis [1986, p. 514].

6.9.3.10 Ryan [1963] had earlier generalized the Riesz–Riesz theorem to vector measures taking their values in a dual Banach space. However, his approach was based on the Pettis integral and has nothing to do with the preceding considerations.

6.9.4 *M*-ideals

Most branches of modern Banach space theory are concerned with *isomorphic* properties. Therefore I stress the *isometric* nature of the concept of an M-ideal. Standard references are [BEHR] and [HARM$^+$].

6.9.4.1 A projection P on a Banach space X is called an

M-**projection** if $\quad \|x\| = \max\{\|Px\|, \|x - Px\|\} \quad$ for all $x \in X$

and an

L-**projection** if $\quad \|x\| = \|Px\| + \|x - Px\| \quad$ for all $x \in X$.

The two concepts are dual to each other; see Alfsen/Effros [1972, p. 140]:

$$P \text{ is an } M\text{-projection} \Leftrightarrow P^* \text{ is an } L\text{-projection},$$
$$P \text{ is an } L\text{-projection} \Leftrightarrow P^* \text{ is an } M\text{-projection}.$$

A closed subspace M of X is said to be an M-**summand** if it is the range of an M-projection. Similarly, one defines an L-**summand** by the property that it is the range of an L-projection. In both cases, the required projections as well as the associated decompositions $X = M \oplus_\infty N$ and $X = M \oplus_1 N$ are uniquely determined; see Cunningham [1960, p. 276] and Alfsen/Effros [1972, pp. 110, 140].

By an M-**ideal** we mean a closed subspace of X whose annihilator $M^\perp$ is an L-summand in X^*. Every M-summand is also an M-ideal. Note, however, that c_0 is an M-ideal in l_∞ that fails to be an M-summand, since it is not complemented.

The definition of an L-projection is due to Cunningham [1960, p. 276]. All other concepts presented above were invented in a seminal paper of Alfsen/Effros [1972] that initiated the theory of M-ideals. Almost simultaneously, Ando [1973, p. 395] developed similar ideas.

6.9.4.2 The theory of M-ideals mainly deals with real Banach spaces. However, the results remain true in the complex case as well. Indeed, Hirsberg [1972, pp. 135–136] showed that the basic concepts have the same meaning if one passes from real to complex scalars; see also [BEHR, pp. 22, 36].

6.9.4.3 One refers to

$$O(x_0, r) := \{x \in X : \|x - x_0\| < r\}$$

as the **open ball** with center $x_0 \in X$ and radius $r > 0$.

According to Alfsen/Effros [1972, p. 98], a closed subspace M of a Banach space X is said to have the **n-ball property** if

$$\bigcap_{k=1}^{n} O(x_k, r_k) \neq \emptyset \quad \text{and} \quad O(x_k, r_k) \cap M \neq \emptyset \quad \text{for } k = 1, \ldots, n$$

imply

$$\bigcap_{k=1}^{n} O(x_k, r_k) \cap M \neq \emptyset.$$

A fundamental theorem of these authors says that a closed subspace is an M-ideal if and only if it has the n-ball property for all $n = 2, 3, \ldots$. They also observed that the 3-ball property suffices, while the 2-ball property does not; see Alfsen/Effros [1972, pp. 98, 120–122].

6.9.4.4 For every Banach space X, the collection of all M-ideals contains the trivial members $\{o\}$ and X. Moreover, Alfsen/Effros [1972, p. 138] proved that the intersection of finitely many M-ideals and the closed linear span of arbitrarily many M-ideals is an M-ideal as well.

We know from [BEHR, p. 24] that apart from the real space l_∞^2, a Banach space cannot contain non-trivial M-ideals and non-trivial L-summands simultaneously.

6.9.4.5 A strictly convex or smooth Banach space does not contain any non-trivial M-ideals. These results go back to Holmes/Scranton/Ward [1975, p. 265] and Behrends [1978, p. 265], respectively.

6.9.4.6 The Banach algebras $C(K)$ are the most prominent examples of Banach spaces that contain many M-ideals. Alfsen/Effros [1972, p. 139] observed that in this case, the M-ideals have the form

$$M_F := \{f \in C(K) : f(t) = 0 \text{ for } t \in F\},$$

where F is any closed subset of the underlying compact Hausdorff space K.

Based on preliminary work of Alfsen/Effros [1972, p. 167], Smith/Ward [1978, p. 347] obtained a non-commutative analogue:

The M-ideals of a $C^\star$-algebra are just the closed two-sided (algebraic) ideals.

6.9.4.7 The previous result implies that the disk algebra $A(\mathbb{T})$ fails to be an M-ideal in $C(\mathbb{T})$.

The M-ideals in $A(\mathbb{T})$ were identified by Hirsberg [1972, p. 144]; see also [HARM$^+$, p. 4]. They have the form

$$M_F := \{f \in A(\mathbb{T}) : f(\zeta) = 0 \text{ for } \zeta \in F\},$$

where F is any *peak set*:

There exists $f \in A(\mathbb{T})$ such that $f(\zeta) = 1$ for $\zeta \in F$ and $|f(\zeta)| < 1$ otherwise,
or
F is a closed Lebesgue null subset of $\mathbb{T}$.
The equivalence of these properties was obtained by the joint efforts of Fatou [1906, p. 393], the brothers Riesz [1916, p. 32], Carleson [1957, p. 448], and Rudin [1956, pp. 808–809]; see also [GAM, pp. 56–60] and [HOF, p. 81].

6.9.4.8 According to Phelps [1960b, p. 238], a closed subspace M of a Banach space X is said to have **property U** if every functional on M admits a *unique* norm-preserving extension to X. Obviously, every M-ideal enjoys property U. It seems that this fact had been overlooked for a while; see Hennefeld [1973].

6.9.4.9 M-ideals turn out to be a useful tool in approximation theory, since they are **proximinal**:

Every member $x \in X$ admits a best approximation $x_0 \in M$,

$$\|x - x_0\| = \inf\{\, \|x - u\| : u \in M\,\}.$$

Note, however, that x_0 need not be unique.

This result was obtained by Alfsen/Effros [1972, p. 120]. At least implicitly, it is also contained in Ando's paper [1973, pp. 396–397]; see Holmes [1976, pp. 392–393].

6.9.4.10 Let X be a predual of $L_1(M, \mathscr{M}, \mu)$. Then Lima [1979, p. 210] showed that the following are equivalent:

$$X \text{ is isometric to some } c_0(\mathbb{I}) \quad \Leftrightarrow \quad X \text{ is an } M\text{-ideal in its bidual.}$$

In particular, we get an *isometric* characterization of c_0, the most prominent predual of l_1.

6.9.4.11 A Banach space X that is an M-ideal in its bidual enjoys many remarkable *isomorphic* properties. For example, X^* has the Radon–Nikodym property, or equivalently, every separable subspace has a separable dual; see Lima [1982, p. 33] and 5.1.4.12. On the other hand, if X is non-reflexive, then it fails to have the Radon–Nikodym property; see Harmand/Lima [1984, p. 257].

6.9.4.12 The next theorem was independently proved by Godefroy/Li and Werner; see [HARM$^+$, pp. 134, 148]:

A separable $\mathscr{L}_\infty$-space that is an M-ideal in its bidual must be isomorphic to c_0.

This result applies to the little Bloch space. Indeed, Werner [1992, p. 347] discovered that $\mathscr{B}_0(\mathbb{D})$, equipped with a suitable norm, is an M-ideal in its bidual $\mathscr{B}(\mathbb{D})$. Since the latter space is isomorphic to l_∞, it follows from 6.1.4.2 that $\mathscr{B}_0(\mathbb{D})$ is a (separable) $\mathscr{L}_\infty$-space and therefore isomorphic to c_0; see 6.7.13.4.

6.9.4.13 Spaces X for which $\mathfrak{K}(X)$ is an M-ideal in $\mathfrak{L}(X)$ have attracted many people. A theorem of Lima [1982, p. 31] says that in this case, X must be an M-ideal in X^{**}.

The first positive result was obtained by Dixmier [1950, p. 396] for Hilbert spaces. Next, Hennefeld [1973, p. 199] proved that $\mathfrak{K}(l_p)$ is an M-ideal in $\mathfrak{L}(l_p)$ if $1<p<\infty$. The same holds for c_0. Later on, Smith/Ward [1978, pp. 348–349] gave a negative answer for l_1 and l_∞. Further counterexamples are the spaces $C[0,1]$ and $L_p[0,1]$ with $1\le p\le\infty$ and $p\ne 2$; see Lima [1979, pp. 209, 215].

6.9.4.14 The way to considerations in general Banach spaces was opened by a paper of Kalton in which he introduced the following properties; see [1993, pp. 148–159]:

(M) $$\limsup_\alpha \|u+x_\alpha\| = \limsup_\alpha \|v+x_\alpha\|$$

for any choice of $u,v\in X$ with $\|u\|=\|v\|$ and every bounded weakly null net (x_α).

(M*) $$\limsup_\alpha \|u^*+x^*_\alpha\| = \limsup_\alpha \|v^*+x^*_\alpha\|$$

for any choice of $u^*,v^*\in X^*$ with $\|u^*\|=\|v^*\|$ and every bounded weakly* null net (x^*_α).

We also need Rosenthal's property; see 5.6.3.8:

(R) Every bounded sequence in X has a weak Cauchy subsequence.

Combining results of Kalton [1993, p. 150], Lima [1982, p. 33], and Kalton/Werner [1995, p. 141] yields that for separable Banach spaces,

$$(\mathbf{M})+(\mathbf{R})=(\mathbf{M}^*).$$

Based on the preceding concepts, the following criterion was established by Kalton/Werner [1995, p. 149] (separable case) and Oja [2000, p. 2819]:

Let X be any Banach space. Then $\mathfrak{K}(X)$ is an M-ideal in $\mathfrak{L}(X)$ if and only if X has the metric compact approximation property and property **(M*)**.

As an outsider, I wonder why the specialists prefer to work with $\mathfrak{K}(X)$ instead of $\overline{\mathfrak{F}}(X)$. In my opinion, the latter case is the more natural one. Indeed, Oja [2000, p. 2821] observed that $\overline{\mathfrak{F}}(X)$ is an M-ideal in $\mathfrak{L}(X)$ if and only if X has the metric approximation property and property **(M*)**.

6.9.4.15 Kalton [1993, p. 156] also showed that $\mathfrak{K}(X)$ is an M-ideal in $\mathfrak{L}(X)$ if and only if it is an M-ideal in the unital algebra generated by $\mathfrak{K}(X)$ and the identity of X.

6.9.4.16 One may also study Banach spaces that are L-summands in their biduals. For example, all preduals of $W^\star$-algebras, and in particular, the classical spaces $L_1(M,\mathscr{M},\mu)$ have this property; see [HARM$^+$, p. 158]. Godefroy [1983] proved a remarkable consequence: weak sequential completeness. Hence non-reflexive spaces that are L-summands in their biduals contain isometric copies of l_1; see 3.5.2.

6.9.4.17 In view of the **Daugavet equation** [1963],

$$\|I+T\| = 1+\|T\| \quad \text{for } T \in \mathfrak{K}\big(C[0,1]\big),$$

$\mathfrak{K}\big(C[0,1]\big)$ is an L-summand in the unital algebra generated by $\mathfrak{K}\big(C[0,1]\big)$ and the identity map of $C[0,1]$. Lozanovskiĭ [1966, стр. 39] showed that $C[0,1]$ can be replaced by $L_1[0,1]$. Hence it is natural to look for further spaces of this kind. Following Kadets/Shvidkoy/Sirotkin/Werner [2000, p. 856], one says that X has the **Daugavet property** if

$$\|I+T\| = 1+\|T\| \quad \text{for all } T \in \mathfrak{L}(X) \text{ with } \operatorname{rank}(T)=1.$$

In this case, $\|I+T\| = 1+\|T\|$ also holds for all weakly compact operators; see Kadets/Shvidkoy/Sirotkin/Werner [2000, p. 857]. For the space $C[0,1]$, this extension of Daugavet's equation had already been discovered by Foiaş/Singer [1965, pp. 441–445] (with the help of Pełczyński); see also Holub [1986]. Kadets/Shvidkoy/Sirotkin/Werner [2000, pp. 861, 867] also showed that every quotient of $L_1[0,1]$ modulo a reflexive subspace has the Daugavet property and that every space with the Daugavet property contains an isomorphic copy of l_1.

A comprehensive presentation can be found in [ABRA$_+$, Chap. 11]; see also a survey of Werner [2001].

6.9.4.18 The Daugavet equation is of consequence to the philosophy of linear approximation.

Let E be any finite-dimensional subspace of $C[0,1]$. Then it follows that there cannot exist any non-zero operator $T : C[0,1] \to E$ such that Tf is a good approximation of f for all $f \in C[0,1]$. Indeed, because of $\|I-T\| = 1+\|T\| > 1$, we find $f_0 \in C[0,1]$ such that $\|f_0 - Tf_0\| > \|f_0\|$.

6.9.5 Stable Banach spaces

6.9.5.1 A fundamental theorem of Aldous [1981, p. 445] says that every infinite-dimensional closed subspace X of $L_1[0,1]$ contains an isomorphic copy of some l_p, where the exponent $1 \le p \le 2$ depends on X.

6.9.5.2 The preceding result inspired Krivine/Maurey [1981] to introduce the following *isometric* concept, which is some kind of an anti-Tsirelson vaccine.

A Banach space X is said to be **stable** if

$$\mathscr{U}\text{-}\lim_m \mathscr{V}\text{-}\lim_n \|x_m + y_n\| = \mathscr{V}\text{-}\lim_n \mathscr{U}\text{-}\lim_m \|x_m + y_n\| \tag{6.9.5.2.a}$$

regardless how we choose the bounded sequences (x_m), (y_n) in X and the ultrafilters $\mathscr{U}$, $\mathscr{V}$ on $\mathbb{N}$.

Note that Krivine/Maurey dealt only with separable Banach spaces. The theory of stable Banach space is presented in Garling's article [1982] and in the monograph [GUER, Chap. 3].

6.9.5.3 The main theorem of Krivine/Maurey [1981, p. 290] states that every infinite-dimensional stable Banach space contains a closed subspace that is isomorphic to some l_p. More precisely, there exists an exponent $1 \le p < \infty$ such that given any $\varepsilon > 0$, we can find a subspace X_ε with $d(X_\varepsilon, l_p) \le 1+\varepsilon$.

Another proof is due to Bu [1989].

6.9.5.4 Krivine/Maurey [1981, pp. 277–278, 280] showed that the classical Banach spaces L_p with $1 \le p < \infty$ are stable. As observed by Garling [1982, p. 167], the same is true for Orlicz spaces L_Φ whenever Φ satisfies the Δ_2-condition; see 6.7.14.5.

6.9.5.5 In view of

$$\lim_{m\to\infty} \big\| e_m + (e_1 + \cdots + e_n) \big\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \big\| e_m + (e_1 + \cdots + e_n) \big\| = 2,$$

the space c_0 fails to be stable. This fact explains why the case $p = \infty$ is excluded from the previous considerations. According to Garling [1982, unpublished preliminary version] and Argyros/Negrepontis/Zachariades [1987, p. 68], this gap can be avoided by using the concept of a **weakly stable** Banach space. To this end, (6.9.5.2.a) is supposed to hold only for all sequences (x_m), (y_n) in X that are contained in some weakly compact subset.

6.9.5.6 Superstable Banach spaces were studied by Raynaud [1983].

6.9.6 Three space problems

6.9.6.1 Let Y be a closed subspace of a Banach space Z. Then the following question is referred to as the **three space problem**:

If one has information about two of the spaces $X := Z/Y$, Y, and Z, what can be said about the third one?

This terminology goes back to Enflo/Lindenstrauss/Pisier [1975, p. 199]; see also [AAA$_{20}$, p. 168].

By far, the most interesting case is that in which a given property $\mathbb{P}$ carries over from X and Y to Z. Then $\mathbb{P}$ is called a **three space property**.

6.9.6.2 The connection between X, Y, and Z can be expressed in a fashionable way by saying that

$$\{o\} \longrightarrow Y \xrightarrow{\;\;J\;\;} Z \xrightarrow{\;\;Q\;\;} X \longrightarrow \{o\}$$

is a short **exact sequence**.

Here J denotes the canonical embedding from Y into Z, and Q denotes the quotient map from Z onto X. Then J is one-to-one, the (closed!) range of J coincides with the null space of Q, and Q is onto. If these conditions are satisfied, one refers to Z as a **twisted sum** of X and Y.

6.9.6.3 The standard reference for this branch of Banach space theory is [CAST$^+$]. A table of more than 150 items tells the reader whether the answer to a concrete three space problem is positive, negative, or open.

The search for three space properties has led to many beautiful, but isolated, results. Of particular significance are some constructions of counterexamples.

6.9.6.4 Kreĭn/Shmulyan [1940, p. 575] discovered the first example of a three space property, namely reflexivity; see 3.4.3.7.

According to Giesy [1966, p. 130] and Enflo/Lindenstrauss/Pisier [1975, p. 203], B-convexity and superreflexivity are three space properties as well. From Edgar [1977, p. 675] we know that the same is true for the Radon–Nikodym property. Being an Asplund space was identified as a three space property by Namioka/Phelps [1975, pp. 742, 744].

Jarchow [1984] successfully applied concepts and techniques from the theory of operator ideals to the study of three space problems.

6.9.6.5 Next, I discuss a few negative examples.

Corson [1961, pp. 13–14] and Lindenstrauss [1972, p. 243] considered the short exact sequence

$$\{o\} \longrightarrow C(\mathbb{T}) \xrightarrow{\;J\;} D(\mathbb{T}) \xrightarrow{\;Q\;} c_0(\mathbb{T}) \longrightarrow \{o\},$$

where $D(\mathbb{T})$ consists of all real-valued functions f on the unit circle $\mathbb{T}$ for which the one-sided limits $f_\pm(z) := \lim_{\varepsilon\searrow 0} f(ze^{\pm i\varepsilon})$ exist and $f(z) = f_+(z)$ for all $z\in\mathbb{T}$. Obviously, $D(\mathbb{T})$ becomes a Banach space with respect to the sup-norm. Moreover, J is the canonical embedding from $C(\mathbb{T})$ into $D(\mathbb{T})$, and the surjection Q takes f to the family $x = (\xi_z)$ with $\xi_z := f_+(z) - f_-(z)$. Then $C(\mathbb{T})$ and $c_0(\mathbb{T})$ are weakly compactly generated, while $D(\mathbb{T})$ is not.

Johnson/Rosenthal [1972, p. 90] showed that every separable Banach space Z contains a subspace Y such that both $X := Z/Y$ and Y have a finite-dimensional Schauder decomposition. This result combined with Enflo's counterexample tells us, among others things, that the approximation property is not a three space property.

6.9.6.6 For $1 < p < \infty$, the **Kalton–Peck space** Z_p consists of all pairs (x,y) of scalar sequences $x = (\xi_k)$ and $y = (\eta_k)$ in l_p such that

$$|||(x,y)|Z_p||| := \|x|l_p\| + \left(\sum_{k=1}^{\infty}\left|\eta_k - \xi_k \log\frac{|\xi_k|}{\|x|l_p\|}\right|^p\right)^{1/p}$$

is finite; Kalton/Peck [1979, pp. 5, 11, 23]. At first glance, this definition yields only a quasi-Banach space. However, a result of Kalton [1978, p. 250] ensures that there exists an equivalent norm. The upshot is a reflexive Banach space which fails to be isomorphic to l_p. Letting $J: y \mapsto (o, y)$ and $Q: (x, y) \mapsto x$, we get a short exact sequence

$$\{o\} \longrightarrow l_p \xrightarrow{\ J\ } Z_p \xrightarrow{\ Q\ } l_p \longrightarrow \{o\}.$$

The Kalton–Peck space provides counterexamples to several problems. Two of them will be described next.

6.9.6.7 According to Enflo/Lindenstrauss/Pisier [1975, p. 199] or [AAA$_{20}$, p. 170], the three space problem for Hilbert spaces is *"apparently due to Palais,"* a well-known specialist in global analysis. This is why the name *"Palais problem"* has become common; see [CAST$^+$, Index on p. 265]. However, when I asked Palais about the origin of his problem the answer was very surprising: he had not even heard of such a question. So one should better speak of the *"non-Palais problem."* Its negative solution was given by Enflo/Lindenstrauss/Pisier [1975, p. 210]. Their approach and the work of Ribe [1979] inspired Kalton/Peck [1979, p. 15] to construct the twisted sum Z_2, which seems to be the most natural counterexample; see also Kalton/Montgomery-Smith [2003, p. 1157].

6.9.6.8 For Z_p with $1 < p \leq 2$, Kalton/Peck [1979, p. 24] showed that the type p constants computed with n vectors are larger than $c \log n$. Hence possessing Rademacher type p is not a three space property.

6.9.6.9 In the setting of quasi-Banach spaces, the strange situation occurs that being a Banach space is not a three space property: there exist twisted sums Z without any equivalent norm such that $X := Z/Y$ and Y are Banach spaces. One may even arrange that $\dim(Y) = 1$. This result was independently obtained by Kalton [1978, p. 272], Ribe [1979, p. 351], and Roberts [1977, p. 79]; see also [KAL$^+$, Chap. 5].

6.9.7 Functors in categories of Banach spaces

6.9.7.1 As it has for many "working mathematicians" [MACL], the language of categories, and in particular, the use of $\longrightarrow$'s proved to be elegant tools in Banach space theory too. However, an attempt to study functors in the category of Banach spaces in their own right fizzled out, since the results were quite formal. The reader may consult the surveys of Mityagin/Shvarts [1964] and Wick-Pelletier [1984] as well as the lecture notes of Michor [MICHO].

6.9.7.2 Covariant **functors** occur when we pass from spaces consisting of scalar-valued sequences or functions to the X-valued setting: $X \mapsto [l_p, X]$ or $X \mapsto [L_p, X]$. The most prominent contravariant functor is given by $X \mapsto X^*$.

6.9.7.3 For a tensor norm α and a Banach ideal $\mathfrak{A}$, the maps

$$(X,Y) \mapsto X \,\widetilde{\otimes}_\alpha Y \quad \text{and} \quad (X,Y) \mapsto \mathfrak{A}(X,Y)$$

are bifunctors.

6.9.7.4 Another category consists of all interpolation couples $\{X_0, X_1\}$ and all couples of operators $\{T_0, T_1\}$ as described in Subsection 6.6.2. Then we have, for example, the real and complex interpolation functors

$$\Phi_{\theta,q} : \{X_0, X_1\} \mapsto (X_0, X_1)_{\theta,q} \quad \text{and} \quad \Phi_\theta : \{X_0, X_1\} \mapsto [X_0, X_1]_\theta;$$

see 6.6.3.8 and 6.6.3.2.

6.9.7.5 Finally, I mention the category whose objects are the operator spaces and whose morphisms are the completely bounded linear operators; see 6.9.16.4.

6.9.8 Local spectral theory

The reader is advised to consult Subsections 5.2.4 and 5.2.5.

6.9.8.1 The **local resolvent set** $\rho(T,x)$ of an operator $T \in \mathfrak{L}(X)$ at a point $x \in X$ is defined to be the union of all open subsets G of the complex plane on which there exists an analytic X-valued function $\boldsymbol{f}_x$ such that

$$(\lambda I - T)\boldsymbol{f}_x(\lambda) = x \quad \text{for all } \lambda \in G.$$

One refers to $\sigma(T,x) := \complement\rho(T,x)$ as the **local spectrum** of T at x.

If $\lambda \in \rho(T)$, then we may take $\boldsymbol{f}_x(\lambda) := R(\lambda, T)x$. Hence $\rho(T) \subseteq \rho(T,x)$ and $\sigma(T,x) \subseteq \sigma(T)$.

6.9.8.2 An operator $T \in \mathfrak{L}(X)$ is said to have the **single-valued extension property** if there exists no analytic X-valued function $\boldsymbol{f} \neq \boldsymbol{o}$ on some open subset G such that

$$(\lambda I - T)\boldsymbol{f}(\lambda) = o \quad \text{for all } \lambda \in G.$$

This happens, in particular, when the spectrum of T contains no interior point.

In a slightly different form, the single-valued extension property was invented by Dunford [1952, p. 564], [1954, p. 327]. According to [DUN_3^+, p. 1932], Kakutani discovered that the backward shift $S_{\overleftarrow{\text{hift}}}$ on l_2 does not have this property; see 6.9.9.12. Nowadays, one infers this fact from a theorem of Finch [1975, p. 61]:

Every surjection with the single-valued extension property is an isomorphism.

As shown by Sine [1964, p. 335], the single-valued extension property implies that

$$\sigma(T) = \bigcup_{x \in X} \sigma(T,x).$$

6.9.8.3 Given $T \in \mathfrak{L}(X)$, then with every closed subset F of the complex plane we associate the **local spectral space**

$$M(T,F) := \{x \in X : \sigma(T,x) \subseteq F\}.$$

Such sets were first considered by Dunford [1952, p. 565], who observed that they are linear. Obviously, $M(T,F)$ is invariant under all operators that commute with T. An operator T is said to have **Dunford's property (C)** if all $M(T,F)$'s are closed; see also Dunford [1958, p. 226].

6.9.8.4 Let M be a closed subspace that is invariant under $T \in \mathfrak{L}(X)$. Then $\sigma(T|M)$ denotes the spectrum of the operator induced by T on M.

Following Foiaş [1963, p. 341], we refer to M as a **spectral maximal space** of T if all invariant closed subspaces M_0 with $\sigma(T|M_0) \subseteq \sigma(T|M)$ are included in M.

6.9.8.5 An operator $T \in \mathfrak{L}(X)$ is called **decomposable** if given any open covering $\{G_1, \dots, G_n\}$ of the spectrum $\sigma(T)$, there exist invariant closed subspaces $M_1, \dots, M_n$ such that $X = M_1 + \dots + M_n$ and $\sigma(T|M_k) \subseteq G_k$ for $k = 1, \dots, n$. In this case, one can additionally arrange that $\sigma(T|M_k) \subseteq \sigma(T)$; see [COL$^+$, p. 23].

Please, note that $X = M_1 + \dots + M_n$ is not assumed to be a direct sum.

The concept of a decomposable operator goes back to Bishop [1959, p. 382] and Foiaş [1963, p. 343]. Bishop used the condition as described above, but he spoke of operators *admitting a duality theory of type 3*. The original definition of Foiaş required $M_1, \dots, M_n$ to be spectral maximal subspaces. Only later and independently of each other, did Albrecht [1979], Lange [1981], and Nagy [1978] achieve a significant simplification by showing that the two approaches are equivalent.

6.9.8.6 The fact that decomposability is carried across between an operator $T \in \mathfrak{L}(X)$ and its dual $T^* \in \mathfrak{L}(X^*)$ in both directions was established by Frunză [1976, p. 320], Eschmeier [1984, pp. 119–120], and Wang/Liu [1984, p. 161]; see also [LAU$^+$, pp. 151, 154, 173].

6.9.8.7 A **spectral capacity** is a map $\boldsymbol{M}$ that assigns to every closed subset F of the complex plane a closed subspace $\boldsymbol{M}(F)$ of a given Banach space X such that the following conditions are satisfied:

(1) $\boldsymbol{M}(\emptyset) = \{o\}$ and $\boldsymbol{M}(\mathbb{C}) = X$.

(2) $\boldsymbol{M}(\overline{G_1}) + \dots + \boldsymbol{M}(\overline{G_n}) = X$ for every open covering $\{G_1, \dots, G_n\}$ of $\mathbb{C}$.

(3) $\bigcap_{k=1}^{\infty} \boldsymbol{M}(F_k) = \boldsymbol{M}\big(\bigcap_{k=1}^{\infty} F_k\big)$ for every sequence of closed subset $F_1, F_2, \dots$.

We refer to $\boldsymbol{M}$ as a *spectral capacity of the operator* $T \in \mathfrak{L}(X)$ if, moreover, $\boldsymbol{M}(F)$ is invariant under T and $\sigma\big(T|\boldsymbol{M}(F)\big) \subseteq F$ for every closed subset F.

The definitions above are due to Apostol [1968, p. 1495], who also observed that every operator with a spectral capacity is decomposable.

6.9.8.8 In [1963, p. 344], Foiaş had shown that for decomposable operators, the local spectral spaces are just the spectral maximal spaces. Using this fact, he was able to prove the converse of Apostol's result; see Foiaş [1968, p. 1541]:

Every decomposable operator T has a unique spectral capacity given by $F \mapsto M(T, F)$.

6.9.8.9 In the preface of [LAU$^+$] the authors claim:

The fact that a decomposable operator possesses a spectral capacity opens the door to a powerful local analysis of its spectrum.

Of course, this credo applies only to operators with a complicated spectrum that should be decomposed into simpler parts.

Unfortunately, even operators with a simple spectrum may behave very badly. Indeed, given any operator $T : X \to Y$, we get a nilpotent operator $S : X \oplus Y \to X \oplus Y$ by letting $S : (x,y) \mapsto (o, Tx)$. Since the equations

$$Tx = b \quad \text{and} \quad S(x,y) = (o,b) \quad \text{with } b \in Y$$

are equivalent, this elementary construction shows that any pathology that occurs for general operators also occurs for operators with a one-point spectrum.

6.9.8.10 Let $C^\infty(\mathbb{C})$ denote the collection of all complex-valued functions φ, defined for $\lambda = \xi + \mathrm{i}\eta \in \mathbb{C}$, that have derivatives of any order with respect to the real variables ξ and η. A locally convex topology is obtained from the semi-norms

$$p_n(\varphi) := \sup\left\{ \left| \frac{\partial^{h+k}\varphi}{\partial\xi^h \partial\eta^k}(\xi + \mathrm{i}\eta) \right| : h+k \le n,\ |\xi|^2 + |\eta|^2 \le n^2 \right\} \quad \text{with } n = 0,1,2,\ldots .$$

Foiaş [1960, p. 148] refers to $T \in \mathfrak{L}(X)$ as a **generalized scalar operator** if there exists a **spectral distribution**. This is a continuous homomorphism $\boldsymbol{F}$ from the ring $C^\infty(\mathbb{C})$ into the ring $\mathfrak{L}(X)$ such that

$$\boldsymbol{F} : 1 \mapsto I \quad \text{and} \quad \boldsymbol{F} : \lambda \mapsto T.$$

Of course, for analytic functions φ, the corresponding operator $\varphi(T) = \boldsymbol{F}(\varphi)$ is given by the Riesz–Dunford–Taylor "operational calculus"; see 5.2.1.2. Nowadays, most authors use the term "functional calculus," and one says that $\boldsymbol{F}$ yields a C^∞-**functional calculus**.

Foiaş [1960, p. 152] showed that in general, $\boldsymbol{F}$ is not uniquely determined by the underlying operator T. For example, the identity map of a Banach space can be obtained from

$$F : \varphi \mapsto \varphi(1)I + \frac{1}{2}\left[\frac{\partial\varphi}{\partial\xi}(1) + \mathrm{i}\frac{\partial\varphi}{\partial\eta}(1)\right]A,$$

where A is any operator with $A^2 = O$.

6.9.8.11 The advantage of C^∞-functions over analytic functions is that the former can be used to construct "partitions of unity." Such partitions are the decisive tool to show that every generalized scalar operator T is decomposable; see [COL$^+$, pp. 65–67]. In terms of the spectral distribution, the associated spectral capacity is given by

$$F \mapsto M(T,F) = \bigcap \big\{ N\big(\varphi(T)\big) : \varphi \in C^\infty(\mathbb{C}),\ \operatorname{supp}(\varphi) \cap F = \emptyset \big\}.$$

Here $N\big(\varphi(T)\big)$ denotes the null space of $\varphi(T)$.

6.9.8.12 The following diagram shows the relationship between various spectral properties of operators:

(implications) (counterexamples)

$$
\begin{tikzcd}[row sep=large]
\boxed{\text{scalar operator}} \arrow[d, shift right, "\text{obvious}"'] \\
\boxed{\text{generalized scalar operator}} \arrow[u, shift right, "/" marking, "f(t)\mapsto tf(t) \text{ on } C[0,1]"'] \arrow[d, shift right, "{[\mathrm{COL}^+,\text{ pp. 65–67}]}"'] \\
\boxed{\text{decomposable operator}} \arrow[u, shift right, "/" marking, "\text{Albrecht [1977, p. 274]}"'] \arrow[d, shift right, "\text{Foiaş [1963, p. 344]}"'] \\
\boxed{\text{Dunford's property (C)}} \arrow[u, shift right, "/" marking, "{\begin{array}{c} f(z)\mapsto zf(z) \text{ on } \mathscr{A}(\mathbb{D}) \\ {[\mathrm{LAU}^+,\text{ pp. 13, 82}]} \end{array}}"'] \arrow[d, shift right, "\text{Laursen/Neumann [1993, p. 485]}"'] \\
\boxed{\text{single-valued extension property}} \arrow[u, shift right, "/" marking, "{[\mathrm{COL}^+,\text{ p. 25}]}"'] \arrow[d, shift right, "\text{obvious}"'] \\
\boxed{\text{arbitrary operator}} \arrow[u, shift right, "/" marking, "S^{\leftarrow}_{\mathrm{hift}} \text{ on } l_2"']
\end{tikzcd}
$$

6.9.8.13 The standard references on local spectral theory are [COL$^+$] and [LAU$^+$]. For further information the reader may consult [AIENA], [ERD$^+$], [ESCH$^+$], [LAN$^+$], and [VAS].

6.9.9 Hankel and Toeplitz operators

6.9.9.1 Every bounded function $\varphi(e^{i\theta}) = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta}$ induces a **multiplication operator**

$$M_\varphi : f \mapsto \varphi f$$

on $L_2(\mathbb{T})$. Passing to the Fourier image yields the **convolution operator**

$$C_a : (\xi_k) \mapsto \Big(\sum_{k\in\mathbb{Z}} \alpha_{h-k}\xi_k\Big)$$

associated with the two-sided sequence $a = (\alpha_k)$.

With respect to the orthogonal decompositions

$$L_2(\mathbb{T}) = H_2(\mathbb{T})^\perp \oplus H_2(\mathbb{T}) \quad \text{and} \quad l_2(\mathbb{Z}) = l_2(\mathbb{N}_0)^\perp \oplus l_2(\mathbb{N}_0), \quad \mathbb{N}_0 := \{0,1,2,\dots\},$$

the operators above can be viewed as $(2,2)$-matrices whose entries are operators of a specific structure:

$$\begin{pmatrix} \begin{array}{c}\text{Toeplitz}\\ \text{operator}\end{array} & & \begin{array}{c}\text{Hankel}\\ \text{operator}\end{array} \\ & \longrightarrow k & \\ \begin{array}{c}\text{Hankel}\\ \text{operator}\end{array} & \downarrow h & \begin{array}{c}\text{Toeplitz}\\ \text{operator}\end{array} \end{pmatrix}.$$

6.9.9.2 Consider the **Riesz projection**

$$P : \sum_{-\infty}^{+\infty} \xi_k e^{ik\theta} \mapsto \sum_{0}^{+\infty} \xi_k e^{ik\theta}$$

from $L_2(\mathbb{T})$ onto $H_2(\mathbb{T})$.

The **Toeplitz operator** T_φ on $H_2(\mathbb{T})$ associated with the function $\varphi \in L_\infty(\mathbb{T})$ is defined by

$$T_\varphi : f \mapsto P(\varphi f).$$

The most common form of a **Hankel operator** is the following:

$$H_\varphi : f \mapsto (I-P)(\varphi f)$$

viewed as a mapping from $H_2(\mathbb{T})$ into $H_2(\mathbb{T})^\perp$. Please, note that slightly modified definitions are in use; see [PARTI, p. 30], [BÖT$^+$, p. 53], and Peetre [1983, p. 295]. Therefore results from different papers should be compared carefully.

The function φ is called the **symbol** of T_φ and H_φ. For Toeplitz operators the correspondence $\varphi \mapsto T_\varphi$ is one-to-one, whereas H_φ determines φ only up to an analytic summand.

If $\varphi \in L_2(\mathbb{T})$, then the mappings $T_\varphi : f \mapsto P(\varphi f)$ and $H_\varphi : f \mapsto (I-P)(\varphi f)$ are obviously defined for trigonometric polynomials, and one may hope that there exist continuous extensions. In the affirmative case, T_φ and H_φ make sense even for certain unbounded symbols.

6.9.9.3 Discrete *"Haplitz operators"* are induced by infinite matrices:

Hankel matrix **Toeplitz matrix**

$$H_a = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots\cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots\cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots\cdots \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad T_a = \begin{pmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \cdots\cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots\cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots\cdots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To the best of my knowledge, Hartman/Wintner [1950] is the first functional analytic paper in which the names *Hankel matrix* and *Toeplitz matrix* were used. This terminology is justified by the fact that Hankel [1861] dealt with finite matrices of the form (α_{h+k-1}). On the other hand, it should be observed that Toeplitz [1910, 1911a] developed his spectral theory for so-called *L*-forms. That is, he studied multipliers: *two-sided* infinite (*Laurent*) matrices (α_{h-k}). Therefore the term *Toeplitz operator* is historically misleading.

6.9.9.4 First of all, I concentrate on the theory of Hankel operators. Besides the little books [PARTI] and [POW], the standard references are [NIK$_2$], [PELL], and [ZHU].

6.9.9.5 Most important is **Nehari's theorem** [1957, p. 154], which says that

$$H_a : (\xi_k) \mapsto \Big(\sum_{k=1}^{\infty} \alpha_{h+k-1} \xi_k \Big)$$

yields an operator on l_2 if and only if there exists a function $\varphi \in L_\infty(\mathbb{T})$ such that

$$\alpha_k = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{\mathrm{i}k\theta} d\theta \quad \text{for } k=1,2,\ldots . \tag{6.9.9.5.a}$$

Moreover, we have $\|H_a\| = \inf \|\varphi|L_\infty\|$, where the infimum ranges over all φ's for which (6.9.9.5.a) holds. A modern version of Nehari's condition requires that the function $\varphi_a(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k=1}^{\infty} \alpha_k \mathrm{e}^{-\mathrm{i}k\theta}$ have bounded mean oscillation. In shorthand, we may write $\varphi_a \in BMO(\mathbb{T})$, where $BMO(\mathbb{T})$ is defined in the same way as $BMO(\mathbb{R})$; see 6.7.12.11.

6.9.9.6 The **Hilbert matrix**

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots\cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots\cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots\cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

can be obtained from the bounded function $\varphi(\mathrm{e}^{\mathrm{i}\theta}) := \mathrm{i}(\theta - \pi)$, whereas the corresponding *BMO*-function $\varphi_{\mathrm{hilb}}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{e}^{-\mathrm{i}k\theta}$ is unbounded.

6.9.9.7 Hartman [1958, p. 863] proved that the Hankel operator H_a is compact if and only if (6.9.9.5.a) holds with a continuous function φ, or equivalently, if and only if $\varphi_a(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k=1}^{\infty} \alpha_k \mathrm{e}^{-\mathrm{i}k\theta}$ has vanishing mean oscillation; see 6.7.12.13. The latter condition may be written as $\varphi_a \in VMO(\mathbb{T})$.

6.9.9.8 The most spectacular result in this area is **Peller's theorem**, which states that the Hankel operator H_a belongs to the Schatten–von Neumann ideal $\mathfrak{S}_p$ if and only if $\varphi_a(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k=1}^{\infty} \alpha_k \mathrm{e}^{-\mathrm{i}k\theta}$ is a member of the Besov space $B^{1/p}_{p,p}(\mathbb{T})$; see 6.3.1.6 and 6.7.13.6. In a first step, Peller [1980, стр. 543, 548, 551] treated the case $1 \le p < \infty$; see also Rochberg [1982, pp. 915–916], [1985, p. 252]. This criterion was extended to $0 < p < 1$ by Peller [1985] and Semmes [1984]. The earliest (either necessary or sufficient) conditions of the nuclearity of Hankel operators are due to Howland [1971].

6.9.9.9 According to Adamyan/Arov/Kreĭn [1971, стр. 35], the s-numbers of a Hankel operator H_a are given by

$$s_n(H_a) = \inf\{\, \|H_a - H_b\| : \operatorname{rank}(H_b) < n \,\}.$$

In other words, it is enough to approximate H_a just by Hankel operators. Further formulas that yield $s_n(H_a)$ are contained in a paper of Butz [1974, p. 304].

An old result of Kronecker [1881], [KRO$^{\bowtie}$, p. 153] can be restated as follows: The Hankel operator H_b associated with a sequence $b = (\beta_k)$ has finite rank if and only if $\psi_b(z) := \sum_{k=1}^{\infty} \beta_k z^{-k}$ extends to a rational function on the complex plane. Moreover, $\operatorname{rank}(H_b)$ equals the number of poles of ψ_b in $\mathbb{D}$ (counting orders); see [PARTI, p. 37].

6.9.9.10 The previous results about Hankel operators on l_2 carry over to Hankel operators H_φ from $H_2(\mathbb{T})$ into $H_2(\mathbb{T})^\perp$ induced by symbols φ such that $\overline{\varphi} \in H_2(\mathbb{T})$:

$$\begin{aligned} H_\varphi \textit{ is bounded} \quad &\Leftrightarrow \quad \varphi \in \mathit{BMO}(\mathbb{T}), \\ H_\varphi \textit{ is compact} \quad &\Leftrightarrow \quad \varphi \in \mathit{VMO}(\mathbb{T}), \\ H_\varphi \textit{ belongs to } \mathfrak{S}_p \quad &\Leftrightarrow \quad \varphi \in B^{1/p}_{p,p}(\mathbb{T}). \end{aligned}$$

6.9.9.11 Besides compact Hankel operators, there also exist Hankel operators with closed infinite-dimensional range; see Clark [1972]. In particular, Power [POW, p. 34] presented a characterization of H_φ's that are partial isometries.

6.9.9.12 Next, I pass to the consideration of Toeplitz operators. Their close relationship to a special kind of singular integral operator is described in [GOH$_1^+$, p. 134], [GOH$_4^+$, p. 60, German edition], and [MICH$^+$, pp. 63–64]; see also 5.2.2.13.

Typical examples on l_2 are the **shift operators**

$$S_{\text{hift}}^{\rightarrow} : (\xi_1, \xi_2, \dots) \mapsto (0, \xi_1, \xi_2, \dots) \quad \text{and} \quad S_{\text{hift}}^{\leftarrow} : (\xi_1, \xi_2, \dots) \mapsto (\xi_2, \xi_3, \dots).$$

Standard references: [BÖT$^+$], [DOU, Chap. 7], [GOH$_4^+$], [NIK$_1$], [NIK$_2$], [ZHU].

6.9.9.13 If $\varphi \in C(\mathbb{T})$, then T_φ is a Φ-operator if and only if φ does not vanish. In this case, the index of T_φ is equal to the negative winding number of φ with respect to the origin. Moreover, we have either $\dim\big[N(T_\varphi)\big] = 0$ or $\operatorname{cod}\big[M(T_\varphi)\big] = 0$.

6.9.9.14 Brown/Halmos [1964, p. 94] proved that O is the only compact Toeplitz operator.

6.9.9.15 A parallel theory is concerned with Hankel and Toeplitz operators on the real half-line $\mathbb{R}_+$:

$$H_a : f(t) \mapsto \int_0^\infty a(s+t) f(t)\, dt \quad \text{and} \quad T_a : f(t) \mapsto \int_0^\infty a(s-t) f(t)\, dt.$$

Devinatz [1967, pp. 82–84] observed that the discrete case and the continuous case can be related via the Cayley transform; see 6.7.12.8 and also [PARTI, pp. 42–47].

Since operators of the right-hand type were first studied by Wiener/Hopf [1931], they are often called **Wiener–Hopf operators**. Many results are due to Kreĭn [1958].

In the setting of Hankel operators, the reader should compare the $\mathfrak{S}_p$-criteria proved by Peller (unit circle) with those of Rochberg (half-line); see 6.9.9.8.

6.9.9.16 So-called Hankel and Toeplitz operators have also been considered on the Bergman space $\mathscr{LA}_2(\mathbb{D})$; see 6.7.13.1. In this case, the Riesz projection 6.9.9.2 must be replaced by the **Bergman projection**

$$P : f(z) \mapsto \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^2}\, dA(z),$$

which yields the decomposition

$$L_2(\mathbb{D}) = \mathscr{LA}_2(\mathbb{D})^{\perp} \oplus \mathscr{LA}_2(\mathbb{D}).$$

The main point is that this modification provides many more symbols. However, the reader should be aware of the fact that the "new" operators

$$T_\varphi : f \mapsto P(\varphi f) \quad \text{and} \quad H_\varphi : f \mapsto (I-P)(\varphi f)$$

differ from the old ones considerably.

6.9.9.17 For symbols φ such that $\overline{\varphi} \in \mathscr{LA}_2(\mathbb{D})$, Axler [1986, pp. 327–329] and Arazy/Fisher/Peetre [1988, pp. 1004, 1015] obtained the following criteria ($1<p<\infty$):

$$\begin{array}{lcl} H_\varphi \textit{ is bounded} & \Leftrightarrow & \overline{\varphi} \in \mathscr{B}(\mathbb{D}), \\ H_\varphi \textit{ is compact} & \Leftrightarrow & \overline{\varphi} \in \mathscr{B}_0(\mathbb{D}), \\ H_\varphi \textit{ belongs to } \mathfrak{S}_p & \Leftrightarrow & \overline{\varphi} \in \mathscr{BA}^{1/p}_{p,p}(\mathbb{D}); \end{array}$$

compare with 6.9.9.10.

Furthermore, Arazy/Fisher/Peetre [1988, pp. 1008–1015] discovered that the only nuclear Hankel operators H_φ with $\overline{\varphi} \in \mathscr{LA}_2(\mathbb{D})$ are obtained from $\varphi = \text{const}$, whereas the condition $\overline{\varphi} \in \mathscr{BA}^1_{1,1}(\mathbb{D})$ implies that H_φ is a member of the Matsaev ideal:

$$\sum_{n=1}^{N} s_n(H_\varphi) = O\big(1+\log N\big).$$

6.9.9.18 Now we enlarge the set of admissible symbols.

For any function $\varphi \in L_2(\mathbb{D})$, let

$$\Phi_r(z) := \inf\left\{ \left(\frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} |\varphi(w) - f(w)|^2 dA(w) \right)^{1/2} : f \in \mathscr{LA}_2(\mathbb{D}) \right\}.$$

Here $\mathbb{D}(z,r) := \{w \in \mathbb{D} : d(w,z) < r\}$ denotes the disk with center z and radius $r>0$, where

$$d(w,z) := \log \frac{|1-w\bar{z}| + |w-z|}{|1-w\bar{z}| - |w-z|}$$

is the hyperbolic metric; see [SHAP, pp. 150–153].

Luecking [1992, pp. 252, 258, 262–263] proved that regardless of the choice of $r>0$, the following criteria hold ($1\le p<\infty$):

$$\begin{array}{rcl} H_\varphi \textit{ is bounded} & \Leftrightarrow & \Phi_r \text{ is bounded,} \\ H_\varphi \textit{ is compact} & \Leftrightarrow & \Phi_r(z)\to 0 \text{ as } |z|\to 1, \\ H_\varphi \textit{ belongs to } \mathfrak{S}_p & \Leftrightarrow & \Phi_r \in L_p\big(\mathbb{D}, \frac{dA(z)}{(1-|z|^2)^2}\big). \end{array}$$

Note that $\frac{dA(z)}{(1-|z|^2)^2}$ is the Möbius invariant measure on $\mathbb{D}$.

6.9.9.19 Toeplitz operators on the Bergman space $\mathscr{L}\!\mathscr{A}_2(\mathbb{D})$ can be obtained from much more general symbols. For any bounded measure μ on $\mathbb{D}$, Luecking [1987, p. 346] defined

$$T_\mu\,:\,f(z)\mapsto \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^2}\,d\mu(z),$$

and he showed that such operators may belong to $\mathfrak{S}_p$ or may even have finite rank; see also [ZHU, Chap. 6].

6.9.9.20 Many results about Hankel and Toeplitz operators on l_2, $H_2(\mathbb{T})$, and $\mathscr{L}\!\mathscr{A}_2(\mathbb{D})$ admit straightforward extensions to l_p, $H_p(\mathbb{T})$, and $\mathscr{L}\!\mathscr{A}_p(\mathbb{D})$. However, some generalizations are less obvious. For example, in the case $1<q<p<\infty$, Peller [1984, p. 217] characterized nuclear Hankel operators from $H_p(\mathbb{T})$ into $H_q(\mathbb{T})$. Nevertheless, a number of interesting problems remain open. Apart from the Hilbert space case, nothing seems to be known about s-numbers.

6.9.10 Composition operators

6.9.10.1 Let φ be any analytic function that takes the open unit disk $\mathbb{D}$ into itself. By Littlewood's subordination principle [1925], the rule

$$C_\varphi : f(w)\mapsto f\big(\varphi(z)\big)$$

yields an operator on every Hardy space $\mathscr{H}_p(\mathbb{D})$. This basic result goes back to Ryff [1966, p. 348] and Nordgren [1968, p. 443]. The latter coined the name **composition operator**.

6.9.10.2 In what follows, I sketch some features of the fascinating interplay between function-theoretic properties of φ and operator-theoretic properties of C_φ. Standard references are [COW$^+$] and [SHAP]. The reader may also consult the surveys of Nordgren [1978] and Cowen [1990].

6.9.10.3 The first condition that ensures compactness of C_φ on $\mathscr{H}_p(\mathbb{D})$ with $1\le p<\infty$ was discovered by (Howard) Schwartz [1969, p. 26]:

$$\int_0^{2\pi} \frac{1}{1-|\varphi(e^{i\theta})|}\,d\theta<\infty, \tag{6.9.10.3.a}$$

where $\varphi(e^{i\theta}) := \lim\limits_{r\nearrow 1}\varphi(re^{i\theta})$ denotes the radial limit, which exists for almost all θ's.

Howard Schwartz [1969, p. 28] also showed that C_φ is compact on $\mathscr{H}_\infty(\mathbb{D})$ if and only if φ maps $\mathbb{D}$ into a disk with a radius strictly smaller than 1.

6.9.10.4 Decisive progress was achieved by Shapiro/Taylor [1973, pp. 481, 494], who proved that condition (6.9.10.3.a) characterizes those φ's that induce Hilbert–Schmidt operators on $\mathscr{H}_2(\mathbb{D})$. In this case, C_φ is p-summing on $\mathscr{H}_p(\mathbb{D})$. Shapiro/Taylor [1973, p. 492] observed that a composition operator is compact on $\mathscr{H}_p(\mathbb{D})$ for *all* finite $p \geq 1$ if it is compact for *some* finite $p \geq 1$. At least for $p > 1$, this result can also be obtained by interpolation; see 6.6.5.1.

6.9.10.5 The following criteria are based on **Nevanlinna's counting function**,

$$N_\varphi(w) := \sum_{\varphi(z)=w} \log \frac{1}{|z|} \quad \text{for } w \neq \varphi(0).$$

Shapiro [1987, p. 381] proved that C_φ is compact on $\mathscr{H}_2(\mathbb{D})$ if and only if

$$\lim_{|w|\to 1} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0,$$

and Luecking/Zhu [1992, p. 1128] characterized the $\mathfrak{S}_p$-operators by

$$\int_{\mathbb{D}} \left[\frac{N_\varphi(w)}{\log \frac{1}{|w|}} \right]^{p/2} \frac{dA(w)}{(1-|w|^2)^2} < \infty.$$

6.9.10.6 Shapiro/Taylor [1973, p. 482] invented a very strong geometric condition by requiring that φ takes $\mathbb{D}$ into a polygon inscribed in the unit circle. In this case, C_φ is a member of all Schatten–von Neumann ideals $\mathfrak{S}_p$.

6.9.10.7 As observed by Caughran/Schwartz [1975, p. 129], the spectrum of a compact composition operator C_φ looks simple; see also [SHAP, pp. 89–96] and (Howard) Schwartz [1969, p. 77].

Based on classical results of Koenigs [1884, pp. 6–7, 16–18] and others, it was proved that φ has just one fixed point $z_0 \in \mathbb{D}$ with $|\varphi'(z_0)| < 1$.

If $\varphi'(z_0) = 0$, then the spectrum consists of 0 and the eigenvalue 1.

If $\varphi'(z_0) \neq 0$, then the spectrum contains in addition the eigenvalues $\varphi'(z_0), \varphi'(z_0)^2, \ldots$.

Every eigenvalue has multiplicity one, and the spectrum is the same for all Hardy spaces $\mathscr{H}_p(\mathbb{D})$ with $1 \leq p < \infty$.

6.9.10.8 A complex number λ is an eigenvalue of the composition operator C_φ if there exists an analytic function $f \neq o$ such that $\lambda f(z) = f\big(\varphi(z)\big)$ for all $z \in \mathbb{D}$. This equation has a long history.

The German mathematician Schröder [1871] was interested in the iterates $\varphi_{n+1}(z) := \varphi_n\big(\varphi(z)\big)$ of a given analytic function φ. Since those iterates are hard to handle, he tried to find substitutions $w = f(z)$ for which the transformed function

$\psi(w) := f\big(\varphi(f^{-1}[w])\big)$ becomes simple, say $\psi(w) = \lambda w$ with some $\lambda \in \mathbb{C}$. This leads to the **Schroeder equation** $\lambda f(z) = f\big(\varphi(z)\big)$, which occurred, in a hidden form, on p. 303 of his paper. For sure, λ was never considered as an eigenvalue. More information about Schröder's work and that of his followers can be found in Alexander's *History of Complex Dynamics* and in a survey of Shapiro [1998].

6.9.10.9 Jarchow [1995] and his students have extended some results about composition operators from Hilbert to Banach spaces. The following example shows that this undertaking is by no means trivial.

Let $C_r : f(z) \mapsto f(rz)$ with $0 < r < 1$. A short glance at the Taylor expansion shows that $s_n(C_r : \mathscr{H}_2 \to \mathscr{H}_2) = r^{n-1}$. However, nothing seems to be known about the various s-numbers of $C_r : \mathscr{H}_p(\mathbb{D}) \to \mathscr{H}_q(\mathbb{D})$ and $C_r : \mathscr{LA}_p(\mathbb{D}) \to \mathscr{LA}_q(\mathbb{D})$.

6.9.11 Methods of summability

Standard references are [ZEL], [ZEL$^+$], [WIL], and [BOOS].

6.9.11.1 By a **method of summability** (Limitierungsverfahren) we mean a rule

$$A : (\xi_k) \mapsto (\eta_h) := \left(\sum_{k=1}^{\infty} \alpha_{hk} \xi_k \right)$$

defined for all scalar sequences (ξ_k) such that the right-hand series converge.

The set $c_A := \{ (\xi_k) : (\eta_h) \in c \}$ is said to be the **convergence domain** of the method A. Here c denotes the Banach space of all convergent sequences equipped with the sup-norm. This definition was suggested by the observation that the transformed sequence (η_h) may have a limit even for some non-convergent sequences (ξ_k).

According to [WIL, p. 3],
*by a historical accident, sequences in c_A are called A-***summable** *instead of the more reasonable A-***limitable**.

The most convincing example comes from the theory of trigonometric series. Du Bois-Reymond [1873, p. 578] discovered 2π-periodic continuous functions f that cannot be reproduced as the pointwise limits of their partial Fourier sums. On the other hand, Fejér's theorem [1904, pp. 59–60] says that the arithmetic means of the partial Fourier sums converge uniformly to f.

6.9.11.2 A method is called **permanent** or **regular** if

$$\lim_{h \to \infty} \eta_h = \lim_{k \to \infty} \xi_k \quad \text{for all } (\xi_k) \in c \subseteq c_A.$$

6.9.11.3 *Allgemeine lineare Mittelbildungen* were first studied in a classical paper of Toeplitz [1911b, p. 117] that contains the following criterion; see also Silverman's thesis [1913]:
An infinite matrix (α_{hk}) yields a permanent method if and only if

$$\sup_h \sum_{k=1}^{\infty} |\alpha_{hk}| < \infty, \qquad \lim_{h \to \infty} \sum_{k=1}^{\infty} \alpha_{hk} = 1, \quad \text{and} \quad \lim_{h \to \infty} \alpha_{hk} = 0 \quad \text{for } k = 1, 2, \dots .$$

I stress the remarkable fact that Toeplitz's proof is based on a preliminary version of the uniform boundedness principle; see 2.4.4.

6.9.11.4 The classical example of arithmetic means,

$$C_1 : (\xi_k) \mapsto \left(\frac{\xi_1+\cdots+\xi_h}{h}\right),$$

goes back to Cesàro [1890]. The following quotation is from [ZEL, p. 100]:

Das Verfahren ist so einfach und naheliegend, daß es schon früh in der Entwicklung der Limitierungstheorie auftritt und oft als Musterverfahren dient.

6.9.11.5 The reader may consult [ZEL$^+$, p. 307] for an index of summability methods that contains 99 entries.

6.9.11.6 Besides "concrete" methods we also have the **Banach limits**, whose existence follows from the Hahn–Banach theorem; see [BAN, p. 34] and 4.1.13:

A toute suite bornée (ξ_k) on peut faire correspondre un nombre $\operatorname*{Lim}_{k\to\infty}\xi_k$ de façon que les conditions suivantes soient remplies:

1) $\operatorname*{Lim}_{k\to\infty}(a\xi_k+b\eta_k) = a\operatorname*{Lim}_{k\to\infty}\xi_k + b\operatorname*{Lim}_{k\to\infty}\eta_k$,
2) $\operatorname*{Lim}_{k\to\infty}\xi_k \geq 0$, *si* $\xi_k \geq 0$ *pout tout* $k = 1,2,\dots$,
3) $\operatorname*{Lim}_{k\to\infty}\xi_{k+1} = \operatorname*{Lim}_{k\to\infty}\xi_k$,
4) $\operatorname*{Lim}_{k\to\infty} 1 = 1$.

Mazur [1929, p. 103] observed that

$$\liminf_{k\to\infty}\xi_k \leq \operatorname*{Lim}_{k\to\infty}\xi_k \leq \limsup_{k\to\infty}\xi_k \quad \text{for all } (\xi_k)\in l_\infty.$$

6.9.11.7 According to Kalton/Wilansky [1976, p. 251], an operator $T\in\mathfrak{L}(X,Y)$ is called **Tauberian** if $T^{**}x^{**}\in Y$ implies $x^{**}\in X$. Note that X and Y are viewed as subspaces of their biduals. This definition mimics the intention of a Tauberian theorem: one looks for conditions that ensure that an A-limitable sequences is also convergent in the ordinary sense. The terminology pays homage to Tauber [1897], who discovered the first result of this kind. A well-known improvement is due to Hardy [1910, p. 304]:

A Cesàro limitable sequence (ξ_k) such that $\xi_k-\xi_{k+1} = O\big(\frac{1}{k}\big)$ is convergent.

6.9.11.8 The invention of the following classes of Banach spaces and operators was also suggested by the theory of summability.

We refer to (x_k) as a **Banach–Saks sequence** if the Cesàro means

$$\frac{x_1+\cdots+x_h}{h}$$

converge in norm. A Banach space X is said to have the **Banach–Saks property** if every bounded sequence contains a Banach–Saks subsequence. The terminology, proposed by Brunel/Sucheston [1975, p. 82], is a tribute to Banach/Saks [1930], who verified this property for all spaces L_p with $1 < p < \infty$. The implications

$$\textit{uniformly convexifiable} \quad \Rightarrow \quad \textit{Banach–Saks property} \quad \Rightarrow \quad \textit{reflexive}$$

were proved by Kakutani [1938a, p. 188] and Nishiura/Waterman [1963, p. 56]. Note that $[l_2, l_1^n]$ has the Banach–Saks property, but fails to be uniformly convexifiable. Schreier [1930] constructed a weakly convergent sequence in $C[0,1]$ that contains no Banach–Saks subsequence. Modifying Schreier's approach, Baernstein [1972] discovered a reflexive space without the Banach–Saks property. Hence the right-hand implication is strict as well. This counterexample is a preliminary version of the famous Tsirelson space discussed in Subsection 7.4.3; see [CASA$^+$, pp. 1–7].

One can also define a **weak** Banach–Saks property by requiring that every weakly convergent sequence contain a Banach–Saks subsequence. For reflexive spaces, we obtain nothing new. However, l_1 has only the weak Banach–Saks property.

As observed by Beauzamy [1980, p. 228], the Banach–Saks property makes sense for operators as well. In this way, we get a closed ideal $\mathfrak{BS}$, which has the factorization property. The fact that $\mathfrak{BS}$ is closed can easily be inferred form a dichotomy due to Erdős/Magidor [1976, p. 232]:
Every bounded sequence in a Banach space has a subsequence such that either *every* or *none* of its subsequences is Banach–Saks.

6.9.12 Lacunarity

6.9.12.1 The meaning of the term **lacunary** is rather vague; see [ZYG, Vol. I, p. 202]:

Lacunary trigonometric series are series in which the terms that differ from zero are "very sparse."

The origin of lacunarity goes back to Hadamard [1892, p. 116], who considered power series

$$\sum_{n=1}^{\infty} \gamma_n \zeta^{k_n} \quad \text{such that } \; k_{n+1}/k_n \geq c > 1.$$

6.9.12.2 For every subset Λ of $\mathbb{Z}$, we let

$$C^{\Lambda}(\mathbb{T}) := \{ f \in C(\mathbb{T}) : \gamma_k(f) = 0 \quad \text{for } k \notin \Lambda \}$$

and

$$L_p^{\Lambda}(\mathbb{T}) := \{ f \in L_p(\mathbb{T}) : \gamma_k(f) = 0 \quad \text{for } k \notin \Lambda \},$$

where $\gamma_k(f)$ denotes the k^{th} Fourier coefficient of f. This definition yields just all closed subspaces that are invariant under translations: $f(t) \mapsto f(t - t_0)$.

6.9.12.3 A subset Λ of $\mathbb{Z}$ is called a **Sidon set** if there exists a constant $B \geq 1$ such that

$$\sum_{k\in\Lambda} |\gamma_k| \leq B \sup\left\{\Big|\sum_{k\in\Lambda} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\Big| : 0 \leq \theta < 2\pi\right\} \tag{6.9.12.3.a}$$

for every trigonometric polynomial whose coefficients vanish out of Λ.

Sidon [1927] proved that Hadamard's condition implies the estimate (6.9.12.3.a) for $\Lambda = \{k_1, k_2, \dots, k_n, \dots\}$, which justifies the naming.

The standard reference is [LOP$^+$].

6.9.12.4 If Λ is a Sidon set, then $f \leftrightarrow \big(\gamma_k(f)\big)$ defines an isomorphism between $C^\Lambda(\mathbb{T})$ and $l_1(\Lambda)$. Hence $C^\Lambda(\mathbb{T})$ has Rademacher cotype 2. According to Pisier [1978d, p. 10], the converse implication is also true. Finally, Bourgain/Milman [1985] discovered the following dichotomy:

Depending on whether Λ is a Sidon set or not, $C^\Lambda(\mathbb{T})$ either has cotype 2 or no finite cotype at all.

6.9.12.5 Let $0 < p < \infty$. According to Rudin [1960, p. 205], a subset Λ of $\mathbb{Z}$ is called a $\Lambda(p)$**-set** if there exist some $r < p$ and a constant $B_{pr} \geq 1$ such that

$$\Big\|\sum_{k\in\Lambda} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\Big| L_p(\mathbb{T})\Big\| \leq B_{pr} \Big\|\sum_{k\in\Lambda} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\Big| L_r(\mathbb{T})\Big\| \tag{6.9.12.5.a}$$

for every trigonometric polynomial whose coefficients vanish out of Λ. Obviously, (6.9.12.5.a) implies that $L_p^\Lambda(\mathbb{T}) = L_r^\Lambda(\mathbb{T})$.

The concept of a $\Lambda(2)$-set goes back to Banach [1930, p. 212] and Sidon [1930, p. 251].

6.9.12.6 Let $\Lambda(p)$ denote the collection of all $\Lambda(p)$-sets. Then

$$\Lambda(p) \supseteq \Lambda(q) \quad \text{whenever } 0 < p \leq q < \infty.$$

A deep theorem of Bourgain [1989, p. 228] asserts that the inclusion above is strict if $2 \leq p < q < \infty$. The case $0 < p < q \leq 2$ seems to be open. We only know from Bachelis/Ebenstein [1974] that if $0 < p < 2$, then every $\Lambda(p)$-set Λ is also a $\Lambda(q)$-set for some $q > p$, which depends on Λ. For further information the reader may consult a survey of Bourgain [2001].

6.9.12.7 Rudin [1960, p. 210] showed that Sidon sets belong to $\Lambda(p)$ for every p. In particular,

$$\Big\|\sum_{k\in\Lambda} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\Big| L_q(\mathbb{T})\Big\| \leq B\sqrt{q}\Big\|\sum_{k\in\Lambda} \gamma_k \mathrm{e}^{\mathrm{i}k\theta}\Big| L_2(\mathbb{T})\Big\| \quad \text{whenever } 2 < q < \infty.$$

The latter condition even characterizes Sidonicity; see Pisier [1978e].

6.9.12.8 The rest of this subsection is devoted to another kind of lacunary system. We describe phenomena that occur when the trigonometric functions $e^{ik_n\theta}$ are replaced by powers t^{λ_n}.

Let $\Lambda=(\lambda_n)$ be any sequence of reals such that $0<\lambda_1<\lambda_2<\cdots$. The **Müntz spaces** $C^\Lambda[0,1]$ and $L_p^\Lambda[0,1]$ with $1\le p<\infty$ are defined to be the closed linear spans of the functions $1,t^{\lambda_1},t^{\lambda_2},\dots$ in $C[0,1]$ and $L_p[0,1]$, respectively.

The famous **Müntz theorem** [1914] says that $C^\Lambda[0,1]$ is a proper subspace of $C[0,1]$ if and only if

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}<\infty;$$

see also Szász [1916]. This condition is, in particular, satisfied for systems with the **Hadamard property**: $\lambda_{n+1}/\lambda_n\ge c>1$.

6.9.12.9 Little is known about the structure of Müntz spaces, even in the case when $\Lambda=\{n^\alpha\}$ with $\alpha>1$.

According to [GUR$^+$, p. 140], there exist at least two non-isomorphic spaces $C^{\Lambda_1}[0,1]$ and $C^{\Lambda_2}[0,1]$. However, one should try to find $2^{\aleph_0}$ sets Λ for which the associated spaces $C^\Lambda[0,1]$ are mutually non-isomorphic. If non-isomorphic is replaced by non-isometric, then this problem has an affirmative solution; see [GUR$^+$, p. 144].

6.9.12.10 A basic result is due to Gurariĭ/Matsaev [1966, стр. 4, 9], who proved that the following are equivalent:

(1) Λ has the Hadamard property.

(2) The functions $1,t^{\lambda_1},t^{\lambda_2},\dots$ constitute a basis of $C^\Lambda[0,1]$.

(3) The functions $1,t^{\lambda_1},t^{\lambda_2},\dots$ constitute a basis of $L_p^\Lambda[0,1]$ for some $1\le p<\infty$.

(4) The functions $1,t^{\lambda_1},t^{\lambda_2},\dots$ constitute a basis of $L_p^\Lambda[0,1]$ for every $1\le p<\infty$.

Hadamard's property implies that $\{t^{\lambda_n}\}$ is a conditional basis of $C^\Lambda[0,1]$ and an unconditional basis of $L_p^\Lambda[0,1]$ for $1\le p<\infty$. Moreover, $C^\Lambda[0,1]$ is isomorphic to c, and $L_p^\Lambda[0,1]$ is isomorphic to l_p. For a moderate generalization, the reader may consult [GUR$^+$, pp. 128–129].

6.9.13 The linear group of a Banach space

6.9.13.1 Let $C(X,Y)$ denote the collection of all continuous functions from a topological space X into a topological space Y. Functions f_0 and f_1 in $C(X,Y)$ are called **homotopic** if there is a continuous function $F:X\times[0,1]\to Y$ such that $f_0(x)=F(x,0)$ and $f_1(x)=F(x,1)$ for all $x\in X$. This definition yields an equivalence relation. The set of homotopy classes is denoted by $C^\sim(X,Y)$.

Continuous deformations of topological objects were invented by Poincaré [1895].

6.9.13.2 A topological space is said to be **contractible** if its identity map is homotopic to a constant map $f:x\mapsto x_0$. In this case, X is arcwise connected.

6.9.13.3 The **linear group** $\mathfrak{LG}(X)$ of a Banach space X consists of all invertible operators in $\mathfrak{L}(X)$, with composition ST as the underlying group operation. In what follows, the linear group is equipped with the topology induced by the operator norm.

6.9.13.4 It came as a surprise when Kuiper [1965] discovered that in contrast to the finite-dimensional case, the linear groups of infinite-dimensional Hilbert spaces are contractible. Subsequently, the same was proved for the classical Banach spaces.

c_0	:	Arlt [1966],
$l_p,\ 1 \le p < \infty$	:	Neubauer [1967],
C, L_1, L_∞ and l_∞	:	Edelstein et al. [1970],
$L_p,\ 1 < p < \infty$	:	Mityagin [1970, стр. 83–87] (in collaboration with McCarthy),
$A(\mathbb{T})$	:	Wojtaszczyk [1979, p. 641].

Douady [1965] showed that the linear group of $l_1 \oplus c_0$ fails even to be connected.

6.9.13.5 The abelian **Grothendieck group** $\mathrm{Groth}(K)$ of a compact Hausdorff space K is defined in terms of *vector bundles* and can be grasped only by those people who have a good command of the underlying "language." Some historical comments were given in [PIER•, pp. 346–349].

The following theorem yields a functional analytic approach to this K-theoretic concept. The main ingredients are Φ-operators; see 5.2.2.7.

Suppose that the Banach space X contains a complemented subspace with a symmetric basis. Then the sequence

$$\{o\} \longrightarrow C^{\sim}\big(K, \mathfrak{LG}(X)\big) \longrightarrow C^{\sim}\big(K, \Phi(X)\big) \longrightarrow \mathrm{Groth}(K) \longrightarrow \{o\}$$

is exact. In other words, the groups

$$\frac{C^{\sim}\big(K, \Phi(X)\big)}{C^{\sim}\big(K, \mathfrak{LG}(X)\big)} \quad \text{and} \quad \mathrm{Groth}(K)$$

are isomorphic.

In the case that X is the infinite-dimensional separable Hilbert space, the preceding result goes back to Jänich [1965, pp. 139–141] and Atiyah [ATI, Appendix]. The adaptation to the Banach space setting was performed by Neubauer [1968, p. 300].

6.9.13.6 If the linear group of X is contractible, then $C^{\sim}\big(K, \mathfrak{LG}(X)\big) = \{o\}$. Hence, subject to Neubauer's conditions, the groups

$$C^{\sim}\big(K, \Phi(X)\big) \quad \text{and} \quad \mathrm{Groth}(K)$$

are isomorphic. This is, in particular, true for c_0 and $C[0,1]$, l_p, and $L_p[0,1]$ with $1 \le p < \infty$. It seems to be unknown whether the same holds for l_∞ and $L_\infty[0,1]$.

For the 1-point space $K = \{o\}$, we have $\mathrm{Groth}\big(\{o\}\big) = \mathbb{Z}$. Then it turns out that the canonical map $C^{\sim}\big(\{o\}, \Phi(X)\big) \to \mathbb{Z}$ is induced by $T \mapsto \mathrm{ind}(T)$.

6.9.13.7 The previous result means that Φ-operators T_0 and T_1 can be connected by a continuous path, inside $\Phi(X)$, if and only if they have the same index. Easy manipulations show that this criterion holds even under the weaker assumption that $\mathfrak{L}\mathfrak{G}(X)$ is connected; see [GOH$_4^+$, p. 143, German edition]. The case of Hilbert spaces is usually attributed to Cordes/Labrousse [1963, p. 716], though it was already treated by Gohberg/Markus/Feldman [1960, стр. 56].

For non-pathological infinite-dimensional spaces X, all sets

$$\Phi_n(X) := \{T \in \Phi(X) : \operatorname{ind}(T) = n\} \quad \text{with } n = 0, \pm 1, \pm 2, \dots$$

are non-empty. On the other hand, we have $\Phi_0(X_{\mathrm{GM}}) = \Phi(X_{\mathrm{GM}})$ for the Gowers–Maurey space; see 7.4.5.2. Moreover, according to Gowers/Maurey [1997, p. 559], there is some X such that $\Phi_{2n}(X) \neq \emptyset$ and $\Phi_{2n+1}(X) = \emptyset$. Probably, one can find spaces X with $\Phi_n(X) \neq \emptyset$ precisely when n is an integral multiple of a given $n_0 \in \mathbb{N}$.

6.9.13.8 The rest of this subsection deals with (linear) isometries from a Banach space X *onto* itself. These operators form a subgroup of $\mathfrak{G}\mathfrak{L}(X)$ that depends on the underlying norm.

To get a first impression, we may look at 2-dimensional real spaces. The Euclidean space is the only one on which the group of isometries has infinitely many members: rotations and reflections. In general, there are only two isometries: $x \mapsto x$ and $x \mapsto -x$. The most illuminating case is $\mathbb{R}^2$ equipped with the sup-norm.

6.9.13.9 The techniques employed in the proof of the Banach–Stone theorem 4.5.5 yield the following criterion:

For every compact Hausdorff space K, the isometries from $C(K)$ onto itself are precisely the operators

$$U : f(s) \mapsto a(t) f(\varphi(t)),$$

where a is a continuous function a such that $|a(t)| = 1$ for $t \in K$ and φ is a homeomorphism of K.

In the case of the interval $[0,1]$ this result is due to Banach; see [BAN, p. 173]. Stone [1937a, p. 469] contributed the general version. Modern proofs use the fact that isometries take extreme points of the closed unit ball into extreme points; see 5.4.1.9.

6.9.13.10 Without a full proof, the isometries of $L_p[0,1]$ with $p \neq 2$ were described in [BAN, p. 178]. It was only in [1958, pp. 461–462] that Lamperti finally filled the remaining gaps. He even treated L_p's over σ-finite measure spaces $(M, \mathscr{M}, \mu)$. The isometries from $L_p(M, \mathscr{M}, \mu)$ onto itself are precisely the following operators U:

Let Φ be an isomorphism of the Boolean σ-algebra $\mathscr{M}/\mathscr{M}_0(\mu)$ discussed in 4.8.2.2. Define $\nu(A) := \mu(\Phi^{-1}(A))$ for $A \in \mathscr{M}$ and suppose that both measures μ and ν are continuous with respect to each other. Fix any μ-measurable function a such that $d\nu = |a|^p d\mu$. Then U is the (unique) continuous linear extension of the rule

$$U : \chi_B \mapsto a\chi_{\Phi(B)},$$

where χ_B and $\chi_{\Phi(B)}$ denote the characteristic functions of B and $\Phi(B)$, respectively.

In the case that M has finite total mass, a representative of the equivalence class $\Phi(B)$ is given by the set of all points at which $U\chi_B$ does not vanish. Moreover, we have $a := U(\mathbf{1})$.

Additional assumptions ensure the more pleasing situation that Φ can be deduced from a measurable map $\varphi : M \to M$ by letting $\Phi(B) := \varphi^{-1}(B)$ for $B \in \mathscr{M}$; see 4.8.2.4 and [ROY, pp. 271–276].

6.9.13.11 The study of isometries of function algebras with the sup-norm was initiated by de Leeuw/Rudin/Wermer [1960]. In particular, they proved that the isometries of $\mathscr{H}_\infty(\mathbb{D})$ have the form

$$U : f(w) \mapsto af\big(\varphi(z)\big),$$

where a is a complex number of modulus 1 and φ is a conformal mapping of $\mathbb{D}$ onto itself. In fact, by the Schwarz lemma,

$$\varphi(z) = b\,\frac{z_0 - z}{1 - \overline{z}_0 z} \quad \text{with } |b| = 1 \text{ and } z_0 \in \mathbb{D}.$$

In the same paper, de Leeuw/Rudin/Wermer showed that the isometries of $\mathscr{H}_1(\mathbb{D})$ look as follows:

$$U : f(w) \mapsto a\varphi'(z)f\big(\varphi(z)\big),$$

with a and φ as above.

These results were independently obtained by Nagasawa [1959].

6.9.13.12 Cima/Wogen [1980] have determined the isometries of Bloch spaces.

6.9.13.13 For much more information the reader may consult [FLEM$^+$].

6.9.14 Manifolds modeled on Banach spaces

6.9.14.1 Analysis and geometry can be performed on

finite-dimensional linear spaces	finite-dimensional manifolds
infinite-dimensional linear spaces	infinite-dimensional manifolds

Usually, a freshman starts in the left upper box. Later on, he may move to the right or downward. The right lower box is the “non plus ultra.”

The finite-dimensional level is skipped in Dieudonné’s *Foundations of Modern Analysis.* Lang promotes the same ideology concerning *differential geometry*; see [LANG_1, Preface]:

It is possible to lay down at no extra cost the foundation of manifolds modeled on Banach or Hilbert spaces rather than finite dimensional spaces. In fact, it turns out that the exposition gains considerably from the systematic elimination of the indiscriminate use of local coordinates $x_1, \ldots, x_n$ and $dx_1, \ldots, dx_n$.

6.9.14.2 Michal was the first who made enthusiastic propaganda for *general differential geometry*. In his address [1939, p. 529] delivered at the Berkeley meeting of the American Mathematical Society in April 1938 he claimed:

We are convinced that the subject of general differential geometry is destined to become one of the great branches of mathematics, comparable to the present status of general (abstract) algebra and general analysis. There is still time for a whole army of young mathematicians to earn their first laurels in general differential geometry while the subject is still in infancy.

His challenge had no immediate effect. However, a vehement development took place in the 1960s. The state of the art in the middle of this period was described by Eells [1966]; see also [CHE$^{\cup}$]. At the ICM 1970 in Nice, three invited lectures dealt with infinite-dimensional manifolds (Anderson, Eells/Elworthy, Kuiper).

6.9.14.3 In what follows, we use the concept of a Fréchet derivative as described in Subsections 5.1.8 and 5.1.9. The reader may also consult Chapter 1 in [LANG$_1$] or [LANG$_2$].

Let U be an open set of a Banach space X, and let $m=1,2,\ldots,\infty$. A function $f: U \to Y$ is said to be of **class** C^m if it has continuous derivatives up to the order m in all of U. We also include the case $m=0$, in which f is continuous.

A one-to-one function f between open subsets is called a C^m*-diffeomorphism* if both f and f^{-1} are of class C^m. For $m=0$, we just get homeomorphisms.

6.9.14.4 By an X**-valued chart** of a Hausdorff space M we mean a pair (U,f), where f is a homeomorphism between an open subset U of M and an open subset of a Banach space X.

An **atlas** of M is a collection of X-valued charts (U_α, f_α) with $\alpha \in \mathbb{A}$ such that the U_α's cover M.

If M is equipped with an atlas for which the *transition functions*

$$\begin{tikzcd} f_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "f_\beta \circ f_\alpha^{-1}"] \arrow[dr, "f_\alpha^{-1}"'] & & f_\beta(U_\alpha \cap U_\beta) \\ & U_\alpha \cap U_\beta \arrow[ur, "f_\beta"'] & \end{tikzcd}$$

are of class C^m, then we speak of a C^m**-manifold modeled on the Banach space** X; in shorthand: C^m-Banach manifold.

Up to minor deviations, Michal's (almost forgotten) paper [1939, pp. 537–538] contains the definition above for $m=1$. His terminology:

1-*differentiable manifold* (*with linear topological coordinates*).

6.9.14.5 The concept of a C^m-Banach manifold makes sense only if there exist non-trivial scalar-valued C^m-functions on the underlying Banach space. The easiest way to obtain such functions is via the norm.

The following diagram illustrates the smoothness of the canonical norms of $L_p[0,1]$ and l_p, respectively.

$$
\begin{array}{cccccccccccccccc}
C^0 & & C^\infty & & C^2 & & C^\infty & & C^4 & & C^\infty & & C^6 & & C^\infty & \\
 & C^1 & & C^2 & & C^3 & & C^4 & & C^5 & & C^6 & & C^7 & & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & \longrightarrow p
\end{array}
$$

As a consequence, there exist partitions of unity with the indicated degree of smoothness. These results, which are best possible, were obtained by Bonic/Frampton [1966, pp. 886–888]; see also an earlier paper of Kurzweil [1954].

6.9.14.6 In the setting of manifolds, analytic concepts are mostly defined by localization.

A continuous function Φ from a C^m-manifold M modeled on X into a C^m-manifold N modeled on Y is said to be of **class** C^m if the local functions $g_\beta \circ \Phi \circ f_\alpha^{-1}$ in the diagram

$$
\begin{tikzcd}
f_\alpha\big(U_\alpha \cap \Phi^{-1}(V_\beta)\big) \arrow[r, "g_\beta \circ \Phi \circ f_\alpha^{-1}"] \arrow[d, "f_\alpha^{-1}"] & g_\beta(V_\beta) \\
U_\alpha \cap \Phi^{-1}(V_\beta) \arrow[r, "\Phi"] & V_\beta \arrow[u, "g_\beta"']
\end{tikzcd}
$$

are of class C^m. Here $(U_\alpha, f_\alpha)_{\alpha\in\mathbb{A}}$ and $(V_\beta, g_\beta)_{\beta\in\mathbb{B}}$ denote atlases of M and N, respectively.

6.9.14.7 Eells [1956, p. 305], [1966, p. 751], recognized that

many of the [non-linear] *function spaces which arise in global geometric mathematics possess a natural infinite dimensional differentiable manifold structure.*

Take, for example, the collection of all C^m-functions from the circle $x^2+y^2=1$ into itself.

6.9.14.8 Every open subset of a Banach space can be viewed as a C^0-Banach manifold. This trivial observation applies, in particular, to the linear group $\mathfrak{GL}(X)$, which is open in $\mathfrak{L}(X)$.

6.9.14.9 The unit sphere $S_H := \{x \in H : \|x\| = 1\}$ of any Hilbert space becomes a C^∞-Hilbert manifold by taking the charts $(S_H \setminus \{x_0\}, f_{x_0})$ with $\|x_0\| = 1$, where f_{x_0} is the stereographic projection onto $H_{x_0} := \{x \in H : (x|x_0) = 0\}$ by means of the rays emanating from x_0.

6.9.14.10 In the rest of this subsection, I present some results about infinite-dimensional Banach manifolds that are in striking contrast to corresponding finite-dimensional phenomena.

The story began when Klee [1953, pp. 22, 30] proved that l_2 is homeomorphic to $l_2 \setminus \{o\}$ and to the unit sphere of l_2. Bessaga [1966, p. 27] found a striking improvement: we even have C^∞-diffeomorphy. These statements are obviously false if l_2 is replaced by l_2^n.

6.9.14.11 The following results are due to Eells/Elworthy [1970a, p. 41], [1970b, p. 465] and Henderson [1970, p. 25].

Every separable infinite-dimensional C^0-Banach manifold can be C^0-embedded as an open subset of l_2.

Every separable infinite-dimensional C^∞-Hilbert manifold can be C^∞-embedded as an open subset of l_2.

Note that the compact unit circle $x^2 + y^2 = 1$ is not homeomorphic to any open subset of a Banach space.

6.9.14.12 The preceding results imply that every separable infinite-dimensional C^0-Banach manifold can be made a C^∞-Hilbert manifold in one and only one way.

According to Milnor [1956], the 7-sphere possesses several distinct C^∞-structures.

6.9.14.13 Only little seems to be known about extensions of the previous results to the setting of Banach spaces.

Bessaga [1966, p. 30] states that l_p is C^m-diffeomorphic to its unit sphere, where the degree of smoothness is the same as that given in 6.9.14.5.

6.9.14.14 Recommendations for further reading:

infinite-dimensional topology: [BES$^+$], [MILL],

infinite-dimensional differential geometry: [KAHN], [LANG$_1$], [LANG$_2$], [SUN$^+$].

6.9.15 Asymptotic geometric analysis

6.9.15.1 At the turn of the century, Milman [1996, 2000, 2003$^\bullet$, 2004$^\bullet$] sketched a new theory that he calls *asymptotic geometric analysis*. His philosophy is based on the fact that between classical geometry in 2 or 3 dimensions and geometry in infinite-dimensional spaces there exists a third possibility:

geometry in *"high-dimensional"* Minkowski spaces.

Here is a concise history of finding an attractive name:

Since the academic year 1983–1984, a selection of lectures delivered at the *Israel Seminar on Geometrical Aspects of Functional Analysis* has regularly been published under the trademark **GAFA**; see p. 723. With a change of meaning, this abbreviation also refers to the Journal **G***eometric* **A***nd* **F***unctional* **A***nalysis* founded in 1991. Tomczak-Jaegermann [1998] used the term *geometric functional analysis*, and the attribute *"asymptotic"* first appeared in the title of the well-known Milman/Schechtman lecture notes [MIL$^+$].

6.9.15.2 Asymptotic geometric analysis may be viewed as a descendant of the local theory of Banach spaces. Originally, one was interested in those properties of an *infinite-dimensional* Banach space that can be described in terms of its *finite-dimensional* subspaces; see p. 288. Now the underlying ideology has changed: specific classes of Minkowski spaces or convex bodies with increasing dimensions are studied in their own right.

6.9.15.3 Next, I present a small collection of relevant concepts and theorems developed within the framework of Banach space theory:

Auerbach bases, Banach–Mazur distances, projection constants, volume ratios, s-numbers and all kinds of numerical parameters associated with ideal norms, ... ,

theorems of Dvoretzky, John, Kadets–Snobar, Grothendieck's inequality, ...

In particular, the *"concentration of measure phenomenon"* had a decisive impact on the creation of asymptotic geometric analysis.

Another source of results is classical geometry, which contributed, for example, the inequalities of Brunn–Minkowski and Santaló as well as the isoperimetric inequality. I also mention Löwner's ellipsoid, whereas John's ellipsoid came from the theory of optimization.

6.9.15.4 In this new theory, geometric problems are studied using the main tool of classical analysis: inequalities. Further powerful techniques are adopted from probability theory and combinatorics; see Section 7.3.

6.9.15.5 For every infinite-dimensional Banach space X, one denotes by $\{X\}_n$ the collection of all n-dimensional spaces E_n with the following property:

Given $\varepsilon > 0$, every finite-codimensional closed subspace of X contains an n-dimensional subspace F_n such that $d(E_n, F_n) \leq 1+\varepsilon$.

According to Milman [2003$^\bullet$, стр. 300], [2004$^\bullet$, p. 231],

the family $\{X\}_n$, $n = 1, 2, \ldots$, *represents the asymptotic linear structure of* X.

For Banach spaces with bases there are similar, but more involved, constructions that finally led to various concepts of *asymptotic* l_p *spaces*. The interested reader may consult the original papers of Milman/Tomczak-Jaegermann [1993, pp. 177–178], Maurey/Milman/Tomczak-Jaegermann [1995, p. 157], and a survey of Odell [2002, pp. 226, 238].

An analogous approach for operators is due to Milman/Wagner [1998].

6.9.15.6 The growing activities in this modern branch of functional analysis are shown by programs of two prominent research centers:

Mathematical Sciences Research Institute (Berkeley, Spring 1996):
Convex Geometry and Geometric Analysis; see [BALL$^\cup$],

Pacific Institute of Mathematical Sciences (British Columbia, Summer 2002):
Asymptotic Geometric Analysis.

6.9.15.7 Let me conclude with a problematic quotation. Having in mind the revolution caused by Gowers–Maurey and the development of *asymptotic geometric analysis*, Milman [2004•, p. 228] (Russian version on p. 681) claims that

the 20-year period between the start of the 1970's and the 1990's was lost and was not necessary for the main line of achievements of the infinite-dimensional Banach space theory. ... Nevertheless a few remarkable results were obtained during this period.

A selection of such results is presented in the preceding 250 pages (Chapter 6).

6.9.16 Operator spaces

According to Pisier [2003, p. 1427],
The notion of operator space is intermediate between that of Banach space and that of $C^\star$-algebra. They could also be called "non-commutative Banach spaces" or else "quantum Banach spaces".

It seems to me that the term "quantized functional analysis," which appeared in the title of Effros's ICM-Lecture [1986], has no deep physical meaning. It just refers to the fact that inspired by Heisenberg's uncertainty principle, one deals with possibly non-commutative operations.

At the end of the 1980s, many preliminary results about "operator systems" and "operator spaces" began to form an autonomous branch of functional analysis; see Pisier [2003, p. 1431]. At present, this theory is well-established; standard references are [EFF$^+$] and [PIS$_4$].

Traditionally, operator spaces are considered in the complex setting. However, one may also use real scalars.

6.9.16.1 Let M_{mn} denote the Banach space of all scalar (m,n)-matrices (α_{hk}). A norm is obtained by viewing (α_{hk}) as an operator from l_2^n into l_2^m. That is, M_{mn} and $\mathfrak{L}(l_2^n, l_2^m)$ are identified. If $m=n$, then we simply write M_n instead of M_{nn}.

Similarly, $M_n(X)$ denotes the linear space of all (n,n)-matrices (x_{hk}) whose entries are taken from a linear space X. Given any norm on X, there exists no canonical way to define a norm on $M_n(X)$. The most remarkable exception is the case that $X=\mathfrak{L}(H)$. Then we may identify $M_n\big(\mathfrak{L}(H)\big)$ and $\mathfrak{L}(H^n)$, where $H^n := [l_2^n, H]$ is the orthogonal sum of n copies of H. A norm $\|(T_{hk})\|_n$ is defined to be the smallest constant $c \geq 0$ such that

$$\Big(\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n} T_{hk} f_k\Big\|^2\Big)^{1/2} \leq c\Big(\sum_{k=1}^{n}\|f_k\|^2\Big)^{1/2} \quad \text{for } f_1,\dots,f_n \in H. \tag{6.9.16.1.a}$$

The theory of operator spaces was born at the moment when Ruan [1988, pp. 217–218] elaborated the most decisive properties of the norms $\|(T_{hk})\|_n$ with $n=1,2,\dots$. His discovery opened the way for an axiomatic approach to be described next.

6.9.16.2 An **operator space** is a linear space X together with a sequence of **matrix norms** $\|\cdot\|_n$ on the matrix spaces $M_n(X)$. The following compatibility conditions are supposed to hold for $m,n=1,2,\dots$:

(M$_1$) $$\left\|\begin{pmatrix}(x_{hk}) & \mathrm{O}\\ \mathrm{O} & (y_{ij})\end{pmatrix}\right\|_{m+n} = \max\big\{\|(x_{hk})\|_m, \|(y_{ij})\|_n\big\}$$
for $(x_{hk})\in M_m(X)$ and $(y_{ij})\in M_n(X)$.

(M$_2$) $$\|(\beta_{ih})(x_{hk})(\alpha_{kj})\|_n \le \|(\beta_{ih})\|\,\|(x_{hk})\|_m\,\|(\alpha_{kj})\|$$
for $(\alpha_{kj})\in M_{mn}$, $(x_{hk})\in M_m(X)$ and $(\beta_{ih})\in M_{nm}$.

In what follows, we consider only the case that all normed linear spaces $M_n(X)$ are complete with respect to $\|\cdot\|_n$.

6.9.16.3 In general, there are various ways to turn a given Banach space X into an operator space. This process is called **quantization**; see [EFF$^+$, p. 47]. It goes without saying that the original norm on $X=M_1(X)$ is supposed to coincide with $\|\cdot\|_1$.

Blecher/Paulsen [1991, p. 266] discovered the extreme cases $X_{\min}$ and $X_{\max}$:

$$\|(x_{hk})\|_n^{\min} := \sup\big\{\|(\langle x^*, x_{hk}\rangle)|l_2^n\to l_2^n\| \,:\, \|x^*\|\le 1\big\}$$
and
$$\|(x_{hk})\|_n^{\max} := \sup\big\{\|(Tx_{hk})\|_n \,:\, \|T:X\to\mathfrak{L}(H)\|\le 1,\ \text{all Hilbert spaces } H\big\};$$

see also Blecher [1992a, p. 16]. The reader should realize that the entries of the (n,n)-matrix (Tx_{hk}) are operators, the norm $\|(Tx_{hk})\|_n$ being defined by (6.9.16.1.a).

For any quantization of X, we have $\|(x_{hk})\|_n^{\min} \le \|(x_{hk})\|_n \le \|(x_{hk})\|_n^{\max}$.

6.9.16.4 Operator spaces may serve as the objects of a category. To this end, we must specify the underlying morphisms.

Every linear map T from a linear space X into a linear space Y induces a sequence of linear maps $T_n:(x_{hk})\mapsto(Tx_{hk})$ acting from $M_n(X)$ into $M_n(Y)$. If X and Y are operator spaces, then it makes sense to require that the T_n's be bounded. Letting

$$\|T_n\| := \|T_n|M_n(X)\to M_n(Y)\|, \quad \text{we have} \quad \|T_1\|\le\|T_2\|\le\cdots\le\|T_n\|\le\cdots .$$

Completely bounded operators are defined by the property that the sequence above is bounded. The collection of these operators becomes a Banach space $\mathfrak{L}_{cb}(X,Y)$ under the norm $\|T\|_{cb} := \sup_n\|T_n\|$. This concept goes back to Stinespring [1955], who defined *completely positive functions* between $C^\star$-algebras. Further historical comments can be found in [PAUL] and in a survey of Christensen/Sinclair [1989].

All special kinds of operators have their counterparts in this new setting: **complete isomorphisms**, **complete isometries**, etc.

6.9.16.5 A decisive starting point of the theory of operator spaces was a generalizations of the Hahn–Banach theorem due to Arveson [1969, p. 149] and Wittstock [1981, pp. 133–134]. Its final version reads as follows; see Effros/Ruan [1988, p. 256]:

Let X_0 be a subspace of an operator space X, and let H be a Hilbert space. Then every completely bounded operator T_0 from X_0 into $\mathfrak{L}(H)$ admits an extension T from X into $\mathfrak{L}(H)$ such that $\|T\|_{\mathrm{cb}} = \|T_0\|_{\mathrm{cb}}$.

6.9.16.6 Following ideas of Blecher/Paulsen [1991, p. 269], Blecher [1992a, p. 18], and Effros/Ruan [1991a, pp. 330–331], we can define an operator space structure on the dual X^* of every operator space X.

To this end, rewrite $X^* := \mathfrak{L}(X,\mathbb{C})$ in the form $M_1(X^*) = \mathfrak{L}_{\mathrm{cb}}(X,M_1)$. Similarly, assigning to every (n,n)-matrix (x^*_{hk}) with $x^*_{hk} \in X^*$ the completely bounded operator $x \mapsto (\langle x^*_{hk}, x\rangle)$ yields a one-to-one correspondence between $M_n(X^*)$ and $\mathfrak{L}_{\mathrm{cb}}(X,M_n)$. Via this identification, $\|\cdot\|_{\mathrm{cb}}$ induces the required matrix norm $\|\cdot\|_n$ on $M_n(X^*)$. Another approach is based on the formula

$$\|(x^*_{hk})\|_n = \sup\left\{ \|(\langle x^*_{hk}, x_{ij}\rangle)\| : (x_{ij}) \in M_n(X),\ \|(x_{ij})\|_n \le 1 \right\},$$

where $\|(\langle x^*_{hk}, x_{ij}\rangle)\|$ denotes the norm of the (n^2,n^2)-matrix $(\langle x^*_{hk}, x_{ij}\rangle)$.

Every Banach space can be viewed as a subspace of its bidual. The same is true for operator spaces, but now the embedding from X into X^{**} is a complete isometry; see Blecher [1992a, p. 18] and Effros/Ruan [1991a, p. 331].

6.9.16.7 As explained in 3.4.2.8, every Banach space admits an isometric embedding into some $C(K)$. Ruan [1988, p. 221] proved an analogous result for operator spaces with $C(K)$ replaced by some $\mathfrak{L}(H)$.

To make this statement meaningful, we first observe that $\mathfrak{L}(H)$ together with the norms defined by (6.9.16.1.a) is an operator space. Moreover, every subspace of an operator space is an operator space as well. Finally, instead of an isometric embedding we get a completely isometric embedding.

6.9.16.8 Ruan's theorem can be used to produce operator spaces. We just need an isometric embedding U from an arbitrary Banach space X into some $\mathfrak{L}(H)$. Then the operator space structure on $\mathfrak{L}(H)$ induces an operator space structure on X, which obviously depends on the choice of U. This explains why every Banach space X admits a huge variety of different operator space structures. Two of them, namely $X_{\min}$ and $X_{\max}$, were already discussed in 6.9.16.3.

6.9.16.9 The Hilbert space l_2 can be embedded into $\mathfrak{L}(l_2)$ by

$$U_{\mathrm{col}} : (\xi_h) \mapsto \begin{pmatrix} \xi_1 & 0 & \dots & 0 & \dots \\ \xi_2 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_n & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad U_{\mathrm{row}} : (\xi_k) \mapsto \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n & \dots \\ 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

For $(x_{hk}) \in M_n(l_2)$, the corresponding matrix norms have the form

$$\|(x_{hk})\|_{\mathrm{col},n} = \Big\| \Big(\sum_{i=1}^{n} (x_{ih}|x_{ik}) \Big) \Big\| \quad \text{and} \quad \|(x_{hk})\|_{\mathrm{row},n} = \Big\| \Big(\sum_{i=1}^{n} (x_{hi}|x_{ki}) \Big) \Big\|,$$

where the right-hand expressions are norms of (n,n)-matrices.

Given any Hilbert space H, the preceding formulas yield the matrix norms of the **column operator Hilbert spaces** H_{col} and the **row operator Hilbert spaces** H_{row}, respectively, which were invented by Effros/Ruan [1991b, p. 258] and Blecher [1992b, p. 78]. If H is infinite-dimensional, then the identity map from H_{col} onto H_{row} fails to be completely bounded. We know from Effros/Ruan [1991b, p. 270] and Blecher [1992b, p. 79] that the classical identification $H^* = \overline{H}$ passes into $(H_{\mathrm{col}})^* = \overline{H}_{\mathrm{row}}$ and $(H_{\mathrm{row}})^* = \overline{H}_{\mathrm{col}}$; see 4.11.3 for the definition of the complex conjugate.

Pisier [PIS$_3$, p. 12] discovered the **self-dual operator Hilbert spaces** OH. Viewing $\big((x_{hk}|x_{ij})\big)$ as an (n^2,n^2)-matrix, he defined the underlying matrix norms by

$$\|(x_{hk})\|_{OH,n} := \big\| \big((x_{hk}|x_{ij})\big) \big\|^{1/2} \quad \text{for } (x_{hk}) \in M_n(H).$$

The attribute "self-dual" is justified by the fact that the operator space dual $(OH)^*$ can be identified with the complex conjugate $\overline{OH}$, an analogue of the classical formula $H^* = \overline{H}$. Moreover, Ol_2^n proved to be a perfect substitute for the Euclidean space l_2^n.

6.9.16.10 Applying the **complex interpolation method** to operator spaces, Pisier [PIS$_3$, pp. 27, 29] showed that

$$[H_{\min}, H_{\max}]_{1/2} = OH \quad \text{and} \quad [H_{\mathrm{col}}, H_{\mathrm{row}}]_{1/2} = OH.$$

6.9.16.11 From the historical point of view, it is of particular interest that a considerable part of operator space theory has been developed by taking Banach space theory as a pattern. The previous presentation shows that most (though not all) concepts and theorems about Banach spaces have a counterpart if one uses completely bounded operators instead of the ordinary ones. Here are some more examples:

Banach–Mazur distance, Löwner ellipsoids, projection constants, nuclear operators, absolutely summing operators, tensor products, ... ,
theorems of John, Kadets–Snobar, Dvoretzky–Rogers, ...

In the preface of [EFF$^+$] this phenomenon is expressed as follows:

It is our conviction that the extraordinary array of techniques developed by Banach space theorists will have many applications in non-commutative analysis, and that conversely, operator space theory will provide Banach space theorists with exciting new vistas for research.

This is why a large number of (young) researchers who were educated in Banach space theory converted to this new and promising discipline.

6.9.17 Omissions

At the beginning of my writing, I had the idea that a *History of Banach Spaces and Linear Operators* could be presented in a book of about 400 pages. As shown by the final result, this was a strong underestimate. Nevertheless, even the present size does not allow the presentation to be complete; many interesting subjects had to be omitted. I mention only few of them.

Let A be any class of separable Banach spaces. Then $X_0 \in \mathsf{A}$ is called a **universal space** of A if any other space in A is isomorphic to a subspace of X_0. Obviously, $C[0,1]$ is universal for all separable Banach spaces. According to Pełczyński [1969a], there exist universal spaces of the class of all spaces with a Schauder basis or with an unconditional basis; see [LIND$_1^+$, p. 92]. Further examples are rare: Szlenk [1968] proved that there is no universal separable reflexive space; see also Bourgain [1980b].

The study of Banach spaces formed by vector-valued functions has attracted many people. Among other things, one may compare properties of a Banach space X with that of $[C,X]$ or $[L_p,X]$. Luckily, such problems are treated in a recent book of (Pei-Kee) Lin [LIN]; see also [CEM$^+$]. Several authors have dealt with properties that are preserved when passing from X and Y to their tensor products $X \tilde{\otimes}_\pi Y$ and $X \tilde{\otimes}_\varepsilon Y$.

The Banach–Stone theorem 4.5.5 extends to the setting of X-valued continuous functions; see [BEHR]. One looks for conditions on the underlying Banach space X that guarantee that the isometry of $[C(K_1),X]$ and $[C(K_2),X]$ implies the homeomorphy of K_1 and K_2.

The reader will find many facts about closed subspaces and quotients of the classical spaces $L_p(M,\mathscr{M},\mu)$ with $1 \le p < \infty$ scattered over this book. However, there is no particular section devoted to this subject. Therefore some important results got lost. For example, I mention a famous criterion of Bretagnolle/Dacunha-Castelle/Krivine [1966, p. 238]; see also [WEL$^+$, pp. 14–24]:
A real Banach X space can be isometrically embedded into some $L_p(M,\mathscr{M},\mu)$ with $1 \le p \le 2$ if and only if the function $x \mapsto \exp\big(-\|x\|^p\big)$ is positive definite on X.

Murray [1945] called a closed subspace M of X **quasi-complemented** if there is a closed subspace N such that $M \cap N = \{\mathrm{o}\}$ and $\overline{M+N} = X$; compare with 4.9.1.8. He also showed that all closed subspaces of a reflexive separable Banach space enjoy this property. Subsequently, Mackey [1946b] removed the assumption of reflexivity. On the other hand, we know from Lindenstrauss [1968] that in non-separable spaces there may exist closed subspaces without quasi-complements. This happens for $c_0(\mathbb{I})$ as a subspace of $l_\infty(\mathbb{I})$ whenever $\mathbb{I}$ is uncountable.

In connection with quasi-complements, Rosenthal [1969, p. 188] raised a problem that is still open; see Mujica [1997] and [FAB$^+$, pp. 377–379]:
Does every infinite-dimensional (non-separable) Banach space have an infinite-dimensional separable quotient?

Johnson/Rosenthal [1972, pp. 78, 85] proved that every infinite-dimensional separable Banach space has a quotient with a Schauder basis.

7

Miscellaneous Topics

7.1 Banach space theory as a part of mathematics

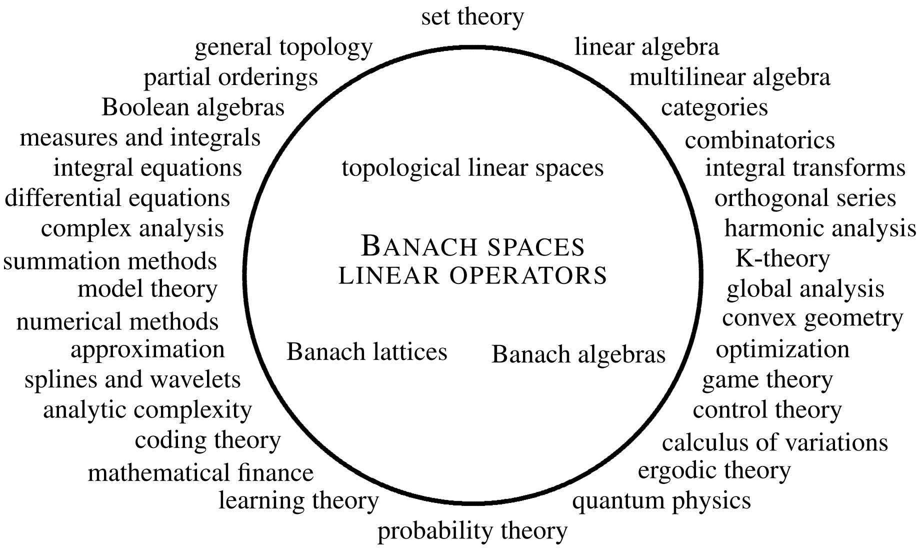
Banach space theory holds an eminent place among the basic disciplines of mathematics. It relies on some elementary facts from

set theory, general topology and *linear algebra.*

Moreover, *measure theory* is needed for constructing the function spaces L_p.

I hope that the following presentation will convince the reader that the relations are two-sided: it is a matter of give-and-take.

First of all, I illustrate how *Banach space theory* is located in the mathematical landscape.



7.1.1 One may distinguish between soft and hard applications of Banach space theory. The adjective *"soft"* indicates that besides the language of Banach spaces and manipulations with the triangle inequality, only some classical results such as the principle of uniform boundedness, the open mapping theorem, or the Hahn–Banach theorem are used. One should also include applications of the fundamentals about compact operators and Fredholm operators.

Examples of *"hard"* applications are Bourgain's solution of the $\Lambda(p)$-problem 6.9.12.6, the relationship between the *UMD* property and the existence of the vector-valued Hilbert transform 6.1.10.4, Schwartz's approach to the Wiener measure 6.8.6.1, and the proof of the reverse Santaló inequality 6.1.11.1. A complete list would be impressive: mainly because of its content, but not because of its length.

Not seldom, translating known concepts and theorems from one theory into another leads to new developments. Therefore besides applications we also have *inspirations*. Typical examples are the relations

$$K \big(\text{compact Hausdorff space}\big) \mapsto C(K) \quad \text{and} \quad \Lambda \big(\text{subset of } \mathbb{Z}\big) \mapsto L_1^{\Lambda}(\mathbb{T}).$$

7.1.2 Since the norm topology of a Banach space is metric, only some elementary facts from *general topology* are required. As a significant exception, I mention the Baire category theorem 2.4.6, which is the decisive tool for proving the open mapping theorem and the uniform boundedness principle (Banach–Steinhaus–Saks approach); see 2.5.3 and 2.4.8. The homeomorphic classification of Banach spaces must be emphasized as a special highlight; see 6.9.2.1.

In dealing with weak topologies, the situation changes dramatically. For example, Tychonoff's theorem 3.4.1.5 implies Alaoglu's theorem 3.4.2.6, which in turn can be used to prove the existence of the *Stone–Čech compactification* 4.5.4. The question of which Banach spaces are weakly Lindelöf has challenged the efforts of many people; see 6.9.1.14.

Of particular importance is the relationship between a compact Hausdorff space K and the corresponding Banach space $C(K)$. Indeed, the famous Banach–Stone theorem 4.5.5 tells us that K is uniquely determined by the isometric structure of $C(K)$. The lattice structure of $C(K)$ gave rise to the invention of Stonean spaces; see 4.8.1.5. Eberlein compactness 6.9.1.6 and Gulko compactness 6.9.1.11 are further concepts based on the interplay between K and $C(K)$.

Here are some more relevant topics:

- Properties of the closed unit ball equipped with the weak topology,
- The topological structure of the Minkowski compacta,
- Contractibility of the linear group of a Banach space,
- Fixed point theorems for non-expansive maps on weakly compact convex sets.

Further information about connections between Banach space theory and general topology can be found in surveys of Negrepontis [1984] and Mercourakis/Negrepontis [1992]; see also Edgar/Wheeeler [1984] and [HART$^{\cup}$].

7.1.3 The starting point of the interplay between Banach space theory and *measure theory* was the classical Riesz representation theorem 2.2.9, the far-reaching ramifications of which are described in Section 4.6.

Particular attention has been paid to measures on topological spaces. Borel, Baire, and Radon measures are the basic tools for developing *probability on Banach spaces*; see Sections 4.6 and 6.8.

The search for conditions on a measure space $(M, \mathscr{M}, \mu)$ that guarantee the duality $L_1(M, \mathscr{M}, \mu)^* = L_\infty(M, \mathscr{M}, \mu)$ led to the concept of localizability 4.3.9.

7.1.4 In the *theory of Banach lattices*, we need another mother-structure in the sense of Bourbaki [1948]: *partial ordering*. *Positivity* plays an important role. *Boolean algebras*, especially *measure algebras*, are the basic tools for representing abstract L_p-spaces; see 4.8.3.3. The key is Maharam's theorem 4.8.2.7.

7.1.5 Only a few tools from *algebra* are used in Banach space theory. Besides Peano's concept of a *linear space*, which was rediscovered in Banach's thesis, we need *multilinear functionals* and *tensor products*. *Radicals* are borrowed from *ideal theory*. The *theory of categories* provides us with a powerful language.

7.1.6 Ultraproduct techniques were adapted from *model theory* and became an elegant tool of the *local theory* of Banach spaces.

7.1.7 Applications of methods from *combinatorics* are discussed in Subsection 7.3.2.

7.1.8 The term *"Integralgleichung"* was coined by du Bois-Reymond [1888, p. 228] in connection with the Dirichlet problem. The idea to solve the equation $\Delta u = 0$ under a boundary condition by writing u as a double-layer potential goes back to Beer; see [NEU, p. 220]. The upshot was an integral equation of the second kind. Historical details can be found in [HEL$^+$, pp. 1345–1347]. Michlin's textbook [MICH, §§ 33–37] contains a beautiful summary of this subject.

The title of Fredholm's first paper [1900],

Sur une nouvelle méthode pour la résolution du problème de Dirichlet,

indicates that the integral equations of *potential theory* inspired his work on *determinant theory*, which, in turn, was one of the main sources of functional analysis.

7.1.9 The Banach space approach for solving the *differential equations*

$$\Delta u = f \quad \text{and} \quad \Delta u + \lambda u = 0$$

is described in Subsection 6.7.11. For more information concerning elliptic equations, the reader may consult [TRI$_1$].

Parabolic and hyperbolic equations, like

$$\Delta u = \frac{\partial u}{\partial t} \quad \text{and} \quad \Delta u = \frac{\partial^2 u}{\partial t^2},$$

can be treated as abstract Cauchy problems; see [KREIN], [PAZY], [ENG$^+$], and Subsection 5.3.4.

7.1.10 The theory of s-numbers yields results about *eigenvalue distributions* of integral and differential operators.

7.1.11 As described in 6.2.4.7, the concept of *entropy numbers* was adopted from *information theory*. The following problem of *coding theory* is in the same spirit: Find large sets of points that are well distinguishable.

The ε-entropy has also played an important role in connection with Hilbert's 13th problem; see 6.7.8.16,

7.1.12 *Khintchine's inequality* (6.1.7.2.a) is a symbol of the trinity

$$\textit{harmonic analysis} \; + \; \textit{Banach space theory} \; + \; \textit{probability theory}.$$

The standard references about this interplay is [KAH]. One may also consult the recent book [LI$^+$] and the survey of Kislyakov [2001].

7.1.13 *Trigonometric series* and, more generally, *orthogonal expansions* are among the favorite mathematical topics related to Banach spaces.

Zygmund wrote in the preface of [ZYG]:

Many basic notions and results of the theory of functions have been obtained by mathematicians while working on trigonometric series.

To begin with, I present a few applications of the basic theorems of Banach space theory to *harmonic analysis*.

- Since the Lebesgue constants tend to ∞, the uniform boundedness principle implies that $\big(\mathrm{e}^{\mathrm{i}kt}\big)_{k\in\mathbb{Z}}$ does not constitute a basis of $C(\mathbb{T})$ and $L_1(\mathbb{T})$; see 2.4.2 and 5.6.4.4.
- The famous Fischer–Riesz theorem 1.5.2 tells us that the periodic Fourier transform
$$F^{2\pi}_{\mathrm{our}} : f \mapsto \big(\gamma_k(f)\big) \quad \text{with} \quad \gamma_k(f) := \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\,dt$$
provides an isometry between $L_2(\mathbb{T})$ and $l_2(\mathbb{Z})$.
- We know from the Riemann–Lebesgue lemma that $F^{2\pi}_{\mathrm{our}}$ acts from $L_1(\mathbb{T})$ into $c_0(\mathbb{Z})$. Since the two spaces fail to be isomorphic, the inverse mapping theorem shows that $F^{2\pi}_{\mathrm{our}}$ cannot be *onto*; see 2.5.6.

One may ask for which subsets Λ of $\mathbb{Z}$ the Banach spaces $C^{\Lambda}(\mathbb{T})$ or $L^{\Lambda}_p(\mathbb{T})$ have a given property; see Godefroy [1988] and [HARM$^+$, pp. 189–199]. In particular, I stress the importance of the concept of *lacunarity* discussed in Subsection 6.9.12. A highlight is the cotype 2 dichotomy for $C^{\Lambda}(\mathbb{T})$ discovered by Bourgain/Milman; see 6.9.12.4. The translation invariant subspaces $L^{\Lambda}_p(\mathbb{T})$ were used to produce non-trivial counterexamples; see 6.3.7.6.

The *theory of multipliers* has developed into a special branch of *harmonic analysis*. The naming refers to the fact that in the Fourier image, convolution operators

$$\varphi(t) \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(s-t)\varphi(t)\,dt$$

are just diagonal (multiplication) operators

$$\big(\gamma_k(\varphi)\big) \mapsto \big(\gamma_k(f)\gamma_k(\varphi)\big).$$

A historical survey is given in [HEW$_2^+$, pp. 412–415].

7.1.14 Banach space theory and *probability theory* are children of the axiomatic method. The fundamental books of Banach and Kolmogoroff appeared almost at the same time. The interplay between both disciplines is fascinating.

- Stochastic processes can be viewed as vector-valued functions; see 6.8.7.4.
- Stochastic processes may induce probabilities on Banach spaces that contain all sample functions. The most important example is the Brownian motion, which yields the Wiener measure; see Subsection 6.8.6.
- Stochastic kernels related to Markov processes define semi-groups of operators; see 5.3.1.5.
- The sequence of the Rademacher functions provides a model of coin tossing.
- Stable laws were used to construct isometric embeddings from l_s into L_p for $1 \le p \le s < 2$; see 6.1.7.21. They also played a crucial role in the early period of the "French" theory of type p.
- Martingales, and in particular, Walsh–Paley martingales have become an indispensable tool in the geometry of Banach spaces; see Subsections 6.8.11 and 6.1.9.5.
- As illustrated in Subsection 7.3.1, probabilistic techniques can be used for proving the existence of exotic spaces and operators.

Readers who are interested in *randomly normed spaces* may consult [HAY$^+$].

7.1.15 *Summability* (Limitierungstheorie) inspired the development of classical Banach space theory. In particular, I stress the fact that Toeplitz [1911b, p. 117] used the gliding hump method to prove his permanence criterion 6.9.11.3. Further functional analytical aspects can be found in a paper of Mazur [1930b] and in Banach's book [BAN, pp. 90–91]. Later on, Mazur/Orlicz [1954, p. 129] observed

that the fields of the summability methods constitute more general linear spaces, to which the authors gave the name B_0-spaces.

Frankly, I know only "soft" applications of Banach space techniques to the theory of summability; see [WIL]. On the other hand, a deep result of Schur [1921, p. 82] about *coercive mappings* (konvergenzerzeugende Transformationen) $A : l_\infty \to c$ implicitly states that weakly convergent sequences in l_1 are norm convergent.

7.1.16 Fredholm determinants and resolvents $\lambda \mapsto R(\lambda, T)$ show that methods from *complex analysis* play an important role in Banach space theory. Liouville's theorem can be used to prove that the spectrum of operators is non-empty. One may also think of inner and outer functions, Blaschke products, as well as the study of boundary values of analytic functions.

The following list presents further remarkable features:

- Geometry of Hardy and Bergman spaces; see Subsections 6.7.12 and 6.7.13.
- The open question whether $\mathscr{H}_\infty(\mathbb{D})$ has the approximation property.
- Vector-valued analytic functions of a complex variable and, in particular, the analytic functional calculus; see Subsections 5.1.7 and 5.2.1.
- Analytic functions defined on a Banach space; see Subsection 5.1.10, [DINE$_1$], [DINE$_2$], and [MUJ].
- Composition operators; see Subsection 6.9.10 and [SHAP].

Besides de Branges's proof of the Bieberbach conjecture [1985], which uses Hilbert spaces, I am not aware of any serious application of Banach spaces theory to classical complex analysis. However, problems about Banach spaces and Banach algebras inspired the development of new complex analytic techniques. Here is a convincing example:

The Gelfand space $\mathbb{M}$ formed by all multiplicative linear (non-trivial) functionals on $\mathscr{H}_\infty(\mathbb{D})$ contains the point evaluations $\delta_\zeta : f \mapsto f(\zeta)$ with $\zeta \in \mathbb{D}$. Kakutani had already conjectured in 1941 that the *"corona"* $\mathbb{M}\backslash\overline{\mathbb{D}}$ is empty. A positive answer was given by Carleson [1962], who, just for this purpose, invented a useful kind of measures on $\mathbb{D}$ that now bear his name; see [DUR, p. 157].

The next quotation is taken from an article of Wermer [1989$^\bullet$, p. 428] about *Function algebras in the fifties and sixties*:

Hoffman's book [HOF] *showed to the world of classical analysts and to the world of functional analysts that they are brothers and sisters rather than strangers* (*as many had thought*).

7.1.17 Banach spaces provide the appropriate basis for *approximation theory*. The main problems are concerned with the existence and uniqueness of best approximation of a given element by elements in a closed subspace or some more general subset.

The famous *Weierstrass theorem* states that every continuous function defined on a closed interval can be uniformly approximated by polynomials. This mathematical jewel has become the starting point of various developments.

Most important was the step from qualitative to quantitative results: We want to know how well a given function can be approximated by polynomials of a prescribed degree. As a classical example I mention the Jackson–Bernstein theorem 6.7.1.3, which finally led to the invention of Besov spaces. Their definition is based on the following rule of thumb; see also 6.7.10.1:

The smoother the function, the better the degree of approximation.

7.1.18 Beginning with Haar's thesis [1910], the search for bases in function spaces had a considerable influence on the study of *splines* and *wavelets*; see 5.6.6.11. Birman/Solomyak [1967] used approximations by piecewise polynomial functions for obtaining upper estimates of widths.

7.1.19 According to the preface of his book *Funktionalanalysis und Numerische Mathematik*, Collatz wanted to show

wie sich in der numerischen Mathematik in neuerer Zeit [read: in the 1950s] *ein Strukturwandel vollzogen hat.*

Besides computers, the invention of functional analytical techniques played a decisive role. Unfortunately, I have seen only 'soft' applications, and the powerful inequalities of local Banach space theory are still ignored. Occasionally, s-numbers of operators have been used.

Almost every textbook on numerical mathematics contains a chapter on *matrix norms*. This concept, due to Faddeeva [FAD] and Ostrowski [1955], is closely related to the concept of an ideal norm. Typical examples are the expressions

$$\big\|(\alpha_{hk}) : l_2^n \to l_2^n\big\|, \quad \sup_{1\le h\le n} \sum_{k=1}^{n} |\alpha_{hk}|, \quad \Big(\sum_{h=1}^{n}\sum_{k=1}^{n} |\alpha_{hk}|^2\Big)^{1/2}, \quad \sup_{1\le h,k\le n} |\alpha_{hk}|.$$

A modern presentation is given in [BEL$^+$].

Of course, approximation processes in all kinds of function spaces play an essential role.

7.1.20 The *theory of analytic complexity* is concerned with the question of how well an infinite-dimensional operator equation can be solved by finite-dimensional (deterministic or stochastic) numerical approximation. Therefore it is not surprising that s-numbers and entropy numbers became important tools. In order to get a first impression of this field, the reader may consult the work of Heinrich [1992, 1998] and Mathé [1990, 1991].

7.1.21 Besides Fredholm's determinant theory, the *calculus of variations* was the most important source of functional analysis. In [1887], Volterra published a series of papers with the title

Sopra le funzioni che dipendono da altre funzioni.

A typical example of a *function depending on other functions* is the Dirichlet integral

$$\iint\limits_{\Omega} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\, dx\, dy;$$

see Hilbert [1904a]. The problem of finding the minimum of this expression for all u's with given boundary values was a decisive stimulus for the invention of compactness and of Sobolev spaces $W_2^1(\Omega)$. The upshot is the basic question of *convex programming*:

Minimize $f(x)$, where $f : C \to \mathbb{R}$ is a convex function f on a convex set C.

7.1.22 The computation of operator norms is a typical case of the opposite problem:

Maximize $f(x)$, where $f : C \to \mathbb{R}$ is a convex function f on a convex set C.

Bauer's maximum principle 5.4.5.4 tells us that under mild conditions, the required solution is attained at some extreme point of C.

7.1.23 In the preface of [LUEN] we find the statement that

a rather large segment of the field of "optimization" can be effectively unified by a few geometric principles of linear vector space theory.

The main feature is a persistent use of duality. The geometric version of the Hahn–Banach theorem turns out to be the most important tool, and convexity plays a dominant role. In the finite-dimensional setting, the standard reference is Rockafellar's *Convex Analysis* [ROCK]. A textbook on *Convexity and Optimization in Banach Spaces* was written by Barbu/Precupanu [BARB$^+$].

7.1.24 The classical reference presenting *control theory* from a functional analytic point of view is [HERM$_2^+$]; see also [KLU$^+$] and [ROL$_2$]. In particular, I refer to the bang-bang principle, which follows from Lyapunov's theorem on vector measures; see 5.4.6.1.

7.1.25 *Minimax theorems* show that functional analytical aspects also occur in *game theory*; see 5.4.5.5.

7.1.26 Concerning applications of Banach space theory to *mathematical finance*, the reader should consult the survey of Delbaen/Schachermayer [2001]. These authors mention that, inter alia, the theorems of Kreĭn–Shmulyan and Bishop–Phelps as well as James's characterization of weakly compact subsets are used.

7.1.27 In the founding period of functional analysis, several concepts were borrowed from Minkowski's *Geometrie der Zahlen* and his *Theorie der konvexen Körper*; see [MIN] and Minkowski [$\leq$1909]. Of particular importance are *Stützebenen* (supporting hyperplanes 5.5.1.2) and *extreme Punkte* (extreme points 5.4.1.1).

Roughly speaking, Choquet's theorem tells us that the extreme points are the elementary building blocks of any convex compact set. The unifying power of this result is fascinating; see Subsection 5.4.2.

Applications of the *local theory* of Banach spaces to *convexity* are described in a survey of Lindenstrauss/Milman [1993].

7.1.28 A workshop on *Asymptotic Geometric Analysis and Machine Learning* took place at the Université de Marne-La-Vallée in 2003. The main object focused on "high-dimensional" geometric, probabilistic and combinatorial problems that occur in *machine learning*. A convincing example of this interplay can be found in a paper of Mendelson [2001], who used the Gauss-summing norm.

7.1.29 The abelian *Grothendieck group* of *K-theory* can be described in terms of Φ-operators; see 6.9.13.5.

7.1.30 *Differentiable manifolds* modeled on Banach spaces are the basic concept of infinite-dimensional *global analysis*; see Subsection 6.9.14.

7.1.31 The most important contribution of functional analysis to *theoretical physics* is the spectral theorem of self-adjoint operators on Hilbert spaces, and its ramifications. Of course, the *theory of* $C^\star$*- and* $W^\star$*-algebras* must be mentioned. Concerning Banach spaces, I refer to the operator-theoretic *ergodic theorems*; see Subsection 5.3.5. A survey of Carl/Schiebold [2000] shows that abstract determinants, discussed in Section 6.5, were successfully used in the *theory of solitons*. To the best of my knowledge, *operator spaces* have not been applied so far.

7.1.32 Without the *axiom of choice*, Banach space theory would become a torso. Rarely, the *continuum hypothesis* and some other axioms of set theory are used. For details the reader is referred to Section 7.5.

7.1.33 Around 1930, several young mathematicians regarded *functional analysis* as very promising and switched to this fashionable area. Prominent example are Garrett Birkhoff and Köthe. This fact caused the polemic interpretation of Wiener quoted on p. 20.

Later on, the movement was mainly in the opposite direction, which proves that working in Banach space theory provides an excellent background. Apart from Grothendieck and Bishop, most researchers migrated from the "abstract" to the "concrete." Here is an alphabetic list of some *converts*:

Alaoglu	$\Rightarrow$	aircraft industry,
Aldous	$\Rightarrow$	statistics,
Beauzamy	$\Rightarrow$	real life mathematics, Irish Math. Soc. Bull. **48** (2002), 43–46,
Bishop	$\Rightarrow$	constructivism,
Bukhvalov	$\Rightarrow$	mathematical finance,
Delbaen	$\Rightarrow$	mathematical finance,
Ghoussoub	$\Rightarrow$	variational calculus,
Goldstine	$\Rightarrow$	computer Ssience, ENIAC
Grothendieck	$\Rightarrow$	homological algebra, algebraic geometry,
Heinrich	$\Rightarrow$	analytic complexity,
Kantorovich	$\Rightarrow$	mathematical economics, computer science,
Karlin	$\Rightarrow$	statistics, game theory,
Rolewicz	$\Rightarrow$	optimal control,
Schachermayer	$\Rightarrow$	mathematical finance,
Schwartz (Jacob)	$\Rightarrow$	computer science,
Singer (Ivan)	$\Rightarrow$	abstract convex analysis,
Tukey	$\Rightarrow$	statistics.

Those working on the border with harmonic analysis and probability theory are not included.

7.2 Spaces versus operators

At the beginning of one of my lectures in Oberwolfach, I asked a provocative question,

What is more important, spaces or operators?

Israel Gohberg, who chaired this meeting, interrupted me and proposed (in his charming way) the following answer,
We know Hilbert spaces, Banach spaces, Orlicz spaces, etc., but we have just operators. Hence spaces are made by men, while operators are made by God.

This "proof" sounds like a part of a parliamentary speech, since politicians prefer to present only those arguments that support their claims. In response, some member of the opposition might have said:
Well! We know Hilbert–Schmidt operators, Hankel operators, Toeplitz operators, etc., but we have just locally convex linear spaces. Hence ... .

This section is intended to be a pleading for a tolerant answer:

We need both spaces and operators!

7.2.1 Solving equations in infinitely many unknowns, integral equations, and differential equations is the classical goal of functional analysis. It does not suffice to consider the underlying rules

$$(\xi_k) \mapsto \left(\sum_{k=1}^{\infty} \alpha_{hk} \xi_k \right), \quad f(t) \mapsto \int_a^b K(s,t) f(t)\, dt, \quad \text{and} \quad u \mapsto \Delta u.$$

We need to specify the elements to which these rules are applied. Though no general procedure is available, in most concrete cases there exists a "natural" domain of definition. Of course, our favorite choice is a Hilbert space. But in some instances, it turns out to be more advantageous to take a Banach space or even a locally convex linear space. In conclusion, we may say that

spaces are the ground on which operators live.

Therefore, from the historical point of view, operators had a certain priority over spaces. This observation is also supported by the fact that the title of Banach's book refers to operators but not to spaces.

In the introduction of their survey on *Scales of Banach spaces* [1966, pp. 85–86], Selim Krein/Petunin take the following view:

The choice of the space in which a problem is studied is partly connected with the subjective aims which the investigator sets himself. Apparently the objective data are only the operators that appear in the equations of the problem. On this account it seems to us that the original and basic concept of functional analysis is that of an operator.

7.2.2 The structure of spaces and their subsets has been studied in its own right, and it seems to me that this aspect dominated the development during the second half of the twentieth century. Apart from the personal interests of the leading researchers, there are objective reasons. In the following, I will try to analyze the situation.

7.2.3 Let me begin my review by listing the most important achievements of *operator theory*.

The highlight is certainly the *Riesz–Schauder theory* of compact operators supplemented by the theorem that states the existence of invariant subspaces; see Subsections 2.6.4 and 5.2.4.

Closely related is the beautiful *theory of Fredholm operators* presented in Subsection 5.2.2. Concerning *perturbation theory*, I refer to [KATO].

Many concrete operator ideals have been studied; see Section 6.3. Of particular importance are their applications to *eigenvalue distributions* described in Section 6.4.

Last but not least, I mention the *theory of semi-groups*, which has become a special branch of functional analysis; see Section 5.3. For more information about recent achievements, the reader may consult the monographs [AREN$^+$] and [ENG$^+$].

7.2.4 In the 1930s, the functional analysts dreamed of a powerful theory that would extend the famous spectral theorem of self-adjoint operators to the setting of Banach spaces. Unfortunately, they ran into several obstacles.

The most obvious defect of non-Hilbertian spaces is the existence of uncomplemented closed subspaces. Nevertheless, Dunford and his pupils tried to overcome these difficulties. The upshot is the theory of scalar type operators and spectral measures, whose range is rather limited; see Subsection 5.2.5. As a consequence, the Romanian school invented "generalized" scalar type operators and spectral capacities. This development has led to the *local spectral theory*, and in particular, to the concept of a decomposable operator; see Subsection 6.9.8.

7.2.5 Obviously, operators can be decomposed only if they have non-trivial invariant subspaces. However, we know that counterexamples exist even in the narrow class of strictly singular operators; see 5.2.4.8.

7.2.6 Another unpleasant phenomenon is described in 6.5.6.3:

There exist "nice" compact operators on infinite-dimensional Hilbert spaces that do not admit an analogue of the Jordan form, which turned out to be so useful in the case of finite matrices.

Luckily enough, the root vector completeness theorem 6.5.6.7 provides sufficient (though quite restrictive) conditions that guarantee a positive result.

In contrast to the misfortune concerning the Jordan form, we have a substitute of the triangular representation; see 5.2.4.6.

7.2.7 Conclusion:

As shown by the historical development, the concept of a Banach space is so general that many pathologies may occur; see Section 7.4. The situation is even worse in the setting of operators. In order to get substantial results, we must restrict our attention to concrete cases in which more structural information is available. Convincing examples are Hankel operators, Toeplitz operators, and composition operators; see Subsections 6.9.9 and 6.9.10.

7.2.8 The rest of this section deals with results from operator theory that have been used for investigating the geometry of Banach spaces, and vice versa. In my opinion, this give-and-take is rather unbalanced. The *"transfer from operators to spaces"* predominates.

7.2.9 At first glance, it seems plausible that good properties of the spaces X and Y imply good properties of the operators acting from X to Y. Unfortunately this is not true in general, since even operators on Hilbert spaces may behave very badly. Nevertheless, there are positive exceptions. For example, the asymptotic behavior of the eigenvalues of nuclear operators strongly reflects the structure of the underlying space; see Subsection 6.4.4.

7.2.10 The problem whether the Fourier transform $F_{\text{our}}: L_p \to L_{p^*}$ extends to spaces of vector-valued functions has led to the concept of Fourier type p; see 6.1.8.2. Asking the corresponding question for the Hilbert transform $H_{\text{ilb}}: L_p \to L_p$, Bourgain and Burkholder discovered the HT spaces; see 6.1.10.4. The definition of K-convexity is based on the same idea; see 6.1.7.12.

A metric concept was invented by Herz [1971, Abstract]:

The class of p-spaces is defined to consist of those Banach spaces B such that linear transformations between spaces of numerical L_p-functions naturally extend with the same bound to B-valued L_p-functions.

7.2.11 The fact that the Hilbert transform does not induce a continuous operator on $L_1(\mathbb{R})$ gave rise to the invention of the Marcinkiewicz space $L_{1,\infty}(\mathbb{R})$ and the Hardy space $H_1^{\text{real}}(\mathbb{R})$.

7.2.12 Subsection 6.3.19 presents a very important feature of the local theory: ideal norms provide numerical parameters that characterize the geometry of finite-dimensional Banach spaces.

7.2.13 Specific components of operator ideals are an inexhaustible source of non-classical Banach spaces. For example, we know from 6.1.5.6 that $\mathfrak{S}_p(l_2)$ fails to have local unconditional structure. According to Pisier [1990], the space $\mathfrak{P}_2(l_p, l_q)$ is superreflexive whenever $1 < p, q \leq 2$. In the remaining cases, (Pei-Kee) Lin [1980] observed that $\mathfrak{P}_2(l_p, l_q)$ fails to be even B-convex.

7.2.14 Finally, I stress the fruitful interplay between space ideals and operator ideals; see Subsection 6.3.13. In this case the relation is indeed two-sided.

Beginning with Grothendieck's work about the Dunford–Pettis property 4.8.5.3, many classes of Banach spaces were defined via operator ideals. Another example is the collection of all spaces satisfying Grothendieck's theorem; see 7.4.1.10.

Conversely, given a certain property of spaces, one may look for appropriate generalizations to the setting of operators. Here is a short list: Radon–Nikodym property, Banach–Saks property, Grothendieck property, superreflexivity, Rademacher type and cotype.

7.3 Modern techniques of Banach space theory

7.3.1 Probabilistic methods

7.3.1.1 Probabilistic proofs of existence are based on a simple idea:

Let M denote a set of mathematical objects. In order to show that there is a member $x \in M$ that enjoys a given property $\mathbb{P}$, we look for a probability P on M such that

$$p := \mathsf{P}\{x \in M : x \text{ has property } \mathbb{P}\} > 0.$$

The reader may think of Brownian motion: almost all paths are nowhere differentiable,

Of course, probabilistic proofs are non-constructive. Ironically, even in the case that p is close to 1, it happens that all concrete examples that come to mind fail to have the desired property. The moral of this observation is that our brain works too regularly. Thus we sometimes need random processes for producing pathologies. Maybe, the fact that randomness can be simulated by deterministic algorithms opens some way for the future.

7.3.1.2 In order to demonstrate the probabilistic method, I prove an important lemma that is due to Kashin [1974, p. 305]:

If $2 \le n \le 2^m$, then there exist m-tuples $x_1, \dots, x_n$ whose entries are ± 1 and such that

$$|(x_h|x_k)| \le 2\sqrt{m \log_2 n} \quad \text{whenever } h \ne k.$$

Starting from $x_1 = (1, 1, \dots, 1)$ and $x_2 = (-1, 1, \dots, (-1)^m)$, we may proceed by induction with respect to n. Assume that $x_1, \dots, x_n$ have the required property. In view of Khintchine's inequality (6.1.7.2.a),

$$\Big(\frac{1}{2^m} \sum_{e \in \mathbb{E}^m} |(e|x_k)|^s\Big)^{1/s} \le B_s \|x_k|l_2^m\| = B_s m^{1/2} \quad \text{for } k = 1, \dots, n.$$

Hence

$$\frac{1}{2^m} \sum_{e \in \mathbb{E}^m} \sum_{k=1}^{n} |(e|x_k)|^s = \sum_{k=1}^{n} \frac{1}{2^m} \sum_{e \in \mathbb{E}^m} |(e|x_k)|^s \le B_s^s\, n\, m^{s/2}.$$

This inequality is the crucial point of our reasoning; it tells us that we can find at least one $e_0 \in \mathbb{E}^m$ such that

$$|(e_0|x_k)|^s \leq \sum_{k=1}^{n} |(e_0|x_k)|^s \leq B_s^s\, n\, m^{s/2}.$$

Since $B_s \leq \sqrt{s/2+1}$, letting $s := \log_2 n$ yields

$$|(e_0|x_k)| \leq 2\sqrt{m(\tfrac{1}{2}\log_2 n + 1)} \leq 2\sqrt{m \log_2(n+1)} \quad \text{for } k = 1, \ldots, n.$$

Finally, enlarging $\{x_1, \ldots, x_n\}$ by $x_{n+1} := e_0$ completes the n^{th} step of the induction.

7.3.1.3 In Banach space theory, probabilistic methods occurred for the first time in the proof of Dvoretzky's theorem on spherical sections; see 6.1.2.3. Dvoretzky [1960] himself used such arguments in a hidden form. The probabilistic nature became clearer in the refinements due to Szankowski [1974] and Figiel [1976b]. The real breakthrough was achieved by Milman's approach via the concentration of measure phenomena.

7.3.1.4 Enflo's original construction of a Banach space without the approximation property is based on combinatorial methods; see also Pełczyński/Figiel [1973]. The significant simplification accomplished by Davie [1973, p. 261] is, however, due to the use of probabilistic techniques. A translation into the language of operators can be found in [PIE$_3$, pp. 135–141].

7.3.1.5 Figiel/Kwapień/Pełczyński [1977, p. 1221] proved the existence of a sequence of n-dimensional spaces E_n whose unconditional basis constants $\mathrm{unbc}(E_n)$ behave like $\sqrt{n}$; see 6.1.5.2. Later on, Szarek [1983, p. 154] obtained the same result for the "ordinary" basis constants $\mathrm{bc}(E_n)$; see 6.1.1.11.

7.3.1.6 For $1 < p < 2$, the existence of almost isometric embeddings of l_p^n into $l_1^{[cn]}$ was established by Johnson/Schechtman [1982, p. 78]. In the case $2 < p < \infty$, Bennett/Dor/Goodman/Johnson/Newman [1977, p. 178] found uniformly isomorphic copies of l_2^n in $l_p^{[n^{p/2}]}$; see 6.1.2.5. A striking improvement of the latter result became the basic tool in Bourgain's solution [1989] of the $\Lambda(p)$-problem.

7.3.1.7 As described in 4.11.3, stochastic as well as deterministic methods can be used to show the existence of complex Banach spaces that are not isomorphic to their complex conjugates.

7.3.1.8 The most spectacular application of probabilistic techniques is due to Gluskin [1981b], who showed that the diameter of the n^{th} Minkowski compactum grows asymptotically like n; see 6.1.1.5. In order to prove this result he considered the collection of all spaces $\big(\mathbb{R}^n, \|\cdot\|\big)$ whose closed unit balls are the absolutely convex hulls of the unit vectors $e_1, \ldots, e_n$ and further vectors $x_1, \ldots, x_{2n}$ randomly chosen on the Euclidean unit sphere $S_{l_2^n}$. The underlying probability is just the $2n$-fold product of the normalized rotation invariant measure on $S_{l_2^n}$.

7.3.1.9 Bennett/Goodman/Newman [1975] were the first to use probabilistic techniques in operator theory. These authors considered operators $A : l_2^n \to l_q^m$ for which the representing matrix has entries ± 1. As a striking application they characterized the (p,q)-summing operators on Hilbert spaces in the case $2 < q < p < \infty$; see 6.3.8.1.

7.3.1.10 Besides random matrices with entries ± 1, one also uses matrices (g_{hk}), where the g_{hk}'s are independent Gaussian random variables. Another example is given by the group of unitary matrices equipped with the Haar measure.

7.3.1.11 Further information concerning probabilistic methods can be found in the surveys of Mityagin [1977], Gluskin [1986], and Mankiewicz/Tomczak-Jaegermann [2003]. Readers who are interested in *random factorizations of operators* may consult the formative paper of Benyamini/Gordon [1981].

7.3.2 Combinatorial methods

7.3.2.1 Given any set M, we denote by $\mathscr{P}_r(M)$ the collection of all subsets X that have exactly r elements. If χ is a map from $\mathscr{P}_r(M)$ onto $\{1,\ldots,s\}$, then $\chi(X)$ may be viewed as the color of X. In this sense, χ is said to be an s-**coloring** of $\mathscr{P}_r(M)$.

We refer to a subcollection $\mathscr{C}$ of $\mathscr{P}_r(M)$ as **monochromatic** if all $X \in \mathscr{C}$ have the same color.

"Finite" **Ramsey theorem** [1929, p. 267]:

If the finite set M is sufficiently large, then for every s-coloring of $\mathscr{P}_r(M)$ we can find a subset M_0 of prescribed size $\text{card}(M_0) = n$ such that $\mathscr{P}_r(M_0)$ is monochromatic. More precisely, $\text{card}(M)$ must exceed the Ramsey number $N(n,r,s)$, which turns out to be *huge*.

Ramsey's seminal paper [1929, p. 264] also contains a result about colorings of $\mathscr{P}_r(M)$ when the underlying set M is infinite.

7.3.2.2 The simplest case of Ramsey's theorem $(n = 2,\ r = 1,\ s = 1,2,\ldots)$ is the **pigeonhole principle** (*Dirichlet's Schubfachprinzip*):

Suppose that $\mathscr{P}_1(1,\ldots,N) = \{1,\ldots,N\}$ is s-colored. If $N > s$, then there exist at least two numbers with the same color.

7.3.2.3 Next, we consider s-colorings of $\mathscr{P}_\infty(\mathbb{N})$, the collection of all infinite subsets of $\mathbb{N}$.

Using the axiom of choice, Erdős/Rado [1952, p. 434] found a 2-coloring such that $\mathscr{P}_\infty(\mathbb{M})$ fails to be monochromatic for all infinite subsets $\mathbb{M}$ of $\mathbb{N}$. Subsequently, positive results were obtained for large classes of colorings defined in topological terms.

The classical topology of $\mathscr{P}_\infty(\mathbb{N})$ is generated by using the intervals

$$\{\mathbb{X} \in \mathscr{P}_\infty(\mathbb{N}) : \mathbb{A} \subseteq \mathbb{X} \subseteq \complement\mathbb{B}\}$$

as a base of open sets. Here $\mathbb{A}$ and $\mathbb{B}$ range over all finite subsets of $\mathbb{N}$. Assume that the monochromatic parts

$$\mathscr{C}_i := \{\mathbb{X} \in \mathscr{P}_\infty(\mathbb{N}) : \chi(\mathbb{X}) = i\} \quad \text{with } i = 1, \dots, s$$

are Borel sets. Then there exists an infinite set $\mathbb{M}_0$ of natural numbers such that $\mathscr{P}_\infty(\mathbb{M}_0)$ is monochromatic; see Galvin/Prikry [1973, p. 195].

This result was generalized and simplified by Ellentuck [1974], who equipped $\mathscr{P}_\infty(\mathbb{N})$ with a stronger topology. He obtained the conclusion above under the weaker assumption that the $\mathscr{C}_i$'s have the **Baire property**:
We can find open sets $\mathscr{G}_i$ for which the symmetric differences $\mathscr{C}_i \triangle \mathscr{G}_i$ are meager.

An example of Galvin/Prikry [1973, p. 197] shows that the preceding condition does not suffice in the case of the classical topology.

7.3.2.4 The first use of Ramsey's theorem in Banach space theory was made by Brunel/Sucheston [1974, p. 295] in their construction of spreading models; see 6.1.3.20. An application in summability theory is presented at the very end of paragraph 6.9.11.8. Most important was Farahat's observation [1974] that Rosenthal's l_1-theorem 5.6.3.8 can be proved by an infinite Ramsey argument.

According to Odell [1980, p. 380],

the general situation in which the Ramsey theorem is usually applied is as follows. A sequence (x_n) in some Banach space is given and a property $\mathbb{P}$ is under consideration. It is desired to pass to a subsequence (x'_n) of (x_n) so that (x'_n) and all its further subsequences have property $\mathbb{P}$.

7.3.2.5 In the case of 2-colorings, Ramsey's theorem yields dichotomies. For example, Rosenthal's l_1-theorem asserts that

every bounded sequence in a Banach space has a subsequence that is either weakly Cauchy or equivalent to the canonical basis of l_1.

Roughly speaking, every infinite-dimensional Banach space X contains an infinite-dimensional subspace X_0 that is either "good" or "bad." In Gowers's dichotomy 7.4.5.8

good means that X_0 has an unconditional basis

and

bad means that X_0 is hereditarily indecomposable.

7.3.2.6 While the "infinite" Ramsey theorem became a powerful tool for dealing with infinite-dimensional Banach spaces, its "finite" version had almost no applications to the local theory. Strangely enough, only one case is known to me: Wenzel [1997b] gave a new proof of a criterion for superreflexivity; see also [PIE$^+$, pp. 392–397].

To illustrate the situation, I present a result whose "finite" Ramsey character is obvious. Unfortunately, no purely finite-dimensional proof has been published so far.

We know from 6.1.3.19 that every spreading model contains an unconditional basic sequence. A finite-dimensional version of this result is due to Maurey/Pisier [1976, p. 57]; see also [MIL$^+$, p. 76]:
Fix $\delta, \varepsilon > 0$ and $n = 1, 2, \ldots$. Let

$$\|x_1\| \le 1, \ldots, \|x_N\| \le 1 \quad \text{and} \quad \|x_h - x_k\| \ge \delta \quad \text{whenever } h \ne k.$$

If N is sufficiently large, then there exist $k_1 < \cdots < k_{2n}$ such that

$$\Big\| \sum_{i=1}^{n} \eta_i (x_{k_{2i}} - x_{k_{2i-1}}) \Big\| \le (2+\varepsilon) \Big\| \sum_{i=1}^{n} \xi_i (x_{k_{2i}} - x_{k_{2i-1}}) \Big\| \quad \text{whenever } |\eta_i| \le |\xi_i|.$$

7.3.2.7 In [1980], Odell published a survey on *Applications of Ramsey theorems to Banach space theory*. The present state of the art is described in Gowers's handbook article [2003]. The reader may also consult [DIE$_2$, pp. 192–211], [GUER, pp. 43–51], [MIL$^+$, Chap. 11], and the recent book [ARG$^+$]. The standard reference on finite Ramsey theory is [GRAH$^+$]; see also Nešetřil [1995].

7.4 Counterexamples (are the salt in the soup)

The term *"counterexample"* has different meanings.

Mathematical objects that do not enjoy a "useful" property are called counterexamples (in a broad sense). For instance, the continuous function $t \mapsto |t|$ fails to be differentiable at 0, and the Dirichlet function is not Riemann integrable on $[0,1]$.

Counterexamples (in a narrow sense) have a special quality: they contradict our intuition or, at least, our hopes. Such counterexamples are mostly obtained by a sophisticated construction and look, therefore, quite pathological. This explains why Hermite [HERM$^\bullet$, Vol. II, p. 318] (original version on p. 681),
turns away with fear and horror from this lamentable plague of continuous functions that do not have a derivative; see Weierstrass [1872].

Bourbaki [BOU$^\bullet$, p. 15] (original version on p. 682), states that
with the research of Riemann himself on integration, and more so with the examples of curves without tangents, constructed by Bolzano and Weierstrass, it is the whole of the pathology of mathematics that was beginning. For a century we have seen so many monsters of this species that we are a bit blasé, and the most weird teratological characters must be accumulated in order still to astound us.

According to Gohberg [1999$^\bullet$, p. 2], Mikhlin expressed his opinion on a geometric counterexample of Mazya as follows:
Your domains are very interesting, but no mother would let her child play in such ravines.

The next quotation is from Diestel [DIE_2, p. 215]:
As any mature mathematician realizes, pathology only highlights the natural limits of a strong and healthy subject. Nonetheless, the onslaught of counterexamples experienced in the early seventies seems to have left an impression that little could be salvaged in the general theory [add: of Banach spaces].

Let me conclude these preliminary comments with the optimistic remark that in some cases counterexamples paved the way for solving problems that have been open since the time of Banach. Indeed, the fact that *all* non-Hilbertian Banach spaces have a certain negative property can be turned in a positive criterion that characterizes Hilbertian spaces. Convincing instances are the complemented subspace problem and the homogeneous subspace problem; see 4.9.1.12 and 7.4.5.9.

Referring to the irregular behavior of particles under Brownian motion, the French physicist Perrin wrote in his book *Les atomes* (1912) (original version on p. 682):
This is a case in which it is indeed natural to think of the continuous functions without derivatives that were invented by the mathematicians and that were wrongfully viewed just as mathematical curiosities, though they can be motivated by experience.

In the following, all subspaces are supposed to be *closed*.

7.4.1 A selection of typical counterexamples

7.4.1.1 In [1873, p. 578], du Bois-Reymond described a 2π-periodic continuous function f whose Fourier series does not converge to $f(t_0)$ at some point t_0. Hence $(\mathrm{e}^{\mathrm{i}kt})_{k\in\mathbb{Z}}$ is not a basis of $C(\mathbb{T})$; see also 5.6.4.4.

Certainly, there are many other classical pathologies that could be translated into the language of Banach spaces. Nevertheless, in my opinion the first genuine counterexample of Banach space theory is due to Helly, who showed that the space of convergent sequences is non-reflexive; see 2.2.4.

7.4.1.2 Even in Hilbert space, the sum $M+N$ of infinite-dimensional subspaces M and N need not be closed; see [STONE, pp. 21–22].

7.4.1.3 The existence of non-complemented subspaces was discussed in 4.9.1.10. Among the main contributors were Banach/Mazur [1933], Murray [1937], and Sobczyk [1941b].

Finally, it turned out that a Banach space in which all subspaces are complemented must be Hilbertian.

7.4.1.4 One of the most famous counterexamples in Banach spaces theory is the **James space** X_{Jam}, which consists of all null sequences $x=(\xi_k)$ such that

$$\|x\| := \sup\Big(|\xi_{k_1}-\xi_{k_2}|^2+|\xi_{k_2}-\xi_{k_3}|^2+\cdots+|\xi_{k_{n-1}}-\xi_{k_n}|^2+|\xi_{k_n}-\xi_{k_1}|^2\Big)^{1/2}$$

is finite; the supremum ranges over all choices of indices $k_1<k_2<\cdots<k_{n-1}<k_n$ and $n=2,3,\dots$; see James [1951] and [LIND_1^+, p. 25].

The spaces X_{Jam} and X_{Jam}^{**} are isometric, but not by means of the canonical embedding; indeed the James space has codimension 1 in its bidual.

Following Civin/Yood [1957], we call a Banach space **quasi-reflexive** if X^{**}/X is finite-dimensional. The James space is the most prominent example.

7.4.1.5 As mentioned in 5.6.3.9, there exist separable Banach spaces with non-separable duals that do not contain any isomorphic copy of l_1.

The first example was the **James tree space** $X_{\mathrm{Jam}}^{\mathrm{tree}}$; see James [1974a] and Lindenstrauss/Stegall [1975]. As indicated by its name, the construction of $X_{\mathrm{Jam}}^{\mathrm{tree}}$ is similar to that of the original James space: the members are scalar families indexed by the elements of a dyadic tree; see 6.1.9.2.

Herman/Whitley [1967] referred to a Banach space as **somewhat reflexive** if every infinite-dimensional subspace contains an infinite-dimensional reflexive subspace. Every quasi-reflexive space is somewhat-reflexive. James showed that $X_{\mathrm{Jam}}^{\mathrm{tree}}$ even has a stronger property: every infinite-dimensional subspace contains an isomorphic copy of l_2.

A **James function space** was studied by Lindenstrauss/Stegall [1975]; see also [LIND$_1^+$, p. 103].

There is another space, discovered by Hagler [1977], such that every infinite-dimensional subspace contains an isomorphic copy of c_0 and whose dual has the Schur property.

7.4.1.6 A sophisticated construction of James [1974b, p. 150] yields a non-reflexive but B-convex Banach space. Introducing an additional parameter, Davis/Lindenstrauss [1976, pp. 188–193] obtained a non-reflexive space that has a prescribed Rademacher type strictly less than 2. Finally, James [1978, pp. 3, 8] showed that one may even achieve type 2.

A more satisfactory example was discovered by Pisier/Xu [1987, p. 189]; see 6.6.5.6. They proved that for $2 < q < \infty$ and $\theta := 1/q^*$, the non-reflexive interpolation space $\big(v_1, l_\infty\big)_{\theta,2}$ has Rademacher type 2 and cotype q.

7.4.1.7 The classical infinite-dimensional Banach spaces are isomorphic to their squares. Hence the question arose whether this is always the case; see [BAN, p. 244]:

As observed by Bessaga/Pełczyński [1960b], the James space yields a counterexample since it has codimension 1 in its bidual. Semadeni [1960] found a "natural" counterexample: $C(\omega_1 + 1)$, the space of continuous functions on the ordinals less than or equal to the first uncountable ordinal; see 4.6.13 and 4.9.3.5.

Finally, Figiel [1972a] showed that for $1 < p < \infty$, the reflexive space $\big[l_p, l_{p_k}^{n_k}\big]$ fails to be isomorphic to its Cartesian square if the n_k's and p_k's are suitably chosen.

7.4.1.8 The Kalton–Peck spaces Z_p were constructed as counterexamples to various three space problems; see Subsection 6.9.6. However, it is conjectured that they have

some other exotic properties. In their survey [2003, p. 1158], Kalton/Montgomery-Smith ask:
Is Z_2 prime? [see 7.4.5.5] *Is Z_2 isomorphic to its hyperplanes*? [see 7.4.5.3].

7.4.1.9 As mentioned in 4.11.3, Bourgain [1986b] and Kalton [1995] found examples of complex Banach spaces that are not isomorphic to their complex conjugates.

7.4.1.10 The Grothendieck theorem 6.3.9.1 says that $\mathfrak{L}(l_1,l_2)=\mathfrak{P}_1(l_1,l_2)$. Lindenstrauss/Pełczyński [1968, p. 287] observed that $\mathfrak{L}(l_1,Y)=\mathfrak{P}_1(l_1,Y)$ holds only when Y is Hilbertian. Moreover, they asked on p. 317 whether $\mathfrak{L}(X,l_2)=\mathfrak{P}_1(X,l_2)$ implies that X must be an $\mathscr{L}_1$-space. This is not true. Kislyakov [1976, стр. 192] and Pisier [1978b, p. 80] showed independently that the class of **spaces satisfying Grothendieck's theorem** is much larger. More precisely,

$$\mathsf{GTh} := \{X : \mathfrak{L}(X,l_2)=\mathfrak{P}_1(X,l_2)\}$$

contains $L_1(M,\mathscr{M},\mu)/N$ whenever N is reflexive. If N is infinite-dimensional, then these quotients fail to be $\mathscr{L}_1$-spaces. In a next step, Bourgain [1984a, pp. 5, 21] added the deep fact that $L_1(\mathbb{T})/H_1(\mathbb{T})\in\mathsf{GTh}$.

For further information, the reader is referred to [PIS$_1$, Chap. 6].

7.4.2 Spaces without the approximation property

7.4.2.1 The story of Enflo's counterexample to the approximation property is told at full length in 5.7.4.15. Details of the underlying construction can be found in [LIND$_1^+$, pp. 87–90] and [PIE$_3$, pp. 135–141].

The fact that $\mathfrak{L}(l_2)$ does not have the approximation property contradicts the idea that counterexamples are always horrible. In this specific case, the space is nice but the proof turns out to be extremely complicated; see Szankowski [1981] and also Pisier [1979a].

7.4.2.2 According to Figiel/Johnson [1973], *the approximation property does not imply the bounded approximation property.* Their proof is the following:

In view of Enflo's counterexample, a result of Lindenstrauss [1971, p. 283] says that there exists a Banach space X with a Schauder basis whose dual does not have the approximation property. Using Szankowski's result from [1981], nowadays we may take $X=\mathfrak{N}(l_2)$. Hence any isomorphic copy of X possesses the bounded approximation property. However, Figiel/Johnson [1973, p. 199] observed that one can find isomorphic copies X_n for which the bounds tend to infinity. Then $[l_2,X_n]$ is the space we are looking for.

7.4.2.3 Johnson [1980] discovered a *non-Hilbertian Banach space all of whose subspaces have the approximation property*.

7.4.2.4 Gluing together an increasing sequence of finite-dimensional spaces, Pisier [1983, p. 197] constructed infinite-dimensional Banach spaces X_{Pis} such that

$$X_{\mathrm{Pis}}\widetilde{\otimes}_\pi X_{\mathrm{Pis}} = X_{\mathrm{Pis}}\widetilde{\otimes}_\varepsilon X_{\mathrm{Pis}} \quad \text{and} \quad \overline{\mathfrak{F}}(X_{\mathrm{Pis}}) = \mathfrak{N}(X_{\mathrm{Pis}}).$$

The left-hand formula disproves a conjecture of Grothendieck [1956b, p. 74].

The **Pisier space** has another remarkable property; see Pisier [1983, p. 201]:
There is a constant $c>0$ such that

$$\|P\| \geq c\sqrt{\operatorname{rank}(P)} \quad \text{for every finite rank projection } P \text{ on } X_{\mathrm{Pis}}.$$

This means that *all* finite-dimensional subspaces are "badly" complemented. So X_{Pis} is "very far" from being Hilbertian. On the other hand, X_{Pis} and its dual have Rademacher cotype 2, which, by Kwapień's theorem, indicates that X_{Pis} is "almost" Hilbertian. This observation reflects the fact that the gap between the two properties

X has Rademacher type 2 and *X^* has Rademacher cotype 2*

is quite large.

For a presentation of the results above the reader is referred to [PIS$_1$, Chap. 10].

7.4.3 Tsirelson's space

This subsection deals with spaces that do not contain isomorphic copies of any l_p with $1 \leq p < \infty$ or c_0; see 5.6.3.11. The standard reference is [CASA$^+$]. The following quotation is taken from the preface of this treatise:
His [read: Tsirelson's] *example opened a Pandora's box of pathological variations, and has had a tremendous effect upon the study of Banach spaces. The original construction has been "dualized, treed, symmetrized, convexified, and modified."*

7.4.3.1 The most convenient approach to **Tsirelson spaces** is due to Figiel/Johnson [1974, pp. 180–181].

Let c_{00} denote the linear space of all scalar sequences $x=(\xi_k)$ having finite support. With every subset $\mathbb{F}$ of $\mathbb{N}$ we associate the projection

$$P_{\mathbb{F}} : (\xi_k) \mapsto \sum_{k\in\mathbb{F}} \xi_k e_k,$$

where e_k is the k^{th} unit sequence.

A collection of finite subsets $\mathbb{F}_1,\dots,\mathbb{F}_m$ of $\mathbb{N}$ is called **admissible** if

$$m \leq \min\mathbb{F}_1 \leq \max\mathbb{F}_1 < \min\mathbb{F}_2 \leq \cdots \leq \max\mathbb{F}_{m-1} < \min\mathbb{F}_m \leq \max\mathbb{F}_m.$$

A sequence of norms on c_{00} is obtained by putting $\|x\|_0 := \|x|c_0\|$ and

$$\|x\|_{n+1} := \max\Big\{\|x\|_n, \sup \tfrac{1}{2}\sum_{h=1}^{m} \|P_{\mathbb{F}_h}x\|_n\Big\},$$

where the supremum ranges over all admissible collections $\{\mathbb{F}_1,\dots,\mathbb{F}_m\}$ of arbitrary length $m=1,2,\dots$.

In view of $\|x|c_0\| = \|x\|_0 \leq \cdots \leq \|x\|_n \leq \cdots \leq \|x|l_1\|$, the expression

$$\|x|X_{\text{Tsir}}\| := \lim_{n\to\infty} \|x\|_n$$

is finite. Completing c_{00} with respect to this norm yields the space X_{Tsir}, which has the desired property, since it does not contain any subsymmetric basic sequence; see Figiel/Johnson [1974, p. 181].

Surprisingly, the definition of X_{Tsir} is relatively simple. Moreover, X_{Tsir} has many good properties; it is reflexive and (e_k) constitutes an unconditional basis.

In his original paper [1974], Tsirelson had constructed the dual space X^*_{Tsir}, which, however, turned out to be less handy.

7.4.3.2 The concept of admissibility goes back to Schreier [1930, p. 59], who considered sets $\{k_1,\dots,k_m\}$ such that $m+1 = k_1 < \cdots < k_m$. His purpose was to construct a weak null sequence in $C[0,1]$ that contains no subsequence whose arithmetic means converge in norm. In modern terminology, Schreier proved that $C[0,1]$ does not have the Banach–Saks property; see 6.9.11.8 and Banach/Saks [1930].

7.4.3.3 According to Casazza/Odell [1982, p. 61], the following definition was suggested by Rosenthal:

An infinite-dimensional Banach space is **minimal** *if it embeds isomorphically into each of its infinite-dimensional subspaces.*

Classical results of Banach and Pełczyński say that l_p with $1 \leq p < \infty$ and c_0 are minimal; see 5.6.3.18. Further examples were not known until the discovery of Casazza/Johnson/Tzafriri [1984, p. 95], who showed that X^*_{Tsir} is minimal as well.

On the other hand, Casazza/Odell [1982, p. 68] proved that X_{Tsir} contains no minimal subspace at all. Therefore minimal spaces cannot be used as elementary building blocks of a general Banach space.

7.4.3.4 According to Figiel/Johnson [1974, p. 187], the r-convexified space $X^{(r)}_{\text{Tsir}}$ with $1 \leq r < \infty$ consists of all sequences $x = (\xi_k)$ such that $\big(|\xi_k|^{r-1}\xi_k\big) \in X_{\text{Tsir}}$. Its norm

$$\big\|x\big|X^{(r)}_{\text{Tsir}}\big\| := \big\|\big(|\xi_k|^r\big)\big|X_{\text{Tsir}}\big\|^{1/r}$$

is uniquely determined by the functional equation

$$\|x|X^{(r)}_{\text{Tsir}}\| = \max\left\{\|x|X^{(r)}_{\text{Tsir}}\|, \sup\left(\frac{1}{2}\sum_{h=1}^{m}\|P_{\mathbb{F}_h}x|X^{(r)}_{\text{Tsir}}\|^r\right)^{1/r}\right\},$$

where the supremum ranges over all admissible collections $\{\mathbb{F}_1,\dots,\mathbb{F}_m\}$ of arbitrary length $m = 1,2,\dots$.

As in the case of $X_{\text{Tsir}} = X^{(1)}_{\text{Tsir}}$, the new spaces $X^{(r)}_{\text{Tsir}}$ with $1 < r < \infty$ do not contain isomorphic copies of any l_p with $1 \leq p < \infty$ or c_0. In addition, they are superreflexive.

7.4.3.5 The space $X_{\text{Tsir}}^{(2)}$ is of particular interest. It has Rademacher type 2 and weak cotype 2 but is not isomorphic to l_2; see [PIS$_2$, p. 205]. Hence there exist "genuine" weak Hilbert spaces (defined in 6.1.11.8).

Johnson [1980, p. 21] discovered another remarkable property; take into account a comment from [CASA$^+$, p. 117]:

Every subspace of every quotient of $X_{\text{Tsir}}^{(2)}$ has a Schauder basis.

It is still unknown whether there exist non-Hilbertian spaces such that every subspace has an unconditional basis.

7.4.4 The distortion problem

For further information the reader is referred to the surveys of Odell [2002] and Odell/Schlumprecht [2003].

7.4.4.1 Let X be an infinite-dimensional Banach space with a fixed norm $\|\cdot\|_{\text{old}}$. Given any equivalent norm $\|\cdot\|_{\text{new}}$ on X, the **distortion constant** is defined by

$$d_{\text{M}}\big(X,\|\cdot\|_{\text{old}},\|\cdot\|_{\text{new}}\big) := \inf_{X_0}\left[\sup\left\{\frac{\|x_1\|_{\text{new}}\,\|x_2\|_{\text{old}}}{\|x_1\|_{\text{old}}\;\|x_2\|_{\text{new}}} : x_1,x_2\in X_0\backslash\{\mathsf{o}\}\right\}\right],$$

where the infimum ranges over all infinite-dimensional subspaces X_0 of X.

An infinite-dimensional Banach space X is called **distortable** if there exists an equivalent norm such that $d_{\text{M}}\big(X,\|\cdot\|_{\text{old}},\|\cdot\|_{\text{new}}\big) > 1$. This is the negation of the following conjecture of Milman [1969, p. 142], [1971a, p. 145]:

Let $r(x)$ be an equivalent norm in a space B. There is a number $a>0$ such that for any $\varepsilon>0$, we can find a subspace $E\subset B$ $(\dim E=\infty)$ on which $r(x)$ is ε-isometric to the original norm, and the ε-isometry is effected by multiplying the original norm by a.

Lindenstrauss/Pełczyński [1971, p. 244] replaced $d_{\text{M}}\big(X,\|\cdot\|_{\text{old}},\|\cdot\|_{\text{new}}\big)$ by

$$d_{\text{LP}}\big(X,\|\cdot\|_{\text{old}},\|\cdot\|_{\text{new}}\big) := \inf_{U}\left[\sup\left\{\frac{\|Ux_1\|_{\text{new}}\,\|x_2\|_{\text{old}}}{\|x_1\|_{\text{old}}\;\|Ux_2\|_{\text{new}}} : x_1,x_2\in X\backslash\{\mathsf{o}\}\right\}\right],$$

the infimum being taken over all isomorphisms U from X (equipped with the old norm) onto some subspace X_0 (equipped with the new norm).

The latter version has led to the **distortion problem** for l_p as formulated in [LIND$_1^+$, p. 97]:

Let X be the space l_p for some $1<p<\infty$, with the usual norm $\|\cdot\|$. Let $|||\cdot|||$ be an equivalent norm on X. Given $\varepsilon>0$, does there exist a subspace Y of X so that $d\big((Y,|||\cdot|||),(X,\|\cdot\|)\big)<1+\varepsilon$?

To the best of my knowledge, the relation between d_{M} and d_{LP} is still unclear for general Banach spaces. Thus we have, in fact, two concepts of "distortion," one in the sense of Milman and the other one in the sense of Lindenstrauss/Pełczyński.

7.4.4.2 For the spaces l_1 and c_0, both versions of the distortion problem have a positive answer; see James [1964c, pp. 547–549] and Milman [1971a, pp. 148–149]. More precisely, James proved the following lemma:

If a normed linear space contains a subspace isomorphic with $l^{(1)}$, then, for any positive number δ, there is a sequence $\{u_i\}$ of members of the unit ball such that

$$(1-\delta)\cdot\sum|a_i| < \Big\|\sum a_i u_i\Big\| \leq \sum|a_i|$$

for all sequences of numbers $\{a_i\}$ that are not all zero.

An analogous lemma holds for c_0. The two "auxiliary results" became the starting points of all further research about distortion.

7.4.4.3 Under the hypothesis that all Banach spaces are non-distortable, Milman [1969, pp. 142, 145], [1971a, pp. 145–147] observed that every Banach space contains an isomorphic copy of some l_p with $1 \leq p < \infty$ or c_0; see also Odell/Schlumprecht [2003, p. 1339]. Since this conclusion fails for the Tsirelson space, Milman's optimistic reasoning had a pessimistic outcome: the existence of distortable spaces.

7.4.4.4 The breakthrough was achieved by Schlumprecht [1991], who constructed a Tsirelson-like Banach space that is arbitrarily distortable. In other words, by a suitable choice of $\|\cdot\|_{\text{new}}$, the distortion constant can be made as large as we please. Later on, Odell/Schlumprecht [1994] were able to show that l_p with $1 < p < \infty$ is arbitrarily distortable. These results are formulated using Milman's concept of distortion. However, according to Odell/Schlumprecht (private communication, Oberwolfach 2003), their method also yields a negative answer to the distortion problem for l_p in the sense of Lindenstrauss/Pełczyński. This observation follows "easily" from the $(1+\varepsilon)$-version of the Bessaga–Pełczyński selection principle 5.6.3.19. Nevertheless, a proof deserves to be published, at least for the convenience of non-experts.

7.4.4.5 At present, it is unknown whether there exists a distortable space X whose distortion constants are bounded:

$$d_{\mathrm{M}}\big(X, \|\cdot\|_{\text{old}}, \|\cdot\|_{\text{new}}\big) \leq c \quad \text{for any equivalent norm } \|\cdot\|_{\text{new}}.$$

Tomczak-Jaegermann [1996b, p. 1076] proved that a Banach space with this property must contain an unconditional basic sequence. As a corollary, she observed that every infinite-dimensional B-convex space X contains an arbitrarily distortable subspace. Under the additional assumption that X has an unconditional basis, this result had earlier been proved by Maurey [1995, p. 136].

7.4.5 The Gowers–Maurey story

To begin with, I quote a remark of Maurey [2003a, p. 1251] about his joint work with Gowers:
The real start was the modification of Tsirelson's space constructed by Schlumprecht [1991].
...
During the summer 1991, after hearing Schlumprecht at the Banach space conference in Jerusalem, we both constructed an example of a space with no unconditional [basic] *sequence.*

In their first paper Gowers/Maurey [1993, p. 852] wrote:
On comparing our examples, we discovered that they were almost identical as were the proofs, so we decided to publish jointly and work together on further properties of the space.

In his handbook article [2003a, p. 1249], Maurey reached the conclusion that
there is no hope for a structure theory of Banach spaces.
...
Despite my rather pessimistic comments above, the results and examples obtained since 1990 represent a significant progress of our understanding of infinite-dimensional Banach spaces.

7.4.5.1 In [1970a, pp. 166–167], Lindenstrauss raised the question

Does every infinite-dimensional Banach space X have a nontrivial representation as a direct sum?

$$X = X_1 \oplus X_2 \quad \text{such that } \dim(X_1) = \infty \text{ and } \dim(X_2) = \infty.$$

Spaces with this property may be called **decomposable**.

According to Gowers/Maurey [1993, p. 852], Johnson observed that

our space is not only not decomposable but does not even have a decomposable subspace. That is, X is **hereditarily indecomposable**.

The famous **Gowers–Maurey space** X_{GM} was defined in [1993, pp. 862–863]. Among others it provides a negative solution of the **unconditional basic sequence problem**; see 5.6.3.23.

Note that X_{GM} has a shrinking and boundedly complete basis; see Gowers/Maurey [1993, p. 869]. Therefore it is reflexive. Ferenczi [1997] constructed a hereditarily indecomposable space that is even superreflexive.

7.4.5.2 On every infinite-dimensional Banach space X there exist non-trivial nuclear operators

$$T = \sum_{k=1}^{\infty} \tau_k x_k^* \otimes x_k \quad \text{with } (\tau_k) \in l_1.$$

Here (x_k^*) and (x_k) are biorthogonal sequences such that $\|x_k^*\|\|x_k\| \le 2$. Scalar multiples of the identity map are another kind of operator on X. In view of these facts, Pisier [PIS_1, p. 147] asked whether there is an infinite-dimensional complex Banach space X such that every operator can be written as $\lambda I + T$ with $\lambda \in \mathbb{C}$ and $T \in \mathfrak{N}(X)$. This problem is still open, even when we replace $\mathfrak{N}(X)$ by the larger ideal $\mathfrak{K}(X)$. However, every hereditarily indecomposable complex Banach space has the desired property with respect to the ideal of strictly singular operators; see Gowers/Maurey [1993, p. 871]. Therefore operators in such spaces are either Fredholm with index zero or strictly singular. The latter result holds in the real case as well; see Maurey [2003a, pp. 1264–1265].

The preceding considerations are perverse, since concrete Banach spaces such as L_p were originally invented in order to solve operator equations. In other words, one was interested in *spaces with many operators*. Now the opposite situation is required: **spaces with few operators**. Nevertheless, this headstand is fascinating!

7.4.5.3 The **hyperplane problem** asked whether every 1-codimensional subspace (closed hyperplane through zero) of any infinite-dimensional Banach space is isomorphic to the whole space; see [BAN, p. 246].

First of all, one has to observe that all 1-codimensional subspaces are isomorphic to each other. Indeed, given different N_1 and N_2, we may choose $x_1 \in N_1 \setminus N_2$ and $x_2 \in N_2 \setminus N_1$. Then $X = \text{span}\{x_1, x_2\} \oplus (N_1 \cap N_2)$. Therefore an isomorphism can be defined by interchanging x_1 and x_2 whereas the elements in $N_1 \cap N_2$ remain fixed.

Clearly, the hyperplane problem has an affirmative answer for l_p. The same is true for all “common” infinite-dimensional Banach spaces. However, Gowers/Maurey [1993, pp. 871–872] discovered counterexamples that show even a stronger pathology. They observed that every hereditarily indecomposable space X is not isomorphic to any of its proper subspaces M.

Suppose that an isomorphism $U : X \to M$ exists, and let $J : M \to X$ denote the natural embedding. Then JU is a semi-Fredholm operator with $\text{ind}(JU) < 0$. Contradiction!

Though published later, the original solution of the hyperplane problem is due to Gowers [1994a]. In contrast to the Gowers–Maurey space, he constructed a counterexample *with* an unconditional basis.

7.4.5.4 Let us recall the **Äquivalenzsatz** (equinumerosity theorem) of set theory, [HAUS_1, p. 48]:

Zwei Mengen, von denen jede einer Teilmenge der anderen äquivalent ist, sind selbst äquivalent.

In 1896, Schröder gave a proof that later turned out to be false; see Korselt [1911]. A correct proof was presented by (Felix) Bernstein in Cantor’s seminar, Halle 1897; see [BOR_1, pp. 104–107] for the first printed version. In fact an earlier proof, dated from 1887, was discovered in Dedekind’s Nachlass. Nevertheless, the result above is quite often quoted as the **Schroeder–Bernstein theorem**. A short historical account was given by Frewer [1981•, pp. 85–86].

One may ask whether an analogous conclusion holds in Banach spaces theory:

Let X and Y be Banach spaces such that X is isomorphic to a subspace of Y and such that Y is isomorphic to a subspace of X. Are X and Y isomorphic?

This is not the case, since Banach/Mazur [1933, p. 106] found a simple counterexample: $X = C[0,1]$ and $Y = C[0,1] \oplus l_1$.

Motivated by the decomposition method, the **Schroeder–Bernstein problem** has been modified as follows; see 4.9.3.4:

Let X and Y be Banach spaces such that X is isomorphic to a *complemented* subspace of Y and such that Y is isomorphic to a *complemented* subspace of X. Are X and Y isomorphic?

Unfortunately, even under these stronger assumptions the answer is NO. Indeed, Gowers [1996a, p. 303] constructed a Banach space X that is isomorphic to its cube but not to its square. Then X is complemented in $X \oplus X$ and $X \oplus X$ is complemented in $X \oplus X \oplus X \cong X$.

The state of the art in the pre-Gowers era was described by Casazza [1989].

7.4.5.5 In a second joint paper [1997], Gowers/Maurey generalized their original construction in order to get spaces with further pathological properties.

According to Lindenstrauss [1970a, p. 164], an infinite-dimensional Banach space is called **prime** if every complemented infinite-dimensional subspace is isomorphic to the whole space. As indicated by the name, it was hoped that the prime spaces could serve as building blocks of general Banach spaces.

In [1960, p. 213], Pełczyński had already proved that l_p with $1 \le p < \infty$ and c_0 are prime, and Lindenstrauss [1967, p. 153] obtained the same result for l_∞. For a long time, no other examples were known. Finally, in the mid 1990s, Gowers/Maurey [1997, p. 556] designed a space X in which all complemented subspaces have either finite dimension or finite codimension. In the latter case, they are isomorphic to X.

7.4.5.6 After the discovery of the Tsirelson space, Rosenthal [1978b, p. 805] modified the original conjecture by asking:

Does every infinite dimensional Banach space contain a subspace which is isomorphic to c_0, isomorphic to l_1, or reflexive and infinite dimensional?

In fact, this problem had already been formulated by James [1974a, p. 738].

A classical result of James 5.6.3.10 implies that a Banach space has the required property if it contains an unconditional basic sequence. Unfortunately, in view of the Gowers–Maurey construction, this assumption need not be fulfilled in general. Hence it was no big surprise when Gowers [1994b] showed that Rosenthal's weakening fails as well.

7.4.5.7 It may happen that spreading models have better properties than the original space; see 6.1.3.19. However, this is not always so. Answering a question of Odell [1980, p. 389], Odell/Schlumprecht [1995, p. 182] constructed a Banach space such that none of its spreading models contains isomorphic copies of l_p with $1 \le p < \infty$ or c_0. Moreover, Androulakis/Odell/Schlumprecht/Tomczak-Jaegermann [2005, p. 678] found a reflexive space with the following property: all spreading models are non-reflexive and non-isomorphic to l_1 and c_0. On the other hand, since every spreading model has an unconditional basic sequence, it must contain either an infinite-dimensional reflexive subspace, an isomorphic copy of l_1, or an isomorphic copy c_0.

7.4.5.8 Next, I present the highlights of this subsection, which show that counter-examples may lead to beautiful *positive* results.

Gowers dichotomy [1996b, p. 1084], [2002, p. 799]:
Every infinite-dimensional Banach space contains a subspace that

either has an unconditional basis	or is hereditarily indecomposable.

Komorowski/Tomczak-Jaegermann dichotomy [1995, p. 217]:
Every infinite-dimensional Banach space contains a subspace that

either is isomorphic to l_2	or has no unconditional basis.

Komorowski/Tomczak-Jaegermann made the additional assumption that the underlying Banach space X is of finite cotype. In the omitted case, X contains the l_∞^n's uniformly. Thus, according to Figiel [1973, pp. 205–209] and Szankowski [1978, p. 129], there exists a subspace that does not even have the approximation property; see also [LIND$_2^+$, p. 111].

7.4.5.9 The preceding dichotomies immediately yield the affirmative solution of the **homogeneous subspace problem**, which had been open since the time of Banach [BAN, pp. 244–245]:

If a Banach space is isomorphic to all of its infinite-dimensional subspaces, then it is isomorphic to l_2.

Otherwise, by the Komorowski/Tomczak-Jaegermann dichotomy, there exists a subspace without an unconditional basis. Hence, by homogeneity, all infinite-dimensional subspaces fail to have an unconditional basis. Now the Gowers dichotomy tells us that we can find a hereditarily indecomposable subspace. Again by homogeneity, the whole space must be hereditarily indecomposable. Therefore it is not isomorphic to any of its proper subspaces; see 7.4.5.3. Contradiction!

For further information, the reader is referred to a survey of Tomczak-Jaegermann [1996a].

7.4.5.10 Very recently, Koszmider [2004] and Plebanek [2004] constructed compact Hausdorff spaces K such that every operator on $C(K)$ has the form $f \mapsto \varphi f + Tf$ with a continuous function φ and a weakly compact operator T. Hence $C(K)$ is a space with few operators, where "few" is meant in a wider sense. The reader should note that an operator from $C(K)$ into any Banach space is weakly compact precisely when it is strictly singular; see Pełczyński [1965, Part I, p. 35].

The property above implies that those $C(K)$'s are "classical" counterexamples to the hyperplane problem. Philosophically speaking, the pathology is shifted from functional analysis to general topology.

7.4.6 A few counterexamples of operator theory

The most striking counterexample of operator theory concerns the invariant subspace problem, whose negative solution is due to Enflo and Read; see 5.2.4.5.

7.4.6.1 As explained at the end of paragraph 5.7.4.14, a counterexample to the approximation problem can be obtained from a nuclear operator T on l_1 that has two seemingly contradictory properties: $T^2 = O$ and $\text{trace}(T) = 1$.

7.4.6.2 The **summation operator**

$$\Sigma : (\xi_k) \mapsto \Big(\sum_{k=1}^{h} \xi_k \Big)$$

acts from l_1 into l_∞. We know from 6.3.16.4 that Σ fails to be weakly compact. On the other hand, Σ is (p,q)-summing whenever $p > q \geq 1$; see Kwapień/Pełczyński [1970, p. 59]. Hence, in contrast to $\mathfrak{P}_p \subset \mathfrak{W}$, we have $\mathfrak{P}_{p,q} \not\subseteq \mathfrak{W}$.

7.4.6.3 The **integration operator** is defined by the rule

$$S : f(t) \mapsto \int_0^s f(t)\,dt.$$

Acting from $L_1[0,1]$ into $C[0,1]$, it induces a compact Volterra operator on $C[0,1]$ and $L_p[0,1]$ with $1 \leq p \leq \infty$. Since S does not have any eigenvector, compact operators in infinite-dimensional Banach spaces need not possess a Jordan normal form.

7.4.6.4 In view of $l_1 \in \mathsf{RN}$, the summation operator $\Sigma : l_1 \to l_\infty$ enjoys the Radon–Nikodym property; see 6.3.18.1. Remarkably, this is not true for the integration operator $S : L_1[0,1] \to L_\infty[0,1]$, since it fails to be representable; see 5.1.3.2.

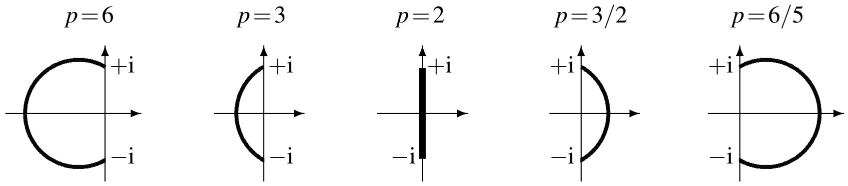
Obviously, Σ factors through S. That is, $\Sigma = BSA$. On the other hand, the preceding observations imply that a factorization $S = V\Sigma U$ is impossible.

7.4.6.5 Certain counterexamples from harmonic analysis can be used to produce strange convolution operators on $C(\mathbb{T})$ and $L_p(\mathbb{T})$; see 6.4.4.2 and 6.7.10.7.

7.4.6.6 Widom [1960, p. 152] proved that for $1<p<\infty$, the spectrum of the operator defined on $L_p(\mathbb{R}_+)$ by

$$H^+_{\text{ilb}} : f(t) \mapsto \frac{1}{\pi}\int_0^{+\infty} \frac{f(t)}{s-t}\,dt$$

is the circular arc in the complex plane with endpoints $\pm$i that passes through $\cot\frac{\pi}{p^*}$ at the real axis:



This natural example shows that the spectrum of an operator may depend on the underlying space. Further information can be found in a survey of Böttcher [1995] that deals with Toeplitz operators.

7.4.6.7 Finally, I stress that there also exist good-natured counterexamples. In order to compare operator ideals $\mathfrak{A}$ and $\mathfrak{B}$, it is quite often successful to look for a diagonal operator $D_t : l_p \to l_q$ such that $D_t \in \mathfrak{A}\backslash\mathfrak{B}$. This method was, for example, used by Carl [1976, p. 261] to show that $\mathfrak{I}_p \neq \mathfrak{P}_p$ if $p \neq 2$.

7.5 Banach spaces and axiomatic set theory

From time to time every mathematician, in particular when he is not a specialist in logic and set theory, gets a guilty conscience and feels just like Cohen [1967, p. 15]:

Mathematics may be likened as a Promethean labor, full of life, energy and great wonder, yet containing the seed of an overwhelming selfdoubt. It is good that only rarely do we pause to review the situation and to set down our thoughts on these deepest questions. To the rest of our mathematical lives we watch and perhaps partake in the glorious procession.

...

This is our fate, to live with doubts, to pursue a subject whose absoluteness we are not certain of, in short to realize that the only "true" science is itself of the same mortal, perhaps empirical, nature as all other human undertakings.

The following comments about set theory are written by an analyst for analysts. I am in the same situation as someone working in numerical mathematics who has decided to write about functional analysis.

7.5.1 Like every rigorous mathematical discipline, Banach space theory is based on set theory. In what follows, I will discuss this relationship in the setting of the **Zermelo–Fraenkel axioms**, (ZF) for short; see Zermelo [1908b].

It seems that the notation (ZF) ignores the significant contributions of Skolem to the creation of axiomatic set theory. Thus we have a similar situation as in the case of the Hahn–Banach theorem; see the concluding remark of 2.3.7.

The main goal is to elucidate the role of the **axiom of choice** (AC), which was stated for the first time when Zermelo [1904] proved the **well-ordering theorem** (WO). Subsequently, it turned out that many mathematicians had used similar processes without realizing the underlying selection principle. Nevertheless, several prominent colleagues did not like the *Auswahlaxiom*. In [1908a, § 2], Zermelo reacted to their criticism.

For more than 30 years, *transfinite induction* had become a main tool. Nowadays, we mostly prefer to use **Zorn's lemma** (ZL) in a form established by Bourbaki [1950b, p. 436]. *The origin of "Zorn's lemma"* was described by Campbell [1978•].

7.5.2 The undefined objects of the Zermelo–Fraenkel theory are **sets**, denoted by lowercase letters $a,b,c,\dots,x,y,z$, which can be compared by the relation $a \in x$; read: a is an element (member) of x. The collection of all sets is our mathematical world, the **universe**. Note, however, that the universe itself fails to be a set.

The set-theoretic axioms express the belief that there exist infinite sets, and they provide a catalogue of admissible manipulations that transform sets into sets.

7.5.3 Besides real or complex scalars, people working in Banach space theory use only a limited number of set-theoretic objects belonging to different levels, and the notation reflects this strategy:
In the basement, we deal with **elements** $x,y,\dots$.
Next, there follow the **spaces** $X,Y,\dots$, $X^*,Y^*,\dots$, $\mathfrak{L}(X,Y)$, and their subsets.
Filters, coverings, and σ-algebras are sets of subsets.

The preceding remark does not mean that we need only a small part of sets. On the contrary, the mapping $\mathbb{I} \mapsto l_1(\mathbb{I})$, where $\mathbb{I}$ is any (index) set, shows that the collection of all Banach spaces is, roughly speaking, as large as the universe of all sets.

7.5.4 In some branches of Banach space theory it is common to deal with **classes**. For example, one considers the collection of *all* reflexive spaces or the collection of *all* weakly compact operators. In this case, the Zermelo–Fraenkel axioms are insufficient. Therefore we should use the von Neumann–Bernays–Gödel approach. However, I prefer the naive point of view as proposed by Devlin [DEV, p. 62]:
Classes can be handled as sets except that the class is never a completed whole to be a possible member of anything else.

7.5.5 Every universe that satisfies the Zermelo–Fraenkel axioms is a **model** of set theory; and every complete normed linear space is a **model** of Banach space theory. Nevertheless, there seems to be a significant difference: while we (have the illusion that we) know many "concrete" Banach spaces such as $\mathbb{R}$, l_2, and $C[0,1]$, nobody has seen any "real" model of set theory. However, each mathematician is convinced that, somewhere between heaven and earth, his very personal model or even an absolute model is gliding.

Quite likely, the last sentence causes protest. As an analyst, I may work in reflexive Banach spaces on Monday and in spaces with the Schur property on Tuesday. With the same right it should be possible to change one's set-theoretic point of view. Using Solovay's model on Wednesday, I ensure that all subsets of $\mathbb{R}$ become Lebesgue measurable, but then the Hahn–Banach theorem is getting lost. Hence I may decide on Thursday that it is more comfortable to live in a world with the axiom of choice, for the rest of the week.

7.5.6 In dealing with axiomatic set theory, the main problem of an analyst is not to use tools that are, by self-limitation, unavailable in the present situation. This is like living with handcuffs or as a vegetarian, deliberately. Thanks to Gödel, we know:

If mathematics is correct at all, then it is also correct with the axiom of choice, the continuum hypothesis, and some more extras.

Hence there is no reason why we should prefer to live in a mathematical world without these useful tools; especially since our decision seems to be free of charge! The reality is different: politicians of all colors claim that they provide us with *the best of all worlds*; and we have to pay for it.

Let us stress the following analogy: when dealing with a given differential or integral equation, we need to choose the underlying Banach space, which should be as simple as possible. Thus, first of all, we try to use a Hilbert space. Further candidates are spaces of continuous or integrable functions, as well as Sobolev spaces. But nobody would have the idea to take a Tsirelson space. This is the pragmatic point of view.

Nevertheless, we construct Banach spaces without the approximation property and various other pathologies. Why??? We want to know the limits of our discipline; set-theorists and logicians want to do the same!

7.5.7 I now consider the simplest Banach space: $\mathbb{R}$.

The real numbers are obtained as the members of a Dedekind complete and totally ordered field, which is uniquely determined (up to isomorphisms) within every fixed model of set theory. However, switching to a different model changes the situation drastically. The most dramatic result is formulated by the

Löwenheim–Skolem theorem; see Skolem [1922, p. 220]:
Wenn eine Zählaussage überhaupt innerhalb irgend eines Denkbereiches erfüllt ist, dann ist sie schon innerhalb eines abzählbar unendlichen Denkbereiches erfüllt.

This means that *inside* this new surrounding, everything is as before. But–looking from *outside*–all sets have become countable; see also McCarty/Tennant [1987]. In particular, $\mathbb{R}^{\text{old}}$ and $\mathbb{R}^{\text{new}}$ are quite different.

We may also take into account that every $x \in \mathbb{R}$ is the supremum of a (non-empty and bounded above) subset of $\mathbb{Q}$, the rational field. Hence $\mathbb{R}$ depends on the decision how "big" the power set of $\mathbb{Q}$ is chosen.

In summary: we realize the frustrating fact that a “universal” $\mathbb{R}$ does not exist. Consequently, two mathematicians speaking about the real field will–most likely–speak about different objects. But this is not as dangerous as it looks at first glance. If, by assumption, both mathematicians are using the same axioms, the same language, and the same logical rules, then they will prove the same theorems; and this is what counts. Moreover, “important” numbers such as

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \quad \text{and} \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

are always available.

7.5.8 Enough of philosophy! Back to real life!

The following considerations are based on the Zermelo–Fraenkel axioms and the belief of their consistency. The use of additional axioms will be stated explicitly.

7.5.9 For some decades, the axiom of choice $\big(\mathsf{AC}\big)$ was used to produce theorems in various fields of mathematics, and the question arose whether this “dubious” tool is really needed. Therefore a new era began when Kelley [1950b] showed that this is so in the case of **Tychonoff’s theorem**:

$\big(\mathsf{TY}\big)$ Any topological product of compact spaces is compact.

We present a streamlined version of Kelley’s proof that $\big(\mathsf{TY}\big) \Rightarrow \big(\mathsf{AC}\big)$:
Let $(M_i)_{i\in\mathbb{I}}$ be any family of non-empty mutually disjoint sets. Then $M := \bigcup_{i\in\mathbb{I}} M_i$ is compact with respect to the (non-Hausdorff) topology whose closed sets are M and $\emptyset$ as well as all finite unions of M_i’s. Form the $\mathbb{I}^{\text{th}}$ power of M, and observe that $C_{\mathbb{F}} := \{(x_i) \in M^{\mathbb{I}} : x_i \in M_i \text{ if } i \in \mathbb{F}\}$ is closed and non-empty for every finite subset $\mathbb{F}$ of $\mathbb{I}$. By the compactness of $M^{\mathbb{I}}$, there exists a family (x_i) in the intersection of all $C_{\mathbb{F}}$’s, and $i \mapsto x_i$ is the required choice function.

For the theory of Banach spaces, Bell/Fremlin [1972, pp. 168–169] obtained a similar result concerning the original **Kreĭn–Milman theorem**:

$\big(\mathsf{KM}^*\big)$ The closed unit ball of a dual Banach space has an extreme point.

$\big(\mathsf{KM}^*\big) \Rightarrow \big(\mathsf{AC}\big)$: Again, let $(M_i)_{i\in\mathbb{I}}$ be any family of mutually disjoint sets. But now we consider the Banach space $[l_1(\mathbb{I}), c_0(M_i)]$. Then the closed unit ball of $[l_\infty(\mathbb{I}), l_1(M_i)] = [l_1(\mathbb{I}), c_0(M_i)]^*$ has an extreme point $x^* = (x_i^*)$. Next, it follows that every x_i^* is an extreme point of $B_{l_1(M_i)}$. However, extreme points of $B_{l_1(M_i)}$ are multiples of unit families and have, therefore, a one-element support in M_i. Hence $i \mapsto \operatorname{supp}(x_i)$ is the required choice function.

The new point of view in the consideration above is that $\big(\mathsf{TY}\big)$ and $\big(\mathsf{KM}^*\big)$ are no longer regarded as theorems but as axioms that may replace $\big(\mathsf{AC}\big)$. In the following the same will be done for various other theorems. As a result we get a certain hierarchy, and it makes sense to say that Alaoglu’s theorem is stronger than the Hahn–Banach theorem, while the Hahn–Banach theorem and Baire’s category theorem are independent of each other. A catalogue of hundreds of those relationships is provided in [HOW$^+$].

The main problem is the following. Given axioms (X) and (Y), one has to find a model of (ZF) in which (X) holds, while (Y) fails. The decisive tool is Cohen's forcing technique.

7.5.10 Let $\mathscr{R}$ be any commutative ring with unit, $\mathbf{1} \neq \mathsf{o}$. Ideals $\mathscr{I}$ in $\mathscr{R}$ are supposed to be proper: $\mathbf{1} \notin \mathscr{I}$.

Using (WO), Krull [1929, p. 732] proved the following result:

(MI) Every commutative ring with unit has a maximal ideal.

On the other hand, Hodges [1979, p. 285] showed that *Krull implies Zorn*. Hence $(\mathsf{AC}) \Leftrightarrow (\mathsf{MI})$.

As a corollary we get

(PI) Every commutative ring with unit has a prime ideal.

The implication $(\mathsf{MI}) \Rightarrow (\mathsf{PI})$ follows from the fact that maximal ideals are prime: Suppose that $\mathscr{I}$ is not prime. Fix elements $a, b \in \mathscr{R}$ such that $a, b \notin \mathscr{I}$ but $a \cdot b \in \mathscr{I}$. Then, in view of $b \in \mathscr{I}_0 := \{x \in \mathscr{R} : a \cdot x \in \mathscr{I}\}$, the ideal $\mathscr{I}_0$ is strictly larger than $\mathscr{I}$.

From now on we deal with a Boolean algebra $\mathscr{B}$, which means that $x^2 = x$ for all elements $x \in \mathscr{B}$; see 4.8.1.1. Then the concept of a prime ideal coincides with the concept of a maximal ideal. Indeed, if the prime ideal $\mathscr{I}$ were contained in a larger ideal $\mathscr{I}_0$, then we could find some $a \in \mathscr{I}_0 \setminus \mathscr{I}$. Hence $a \cdot (\mathbf{1} - a) = \mathsf{o} \in \mathscr{I}$ implies $\mathbf{1} - a \in \mathscr{I} \subset \mathscr{I}_0$, which in turn yields $\mathbf{1} \in \mathscr{I}_0$. Contradiction!

As explained in 4.8.1.1, we may look at Boolean algebras from the algebraic as well as from the lattice-theoretic point of view. In this sense, Stone's [1936, p. 100] **Boolean prime ideal theorem** is only a special case of Krull's theorem:

(BPI) Every Boolean algebra has a prime ideal.

Obviously, $(\mathsf{AC}) \Rightarrow (\mathsf{BPI})$. However, Halpern/Lévy [1967, p. 83] constructed a model in which (BPI) holds, but (AC) fails. The equivalence $(\mathsf{BPI}) \Leftrightarrow (\mathsf{PI})$ was announced in a *Preliminary report* of Scott [1954]. Only Rav [1977, pp. 156–158] and Banaschewski [1983, p. 194] provided belated proofs.

The map $x \mapsto x'$ defines a duality in every Boolean algebra $\mathscr{B}$. In this way, ideals pass into filters $\mathscr{I}' := \{x' : x \in \mathscr{I}\}$. It is common to speak of ultrafilters instead of maximal filters. In this terminology, (BPI) implies the **ultrafilter theorem**:

(UF) On any set, every filter is contained in an ultrafilter.

We even have equivalence: $(\mathsf{BPI}) \Leftrightarrow (\mathsf{UF})$.

7.5.11 By a measure on a Boolean algebra $\mathscr{B}$ we always mean a *finitely* additive mapping $\mu : \mathscr{B} \to [0,1]$:

$$\mu(a \vee b) = \mu(a) + \mu(b) \quad \text{for all } a, b \in \mathscr{B} \text{ with } a \wedge b = 0.$$

Then $\mu(\emptyset) = 0$. We further assume that $\mu(\mathbf{1}) = 1$.

It is useful to look at homomorphisms from an arbitrary Boolean algebra into the simplest and most important Boolean algebra $\{0,1\}$. These are just the $\{0,1\}$-valued measures μ. Moreover, there is a one-to-one correspondence to the collection of all prime ideals:

$$\mathscr{I} = \{\, x \in \mathscr{B} : \mu(x) = 0 \,\} \quad \text{and} \quad \mu(x) = \begin{cases} 0 & \text{if } x \in \mathscr{I}, \\ 1 & \text{if } x \notin \mathscr{I}. \end{cases}$$

Hence (BPI) is equivalent to the following statement:

$(\mathsf{M}_{\{0,1\}})$ Every Boolean algebra has a $\{0,1\}$-valued measure.

This result, which goes back to Ulam [1929] and Tarski [1930, p. 49], can be viewed as the historical root of (BPI) and (UF), though these theorems were independently obtained by Stone [1936, p. 100] and Cartan [1937, pp. 778–779], respectively. All authors had based their proofs on transfinite induction: (WO). For further information, the reader is referred to the comments of Stone [1938, pp. 811–812].

There is a continuous counterpart of $(\mathsf{M}_{\{0,1\}})$:

$(\mathsf{M}_{[0,1]})$ Every Boolean algebra has a $[0,1]$-valued measure.

Note that $(\mathsf{M}_{\{0,1\}}) \underset{\not\Leftarrow}{\Rightarrow} (\mathsf{M}_{[0,1]})$; see Pincus [1972, p. 205].

7.5.12 Next, we state four variants of **Tychonoff's theorem**; see 3.4.1.5:

(TY) Any topological product of compact spaces is compact.
(TYH) Any topological product of compact Hausdorff spaces is compact.
$(\mathsf{TY}_{[0,1]})$ Any topological power $[0,1]^{\mathbb{I}}$ is compact.
$(\mathsf{TY}_{\{0,1\}})$ Any topological power $\{0,1\}^{\mathbb{I}}$ is compact.

Obviously, $(\mathsf{TY}) \Rightarrow (\mathsf{TYH}) \Rightarrow (\mathsf{TY}_{[0,1]}) \Rightarrow (\mathsf{TY}_{\{0,1\}})$. We stress that (TYH) follows from (UF), while (TY) requires the full axiom of choice.

$(\mathsf{TY}_{\{0,1\}}) \Rightarrow (\mathsf{M}_{\{0,1\}})$: Let $[\mathscr{F}]$ be the Boolean subalgebra generated by any finite subset $\mathscr{F}$ of the Boolean algebra $\mathscr{B}$. Denote by $C_{\mathscr{F}}$ the closed set of all $\mu \in \{0,1\}^{\mathscr{B}}$ whose restrictions to $[\mathscr{F}]$ are $\{0,1\}$-valued measures. Obviously, $C_{\mathscr{F}}$ is non-empty. Hence, by the compactness of $\{0,1\}^{\mathscr{B}}$, there is a μ in the intersection of all $C_{\mathscr{F}}$'s, and this function is a $\{0,1\}$-valued measure on all of $\mathscr{B}$.

7.5.13 In what follows, (HB) stands for the analytic **Hahn–Banach theorem**, as stated in 2.3.5. Its first proofs were based on transfinite induction: (WO). Łoś/Ryll-Nardzewski [1951, p. 237] derived (HB) from (TYH), or (BPI). The real significance of this observation became clear when Halpern/Levy discovered that (BPI) is indeed weaker than (AC); see 7.5.10. The most important contributions concerning the status of the Hahn–Banach theorem are due to Luxemburg [1969b, p. 131], who showed that (HB) and $(\mathsf{M}_{[0,1]})$ are equivalent. We only sketch his proof of $(\mathsf{M}_{[0,1]}) \Rightarrow (\mathsf{HB})$:

Fix a sublinear functional $p : X \to \mathbb{R}$ and a linear form $\ell_0 : X_0 \to \mathbb{R}$ such that $\ell_0(x) \le p(x)$ on a subspace X_0 of X. Let $\mathbb{I}$ denote the collection of all pairs (ℓ, M), where ℓ is a linear form on a subspace $M \supseteq X_0$ such that

$$\ell(x) \le p(x) \quad \text{for all } x \in M \quad \text{and} \quad \ell(x) = \ell_0(x) \quad \text{for all } x \in X_0.$$

Helly's extension lemma implies that

$$F_{x_1,\dots,x_n} := \{ (\ell,M) \in \mathbb{I} : x_1,\dots,x_n \in M \}$$

is non-empty for any choice of $x_1,\dots,x_n \in X$. Hence there exists a filter $\mathscr{F}$ that contains all $F_{x_1,\dots,x_n}$'s.

Next, we apply the main construction from non-standard analysis. Consider in $l_\infty(\mathbb{I})$ the subspace N formed by all $(\xi_{(\ell,M)})$'s with $\{(\ell,M) \in \mathbb{I} : \xi_{(\ell,M)} = 0\} \in \mathscr{F}$. The quotient $l_\infty(\mathbb{I})/N$ is a linear lattice whose elements $(\xi_{(\ell,M)})^{\mathscr{F}}$ are equivalence classes modulo N. Note that $\mathbf{1} = (1)^{\mathscr{F}}$ is an order unit. The canonical map $j : \xi \mapsto (\xi)^{\mathscr{F}}$ yields an embedding from $\mathbb{R}$ into $l_\infty(\mathbb{I})/N$.

Let $\Phi(x) := (\xi_{(\ell,M)})^{\mathscr{F}}$, where $\xi_{(\ell,M)} := \ell(x)$ if $x \in M$ and $\xi_{(\ell,M)} := 0$ otherwise. Put $j(\xi) := \xi \cdot \mathbf{1}$, and denote the canonical embedding from X_0 into X by J. Then we have the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\ell_0"] \arrow[d, "J"'] & \mathbb{R} \arrow[d, "j"] \\ X \arrow[r, "\Phi"'] & l_\infty(\mathbb{I})/N \end{tikzcd},$$

which shows that Φ can be viewed as an extension of ℓ_0. This result was achieved at the cost of enlarging the codomain; $\mathbb{R}$ is replaced by $l_\infty(\mathbb{I})/N$. In order to get an $\mathbb{R}$-valued extension, we need a positive linear form $\wp : l_\infty(\mathbb{I})/N \to \mathbb{R}$ with $\wp(\mathbf{1}) = 1$. Such a $\wp$ could be defined by putting $\wp((\xi_{(\ell,M)})^{\mathscr{F}}) := \mathscr{U}\text{-}\lim \xi_{(\ell,M)}$, where $\mathscr{U}$ is any ultrafilter larger than $\mathscr{F}$. But this would require (UF) or $(\mathsf{M}_{\{0,1\}})$. Luckily enough, $\wp$ can also be obtained from a $[0,1]$-valued measure on a suitable Boolean algebra. Hence the strictly weaker axiom $(\mathsf{M}_{[0,1]})$ suffices.

7.5.14 Recently, Luxemburg/Väth [2001, p. 268] showed that the Hahn–Banach theorem is equivalent to a seemingly much weaker statement:

(LV) $X^* \neq \{o\}$ for all Banach spaces $X \neq \{o\}$.

7.5.15 From Subsection 3.4.2, we recall **Alaoglu's theorem**:

(AL) The closed unit ball of a dual Banach space is weakly* compact.

Note that (AL) easily follows from $(\mathsf{TY}_{[0,1]})$. The converse implication is also true, since $[0,1]^{\mathbb{I}}$ can be viewed as a closed subset of $B_{l_\infty(\mathbb{I})}$.

7.5.16 Nowadays, the **Kreĭn–Milman theorem** is stated in the following form:

(KM) Every compact convex subset of a locally convex linear space has an extreme point.

Obviously, $(\mathsf{AL}) + (\mathsf{KM})$ implies the original version (KM^*) and, therefore, (AC). Pincus [1972, pp. 224, 225, 230] constructed a model in which (HB) is true, while (BPI), (AL), and (KM) are false.

7.5.17 We now give a summary of results. Axioms in each box are equivalent. The different levels can most clearly be distinguished by looking at the irreversible relations $(\mathsf{MI}) \underset{\not\Leftarrow}{\Rightarrow} (\mathsf{PI})$, $(\mathsf{TY}) \underset{\not\Leftarrow}{\Rightarrow} (\mathsf{TYH})$, and $(\mathsf{M}_{\{0,1\}}) \underset{\not\Leftarrow}{\Rightarrow} (\mathsf{M}_{[0,1]})$.

$$\boxed{\begin{array}{cccc} & (\mathsf{AC}) \quad (\mathsf{WO}) & (\mathsf{ZL}) & \\ (\mathsf{MI}) & (\mathsf{TY}) & (\mathsf{KM}^*) & (\mathsf{AL})+(\mathsf{KM}) \end{array}}$$

$$\Downarrow \ \not\Uparrow \ \text{Halpern/Lévy [1967]}$$

$$\boxed{\begin{array}{cccc} (\mathsf{PI}) & (\mathsf{BPI}) & (\mathsf{UF}) & (\mathsf{AL}) \\ (\mathsf{TYH}) & (\mathsf{TY}_{[0,1]}) & (\mathsf{TY}_{\{0,1\}}) & (\mathsf{M}_{\{0,1\}}) \end{array}}$$

$$\Downarrow \ \not\Uparrow \ \text{Pincus [1972]}$$

$$\boxed{\begin{array}{cc} (\mathsf{HB}) & (\mathsf{M}_{[0,1]}) \end{array}}$$

(AC) axiom of choice (WO) well-ordering theorem (ZL) Zorn's lemma
(MI) maximal ideal theorem (PI) prime ideal theorem (7.5.10)
(BPI) Boolean prime ideal theorem (UF) ultrafilter theorem (7.5.10)
$(\mathsf{M}_{\{0,1\}})$, $(\mathsf{M}_{[0,1]})$ existence of Boolean measures (7.5.11)
(TY), (TYH), $(\mathsf{TY}_{[0,1]})$, $(\mathsf{TY}_{\{0,1\}})$ various Tychonoff theorems (7.5.12)
(HB) Hahn–Banach theorem (7.5.13)
(AL) Alaoglu's theorem (7.5.15)
(KM^*), (KM) Kreĭn–Milman theorems (7.5.9, 7.5.16)

7.5.18 We list two more consequences of the axiom of choice, which have already been discussed in 1.2.2 and 1.5.10, respectively:

$(\mathsf{B}_{\text{alg}})$ Every real or complex linear space has a Hamel basis.
$(\mathsf{B}_{\text{orth}})$ Every real or complex Hilbert space has an orthonormal basis.

It is unknown whether $(\mathsf{B}_{\text{alg}}) \overset{?}{\Rightarrow} (\mathsf{AC})$ or $(\mathsf{B}_{\text{orth}}) \overset{?}{\Rightarrow} (\mathsf{AC})$. Some partial results were obtained by Bleicher [1964] and Halpern [1966]. Blass [1984, p. 31] showed that the axiom of choice follows if we assume that every linear space over an *arbitrary* field has a basis.

7.5.19 In calculus courses, quite often and tacitly, we pick sequences of points x_n without stating an explicit rule. Sometimes the x_n's are chosen step by step. This requires the **axiom of dependent choices**, which was formulated for the first time by Bernays [1942, p. 86]; its name is due to Tarski [1948, p. 96]:

(DC) Let R be a relation on a set M such that $R_x := \{y \in M : xRy\}$ is non-empty for all $x \in M$. Then, given $x_0 \in M$, there exists a sequence of elements $x_1, x_2, \ldots \in M$ such that x_0Rx_1, $x_1Rx_2, \ldots$.

Clearly, if f is a choice function of the family $(R_x)_{x\in M}$, then the desired sequence can be obtained recursively: $x_1 = f(x_0)$, $x_2 = f(x_1), \dots$. Hence $(\mathsf{AC})\Rightarrow(\mathsf{DC})$. The converse implication fails; Mostowski [1948, p. 127]. The axioms (DC) and (HB) are independent of each other. This means that there are models in which one of these axioms is true and the other one is false.

A typical application of (DC) is made in the proof of **Baire's category theorem**:

(BC) Every complete metric space is of second category.

Blair [1977] showed that even equivalence holds: $(\mathsf{DC})\Leftrightarrow(\mathsf{BC})$.

7.5.20 Recall from 2.5.3 that (BC) is the main tool for proving the **closed graph theorem** (CG), which in turn implies the **uniform boundedness principle** (UBP).

$(\mathsf{CG})\Rightarrow(\mathsf{UBP})$: Assume that $\mathscr{B}$ is a pointwise bounded subset of $\mathfrak{L}(X,Y)$. Then $B : x \mapsto (Tx)_{T\in\mathscr{B}}$ defines a closed linear mapping from X into $[l_\infty(\mathscr{B}),Y]$. Hence $\sup\{\|Tx\| : T \in \mathscr{B}\} \le c\|x\|$ for some constant $c > 0$ and all $x \in X$.

Quite often the **axiom of countable choice** suffices:

(AC^ω) Every sequence of non-empty sets has a choice function.

Sometimes we are in a situation that requires no additional axioms. For example, in a separable topological space, for every subset A with non-empty interior we can make a well-defined choice. To this end, let $\{x_1,x_2,\dots\}$ be any dense sequence and put $f(A):=x_n$, where x_n is the first element that belongs to A.

This observation can be used to prove (BC) in the separable case, which, in tandem with (AC^ω), yields the uniform boundedness principle. Hence (AC^ω) implies (UBP), and (DC) is not required.

It seems to be unknown whether $(\mathsf{CG})\overset{?}{\Rightarrow}(\mathsf{DC})$ or $(\mathsf{UBP})\overset{?}{\Rightarrow}(\mathsf{AC}^\omega)$. Brunner [1987] showed that $(\mathsf{UBP})\Rightarrow(\mathsf{MC}^\omega)$, the latter is the axiom of countable multiple choice. In particular, no "effective" proof of $(\mathsf{UBP})\overset{?}{\Rightarrow}(\mathsf{CG})$ is available. Only if this implication were false would Dieudonné [DIEU$_2^\bullet$, p. 142] be justified in saying that (CG) is *much deeper* than (UBP). Otherwise, such a statement just means that all existing proofs of (CG) are more complicated than those of (UBP).

7.5.21 Choosing one point in every set $(x+\mathbb{Q})\cap(0,\frac{1}{2})$, Vitali [1905b] proved a surprising fact; see also [VIT$^{\bowtie}$, pp. 233–235]:

$(\text{non-}\mathsf{LM})$ There is a subset of $\mathbb{R}$ that fails to be Lebesgue measurable.

Much later, Foreman/Wehrung [1991] observed that $(\text{non-}\mathsf{LM})$ already follows from (HB).

The **Banach–Tarski paradox** [1924] is even more shocking; [WAG]:

(BT) The unit ball of $\mathbb{R}^3$ admits a decomposition into finitely many pieces (five suffice) that can be glued together to yield two unit balls.

Nowadays, we know that (HB)$\Rightarrow$(BT); Pawlikowski [1991]. If every subset of $\mathbb{R}$ is Lebesgue measurable, then so is every subset of $\mathbb{R}^3$, and (BT) must be false. Hence (BT)$\Rightarrow$(non-LM).

It is strange that exotic models of set theory are required to exclude the existence of exotic sets as described above. The most famous construction is due to Solovay [1970], who found *a model of set-theory in which every set of reals is Lebesgue measurable*. Consequently, (HB) must fail. As a matter of poetic justice, people living in this world have no problems with linear maps on Banach spaces: they get their continuity for free; Garnir [1973] and Wright [1976].

An ultrafilter $\mathscr{U}$ on a set M is called **principal** if it consists of all subsets containing a fixed point $x_0 \in M$. These are the only ultrafilters ever seen by a mathematician. Blass [1977] constructed a model in which all ultrafilters are principal. Hence ultrafilters cannot be used to characterize compactness, which is a horror for every topologist. One may also achieve that *the dual space of L_∞ is L_1*; Väth [1998]. In this model, l_1 and l_∞ are reflexive, whereas c_0 is not. Frankly, I feel like *Alice in Wonderland*. Of course, referring to the Banach–Tarski paradox, a supporter of Solovay's point of view could say just the same. Make your choice!

At some university there was a professor of numerical mathematics who believed that

every closed subspace of a Banach space is complemented.

He had good company, since Kantorovich/Akilov [KAN_1^+, German transl., p. 377] claimed (certainly a slip):

daß für die Existenz einer linearen Linksinversen [see p. 139] *die Gültigkeit der Ungleichung $\|U(x)\| \geq m\|x\|$ mit einer von $x \in X$ unabhängigen positiven Konstanten m notwendig und hinreichend ist.*

Don't smile! Probably, 99.9% of all analysts would agree that

the union of countably many countable sets is countable.

However, this is no theorem of (ZF). Thus axiomatic set theory is full of surprises, but also provides deep insights.

7.5.22 Finally, I mention that the continuum hypothesis, Martin's axiom, and the existence of measurable cardinals play an essential role in general topology and measure theory; see Subsection 6.9.1, Gardner/Pfeffer [1984], and [TAL].

7.5.23 References for further reading:
[$\text{BARW}^\cup$], [BUS], [CIE], [DEV], [HOW^+], [JECH_1], [JECH_2], [POT], [RUB^+], [WAG].

Especially, I recommend Moor's book
Zermelo's Axiom of Choice: *Its Origins, Development and Influence*, [$\text{MOO}^\bullet$].

8

Mathematics Is Made by Mathematicians

For the ones they are in darkness
And the others are in light.
And you see the ones in brightness
Those in darkness drop from sight.

Bertholt Brecht
The Threepenny Opera
(original version on p. 682)

This chapter is dedicated
to those mathematicians
–named or unnamed–
who contributed to the
theory of Banach spaces.

In the last chapter of this book, I intend to describe more than just life and work of world-famous mathematicians whose names can be found in any *Who's Who in Science*. Since it was impossible to mention all colleagues who contributed to Banach space theory, the result is a subjective selection, heavily influenced by my knowledge and taste. I hope that the readers will accept my sincere apologies for any omissions and misjudgments.

I did not follow Halmos [HAL$^\bullet$, p. 305], who proposed a rating of mathematicians and considered himself as a candidate for rank four. However, I have tried to paint a colorful picture formed by quite different characters:

- renowned mathematicians who worked in Banach space theory for a limited period of their life or obtained relevant results in passing:
  Wiener, von Neumann, Kolmogorov, Gelfand, Grothendieck, ... ,
- dominating figures:
  Banach, Pełczyński, Lindenstrauss, ... ,
- those who proved celebrated classical theorems:
  Alaoglu, Kreĭn/Milman, Kadets, Dvoretzky, ... ,
- leading researchers of the modern period (alphabetical order):
  Bourgain, Johnson, Maurey, (Vitali) Milman, Pisier, ... ,
- those who produced outstanding counterexamples:
  James, Enflo, Tsirelson, Pisier, Gowers/Maurey, ... ,
- those who wrote influential books:
  Banach, Day, Diestel/Uhl, Lindenstrauss/Tzafriri, ...

In particular, I would like to emphasize the role of those who have done their best, loving mathematics as much as the most famous of our colleagues. Trees have thick and thin branches, but without leaves they cannot survive. The same idea was expressed by P.J. Davis, [ALB$^\cup$, p. xv]:
The giants stand on the shoulders of lesser giants, and the whole pyramid would collapse without the firm support of the community.

These statements may explain the spirit of the present chapter.

8.1 Victims of politics

Mathematics is a part of the real world, and mathematicians cannot escape politics. The following list is therefore devoted to the memory of those colleagues who were killed in wars, murdered under totalitarian regimes, or committed suicide for political reasons.

Артеменко, Александр П.	(1909–1944)
Auerbach, Herman	(1901–1942)
Егоров, Дмитрий Ф.	(1869–1931)
Eidelheit, Maks	(1911–1943)
Doeblin, Wolfgang	(1915–1940)
Гантмахер, Вера Р.	(1909–1942)
Gâteaux, René	(1889–1914)
Глазман, Израиль М.	(1916–1968)
Hartogs, Friedrich	(1874–1943)
Hausdorff, Felix	(1868–1942)
Kaczmarz, Stefan	(1895–1939)
Kerner, Michał	(1902–1943)
Łomnicki, Antoni	(1881–1941)
Marcinkiewicz, Józef	(1910–1940)
Noether, Fritz	(1884–1941)
Saks, Stanisław	(1897–1942)
Schauder, Juliusz	(1899–1943)
Schreier, Józef	(1909–1942)
Сирвинт, Юрий Ф.	(1913–1942)
Шмульян, Витольд Л.	(1914–1944)
Шнирельман, Лев Г.	(1905–1938)
Sternbach, Ludwik	(1905–1942)
Юдин, Абрам И.	(1919–1941)
Tauber, Alfred	(1866–1942)

One may add I. Schur (1875–1941), O. Toeplitz (1881–1940), and MANY OTHERS who left their homeland because of political persecution.

The names of the victims listed above are taken from [aaa_1], [aaa_2], and [aaa_3]. For further information, I refer to [KAŁ$^\bullet$, pp. 86–89], [KUR$^\bullet$, pp. 80–90], [ULAM$^\bullet$, pp. 39–43], a survey of Orlicz [1988$^\bullet$, pp. 1635–1641], the reports of Dresden [1942$^\bullet$], and Pinl [1969$^\bullet$], … , [1976$^\bullet$], as well as to the catalogs prepared by Sigmund [SIG$^\bullet$] and Brüning/Ferus/Siegmund-Schultze [BRÜ$^{+\bullet}$].

Transliterations of the Russian names can be found on pp. 656–659.

8.2 Scientific schools in Banach space theory

This section is devoted to scientific schools in the field of Banach space theory. At least in the past, those schools were mainly working on a local basis, in particular, during the time of the Cold War.

It is my sincere intention that the following presentation does not contain any kind of chauvinism. After living in Germany under three political regimes, I came to the conclusion that I should consider myself just as a mathematician.

Mathematical research is, above all, a randomed process. It makes no sense to look for connections that, in reality, never existed. One Ph.D. student grew up in a scientific school directed by an eminent teacher and the other one, quite isolated, caught fire when reading an interesting paper on our subject.

8.2.1 Poland

Because Germans were responsible for the dark period in the history of Banach spaces, I would like to begin with a very personal statement:
In 1960, when I was a young assistant, Stanisław Mazur visited Berlin; we had a short mathematical discussion, and as a result, he invited me to a conference that took place in Jabłonna. Here I met many famous old mathematicians such as Mazur, Orlicz, Szőkefalvi-Nagy, Nikolskiĭ, Sobolev, and Stone, as well as the young Polish generation. Since that time, Warszawa has been the "City of Banach Spaces" for me, and I am thankful that, despite the past, I have been accepted as a mathematician and friend.

8.2.1.1 Throughout the following, the reader should keep in mind a quotation from Zygmund [1987•, p. 37]:

One might say that [add: from 1918 to 1939] *Warsaw mathematics was dominated by topology and real variable, whereas the Lwów school excelled in functional analysis; but the statement would be more emphatic than precise, since each school was from the outset strongly influenced by the other, both through personal contacts and through scientific collaboration. The personal friendship of Banach and Sierpiński was of great consequence in this respect.*

8.2.1.2 After World War I, Poland was reconstituted as a republic. In the former Russian part, the University of Warszawa had to be restored completely. The most important centers of Polish mathematics were located in the former Austrian part (Galicia):

the Jagiellonian University in Kraków (founded in 1364),
the Jan Kazimierz University in Lwów (founded in 1661),
the Polytechnical Institute in Lwów (founded in 1877).

The University of Poznań opened its doors in 1919, and one year later, the University of Wilno became Polish.

8.2.1.3 In 1918, Z. Janiszewski (1888–1920) published an article, *On the needs of mathematics in Poland*. He proposed to concentrate on a small selection of disciplines in which Polish mathematicians already had international reputations; see [KUR$^\bullet$, pp. 30–33]:

set theory and topology, foundations of mathematics and mathematical logic.

He further suggested establishing a journal specializing in these fields. Despite Janiszewski's untimely death during an epidemic of influenza, his program was realized, mainly at the University of Warszawa. The first volume of the new journal *Fundamenta Mathematicae* appeared in 1920 edited by S. Mazurkiewicz (1888–1945) and W. Sierpiński (1882–1969).

8.2.1.4 The *topology* and *real analysis schools* of Warszawa developed well. In 1937, K. Kuratowski (1896–1980) published his *Topologie* [KUR]. As further highlights I mention the monographs *Theory of the Integral* [SAKS] and *Trigonometric Series* [ZYG]. Both authors, S. Saks (1897–1942) and A. Zygmund (1900–1992), were Ph.D. students of S. Mazurkiewicz. In 1930, Zygmund was appointed professor at the University of Wilno, where he *discovered* J. Marcinkiewicz (1910–1941). Their fruitful collaboration was stopped by World War II: Zygmund emigrated to the USA, and Marcinkiewicz was slaughtered; it seems to be unknown whether in Kharkov or in the Katyń Forest (Soviet Union).

8.2.1.5 The *Polish functional analysis school* was founded in Lwów by H. Steinhaus (1887–1972) and S. Banach (1892–1945). Since 1929, the two acted as the first editors of the famous *Studia Mathematica*. This journal has become the most important voice of Banach space theory. One hundred thirty-six `items` from our bibliography were published in Studia, while no other journal achieved more than 93.

Those who like tradition will enjoy the following genealogical tree:

F. Klein
(1849–1925)
↓
F. von Lindemann
(1852–1939)
↓
D. Hilbert
(1862–1943)
↓
H. Steinhaus
(1887–1972)
↓
S. Banach
(1892–1945)
↙ ↘

S. Mazur (1905–1981) W. Orlicz (1903–1990)

To be honest, I must confess a little bit of cheating: in fact, the formal supervisor of Banach's thesis was "professor" Łomnicki and not "docent" Steinhaus.

8.2.1.6 The period from the beginning of the 1920s until the outbreak of World War II was extremely successful. Many talented mathematicians grew up in Lwów:

O. Nikodym (1887–1974), emigrated to the USA in 1947,
S. Kaczmarz (1895–1939), victim of the Nazi terror,
J. Schauder (1899–1943), victim of the Nazi terror,
H. Auerbach (1901–1942), victim of the Nazi terror,
W. Orlicz (1903–1990),
S. Mazur (1905–1981),
M. Kac (1909–1985), emigrated to the USA in 1938,
J. Schreier (1909–1942), victim of the Nazi terror,
S. Ulam (1909–1984), emigrated to the USA in 1936,
M. Eidelheit (1911–1943), victim of the Nazi terror,
L. Sternbach (1905–1943), victim of the Nazi terror.

8.2.1.7 The Lwów functional analysts liked to discuss mathematics in the *Café Szkocka*. One day in 1935, Banach proposed to collect the open problems in a notebook. This notebook, which later became famous under the name *Scottish Book*, was kept by a waiter, who, as remembered by Ulam [MAUL•, p. 8],
knew the ritual–when Banach or Mazur came in it was sufficient to say "The book please" and he would bring it with the cups of coffee [followed by a "few" glasses of vodka].

In [KAŁ•, p. 67] one can read:

It was said and written in Poland and abroad that the café work culture constituted the "Polish way of doing mathematics," a phenomenon of teamwork in unorthodox places.

According to my own experience, the teamwork survived, but "more serious" and "less expensive" places were chosen.

8.2.1.8 Lwów (Lemberg, Львів) has a colorful history: founded in 1256, it was captured by Poland in 1340, passed to Austria in 1772, and was retaken by Poland in 1918. The city was formally ceded to the USSR in 1945.

Based on the Ribbentrop–Molotov Treaty, Lwów had been occupied by the Soviets from September 1939 to June 1941. In this period, Banach, Mazur, Orlicz, Saks, and Knaster remained members of the Physical-Mathematical Faculty of the former Jan Kazimierz University, which now went by the name of the Ukrainian poet Ivan Franko. Then there came the horrible Nazi time during which large parts of the Polish intelligentsia were killed. In July 1944, the Red Army reentered Lwów. After 1945 most Polish mathematicians were repatriated and left the city.

8.2.1.9 The list in 8.2.1.6 shows that World War II had disastrous consequences for Polish mathematics. Since Banach passed away in August 1945, only three functional analysts remained for a new beginning:

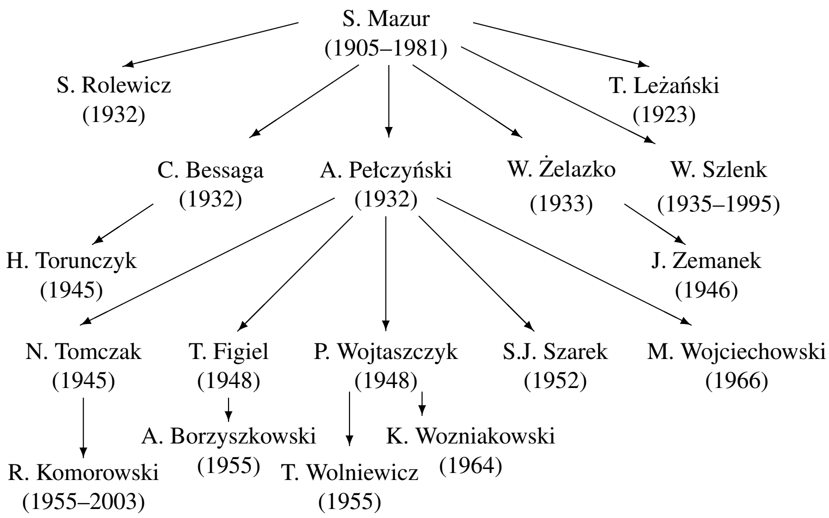
Hugo Steinhaus, Stanisław Mazur, Władysław Orlicz.

Despite their misfortune, Polish mathematicians made it a point of honor to continue Lwów's tradition.

8.2.1.10 Steinhaus moved to Wrocław, where he created a fruitful scientific atmosphere; see [KUR•, pp. 173–179]. His interest was mainly devoted to applications of mathematics in various fields. He also founded a school in the theory of real functions and the calculus of probability, which includes famous experts like C. Ryll-Nardzewski (1926) and K. Urbanik (1930).

An essential role in developing the University of Wrocław was played by E. Marczewski (1907–1976); born as E. Szpilrajn, he changed his name during the time of the Nazi occupation.

8.2.1.11 After World War II, the most important branch of Banach space theory developed in Warszawa under Mazur's direction:



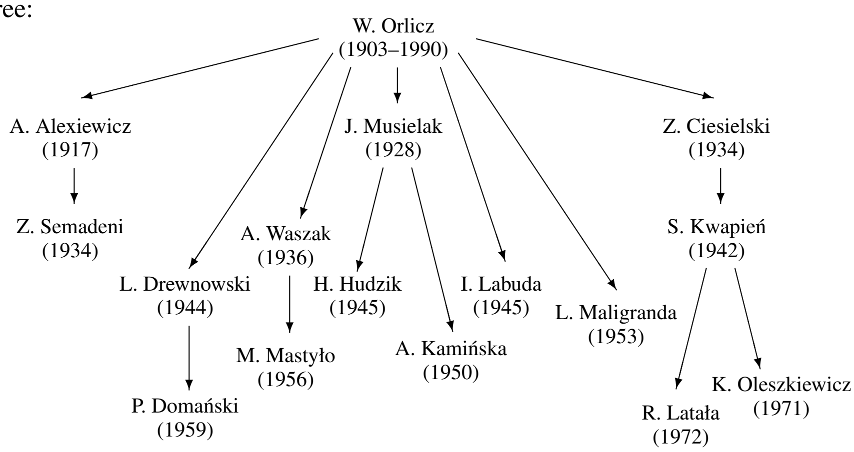
Further pupils of A. Pełczyński are N.J. Nielsen (1943; Denmark) and A. Szankowski (1945). The offspring of Mazur's school also includes W. Bogdanowicz (1933, now V. Bogdan) and his Ph.D. student J. Diestel (1943).

8.2.1.12 Without any doubt,

Aleksander Pełczyński

and his contemporary Joram Lindenstrauss (Israel; see also 8.2.4) have become the dominating figures in the renaissance of Banach space theory.

8.2.1.13 In 1937, Orlicz was nominated associate professor at Poznań University. With an interruption during World War II, he stayed there until his retirement in 1974. Orlicz is well known as the father of a class of function spaces that bear his name; see Subsection 6.7.14. Only a few of his 39 Ph.D. students are represented in the following tree:



Beginning in 1986, the functional analysts from Poznań organized a successful series of international conferences on *Function Spaces*.The seventh conference (Poznań 2003) was dedicated to the 100th birthday of Władysław Orlicz; see [CIE$^\cup$].

8.2.1.14 R. Sikorski (1920–1983) received his Ph.D. under K. Kuratowski for a thesis on Boolean algebras (Warszawa, 1949) and wrote a famous Ergebnisbericht on the same subject [SIK]. However, he also contributed to determinant theory of abstract operators; see 6.5.3.4. R. Sikorski supervised the thesis of W. Słowikowski (1932), who on his part was the teacher of P. Mankiewicz (1943). Hence not all Polish Banach space people are direct descendants of Banach.

8.2.1.15 Extremely important for Polish mathematics was the founding of the Mathematical Institute of the Polish Academy of Sciences in 1948. It is mainly located in Warszawa, but has branches in Gdańsk, Kraków, Poznań, and Wrocław.

Another great achievement is the *Banach Center* founded in 1972, which for many years was a meeting place of mathematicians from East and West.

8.2.1.16 In 1960, as a tribute to Stefan Banach, a *Conference on Functional Analysis* was organized in Jabłonna (near Warszawa). An *International Colloquium on Nuclear Spaces and Ideals in Operator Algebras* took place in Warszawa (1969). A particular highlight was the ICM 1983 in Warszawa, where Pełczyński delivered a plenary lecture [1983]. I also mention the *Polish East German Seminars*: Nowy Sącz (1978), Wisła (1980), Jachranka (1985), and Jabłonna (1987); the meetings in the GDR are listed in 8.2.8.12. A special event was the *Conference on Functional Analysis in Honor of Aleksander Pełczyński*, Będlewo 2002; see Vol. 159 of Studia Mathematica.

8.2.1.17 A. Pełczyński's lecture notes on *Banach Spaces of Analytic Functions and Absolutely Summing Operators* (1977) and his surveys [1979, in collaboration with C. Bessaga], [1980], [1983] had a strong impact on the development of modern Banach space theory. I also mention the monograph [BES$^+$].

S. Rolewicz and his wife, Danuta, wrote a very useful book on *Equations in Linear Spaces* (1968). *Banach Spaces of Continuous Functions* (1971) were treated by Z. Semadeni. Last but not least, I emphasize P. Wojtaszczyk's successful undertaking to write a book on *Banach Spaces for Analysts* (1991).

8.2.1.18 The preceding presentation is based mainly on Kuratowski's authentic account [KUR$^\bullet$], Banach's biography [KAŁ$^\bullet$], and some chapters of Ulam's autobiography [ULAM$^\bullet$]. I also refer to articles of Orlicz [1988$^\bullet$] and Köthe [1989$^\bullet$]. The *Scottish Book* [MAUL$^\bullet$] was edited by Mauldin.

The following *Collected* or *Selected Works* of Polish mathematicians (related to Banach space theory) have been published: [BAN$^\bowtie$], [BORS$^\bowtie$], [JANI$^\bowtie$], [MAR$_1^\bowtie$], [MAR$_2^\bowtie$], [ORL$^\bowtie$], [SCHAU$^\bowtie$], [STEIN$^\bowtie$], [ULAM$^\bowtie$], [ZYG$^\bowtie$].

8.2.2 USA

8.2.2.1 In the period from 1906 to 1922, E.H. Moore (1862–1932) developed his *general analysis*; see Bolza [1914$^\bullet$] and Siegmund-Schultze [1998$^\bullet$]. He will always be remembered for creating the concept of a directed system or net, an object that is also called a *Moore–Smith sequence*; see 3.2.2.1.

A critical evaluation given by Garrett Birkhoff [1977$^\bullet$, p. 32] reads as follows:

Though very original, Moore's ideas had little influence at the time, probably because they were so abstract and often couched in Peanese symbols.

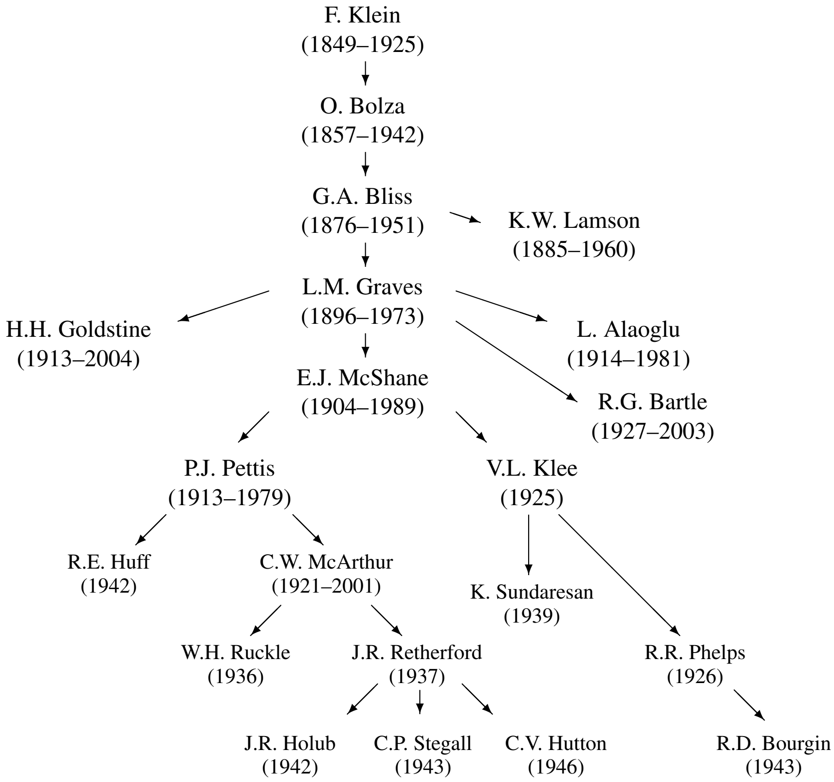
...

Even more notable than Moore's own research achievements were those of his Ph.D. students.

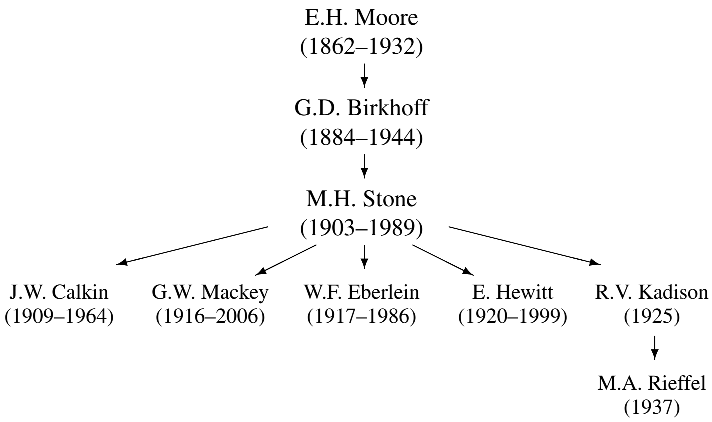
As a result of Moore's initiative, the University of Chicago (founded in 1891) became the first center of functional analysis in the USA.

8.2.2.2 Among Moore's 30 Ph.D. students was T.H. Hildebrandt (1888–1980), who became a leading figure in the theory of "linear normed complete spaces"; see 5.1.8.8. He taught at the University of Michigan (Ann Arbor) from 1909 to 1957, since 1923 as a full professor. His most prominent pupils were R.S. Phillips (1913–1998) and C.E. Rickart (1913–2002).

8.2.2.3 Now I present an impressive genealogical tree, whose trunk (Bliss and Graves) was certainly influenced by E.H. Moore:



8.2.2.4 Here is another functional analytic tree:



With the exception of Kadison, the Stone students mentioned above received their Ph.D.s from Harvard.

The impact of Stone's book

Linear Transformations in Hilbert Space (1932)

is comparable to that of Banach's book, which appeared in the same year.

8.2.2.5 The Institute for Advanced Study at Princeton was founded in 1933. Among the first permanent members were J. von Neumann (1903–1957) and H. Weyl (1895–1955).

J. von Neumann's impact on Banach space theory is described in 8.3.4.

It seems that H. Weyl did not like axiomatic theories very much. His personal love was the study of eigenvalue problems [1950]. In particular, we owe to him a fundamental inequality [1949] that became the pattern of many deep results about eigenvalue distributions; see 6.4.2.2.

S. Bochner (1899–1982) was a professor at Princeton University from 1933 to 1968. He invented, inter alia, the concept of the Bochner integral; see 5.1.2.4. Among his Ph.D. students were A. Sobczyk (1915–1981), B.R. Gelbaum (1922), S. Karlin (1924), and C.S. Herz (1930).

8.2.2.6 In 1925, J.D. Tamarkin (1888–1945) emigrated from Leningrad to the USA. Two years later, he was called to Brown University. Szegő stressed the decisive role of Tamarkin in the preface to the revised edition of [SZE]:

Since his untimely death in 1945 his name is not too frequently mentioned. It is justified and probably necessary to remind the younger mathematical generation, in the rush of modern developments, how much American mathematics owes to his great energy and far-sighted intelligence.

A long list of coauthors of Tamarkin can be found in an obituary written by Hille [1947•, p. 453]. From the viewpoint of this text, his joint work with Hille on linear integral equations is of special significance; see Hille/Tamarkin [1931], [1934]. Most importantly, he stimulated

J.A. Clarkson,	N. Dunford,	M.M. Day
(1906–1970)	(1906–1986)	(1913–1992)

to deal with Banach space theory.

Clarkson's work [1936] on uniform convexity was quite influential. However, to the best of my knowledge, he did not have any pupils.

The impetus of Dunford and Day on the development of Banach space theory in the USA will be described in the following paragraphs.

8.2.2.7 Dunford was hired in 1936 by Brown University and made full professor in 1943. He realized his program of extending spectral theory to non-self-adjoint operators in collaboration with W.G. Bade (1924), R.G. Bartle (1927–2003), and J.T. Schwartz (1930). The upshot was the three-volume treatise

Dunford–Schwartz, Part I: 1958, Part II: 1963, Part III: 1971.

Its writing took approximately 20 years. During that time the project was supported by the Office of Naval Research. This explains why, according to Rota [1996•, p. 48],

there is a persistent rumor, never quite denied, that every nuclear submarine on duty carries a copy of "Linear Operators."

8.2.2.8 After receiving his Ph.D. from Brown University in 1939, M.M. Day became a member of the Mathematical Faculty of the University of Illinois (Urbana-Champaign), where he stayed until his retirement in 1983. Day is well known for the Ergebnisbericht, *Normed Linear Spaces* (1958), the first (post-Banach) monograph on geometry of Banach spaces. He directed a school devoted to all forms of rotundity and convexity. The offspring includes D.B. Goodner (1913–1995), D.W. Dean (1931), D.F.R. Cudia ($\sim$1935), M.A. Smith (1947), and R.E. Megginson (1948).

Other functional analytic fields studied at the University of Illinois were UMD spaces (D.L. Burkholder) and vector measures (J.J. Uhl).

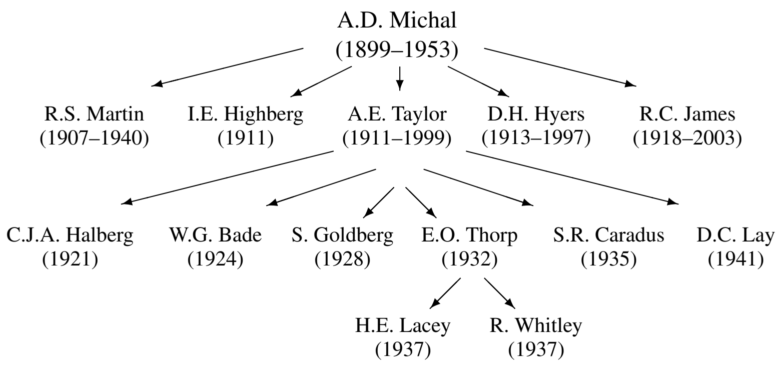
8.2.2.9 Hille (1894–1980) went to Yale in the mid 1930s, and wrote there his book *Functional Analysis and Semi-Groups* (1948). Among his Ph.D. students were H.F. Bohnenblust (1906–2000), I.E. Segal (1918–1998), and C. Ionescu Tulcea (1923).

After the end of World War II, S. Kakutani (1911–2004) came to Yale from Japan and C.E. Rickart (1913–2002) from Michigan. Thus, at the beginning of the 1950s, Yale had developed into a major center of functional analysis.

8.2.2.10 Garrett Birkhoff (1911–1996), who served as a professor at Harvard for 45 years, had a broad mathematical spectrum. His *Lattice Theory* (1940) was the first book in which Banach lattices occurred (Chapter VII). He also contributed to a better understanding of convergence. Concerning the Birkhoff integral, I refer to 5.1.2.6. Among his Ph.D. students were R.F. Arens (1919–2000), Chandler Davis (1926), and G.A. Edgar (1949).

8.2.2.11 In 1929, A.D. Michal was appointed associate professor at the California Institute of Technology. There he founded the first functional analytic school on the West Coast of the USA. At the beginning, Michal directed his interest to infinite-dimensional differential geometry [1939]; see 6.9.14.2. As an indispensable tool, he needed derivatives of vector-valued functions. Michal and his pupils also made substantial contributions to the theory of vector-valued polynomials and vector-valued analytic functions; see Subsections 5.1.9 and 5.1.10. A book on these subjects (with a preface by Fréchet) was published posthumously: [MICHA].

It seems to me that Michal has been underestimated by many of his American colleagues. However, a short glance at the following impressive tree shows Michal's actual significance.



I stress the different abilities of Michal's most important students: Taylor is well known for his textbooks as well as for his work on spectral theory and analyticity, whereas James became famous as an ingenious and sophisticated expert of reflexivity.

8.2.2.12 As already mentioned, J.D. Tamarkin (1888–1945) emigrated from Leningrad to the USA. The same happened with G.G. Lorentz (1910–2006) who came into this country via Germany (1944–1949) and Canada (1949–1953); see [LORZ$^{\bowtie}$, Vol. I, p. xix]. He received two Ph.D.s: one from Leningrad under G.M. Fichtenholz/ L.V. Kantorovich and another one from Tübingen under K. Knopp. Lorentz wrote a book on *Approximation of Functions* (1966), which opened up the concepts of *widths* and *entropy* to the "Free World" and has become a standard reference; see also Lorentz [1966]. After professorships at Wayne State University in Detroit and at Syracuse University, he finally went to the University of Texas. The *Austin school of approximation theory* became famous. Among his Ph.D. students were K.L. Zeller (1924–2006), P.L. Butzer (1928), L. Sucheston (1926), and R. Sharpley (1946).

8.2.2.13 The year 1958 was exceptional, since there appeared three monographs:

- M.M. Day: *Normed Linear Spaces*,
- N. Dunford, J.T. Schwartz: *Linear Operators*, Part I,
- A.E. Taylor: *Introduction to Functional Analysis*.

Thus one may say that the classical theory of Banach spaces was brought to a close. In the subsequent period, besides the authors mentioned above, only few mathematicians were working in this field: V.L. Klee and R.C. James. The growing of a new generation of Banach space people took several years. A very decisive impetus came from the outside: C. Bessaga and A. Pełczyński (Poland) as well as J. Lindenstrauss (Israel) were among the catalysts.

The following table shows that the development was rather heterogeneous:

E. Artin (1898–1962)	$\Rightarrow$	K. de Leeuw (1930)	$\Rightarrow$	H.P. Rosenthal (1940)	$\Rightarrow$	J.N. Hagler T. Starbird
A.E. Schild (1921)	$\Rightarrow$	J.A. Dyer (1932)	$\Rightarrow$	W.B. Johnson (1944)	$\Rightarrow$	E.W. Odell D.E. Alspach L.E. Dor
S. Mazur (1905–1981)	$\Rightarrow$	V.M. Bogdan (1933)	$\Rightarrow$	J. Diestel (1943)	$\Rightarrow$	P.N.Dowling C.J. Lennard
	$\Rightarrow$	M.M. Rao (1929)	$\Rightarrow$	J.J. Uhl (1940)	$\Rightarrow$	M. Girardi E. Saab P. Saab
R.E. Langer (1894–1968)	$\Rightarrow$	L.R. Bragg (1931)	$\Rightarrow$	W.J. Davis (1940)	$\Rightarrow$	Pei-Kee Lin
H.S. Wall (1902–1971)	$\Rightarrow$	J.R. Dorroh (1937)	$\Rightarrow$	D.R. Lewis (1944)	$\Rightarrow$	
(Ky) Fan (1914)	$\Rightarrow$	Bor-Luh Lin (1935)	$\Rightarrow$	P.G. Casazza (1944)	$\Rightarrow$	
F. Smithies (1912–2002)	$\Rightarrow$	D.J.H. Garling (1937)	$\Rightarrow$	N.J. Kalton (1946)	$\Rightarrow$	B. Randrian- antoanina
F. Smithies (1912–2002)	$\Rightarrow$	S. Simons (1938)	$\Rightarrow$	T.J. Leih ($\sim$1946)	$\Rightarrow$	
S. Kakutani (1911–2004)	$\Rightarrow$	A. Beck (1930)	$\Rightarrow$	D.P. Giesy ($\sim$1937)	$\Rightarrow$	
W. Hoeffding (1914–1991)	$\Rightarrow$	D.L. Burkholder (1927)	$\Rightarrow$	B.J. Davis ($\sim$1941)	$\Rightarrow$	
D.H. Tucker (1930)	$\Rightarrow$	R.K. Goodrich (1941)	$\Rightarrow$	H.B. Maynard (1943)	$\Rightarrow$	

Due to lack of space, the fourth row is necessarily incomplete. For example, thus far J. Diestel and W.B. Johnson have supervised 17 and 12 Ph.D. students, respectively.

8.2.2.14 With the exception of the California Institute of Technology, where W.A.J. Luxemburg (1929) was appointed in 1962, modern Banach space theory did not become popular at the first-rate academic institutions of the USA. However, it was intensively developed at some other places (alphabetic order):

University of California (Los Angeles): A.E. Taylor (1911–1999).

Case Western Reserve University (Cleveland): W.A. Woyczyński (1943), J. Szarek (1952), E. Werner (1958).

University of Denver: J.N. Hagler (1945), A. Arias (1961).

University of Illinois (Urbana-Champaign): M.M. Day (1913–1992), D.L. Burkholder (1927), H.P. Lotz (1934), N.T. Peck (1937–1996), J.J. Uhl (1940), M. Junge (1962).
In honor of Day's retirement, a conference on *Geometry of Normed Linear Spaces* (1983) took place; [BARTL$^{\cup}$].

University of Iowa (Iowa City): Bor-Luh Lin (1935).
Workshops on *Banach Space Theory* were organized in 1981 and 1986; see [LIN$_1^{\cup}$], [LIN$_2^{\cup}$]. Bor-Luh Lin (1935) also strongly supported a similar workshop in Merida (Venezuela, 1992); see [LIN$_3^{\cup}$].

Kent State University: J. Diestel (1943), R.M. Aron (1944), P. Enflo (1944), V.I. Gurariĭ (1935–2005), V.I. Lomonosov (1946), A.M. Tonge (1948).
A conference on *Banach Spaces of Analytic Functions* (1976) as well as two conferences on *Banach Space Theory* (1979, 1985) took place; see [BAK$^{\cup}$].

Louisiana State University (Baton Rouge): J.R. Retherford (1937).
In 1964, the first USA conference specializing in Banach spaces was organized by Retherford. A symposium on *Infinite-Dimensional Topology* followed in 1967; see [AND$_1^\cup$].

University of Maryland (College Park): J.W. Brace (1926), S. Goldberg (1928), D.C. Lay (1941).

Memphis State University: J.E. Jamison (1943), A. Kaminska (1950), Pei-Kee Lin (1952).
A conference on *Trends in Banach Spaces and Operator Theory* was held in 2001; see [KAM$^\cup$].

University of Missouri (Columbia): P.G. Casazza (1944), N.J. Kalton (1946), E. Saab (1946), P. Saab (1952), A.L. Koldobskiĭ (1955).
Conferences: *Banach Spaces* (1984) and *Interaction between Functional Analysis, Harmonic Analysis, and Probability* (1994); see [KAL$_1^\cup$], [KAL$_2^\cup$].

Ohio State University (Columbus): L. Sucheston (1926), B.S. Mityagin (1937), W.J. Davis (1940), W.B. Johnson (1944), D.R. Lewis (1944), G.A. Edgar (1949) as well as a special creation: Peter Ørno.

Oklahoma State University (Stillwater): D.E. Alspach (1950).
Location of the *Banach Space Archive*.

University of Oregon (Eugene): B. Yood (1917–2004), K.T. Smith (1926–2000).

Pennsylvania State University (University Park): P.D. Morris (∼1939), R.E. Huff (1942).

University of South Carolina (Columbia): J.W. Roberts (∼1943), S.J. Dilworth (1959), M. Girardi (1963), G. Androulakis (1967).

University of Texas at Austin: G.G. Lorentz (1910–2006), H.P. Rosenthal (1940), E. Odell (1947).
The lectures presented at the *Texas Functional Analysis Seminar* were published from 1982 to 1989 in the *Longhorn Notes*; see p. 723.

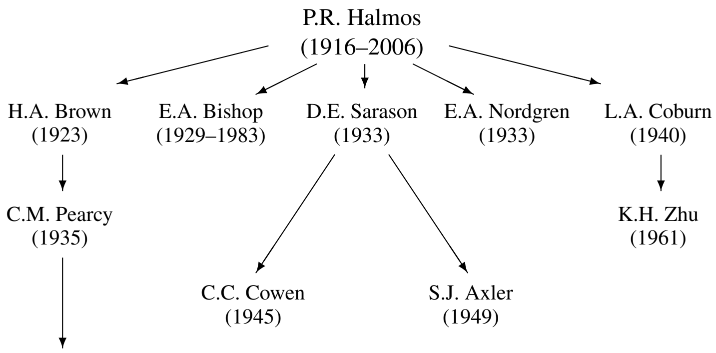
Texas A & M (College Station): H.E. Lacey (1937), W.B. Johnson (1944), D.L. Lewis (1944), G. Pisier (1950), T. Schlumprecht (1954).
Since 1985, a *Workshop on Banach Space Theory* was organized each summer. Now linear algebra and probability are included as well. Another important annual event is the *Informal Regional Functional Analysis Seminar*.

University of Washington (Seattle): Z.W. Birnbaum (1903–2000), V.L. Klee (1925), R.R. Phelps (1926), I. Namioka (1928), B. Grünbaum (1929).
There also exists a fictitious mathematician, John Rainwater, whose biography was written by Phelps [2002•].

One should take into account that each of these centers has had its ups and downs. For example, the most productive period at Ohio State University was from 1970 to 1984. Then W.B. Johnson moved to Texas A&M. In 1979, H.P. Rosenthal became a full professor at Austin. Since that time, the core of Banach space theory within the USA has been located in Texas. Mainly due to the efforts of N.J. Kalton and P.G. Casazza, the University of Missouri (Columbia) also developed into an excellent place for Banach spaces.

8.2.2.15 For various reasons, it is a must to mention J. Diestel. First of all, he is the author or coauthor of some well-known books. Most influential was the treatise on *Vector Measures* (1977) written in collaboration with J.J. Uhl. Secondly, he organized many "Banach space events." Last but not least, thanks to his initiative Kent State University awarded three *doctores honoris causa* to Banach space experts: R.C. James, A. Pełczyński, and J. Lindenstrauss.

8.2.2.16 In the USA, modern operator theory was initiated mainly by P.R. Halmos. Since he preferably worked in the Hilbert space setting, most of his studies are irrelevant for the purpose of the present text. Nevertheless, I include the following (rather incomplete) genealogical tree:



Among the Ph.D. students of Pearcy were N. Salinas (1940–2005), L.A. Fialkow (1946) and H. Bercovici (1953).

Besides operator theory, Halmos also contributed to many other areas such as ergodic theory and Boolean algebra. His books and the article *How to write mathematics* have influenced several generations of young mathematicians. Most famous is his *Measure Theory* (1950).

8.2.2.17 Next, I list some researchers whose major subjects were concrete operators on spaces of analytic functions.

$$\underset{(1927\text{–}1989)}{\text{A.L. Shields}} \Rightarrow \underset{(1940)}{\text{J.H. Shapiro}} \Rightarrow \underset{(1953)}{\text{B.D. MacCluer.}}$$

Concerning composition operators, I also mention D.H. Luecking (1950), who was a pupil of L.A. Rubel (1928–1995).

H. Widom (1932) in Santa Cruz is a specialist in singular integral operators.

8.2.2.18 *Zygmund's school of harmonic analysis* had a decisive impact on interpolation theory and the theory of function spaces; see Coifman/Strichartz [1989•].

8.2.2.19 Unfortunately, mathematicians are not free of political or racial discrimination, even in the so-called "Free World." In his recollections, Phillips [1984•, p. 31] reported that in summer 1941, Ted Martin unofficially offered him the position of an instructor at M.I.T. However, Phillips was Jewish. *This put the chairman of the mathematics department in an "impossible" position.*

An authentic picture is given by Chandler Davis [1988•, p. 413]:

Between 1947 and 1960 it was even harder than usual for left-wingers in the United States to get by. If you were active on the left, or were thought to be, there were more ways then than now that you could be arrested or threatened with arrest, or have civil rights such as the right to travel abroad withdrawn.

J.L. Kelley (1916–1999) was fired by the University of California (Berkeley) in 1950 for refusing to sign a loyalty oath; see Kelley [1989•]. After intermediate stays at Tulane and Kansas, he returned in fall 1953. Kelley, a student of G.T. Whyburn (1904–1969), made substantial contributions to general topology, set theory, measure theory, and also to Banach space theory. In particular, he is well known through the books *General Topology* (1955) and *Linear Topological Spaces* (1963, in collaboration with his pupil Namioka).

8.2.2.20 The preceding presentation is, inter alia, based on articles collected in *The Bicentennial Tribute to American Mathematics 1776–1976* [AAA_{04}] and *A Century of Mathematics in America* [AAA_{05}]. Furthermore, the reader is referred to articles by Garrett Birkhoff [1977•], Duren [1976•], and Mac Lane [1989•].

8.2.3 The former Soviet Union, mainly Russia and Ukraine

8.2.3.1 In the Soviet Union, functional analysis was constituted as an individual mathematical discipline in the mid 1930s. In Vol. **1** of *Успехи Математических Наук*, which appeared in spring 1936, L.A. Lyusternik (1899–1981) published a survey on *Basic concepts of functional analysis.* This article [1936] became the origin of the famous book *Элементы функционального анализа* (1951) written by him in collaboration with his pupil V.I. Sobolev (1913–1995).

The first conferences on the new discipline took place in Moscow (1937) and Kiev (1940). Moscow, Leningrad (now Sankt Petersburg), Odessa, Kiev, Kharkov, Kishinev, and Voronezh became the main centers.

A typical feature of Soviet functional analysis is its close connection with applications. Banach space theory was mainly considered as an auxiliary device to deal with orthogonal expansions, integral equations, and differential equations, approximation theory, and numerical mathematics. A singular point is the *Kadets school* in Kharkov, where Banach spaces are studied in their own right.

Mathematical research took place at the institutes of the Academy of Sciences of the USSR and its branches as well as at the universities and colleges (VUZ=Высшее Учебное Заведение).

Unfortunately, the history of mathematics in the Soviet Union is inseparably connected with anti-Semitism. Many famous functional analysts were unable to obtain positions at first-rate institutions; they were compelled to work at colleges (with heavy teaching duties). The most important academic degree (д-р. физ.-мат. наук) had to be approved by the *Higher Certification Commission* sitting in Moscow. It happened that this committee rejected very good theses of Jewish mathematicians; examples are A.S. Markus (Kishinev), Yu.L. Shmulyan (Odessa), and G.Ya. Lozanovskiĭ (Leningrad). The reader is referred to Gohberg's first-hand report [1989•].

8.2.3.2 Most Muscovite pioneers of functional analysis were members of *Luzin's school* (Luzitania):

N.N. Luzin
(1883–1950)

A.N. Kolmogorov, (1903–1987)	D.E. Menshov, (1892–1988)	A.Ya. Khintchine, ((1894–1959))	P.S. Aleksandrov, (1896–1982)
I.M. Gelfand, (1913)	S.M. Nikolskiĭ, (1905)	S.B. Stechkin, (1920–1995)	A.N. Tikhonov. (1906–1993)

8.2.3.3 In the early 1930s, there were two copies of Banach's monograph in Moscow, owned by A.N. Kolmogorov and A.I. Plesner. The latter based his seminar on this book and became the founder of the *Moscow Functional Analysis Seminar*. The list of participants included L.A. Lyusternik (as a partner) as well as Kolmogorov's famous Ph.D. students I.M. Gelfand and S.M. Nikolskiĭ.

A.I. Plesner (1900–1961), who earned his Ph.D. at the University of Gießen, is well known for a monograph on *Spectral Theory of Linear Operators*, which extends previous surveys; see [PLE], [1941] and [1946]. Some of his work was written in collaboration with or edited by his pupil V.A. Rokhlin (1919–1984). According to Lyusternik [1936, стр. 115] and the Russian folklore, Plesner discovered the first criterion of reflexivity for Banach spaces; see 3.6.8. Most likely, his unpublished proof was presented in the seminar mentioned above.

8.2.3.4 Certainly, the theory of topological linear spaces was not the favorite discipline of A.N. Kolmogorov. Nevertheless, he wrote some short but stimulating papers on this subject:

- a criterion of compactness in L_p; see [1931b],
- a characterization of normable topological linear spaces: [1934],
- the first use of Borel measures on Banach spaces: [1935],
- Kolmogorov widths: [1936],
- the concept of approximative dimension: [1958].

His joint work [1959] with V.M. Tikhomirov (1934) on ε-entropy and ε-capacity is also extremely important; see 6.7.8.15.

Another remarkable achievement of A.N. Kolmogorov is the two-volume textbook *Элементы теории функций и функционального анализа* (1954/1960) that he wrote in collaboration with his pupil S.V. Fomin (1917–1975); see [KOL$^+$].

8.2.3.5 I.M. Gelfand is the most prominent pupil of A.N. Kolmogorov. His thesis had the title *Abstrakte Funktionen und lineare Operatoren* [1938]. Gelfand's mathematical spectrum is impressive; it ranges from Banach space theory to computational mathematics, physiology, and biology.

In the beginning, the famous *Gelfand Seminar* (founded on his thirtieth birthday, September 1943) dealt with the theory of Banach algebras. The leading contributors were

G.E. Shilov (1917–1975), the first Ph.D. student of I.M. Gelfand,

D.A. Raĭkov (1905–1980), a student of A.Ya. Khintchine,

N.Ya. Vilenkin (1920–1991), a student of A.G. Kurosh,

M.A. Naĭmark (1909–1978), the first Ph.D. student of M.G. Kreĭn.

The results have been summarized in two monographs, [GEL$^+$] and [NAI].

The next highlight was the theory of generalized functions, which culminated in the multi-volume treatise *Обобщенные функции* (1958–1962); see [GEL$_4^+$]. The program of this seminal work was sketched in [1956]; Gelfand's coauthors were G.E. Shilov, N.Ya. Vilenkin and M.I. Graev. However, the younger generation also made substantial contributions. I mention R.A. Minlos (1931), a pupil of I.M. Gelfand, as well as A.G. Kostyuchenko (1931) and B.S. Mityagin (1937), both pupils of G.E. Shilov.

8.2.3.6 The *Moscow Seminar on the Theory of Functions of Several Variables* has been held under the direction of S.M. Nikolskiĭ, V.I. Kondrashov (1919), L.D. Kudryavtsev (1923) since 1956. The list of active participants includes O.V. Besov (1933) and P.I. Lizorkin (1922–1993). Thus one may claim that Besov spaces and (one half of) the Lizorkin–Triebel spaces were born in this seminar.

Certainly, the classical work of S.N. Bernstein (1880–1968) has been an important source of inspiration. The same can be said about Sobolev's book [SOB].

A milestone was Nikolskiĭ's monograph *Approximation of Functions of Several Variables* (1969). In 1960, his pupil A.F. Timan (1920–1988) has published a similar book with emphasis on the 1-dimensional case. Comparing [NIK] and [TIM] shows how and why the concept of a function space became dominant.

8.2.3.7 The most prominent contributor to the theory of widths was certainly V.M. Tikhomirov, a pupil of A.N. Kolmogorov. However, there were many other leading Soviet mathematicians working in this field, among them M.S. Birman (1928) and M.Z. Solomyak (1931) in Leningrad, B.S. Mityagin (1937) and R.S. Ismagilov (1938) in Moscow, and Yu.I. Makovoz (1937) in Minsk. From the next generation, I mention

V.E. Maĭorov (1946), a student of V.M. Tikhomirov in Moscow,

B.S. Kashin (1951), a student of P.L. Ulyanov in Moscow,

E.D. Gluskin (1953), a student of M.Z. Solomyak in Leningrad.

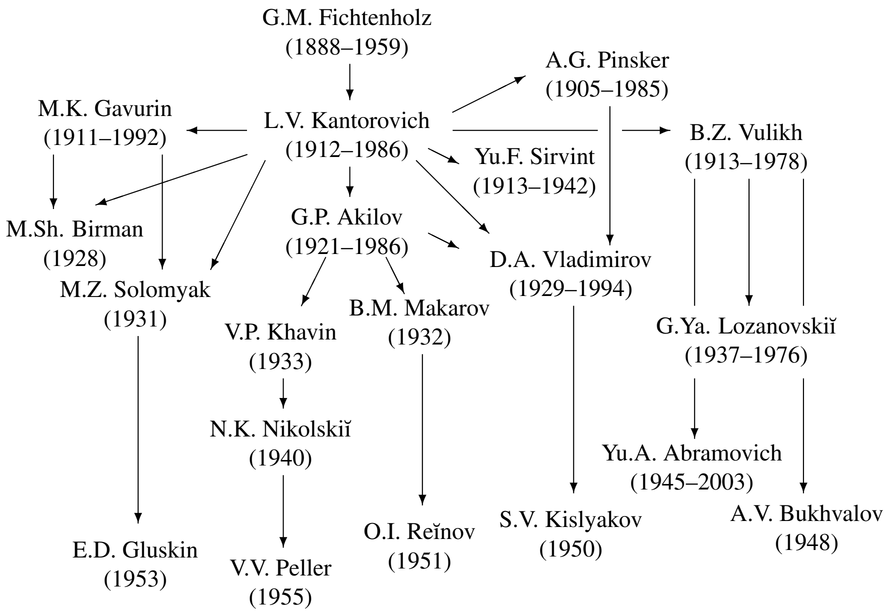
8.2.3.8 In Moscow there is also a strong group working on trigonometric series. Its nestor was D.E. Menshov (1892–1988). He had an equal partner, N.K. Bari (1901–1961), who wrote the famous book *Тригонометрические ряды* (1961); see [BARI$^+$]. After her untimely death, P.L. Ulyanov (1928) became a professor at the Moscow State University. The offspring of this school includes A.M. Olevskiĭ (1939), E.M. Nikishin (1945–1986), S.B. Bochkarev (1941), and B.S. Kashin (1951).

8.2.3.9 Last but not least, I stress the merits of P.S. Alexandrov and his topological school in clarifying the concept of compactness. His pupil A.N. Tikhonov discovered the product topology and laid the foundation for all investigations concerned with weak topologies. L.S. Pontryagin (1908–1988) and L.G. Shnirelman (1905–1938) characterized the dimension of metric spaces in terms of ε-entropy.

8.2.3.10 The first seminar on functional analysis at the University of Leningrad was organized in 1933 under the leadership of V.I. Smirnov (1887–1974). Among the participants were G.M. Fichtenholz, L.V. Kantorovich, N.M. Gyunter, S.G. Mikhlin, A.A. Markov (junior), M.K. Gavurin, and B.Z. Vulikh. In the very beginning, the work of this seminar had a purely educational character; in particular, Banach's book was studied; see [1987$^\bullet$]. But, almost immediately, some new results were produced. For example, Fichtenholz/Kantorovich [1934] identified the dual of $L_\infty[0,1]$.

Without any doubt, V.L. Kantorovich was the dominating figure for many years to come. At the age of 20, he became a professor at the Leningrad University and stayed there until his move to Novosibirsk in 1960. Concerning functional analysis, his main contributions were devoted to the theory of Banach lattices and to application in numerical mathematics. A highlight was the monograph *Функциональный анализ в полуупорядоченных пространствах* (1950) that he wrote in collaboration with B.Z. Vulikh and A.G. Pinsker; see [KAN$_2^+$]. Unfortunately, there is no English translation. Vulikh's book [VUL] may serve as a substitute. I also mention a well-known textbook on *Functional Analysis in Normed Spaces* (1959) written by Kantorovich/Akilov.

The radiating power of Kantorovich is demonstrated by a genealogical tree:



A.M. Vershik (1933), another Ph.D. student of G.P. Akilov, must be mentioned as well.

8.2.3.11 Besides the members of Kantorovich's school, many other prominent mathematicians were working at Leningrad University.

S.G. Mikhlin (1908–1990) was the first to treat singular integral equations from a functional analytic point of view. I.C. Gohberg considers him as one of his teachers. S. Prößdorf (1939–1998), an East German Ph.D. student and coauthor of Mikhlin, transferred his ideas to Chemnitz (at that time, Karl-Marx-Stadt) and laid the foundation for a well-known school; see 8.2.8.13.

Another functional analyst from Leningrad is V.G. Mazya (1937), who moved to Sweden after the перестройка (perestroĭka). Very close to Mikhlin, but not his Ph.D. student, he made substantial contributions to the theory of Sobolev spaces; see [MAZ].

I do not know whether A.A. Markov (1903–1979) and A.D. Aleksandrov (1912–1999) ever collaborated. In any case, both proved representation theorems for functionals on $C_b(M)$; see 4.6.5 and 4.6.6.

In the course of his life, the famous number theorist Yu.V. Linnik (1915–1972) became interested in Stochastics. Among his pupils in this new field were I.A. Ibragimov (1932) and V.N. Sudakov (1934). Both superviswd V.S. Tsirelson (1950), who, as an *outsider*, solved one of the most challenging problems about Banach spaces; see 5.6.3.11 and Subsection 7.4.3.

8.2.3.12 After his studies in Paris, S.N. Bernstein (1880–1968) became a professor at the University of Kharkov in 1907. His contributions to approximation theory inspired several generations of Soviet mathematicians. Since Bernstein went to Leningrad in 1933, he did not belong to the founding fathers of the *Ukrainian functional analytic school*. These were

M.G. Kreĭn,	N.I. Akhiezer,	B.Ya. Levin.
(1907–1989)	(1901–1980)	(1906–1993)

The life and work of M.G. Kreĭn will be described in Subsection 8.3.5. Some information about N.I. Akhiezer (Kharkov) and B.Ya. Levin (Odessa and Kharkov) is given in the following two paragraphs.

8.2.3.13 Though mainly interested in approximation theory, spectral theory in Hilbert spaces and the calculus of variations, N.I. Akhieser had a strong impact on the development of Banach space theory in Ukraine. He worked in Kharkov from 1933 to his death, interrupted only by World War II. Of special importance was his collaboration with M.G. Kreĭn in the theory of moments; see [AKH$^+$].

8.2.3.14 In 1949, B.Ya. Levin, well known for his treatise on *Distribution of Zeros of Entire Functions* (1956), moved from Odessa to Kharkov. Among his new Ph.D. students was M.I. Kadets (1923) who became, for a long period, the leading figure in Banach space theory not only in Kharkov, but in the whole USSR. He was mainly influenced by reading the Ukrainian translation of Banach's book [BAN], which had appeared in 1948: *Курс функціонального аналізу*.

Throughout his scientific career, M.I. Kadets was subject to anti-Semitism. As a consequence, he never obtained a professorship at the University of Kharkov; his places of work were various colleges (ВУЗ's) in Kharkov. In 1965, Kadets became the chair of higher mathematics at the Харковский институт инженеров коммунального строительства (Kharkov Institute of Communal Construction Engineering).

The most striking result of M.I. Kadets is his solution of the Banach–Fréchet problem: all separable infinite-dimensional Banach spaces are homeomorphic; see 5.5.4.1 and 5.6.1.5. My personal favorite is, however, the Kadets-Snobar theorem; see 6.1.1.7.

M.I. Kadets had many prominent Ph.D. students: V.I. Gurariĭ (1935–2005), M.I. Ostrovskiĭ (1960), S.L. Troyanski (1944; Bulgaria), G.A. Alexandrov (1950; Bulgaria), S. Heinrich (1950; Germany) and, K.-D. Kürsten (1950; Germany). His son Vladimir Mikhailovich (1960), who earned his Ph.D. under N.S. Landkof (1915) from Rostov-on-Don State University, also became a leading expert in Banach space theory. Father and son wrote a joint book on *Rearrangements of Series in Banach Spaces* (1988).

V.P. Fonf (1949), who, for political reasons, received his Ph.D. from Sverdlovsk and his habilitation from Warsaw, also considers M.I. Kadets as his teacher.

8.2.3.15 Among the Ph.D. students of B.Ya. Levin were V.I. Matsaev (1937) and V.D. Milman (1939).

8.2.3.16 Because of his Jewish origin, I.C. Gohberg (1928) was not accepted as an official Ph.D. student. Thus he wrote a thesis in his spare time and defended it at the Leningrad Pedagogical Institute (1954); the examiners were G.P. Akilov and S.G. Mikhlin. However, his development into a leading expert in operator theory was mainly stimulated by M.G. Kreĭn.

After working at teachers colleges in Soroki (1951–1953) and Beltsy (1953–1959), Gohberg moved to the Mathematical Department of the Moldavian Academy of Science in Kishinev, where he stayed until his emigration to Israel in 1974. In this city the famous *Gohberg school on operator theory* grew up. Members of the first generation were A.S. Markus (1932), N.Ya. Krupnik (1932), and I.A. Feldman (1933). The number of pupils of the second generation is much larger; see Gohberg [1989•, p. 43].

I also stress the joint work of Markus (Kishinev) and Matsaev (Kharkov).

8.2.3.17 In 1953/54 two prominent functional analysts, M.A. Krasnoselskiĭ (1920–1997) and S.G. Kreĭn (1917–1999, the younger brother of M.G. Kreĭn), moved from Kiev to Voronezh, where they founded a famous school. Typically, Kreĭn had to work for some periods (1954–1964 and 1972–1986) at the Леснотехнический институт (Forestry Technical Institute). While Krasnoselskiĭ became famous as a master of non-linearity, Kreĭn made also significant contributions to the linear theory; see [KREIN] and [KREIN$^+$]. Among others, he was the editor of a *Dictionary on Functional Analysis* [KREIN$^\cup$]. Most impressively, Kreĭn had 83 Ph.D. students, a number that is worthy of being included in the *Guinness Book of Records*. His best pupils in Banach space theory are Yu.I. Petunin (1937), E.M. Semenov (1940), and V.I. Ovchinnikov (1947).

S.G. Kreĭn also stimulated Yu.A. Brudniĭ (1934) and N.Ya. Kruglyak (1949) to deal with interpolation theory. M.A. Krasnoselskiĭ and Ya. B. Rutitskiĭ (1922) wrote the first book on *Convex Functions and Orlicz Spaces* (1958). Further information about the *Voronezh school* can be found in a report of Krasnoselskiĭ et al. [1964•].

8.2.3.18 The development in Львiв (the former Lwów) after 1945 is described in an article of Lyantse/Plichko/Storozh [1992•]. Only very few people were interested in Banach spaces. V.V. Shevchik (1949), who worked there from 1978 to 1981, emigrated to Germany in 1995, and A.N. Plichko (1949) went to Kirovograd in 1995. At present, the main fields of research in functional analysis are operator theory in Hilbert spaces (V.E. Lyantse, 1920) and general topology in Banach spaces (T. Banakh).

8.2.3.19 Tbilisi (Tiflis), the capital of Georgia, has a great mathematical tradition. In modern times, the outstanding contributions of N.I. Musheliskhvili (1891–1976) on singular integral equations must be emphasized. Concerning the subject of this text, I refer to N.N. Vakhania (1930), who, in collaboration with his pupils V.I. Tarieladze (1949) and S.A. Chobanyan (1942), wrote a monograph on *Probability Distributions in Banach spaces* (1985); see also [VAKH].

8.2.3.20 In the academic year 1980/81, E. Oja (1948) visited P. Billard at the University of Marseille and became interested in Banach spaces. Back in Tartu, she continued her work on this subject. Thanks to her activities, a group of young Banach space people is now successfully working in Estonia: Põldvere (1968), Ausekle (1970), Haller (1975).

8.2.3.21 Besides the emigrants to Israel, who are listed in 8.2.4.4, the former Soviet Union lost many functional analysts to other countries.

J.D. Tamarkin (in 1925 to the USA),
G.G. Lorentz (in 1944 via Germany and Canada to the USA),
B.S. Mityagin (in 1979 to the USA),

A real exodus happened after the перестройка (perestroĭka):

I.M. Gelfand (now at Rutgers University, New Jersey, USA),
V.I. Gurariĭ (was at Kent State University, Ohio, USA),
A.L. Koldobskiĭ (now at the University of Missouri, Columbia, USA),
N.Ya. Kruglyak (now in Luleå, Sweden),
V.I. Lomonosov (now at Kent State University, Ohio, USA)
Yu.I. Makovoz (now at the University of Massachusetts, Lowell, USA),
V.G. Mazya (now in Linköping, Sweden),
N.K. Nikolskiĭ (now in Bordeaux, France),
M.I. Ostrovskiĭ (now at the Catholic University of America, Washington, D.C.),
V.V. Peller (now at Michigan State University, East Lansing, USA).

8.2.3.22 In countries of the former Eastern Bloc, in particular in the Soviet Union, scientists had a very high reputation. In my opinion this can be explained by the fact that "having money" was only a necessary but not a sufficient condition for a comfortable life. Hence personalities were evaluated by other criteria.

As a consequence, there is considerable biographical information. First of all, I recommend the volumes *Математика в СССР*, [AAA_{07}], [AAA_{08}], [AAA_{09}], and [AAA_{10}]. Of particular interest are the surveys of Kreĭn/Lyusternik [$1948^\bullet$] and M.A. Krasnoselskiĭ et al. [$1958^\bullet$], [$1964^\bullet$]; see also Lyusternik [$1967^\bullet$]. Moreover, the Journals *Успехи Математических Наук* and *Математика в Школе* provide a nearly complete picture of the mathematical life in the former Soviet Union. Concerning anti-Semitism, one has to read between the lines and to speculate about the articles that have never been written. Lorentz [$2002^\bullet$] gave a detailed description of the political persecution during the Stalin era.

8.2.4 Israel

8.2.4.1 The history of functional analysis in Israel can be described quite simply. There are two essential roots (with non-empty intersection):

the Dvoretzky–Lindenstrauss school
and
the immigrants from Ukraine and Russia.

8.2.4.2 The parents of A. Dvoretzky (1916) immigrated from Ukraine to Israel in 1922. He enrolled at Hebrew University, which had been founded in 1925, and earned his Ph.D. there in 1941 under M. Fekete (1886–1957). His first student in Banach space theory was B. Grünbaum (1929), who became a well-known specialist in convex geometry and proved some results on projection constants. Grünbaum is now working at the University of Washington (Seattle). Most important for the development in the country were Dvoretzky's Ph.D. students J. Lindenstrauss (1936) and D. Amir (1933). The decisive breakthrough was achieved when Dvoretzky proved the theorem on spherical sections, around 1960.

8.2.4.3 In his survey

Some aspects of the theory of Banach spaces,

Lindenstrauss [1970a] proposed a guiding program. Since 1968, he has supervised many Ph.D. students. This phenomenon is described as *the industry of Lindenstrauss*, a term coined in [DAY, 3rd edition, p. 183]. As of now, the following have become professors (the first number in parentheses gives the year of birth):

A. Lazar (1936; University of Tel Aviv),
M. Zippin (1939; Hebrew University, Jerusalem),
Y. Gordon (1940; Technion, Haifa),
J. Arazy (1942; University of Haifa),
Y. Benyamini (1943; Technion, Haifa),
Y. Sternfeld (1944–2001; University of Haifa),
I. Aharoni (1945; Jerusalem College for Women),
G. Schechtman (1947; Weizmann Institute, Rehovot).

Among the younger ones are M. Rudelson (1965) and A. Naor (1975).

8.2.4.4 On the other hand, there has been a permanent flow of immigrants from the former Soviet Union:

A. Dvoretzky (1916; from Ukraine in 1922 as a child),
V.D. Milman (1939; from Moscow in 1973),
L.E. Lerer (1943; from Kishinev in 1973),
D.P. Milman (1913–1982; from Odessa in 1974),
I.C. Gohberg (1928; from Kishinev in 1974),
M.S. Livsic (1917; from Kharkov via Tbilisi in 1978),
E.D. Gluskin (1953; from Leningrad in 1989),

N.Ya. Krupnik (1932; from Kishinev in 1990),
A.S. Markus (1932; from Kishinev in 1990),
V.I. Matsaev (1937; from Kharkov in 1990),
M.Z. Solomyak (1931; from Sankt Petersburg in 1991),
I.A. Feldman (1933; from Kishinev in 1991),
Yu.A. Brudnyi (1934; from Yaroslavl in 1991),
A.M. Olevski (1939; from Moscow in 1991),
B.S. Tsirelson (1950; from Sankt Petersburg in 1991),
S.M. Gorelik (1947; from Moscow in 1992),
Yu.I. Lyubich (1931; from Kharkov in 1992),
V.E. Maiorov (1946; from Moscow in 1992),
V.P. Fonf (1949; from Kharkov in 1993).

Concerning Banach space theory, the most important immigrant is Vitali Milman, who became famous thanks to his contributions related to Dvoretzky's theorem. Even more renowned, because of the Kreĭn–Milman theorem, is his father David Milman. Still living in Soviet Union, Vitali wrote two surveys in *Успехи Математических Наук*; see [1970], [1971a]. After his arrival in Israel, a *school of geometric functional analysis* grew up in Tel Aviv. Milman's subject gradually changed into a new direction, which he calls *Asymptotic Geometric Analysis*; see 6.9.15. Since 1991, there has appeared the journal Geometric And Functional Analysis whose founding was initiated by him. Among his Ph.D. students were H. Wolfson (1951), A. Litvak (1969), S. Dar (1970), S. Alesker (1972), R. Wagner (1973), and S. Artstein (1978).

8.2.4.5 To complete the picture, I present a list of immigrants from other countries:

P.D. Saphar (1934; from France in 1970), Ph.D. supervisor of S. Reisner (1943),
L. Tzafriri (1936; from Romania in 1961), Ph.D. supervisor of A. Altshuler (1940),
H. Dym (1938; from USA in 1970); see [ALP$_2^{\cup}$, pp. 1–17],
A. Szankowski (1945; from Poland via Denmark in 1980),
A. Pinkus (1946; from Canada in 1970),
M. Cwikel (1948; from Australia in 1970).

8.2.4.6 In 1939, because of the Nazi terror, two old mathematicians emigrated from Germany: I. Schur (1875–1941) and O. Toeplitz (1881–1940). However, neither had an impact on the development in Israel. Mentally and physically broken, they passed away in 1941 and 1940, respectively.

8.2.4.7 At present, the number of sabras (native-born Israelis) among Banach space people is still small. The earliest example known to me is S.R. Foguel (1931), a Ph.D. student of N. Dunford (1906–1986) at Yale. He had an outstanding pupil: L. Tzafriri (1936). J. Lindenstrauss (1936), whose parents emigrated from Berlin, was born in Tel Aviv. I further mention A. Pazy (1936), who earned his Ph.D. from Hebrew University under S. Agmon (1921).

8.2.4.8 A seminal contribution to functional analysis is the two-volume treatise of Lindenstrauss/Tzafriri on *Classical Banach Spaces*, which appeared in 1977/79; see also the preliminary version [LIND_0^+]. Originally, the authors planned to write four volumes. However, the development was too vigorous. Thus, in order to finish this project, a complete rewriting would have been necessary. The stimulating power of [LIND_1^+] and [LIND_2^+] cannot be overestimated.

Another important achievement of Banach space theory in Israel is the treatise of Milman/Schechtman on *Asymptotic Theory of Finite Dimensional Normed Spaces.*

8.2.4.9 An *International Symposium on Linear Spaces* [AAA_{19}] was organized in Jerusalem (1960). Likewise at Hebrew University, conferences on Banach space theory took place in 1991 and 2005, devoted to the 75th birthday of Dvoretzky and in honor of Lindenstrauss–Tzafriri, respectively.

8.2.4.10 The *Israel Journal of Mathematics*, founded in 1963, contains many famous papers on Banach space theory. Lectures delivered at the *Israel Seminar on Geometric Aspects of Functional Analysis* have been published since 1983 under the short title **GAFA**; see p. 723.

8.2.5 France

8.2.5.1 French mathematicians contributed substantially to the foundation of functional analysis. First of all, we must refer to J. Hadamard (1865–1963, who studied continuous linear functionals on $C[a,b]$. He also coined the term *calcul fonctionnel*; see [1912]. Among his pupils were M. Fréchet (1878–1973), R. Gâteaux (1889–1914), and P. Lévy (1886–1971).

Extremely important was the French school of real analysis:

É. Borel, (1871–1956) R. Baire, (1874–1932) H. Lebesgue. (1875–1941)

Indeed, the theory of function spaces is unthinkable without Lebesgue's integral.

8.2.5.2 The term *analyse fonctionnelle* first appeared in the title of Lévy's monograph *Leçons d'analyse fonctionnelle* (1922). However, this treatise had only little impact on the later development. An exception is the *concentration of measure phenomenon* established in the second and extended edition (1951).

8.2.5.3 According to Mandelbrojt/Schwartz [1965•, p. 118],

Mathematical life in Paris in the twenties and early thirties was for the large part described by two words: SÉMINAIRE D'HADAMARD.

8.2.5.4 In the early 1940s, J. Dieudonné (1906–1992) and R. Fortet (1912–1998) were the first in France to study Banach spaces. Subsequently, due to Bourbaki's general standpoint, Banach spaces were regarded as a special class of locally convex linear spaces, which did not deserve much interest in its own right. An extraordinary exception is Grothendieck's *Résumé* written in 1953; see 5.7.2.13 and 6.3.11.8.

8.2.5.5 According to Pełczyński [1983, p. 238], *modern history* [add: of Banach space theory] *starts with the Great French Revolution in Banach spaces.* The *Séminaire Laurent Schwartz 1969/70* on *Applications radonifiantes* may be compared with the Storming of the Bastille. However, its nickname *Le Séminaire Rouge* has no political meaning; the cover of the printed version was red; see Maurey [1985•, p. 38]. In the following 12 years, the notes of this seminar (published under different titles) reflect the vigorous development of Banach space theory; see p. 722.

Here is a selection of French contributors (first appearance):

1969/70: P. Assouad, A. Badrikian, S. Chevet, P. Saphar, L. Schwartz,
1971/72: D. Dacunha-Castelle, B. Maurey,
1972/73: B. Beauzamy, J.-T. Lapresté, A. Nahoum, G. Pisier,
1973/74: A. Brunel, J.L. Krivine,
1977/78: M. Talagrand,
1978/79: J.B. Baillon,
1979/80: S. Guerre, M. Levy,
1980/81: G. Godefroy.

The main result of this period was the theory of type and cotype, which is inseparably connected with the names of Maurey and Pisier; see 6.1.7.9 and 6.1.7.26.

8.2.5.6 The following tree shows the Banach space component of the *Schwartz school*. However, the arrows should not be taken too seriously, since among the younger generation there was a real teamwork, and roughly speaking, everybody influenced everybody. Pisier told me that he was decisively inspired by his collaboration with Maurey.

L. Schwartz (1915–2002)

B. Maurey (1948) | B. Beauzamy (1949) | J.-T. Lapresté (1949) | G. Pisier (1950)

V. Ferenczi (1970) | S. Guerre (1953) | M. Levy (∼1955) | A. Pajor (1949) | Quanhua Xu (1961)

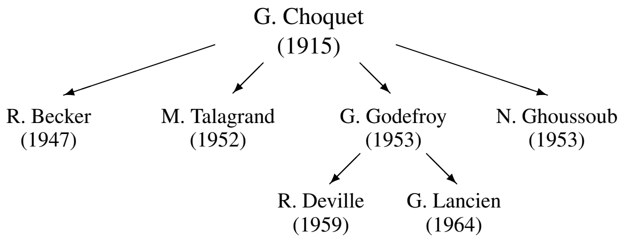
A. Badrikian (1933–1994), though a Ph.D. student of R. Fortet, was greatly stimulated by L. Schwartz as well. Badrikian supervised S. Chevet (1941). Another Ph.D. student of L. Schwartz (and J. Dixmier) was P. Saphar (1934), who became a professor at the Technion Haifa.

8.2.5.7 In 1982, B. Beauzamy published the first elementary *Introduction to Banach Spaces and their Geometry*, a very useful counterpart of *Lindenstrauss–Tzafriri.*

Pisier's lectures on *Probabilistic Methods in the Geometry of Banach Spaces* and *Factorization of Linear Operators and Geometry of Banach Spaces* as well as his book *The Volume of Convex Bodies and Banach Space Geometry* (1989) had an enormous impact.

Here are some further Banach space books of French mathematicians: [BEAU$_3$], [BEAU$^+$], [GUER], [HAY$^+$], [PAJ], and, very recently, [LI$^+$].

8.2.5.8 G. Choquet (1915), a Ph.D. student of A. Denjoy (1884–1974), also supervised several Banach space pupils:



Deville/Godefroy/Zizler wrote a remarkable book on *Smoothness and Renormings in Banach Spaces* (1993). H. Brézis (1944), another Ph.D. student of Choquet, made substantial contributions to *Analyse fonctionnelle. Théorie et applications* (1983).

8.2.5.9 A typical feature of "French" Banach space theory is its close connection with probability and harmonic analysis; the term "vector-valued random trigonometric series" illustrates this interplay. J.-P. Kahane (1926) wrote an influential textbook on this subject, [KAH]. The treatise [MAR$^+$] is more advanced.

8.2.5.10 In the late 1960s, but only for a short while, the probabilists J. Bretagnolle (1937) and D. Dacunha-Castelle (1937) made important contributions to the theory of Banach spaces; [AAA$_{17}$]. Both were pupils of R. Fortet (1912–1998), a Ph.D. student of M. Fréchet (1878–1973). They had a successful collaboration with J.-L. Krivine (1939). The latter was a pupil of G. Kreisel (1923), who introduced him to model theory. Krivine applied this knowledge to functional analysis and probability. His main achievements in Banach space theory are the famous Krivine theorem and the concept of "ultraproducts," invented in a joint paper with D. Dacunha-Castelle; see 6.1.3.6. One of his pupils, J. Stern (1949), wrote his first papers on the same subject. Among Krivine's Ph.D. students there is also a permanent Banach space specialist: Y. Raynaud (1952).

8.2.5.11 Last but not least, I mention that M. Talagrand and M. Ledoux (1958), a pupil of X. Fernique (1934), wrote a fine treatise on *Probability in Banach Spaces* (1991). Another book of M. Ledoux is devoted to *The Concentration of Measure Phenomenon* (2001).

8.2.5.12 A *Colloque d'analyse fonctionnelle* took place in Bourdeaux (1971), and a *Colloque en l'honneure de Laurent Schwartz* was held at École Polytechnique (1983). Furthermore, the Université de Marne-La-Vallée organized a *Congrès international sur la convexite* (1994); see also 7.1.28.

8.2.6 United Kingdom

8.2.6.1 According to [YOU•, p. 293], *a second golden age that surpassed the age of Newton* began at Cambridge in 1931 when Hardy became Sadleirian professor of pure mathematics. *Cambridge was suddenly at least the equal of Paris, Copenhagen, Princeton, Harvard and of Warsaw, Leningrad, Moscow*. Now G.H. Hardy (1877–1947) and J.E. Littlewood (1885–1877), whose famous partnership dates back to 1911, were working at the same university. Though classical analysts, they prepared the way for modern developments.

8.2.6.2 With respect to functional analysis, F. Smithies (1912–2002) was the most important of Hardy's pupils. Besides his well-known book on *Integral Equations* (1958), Smithies wrote only a small number of research papers. However, he was a highly gifted teacher. The list of his 32 Ph.D. students looks quite impressive. Without exaggeration one can claim that Smithies is the father of British functional analysis. Among his pupils were A.F. Ruston (1920–2005), J.R. Ringrose (1932), and D.J.H. Garling (1937). In addition, F.F. Bonsall (1920) acknowledged that he was strongly influenced by him. Further Ph.D. students of Smithies are spread over the world:

Ireland	:	T.T. West (1938),
Australia	:	A.P. Robertson (1925–1995), W.J.R. Robertson (1927),
USA	:	S.J. Bernau (1937), S. Simons (1938),
Austria	:	J.B. Cooper (1944),
China	:	Shih-Hsun Chang (1900–1985).

The next generation of functional analysts from Cambridge was educated by D.J.H. Garling. Among his Ph.D. students were D.H. Fremlin (1942), G. Bennett (1945), N.J. Kalton (1946), R.G. Haydon (1947), D.J. Aldous (1952), S.J. Dilworth (1959), and S.J. Montgomery-Smith (1963).

8.2.6.3 A.F. Ruston deserves a special place among Smithies's pupils. A few years before A. Grothendieck, he developed a theory of direct products and discovered the concept of a trace class operator in Banach spaces; see 5.7.3.5. His book on *Fredholm Theory in Banach Spaces* (1986) contains a very personal chapter *Notes and comments*. For example, on p. 232 one can read:

The relevance of complete uniformity occurred to me on New Year's Day, 1956.

Though an Englishman, he was quite happy to live in Bangor (Wales): the book mentioned above has the Welsh subtitle *Theori Fredholm yng ngofodau Banach.*

8.2.6.4 As a supplement to [GOH$_3^+$], J.R. Ringrose published a little, but extremely nice book on *Compact Non-Self-Adjoint Operators* (1971). However, his main interest was concentrated upon Banach algebras. H.R. Dowson (1939), a pupil of Ringrose, is the author of a treatise on *Spectral Theory of Linear Operators* (1978) designed in the spirit of *Dunford–Schwartz*, Part III.

8.2.6.5 F.F. Bonsall (1920) had 22 Ph.D. students. In collaboration with J. Duncan (1938) he wrote a standard monograph on *Complete Normed Algebras* (1973). His pupil G.J.O. Jameson (1942) published an elementary text on *Topology and Normed Spaces* (1974) as well as a booklet on *Summing and Nuclear Norms in Banach Space Theory* (1987).

8.2.6.6 For many years, B. Bollobás (1943) was the adlatus of J.E. Littlewood at Trinity College. The former is well known for his contributions to graph theory. He also wrote an elementary textbook on *Linear Analysis* [BOL] and supervised several extraordinary students in Banach spaces theory: T.K. Carne (1953), J.R. Partington (1955), C.J. Read (1958), K.M. Ball (1960), and W.T. Gowers (1963).

At the International Congress of Mathematicians in Berlin (1998), Timothy **Gowers** was awarded the Fields Medal.

8.2.6.7 In 1931, E.C. Titchmarsh became the successor of G.H. Hardy at Oxford. According to Cartwright [1964•, p. 560]:

He himself never used even the terminology of linear spaces in his proofs though their use would have shortened his work, but in [TIT, Part II] *he translated some of his results into operator form.*

Titchmarsh's Banach space pupils are presented in the following tree:

- E.C. Titchmarsh (1899–1963)
  - J.B.L. Cooper (1915–1979)
    - R.E. Edwards (1926–2000)
    - D.E. Edmunds (1931)
  - F.V. Atkinson (1916–2003)

R.E. Edwards went to Australia, and F.V. Atkinson taught in Nigeria, Australia and Canada. Thus, taking into account the Ph.D. students of F. Smithies mentioned in 8.2.6.2, one may state that several British mathematicians found jobs in the former colonies.

8.2.6.8 Unfortunately, R.E.A.C. Paley (1907–1933), the most promising student of Littlewood, was killed by an avalanche while skiing near Banff, Alberta.

8.2.6.9 C.A. Rogers (1920), a pupil of H. Davenport (1907–1969), is the nestor of *convex geometry* in England. The Dvoretzky–Rogers lemma 6.1.2.1 was discovered during a visit at the Institute for Advanced Study; see Dvoretzky/Rogers [1950].

8.2.6.10 A.M. Davie (1946), a former student of Cambridge who got his Ph.D. under A. Page from Dundee, became famous for his decisive simplification of Enflo's counterexample to the approximation problem.

8.2.7 Further countries

8.2.7.1 Australia

Functional analysis in Australia was founded by British mathematicians. R.E. Edwards (1926–2000) arrived in Canberra in 1961, and A.P. Robertson (1925–1995) was appointed at Murdoch University in 1973. His wife, Wendy Robertson (1927), became a senior staff member of the University of Western Australia. Edwards wrote several well-known books, among them a huge treatise on *Functional Analysis* (1965). The little Cambridge Tract of the Robertson couple on *Topological Vector Spaces* (1964) is also highly appreciated.

Nevertheless, Banach space theory lives in the shadow. As singular points, I mention B. Sims (1947), P. Dodds (1942), a pupil of Luxemburg at the California Institute of Technology, and the Erdős-like cosmopolite D. Yost (1955, Melbourne) who received his Ph.D. from the University of Edinburgh under A.G. Robertson (no relative of A.P. Robertson).

8.2.7.2 Austria

The birth of Banach space theory is inseparably connected with such famous Austrian names as H. Hahn (1879–1934), E. Helly (1884–1943), and J. Radon (1887–1956). Moreover, L. Vietoris (1891–2002) made substantial contributions to the foundation of general topology; see 3.2.2.8.

Due to the Nazi occupation and World War II, there was a long break. The modern era started only in the early 1960s when the probabilist L. Schmetterer (1919–2004), who earned his Ph.D. under Radon, became a full professor at the University of Vienna. He introduced Banach spaces in his lectures. A next step was accomplished by appointing J. Cigler (1937), a pupil of E. Hlawka (1916). Among Cigler's Ph.D. students were P. Michor (1949), V. Losert (1952), and G. Racher (1952). In 1975/76, the young Viennese generation studied Grothendieck's *Résumé* in a seminar; there even is an (unpublished, German) manuscript edited by V. Losert.

Beginning in the mid 1970s and stimulated from abroad, modern Banach space theory was developed at the University of Linz by

C.P. Stegall (1943), a Ph.D. student of J.R. Retherford,

J.B. Cooper (1944), a Ph.D. student of F. Smithies,

W. Schachermayer (1950), strongly influenced by L. Schwartz and A. Badrikian,

P.F.X. Müller (1960), stimulated by P. Wojtaszczyk.

The colleagues from Linz organized a conference on *Geometry of Banach Spaces* at Strobl (1989); see [MÜLL$^{\cup}$].

8.2.7.3 Belgium

In Belgium, functional analysis was initiated by H.G. Garnir (1921–1985) and L. Waelbroeck (1929).

Until his untimely death, the *Liège school* was directed by Garnir. In collaboration with his pupils M. De Wilde (1940) and J. Schmets (1940) he wrote a three-volume treatise on *Analyse fonctionnelle*, which stressed the following point of view, [GARN$^+$, Part I, Introduction] (original version on p. 682):

We have only used such constructive kinds of reasoning, generally admitted in analysis, that are based on a correct language, traditional logic, naive set theory, the properties of integers and the axiom of countable choice.

We have avoided to apply the axiom of non-countable choice or its equivalent forms (*axioms of Zermelo, Zorn,...*). *In the past, there was a controversial discussion about this axiom and nowadays it is used without conviction or just by routine.*

At the Vrije Universiteit Brussels, Waelboeck worked on bornologies. He influenced F. Delbaen (1946) who obtained his Ph.D. in mathematical economics. In addition to his papers, the latter made a decisive contribution to Banach space theory: J. Bourgain (1954). In an e-mail to the author, Delbaen stated:

Jean Bourgain was my student for a while. Not for very long, because after a couple of months (*this guy was moving fast!!*) *I could learn more from him than he from me.*

At the International Congress of Mathematicians in Zürich (1994), Jean **Bourgain** was awarded the Fields Medal.

Another group of functional analysts is working in Mons. Among them, C. Finet (1956) who received her Ph.D. at Paris under Beauzamy (Thèse de troisième cycle) and Godefroy (Thèse d'Etat). The colleagues from the Université Mons-Hinaut organized three Banach space conferences (1987, 1992, and 1997). The last one stressed the interplay between *Analysis and Logic*, [FIN$^{\cup}$]. G. Lumer (1929–2005), who obtained his Ph.D. from the University of Chicago under I. Kaplansky (1917), has been a professor in Mons since 1973.

8.2.7.4 Brazil

The development of functional analysis in Brazil was strongly influenced by the Bourbakists. In the second half of the 1940s, J. Dieudonné and A. Weil were visiting professors at the University of São Paulo, each of them for two years. Next, Grothendieck stayed there in 1953/54 and wrote his famous *Résumé.* The University of Rio de Janeiro had good contacts with the University of Chicago. For example, in 1947, M. Stone delivered a three month course on *Rings of Continuous Functions*.

L. Nachbin (1922–1993) became the leading figure. He was not only an excellent researcher, but also an outstanding organizer, who played a decisive role in founding the Instituto de Matemática Pura e Aplicada (IMPA) in Rio de Janeiro. Furthermore, he edited the well-known series *Notas de Matemática*, which later went under the name *North-Holland Mathematical Studies*.

Among his Ph.D. students from South America were J.A. Barroso (1928), J.B. Prolla (1935), M.C. Matos (1939), G.I. Zapata (1940), and J. Mujica (1946).

Nachbin also was a professor at the University of Rochester (USA), where he supervised C. Gupta (1939), R.M. Aron (1944), and P.J. Boland (1944). All of them stayed in Brazil for longer periods. Similarly, S. Dineen (1944) got his Ph.D. from the University of Maryland, but he wrote his thesis under Nachbin in Brazil.

8.2.7.5 Bulgaria

In Bulgaria, Banach space theory is mainly represented by S.L. Troyanski (1944) and his Ph.D. students; among them R.P. Maleev (1943) and D.N. Kutzarova (1956). Troyanski himself was a pupil of M.I. Kadets (1923) at the University of Kharkov. His theorem on local uniform convexification is a classic; see 5.5.3.5. Kadets also supervised G.A. Aleksandrov (1950). Finally, I mention G.E. Karadzhov (1946). His contributions to interpolation theory and s-numbers were initiated by M.Z. Solomyak (1931).

8.2.7.6 Canada

Functional analysis was imported into Canada from various countries. The major branch (irrelevant for this text) is spectral theory for matrices and operators in Hilbert spaces.

Chandler Davis (1926), a pupil of Garrett Birkhoff at Harvard, emigrated from the USA in 1962 as a result of *"The purge"*; see Davis [1988•]. P. Lancaster (1929), born in England, received his Ph.D. from the University of Singapore. He went to Calgary in 1962. A book on *Invariant Subspaces* (1973) was written by H. Radjavi (1935) and P.M. Rosenthal (1941). The latter earned his Ph.D. from the University of Michigan; his teacher was P. Halmos.

I also mention F.V. Atkinson (1916–2003), who was appointed professor at the University of Toronto in 1960, after staying in Nigeria (1948–1955) and Australia (1956–1960). To D.W. Boyd (1941) we owe the concept of the index that now bears his name; see 6.6.7.10.

Here is a list of Banach space people:

- P.J. Koosis (1929, USA), a pupil of J.L. Kelley at the University of California (Berkeley),
- J.H.M. Whitfield (1939, USA), a pupil of E. Leach at Case Western Reserve University,
- V. Zizler (1943, Czechoslovakia),
- N. Tomczak–Jaegermann (1945, Poland), a pupil of A. Pełczyński at Warsaw,
- N. Ghoussoub (1953, Lebanon), a pupil of G. Choquet at Paris.

Tomczak-Jaegermann's monograph on *Banach–Mazur Distances and Finite-Dimensional Operator Ideals* (1989) has become a standard reference, and Koosis wrote a well-known book on *H_p Spaces* (1980). Zizler is a coauthor of [DEV$^+$], [FAB$^+$] and [HAB$^+$]; see the next paragraph.

A CMS Annual Seminar on *Banach Spaces and Geometry of Convex Bodies* was held in Banff (1988); see also 6.9.15.6.

Finally, I cannot resist mentioning that P.M. Rosenthal is not only a mathematician but also a lawyer, who mainly represents "leftist" demonstrators. On his homepage he reports about the following conversation:

A professor of law asked me, "what do you think about the relationship between mathematics and law?"
"I don't see much relation," I replied.
"Well," he said, "Kant wrote that the only two true sciences are mathematics and law."
"He was right about mathematics," I answered.

8.2.7.7 China

In the late 1940s, (Chao-Chih) Kwan (1919–1982) visited Fréchet in Paris and became interested in functional analysis. After his return to Beijing, he offered courses and seminars on this subject. Thus Kwan may be considered the founder of functional analysis in **China**. Further mathematicians from the pioneering era were (Yuan-Yung) Tseng (1903–1994), who earned his Ph.D. under Moore from Chicago, and (Dao-Xing) Xia (1930), a student of Gelfand in Moscow. The latter wrote a text on *Measure and Integration Theory on Infinite-Dimensional Spaces* (1972). The first Chinese dealing with spectral theory of non-self-adjoint operators was (Shih-Hsun) Chang (1900–1985). Supervised by Smithies, he obtained the Ph.D. from the University of Cambridge. After a stay in Princeton, where he met Weyl, Chang returned to China (around 1950) and was appointed professor at the University of Sichuan, Chengdu.

Today, the University of Harbin has developed into a center of functional analysis, the main object being Orlicz spaces; see the monograph [CHEN]. This group includes: (Shutao) Chen (1950), (Tingfu) Wang (1933–2000), (Yuwen) Wang (1950) and (Congxin) Wu (1935). (Zhong-Dao) Ren (1938), from Xiangtan University, also wrote a book on *Geometry of Orlicz Spaces* (1991), in collaboration with M.M. Rao.

A short glance at Mathematical Reviews or Zentralblatt shows that in resent years, several young Chinese mathematicians started to work in Banach space theory. Let me mention (Shanquan) Bu (1963), Tsinghua University in Beijing, who obtained his Ph.D. from Paris under Maurey in 1990, and (Lixin) Cheng (1959), Xiamen University in Fujian.

The preceding comments are mainly based on information borrowed from the preface of [LI$^\cup$].

8.2.7.8 Czechoslovakia

The topologist E. Čech (1893–1960) was the founder of modern mathematics in Czechoslovakia. He also influenced M. Katětov (1918–1995), who had considerable impact on the development of functional analysis in this country. Among his Ph.D. students were V. Pták (1925–1999), Z. Frolík (1933–1989), and K. John (1942). Due to political discrimination during the communistic era, several Banach space mathematicians emigrated:

V. Zizler (1943), a pupil of J. Kolomý (1934–1993), is now in Edmonton (Canada), where P. Hájek (1968) obtained his Ph.D. Another young Czech, P. Habala (1968), worked there under N. Tomczak-Jaegerman. D. Preiss (1947) became a professor in London.

A remarkable achievement of Zizler's school is a textbook on *Functional Analysis and Infinite-Dimensional Geometry* (2001), which had a forerunner published by Charles University (Prague, 1996).

A. Kufner (1934), who earned his Ph.D. under J. Nečas (1929–2002), founded a research group on *Function Spaces*. In collaboration with his pupils O. John (1940) and S. Fučik (1944–1979), he wrote a book about this subject; see [KUF$^+$].

In 1972, Z. Frolík founded the famous *Winter Schools on Abstract Analysis*. More than 30 meetings have taken place in various parts of the country. It is remarkable that these activities survived all political changes: once the Austrian colleagues from Linz helped out (by organizing the 20th school at Strobl), but regardless, the Czech mathematicians continued the series successfully.

Other well-known events (initiated by Čech) are the Prague symposia on *General Topology and its Relations to Modern Analysis and Algebra*, which took place in $1961+5(n-1), n=1,\dots,9$ (presently).

8.2.7.9 Denmark

According to Hardy/Littlewood/Pólya [HARD$^+$, p. 71],

the foundations of the theory of convex functions are due to Jensen [1906].

The tradition of J.L.W.V. Jensen (1859–1925) was continued by T. Bonnesen (1873–1935) and W. Fenchel (1905–1988), who wrote a classical Ergebnisbericht, *Theorie der konvexen Körper* (1934). I also mention that B. Jessen (1907–1993) proved Minkowski's inequality for integrals. However, it seems that this old generation had only little impact on the development of functional analysis in Denmark.

E.T. Poulsen (1931) produced a simplex that bears his name. In Odense, Banach space theory goes back to E. Asplund (1931–1974). N.J. Nielsen (1943) received his Ph.D. from Poland under the supervision of A. Pełczyński. U. Haagerup (1949), a student of G.K. Pedersen (1940–2004), is well known for important contributions to $C^\star$-algebras, and Banach space people appreciate his early work concerning the best constants in Khintchine's inequality. K.B. Laursen (1942) wrote *An Introduction to Local Spectral Theory* (2000, coauthor: M. Neumann). Last but not least, I stress the important role played by J. Hoffmann-Jørgensen (1942) in creating the concepts of type and cotype.

In memory of Asplund, a *Seminar on Random Series, Convex Sets and Geometry of Banach Spaces* was organized in Århus (1974); see [AAA_{20}].

8.2.7.10 Finland

K. Vala (1930–2000) initiated Banach space theory in Finland. Among his students were H. Apiola (1942), K. Astala (1953), and J. Taskinen (1966). A member of the next generation is H.-O. Tylli (1958), a pupil of Astala. It should also be mentioned that the thesis of T. Ketonen (1950) was based on a problem suggested by Enflo; see Ketonen [1981].

The functional analysts from Helsinki are also interested in locally convex linear spaces. By far, the most important contribution along these lines is due to Taskinen [1986], who produced a counterexample to Grothendieck's *Problème des Topologies*; see [GRO_1, Chap. I, pp. 33–34].

8.2.7.11 Greece

As everybody knows, mathematics had a great tradition in ancient Greece. Then there came a long break. Most Greek mathematicians used to be educated abroad. The situation changed only in the 1930s, when D. Kappos (1904–1995) returned from Germany, where he was strongly influenced by his teacher and compatriot C. Carathéodory.

The initiator of Banach space theory was S. Negrepontis, who received his Ph.D. from the University of Rochester:

E. Hewitt	$\Rightarrow$	W.W. Comfort	$\Rightarrow$	S. Negrepontis	$\Rightarrow$	S. Argyros	$\Rightarrow$	I. Deliyanni.
(1920–1999)		(1933)		(1939)		(1950)		(1964)

At the beginning, Negrepontis worked on general topology. In collaboration with Comfort, he wrote a well-known book on *The Theory of Ultrafilters* (1974). I also mention his surveys on *Banach spaces and topology*; see Negrepontis [1984] and Mercourakis/Negrepontis [1992]. The basic idea of Negrepontis was to apply techniques from infinite combinatorics. Around 1980 he had several Ph.D. students, among them Argyros who made substantial contributions to non-separable as well as to hereditarily indecomposable spaces.

A Banach space conference was organized in Herakleion (Crete, 1979).

8.2.7.12 Hungary

According to Hersh/John-Steiner [1993•, p. 15], the Hungarian analyst J. Horváth said in an interview:

You can name the day in 1900 when Fejér sat down and proved his theorem on Cesàro sums of Fourier series. That was when Hungarian mathematics started with a bang.

The pioneers were F. Riesz (1880–1956), L. Fejér (1880–1959), and A. Haar (1885–1933). While Riesz played a dominating role in the early period of Banach spaces, the other two laid the foundation for the theory of summation and the theory of wavelets, respectively. Every mathematician has heard about the *Haar measure*.

B. Szőkefalvi-Nagy (1913–1998), the most important pupil of Riesz, was mainly interested in Hilbert spaces. This explains why Banach spaces did not become a favorite subject of Hungarian mathematics.

Fejér had many famous Ph.D. students: M. Riesz (1886–1969), M. Fekete (1886–1957), G. Pólya (1887–1985), G. Szegő (1895–1985), O. Szász (1884–1952), S. Szidon (Sidon) (1892–1941), J. von Neumann (1903–1957).

The reader may also consult [HOR$^\cup$].

8.2.7.13 India

Functional analytic research was initiated by R. Vaidyanathaswamy (1894–1960), who received his Ph.D. from St. Andrews University. After his studies in the United Kingdom he returned to India in 1925. His school at the University of Madras became very important for the modern development of mathematics in that country. Vaidyanathaswamy is the author of an influential *Treatise on Set Topology* (1949). Among his students were K. Chandrasekharan (1920) and S. Swaminathan (1926). I also mention A.K. Roy (1939) and V.S. Sunder (1952), pupils of K. Hoffmann and P.R. Halmos, respectively. The team P.K. Kamthan (1938–1990) and M. Gupta (1950) wrote several textbooks on functional analysis.

Of particular importance is the Indian school of probability, founded by C.R. Rao (1920). His Ph.D. students V.S. Varadarajan (1937), K.R. Parthasarathy (1938), and S.R.S. Varadhan (1940) made, among others things, substantial contributions to topological measure theory. Varadarajan published his influential thesis [1961] in Мат. Сборник, since Prokhorov was the foreign referee.

8.2.7.14 Ireland

The leading figures of functional analysis in Ireland are S. Dineen (1944), a pupil of L. Nachbin (1922–1993), and T. West (1938), a pupil of F. Smithies (1912–2002). The former is a specialist in infinite-dimensional holomorphy, [DINE$_1$], [DINE$_2$]. R.M. Aron (1944), another pupil of L. Nachbin, is interested in the same field. Before moving to Kent (Ohio), Aron stayed at Dublin from 1972 to 1982. His Ph.D. student

R.A. Ryan (1953) became an expert in *Tensor Products of Banach Spaces* (2002). B.A. Barnes (1938, USA), G.J. Murphy (1948), M.R.F. Smyth (1946) and West wrote a booklet on *Riesz and Fredholm Theory in Banach Algebras* (1982).

8.2.7.15 Italy

Mathematicians from Italy played a decisive role in the prenatal period of functional analysis: Giulio Ascoli (1843–1896), C. Arzelà (1847–1912), S. Pincherle (1853–1936), and especially V. Volterra (1860–1940). Despite this great tradition, only minor attention was paid to the abstract theory of Banach spaces.

We owe to R. Cacciopoli (1904–1959) the final version of the Banach fixed point theorem, Guido Ascoli (1887–1957) worked about convex bodies, and E. Gagliardo (1930) made substantial contributions to interpolation theory and function spaces.

Concerning the present time, I can mention only some singular points: P. Aiena, F. Altomare (1951), P.L. Papini (1943), G. Metafune (1962), V.B. Moscatelli (1945), and P. Terenzi (1940).

Some highly appreciated conferences on *Functional Analysis and Approximation Theory* were organized in Acquafredda di Maratea (1989, 1992, 1996, 2000, 2004).

8.2.7.16 Japan

The founders of functional analysis in Japan were M. Nagumo (1905), H. Nakano (1909–1974), K. Yosida (1909–1990), and S. Kakutani (1911–2004). Yosida became famous mainly because of the Hille–Yosida theorem and his *Functional Analysis* (1965). To Kakutani we owe, among others things, the representation theorems for L- and M-spaces. Partly in cooperation, Kakutani and Yosida developed an operator-theoretic approach to ergodic theory wherein weak compactness is first studied in relation to operators; see 4.8.5.2 and 5.3.5.4.

Another famous researcher was T. Kato (1917–1999), who wrote a well-known book on *Perturbation Theory for Linear Operators* (1966). As a by-product he discovered the concept of a strictly singular operator.

We owe to Nakano three books on functional analysis. His treatise *Modulared Semi-Ordered Linear Spaces* (1950) is devoted to an abstract version of Orlicz spaces and can therefore be considered as a forerunner of [KRA$^+$].

I. Amemiya (1923–1995) and K. Shiga (1930) are well known for their attempt to popularize Grothendieck's *Résumé*.

The next generation includes K. Miyazaki (1930–2001), T. Ando (1932), and T. Shimogaki (1932–1971). Replacing l_p by $l_{p,q}$, Miyazaki generalized the concepts of nuclear, absolutely summing, and factorable operators. Ando made substantial contributions to the theory of Banach lattices and to spectral theory; later he switched to Hilbert spaces. As a coauthor of Lorentz, Shimogaki wrote some remarkable papers on interpolation of operators.

The theory of operator algebras has become the main subject of Japanese functional analysis. This tradition goes back to M. Nagumo, who, anticipating the Russian school, invented the concept of a linear metric ring. There are many famous names such as S. Sakai (1928), M. Takesaki (1933), and M. Tomita (1924).

Modern Banach space theory has played only a minor role. Let me just mention M. Matsuda (1946) and M. Kato (1948).

8.2.7.17 Netherlands

In the Netherlands, the name of A.C. Zaanen (1913–2003) stands as synonymous with functional analysis. His *Linear Analysis* (1953) was one of the first textbooks in this field. Mainly interested in function spaces, Zaanen wrote a two-volume treatise on *Riesz Spaces* (1971, 1983), the first volume in collaboration with his Ph.D. student W.A.J. Luxemburg (1929). The latter became a professor at the California Institute of Technology. Luxemburg combined functional analysis and non-standard analysis; see 6.1.3.9. In the Netherlands, Zaanen's tradition was carried on by M.A. Kaashoek (1937) and his pupils; see [BART$^{\cup}$].

Geometry of Banach spaces was represented only by D. van Dulst (1938), who wrote two books on this subject: [DUL$_1$] and [DUL$_2$]. Supervised by C.G. Lekkerkerker (1922–1999), he received his Ph.D. from the University of Amsterdam.

Finally, I mention that A.F. Monna (1909–1995) was the first to consider *Functional Analysis in Historical Perspective* (1973).

8.2.7.18 Norway

Though an operator algebraist, the most important Norwegian representative of Banach space theory is E.M. Alfsen (1930), the author of a well-known book on *Compact Convex Sets and Boundary Integrals* (1971). In a joint paper with E.G. Effros (1935, USA), he introduced the powerful concept of an M-ideal; see 6.9.4.1. Among his pupils were Å. Lima (1942) and G. Olsen (1946).

8.2.7.19 Romania

One of the founding fathers of functional analysis in Romania was T. Lalesco (1882–1929), who wrote the first book on *Équations intégrales* (1912). In 1935, A. Ghika (1902–1964) became a lecturer on functional analysis in Bucharest, and he decisively stimulated the subsequent development. Another initiator of functional analysis was G. Călugăreanu (1902–1976), who gave introductory courses at the University of Cluj in the 1950s.

One must also mention S. Stoïlow (1887–1961), O. Onicescu (1892–1983), and M. Nicolescu (1903–1975). The latter acted, from 1966 to 1975, as president of the Romanian Academy of Sciences, a lucky chance for mathematics.

The best known of Nicolescu's Ph.D. students is C. Foiaş (1933), who was mainly influenced by his collaboration with B. Szőkefalvi-Nagy. Their joint book on *Analyse harmonique des opérateurs de l'espace de Hilbert* (1967) had a great impact on operator theory (not only in Hilbert, but also in Banach spaces). He emigrated to France in 1978 and became finally a distinguished professor at Indiana University (Bloomington). Among the pupils of Foiaş were L. Zsidó (1946) and D. Voiculescu (1949). Together with I. Colojoară (1930), Foiaş wrote a remarkable monograph on *Generalized Spectral Operators* (1968). The same subject was treated in the thesis of C. Apostol (1936–1987). The *Journal of Operator Theory*, founded in 1979 at the INCREST (Bucharest), is highly esteemed.

Ivan Singer (1929) started to work on Banach spaces in 1954. His two-volume treatise on *Bases in Banach Spaces* (1970/81) has become a standard reference. Furthermore, he laid the foundation of *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces* (1970, Romanian edition in 1967).

Another achievement of Romanian functional analysis is the first monograph on *Vector Measures* (1966) by N. Dinculeanu (1925), who was a Ph.D. student of Onicescu.

While living in the USA, the Ionescu Tulcea couple wrote a well-known book on *The Theory of Lifting* (1969). Both received their Ph.D.s from Yale, Alexandra (1935) under Kakutani and Cassius (1923) under Hille.

From the younger generation, I mention N. Popa (1943), a pupil of R. Cristescu (1928) as well as C. Niculescu (1947) and D. Vuza (1955).

A series of highly estimated international conferences on *Operator Theory* have been organized by the people from the University of Timişoara. Beginning in 1976, these conferences took place annually and, since 1988, every second year. Number 20 was reached in 2004.

A Romanian–GDR Seminar on *Banach Space Theory and its Applications* came off in Bucharest (1981); see [PIE$^{\cup}$].

8.2.7.20 South Africa

In South Africa, Banach space theory was initiated by J.J. Grobler (1943), a Ph.D. student of A.C. Zaanen (1913–2001) at Leiden, and J. Swart (1948), a Ph.D. student of H. Jarchow (1941) at Zürich. Both had several pupils, among them P. van Eldik (1945), J. Engelbrecht (1946), H. Raubenheimer (1952), J.H. Fourie (1951), and A. Ströh (1965).

International Conferences on Abstract Analysis (1993, 1996, 2000) have been organized in the Kruger National Park.

8.2.7.21 Spain

M. Valdivia (1928) started his scientific career as an agricultural engineer. At the age of 31, he enrolled at the Faculty of Mathematics in Madrid. He wrote a thesis under the supervision of D. Ricardo San Juan, and obtained his Ph.D. in 1963. Valdivia then became a professor in Valencia, where he founded the *functional analytic school of Spain*. Though Valdivia's interest is mainly directed to locally convex linear spaces, he also contributed to Banach space theory. Almost all of his 31 Ph.D. students are now professors.

Further nestors of Spanish functional analysis were A. Plans (1922–1998) and B. Rodríguez-Salinas (1925). Both had many Ph.D. students.

Plans: V. Onieva (1938–1988), E. Martín-Peinador (1948), A. Reyes (1951–1983), A. Rodés (1953), … .

Salinas: F. Bombal (1944), P.J. Guerra (1951), F.L. Hernández (1953), … .

From the younger generation, I mention J.L. Rubio de Francia (1949–1988) and J. García-Cuerva (1949), a pupil of R.R. Coifman ($\sim$1938). They founded a powerful research group working on the boundary between harmonic analysis and geometry of Banach spaces. Their offspring includes O. Blasco (1959) and J.L. Torrea (1955). Another group, directed by F. Cobos (1956), deals with interpolation theory.

As a conclusion, it may be said that Spain has a very high percentage of functional analysts among its professors of mathematics.

8.2.7.22 Sweden

I. Fredholm (1866–1927) and H. von Koch (1870–1924) were Ph.D. students of G. Mittag-Leffler (1846–1926). Based on earlier work of Poincaré and Hill, they created (under different viewpoints) the theory of infinite determinants and laid, in this way, the foundation for operator theory in Banach spaces. Since both passed away quite untimely, they had only little impact on the development of functional analysis in Sweden.

Marcel Riesz (1886–1969), the younger brother of Frigyes, played a decisive role. In 1926 he was appointed professor at Lund, where he had prominent Ph.D. students: G.O. Thorin (1912), L. Gårding (1919), and L. Hörmander (1931). He also supervised E. Hille (1894–1980). My coauthor A. Persson (1933) was a pupil of Gårding.

J. Peetre (1935), whose parents emigrated from Estonia in 1944, wrote his thesis under the direction of L. Gårding and Å. Pleijel. We owe to him the modern version of the real interpolation method. The offspring of Peetre's school includes T. Holmstedt (1938), J. Bergh (1941), J. Löfström (1941), G. Sparr (1942), B. Jawerth (1952), and P. Nilsson (1955).

Further outstanding analysts are T. Carleman (1892–1942) and L. Carleson (1928), a pupil of A. Beurling (1905–1986).

The most important Swedish contribution to modern Banach space theory is Enflo's counterexample to the approximation problem. P. Enflo (1944) received his Ph.D. from the Stockholm University under H. Rådstrøm (1919–1970) and went afterward to the USA. Another Ph.D. student of Rådstrøm was M. Ribe (1945).

The monograph of Bergh/Löfström on *Interpolation Spaces* (1976) has become a standard reference. Gårding wrote a book on *Mathematics in Sweden before 1950* (1998).

8.2.7.23 Switzerland

In Switzerland, Banach space theory has a singular point: H. Jarchow (1941), who obtained his Ph.D. under H.H. Keller (1922) from the University of Zürich. He wrote books on *Locally Convex Spaces* (1981) and *Absolutely Summing Operators* (1995, coauthors: J. Diestel, A. Tonge). Among his students were J. Swart (1948, South Africa), U. Matter (1955), and V. Mascioni (1962). Jarchow organized a conference on *Local Theory of Banach Spaces and Related Topics* (Ascona, 1993).

S. Chatterji (1935) earned his Ph.D. from Michigan State University for a thesis on *Martingales of Banach-valued random variables.*

8.2.8 Germany

Due to my first-hand knowledge about the development of functional analysis in Germany, this subsection has become too long. I hope that the reader will forgive this disproportion.

8.2.8.1 Concerning the prehistory of Banach space theory in Germany, I refer to the famous inequalities that bear the names of O. Hölder (1859–1937) and H. Minkowski (1864–1909).

The Hellinger/Toeplitz report [HEL$^+$] from 1927 clearly shows that until the end of the 1920s, abstract spaces and operators did not play a role. Indeed, Hilbert's *Göttingen school* was mainly interested in concrete spaces: l_2 and L_2.

Hausdorff's contribution *Zur Theorie der linearen metrischen Räume* (1931) remained an isolated case.

8.2.8.2 In 1929, at the recommendation of Emmy Noether, G. Köthe (1905–1989) went to Bonn as an assistant of O. Toeplitz (1881–1940). This was the beginning of a fruitful collaboration that led to the theory of *perfect sequence spaces* (vollkommene Folgenräume). Their most important paper appeared in 1934. Its title *Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen* shows that the spirit of Göttingen still survived. Nevertheless, the basic concept of a dual system had already occurred in an implicit form; see 3.3.2.5. The final step was done by J. Dieudonné and G.W. Mackey, who translated the underlying ideas into the abstract language of locally convex linear spaces.

Köthe himself wrote a two-volume treatise on *Topological Vector Spaces* (1960/1979). These books and the strong personality of Köthe were the reasons why the theory of locally convex linear spaces and its application to partial differential operators became the favorite subject of (West) German functional analysts. Nowadays, the most important representatives of this school are D. Vogt (1941), S. Dierolf (1942), R. Meise (1945), and K.D. Bierstedt (1945).

8.2.8.3 As a result of World War II, Germany was divided into two parts. The official names were Bundesrepublik Deutschland and Deutsche Demokratische Republik. In what follows, I simply speak of "West" and "East."

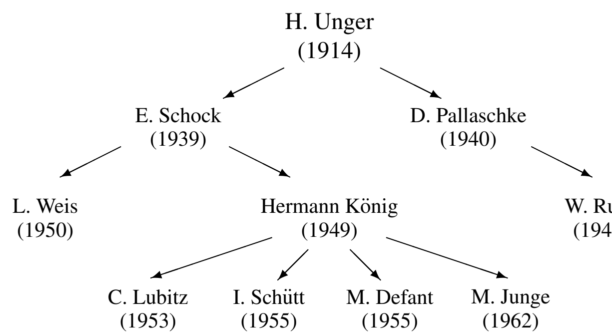
The two ways of German mathematics in the period from 1945 to 1990 will now be discussed.

8.2.8.4 The postwar generation in West Germany includes

H.G. Tillmann (1924), a pupil of H. Ulm (1908–1975),
K. Zeller (1924–2006), a pupil of G.G. Lorentz (1910–2006),
H.H. Schaefer (1925–2005), a pupil of E. Hölder (1901–1990),
H. Bauer (1928–2002), a pupil of O. Haupt (1887–1988),
P.L. Butzer (1928), a pupil of G.G. Lorentz (1910–2006),
Heinz König (1929), a pupil of K.H. Weise (1909–1990).

All of them made important contributions to functional analysis, but not to Banach space theory. A remarkable exception is Schaefer's monograph on *Banach Lattices and Positive Operators* (1974). He also supervised several Ph.D. students in Germany as well as in the USA who worked on Banach spaces and Banach lattices; see 8.2.8.8. Butzer used Banach space techniques in approximation theory; see [BUT_1^+]. I also mention two pupils of Heinz König: P. Meyer-Nieberg (1944) wrote a text on *Banach Lattices* (1991) and M.M. Neumann (1950) is the author of a comprehensive book on *Local Spectral Theory* (2000, coauthor: K.B. Laursen). Tillmann's pupil J. Eschmeier (1956) also made a substantial contribution: *Spectral Decompositions and Analytic Sheaves* (1998, coauthor: M. Putinar).

8.2.8.5 Though having a doctorate in engineering, H. Unger liked functional analysis very much. He wrote an elementary textbook on this subject, [UNG^+]. More important were his efforts as a teacher:



Another achievement must be mentioned: Unger was one of the initiators of the *Sonderforschungsbereich 72* in Bonn (1971–1986). In this way, he obtained money from the Deutsche Forschungsgemeinschaft for employing young people and for supporting foreign visitors. The wisely chosen headline *Approximation und Mathematische Optimierung in einer anwendungsbezogenen Mathematik* gave free play for including functional analysis. Thus J. Lindenstrauss, L. Tzafriri, J.R. Retherford, and many others could be invited.

8.2.8.6 The most spectacular German contribution to Banach space theory was made by T. Schlumprecht (1954), a pupil of J. Batt (1933). His construction of an arbitrarily distortable norm on a *Tsirelson like Banach space* gave the decisive impetus for solving several long-standing open problems; see 7.4.4.4. Schlumprecht has worked in the USA since 1988; he is now a professor at Texas A & M, College Station.

8.2.8.7 The following chairs (Lehrstühle) in West Germany are concerned with Banach space theory (alphabetic order).

Berlin:
E. Behrends (1946) founded a group that mainly dealt with M-ideals; see [BEHR], [HARM$^+$].
The following "teacher $\Rightarrow$ pupil" chain is remarkable:
F. Klein (1849–1925) $\Rightarrow$ F. von Lindemann (1852–1939) $\Rightarrow$
D. Hilbert (1862–1943) $\Rightarrow$ E. Schmidt (1876–1959) $\Rightarrow$
A. Dinghas (1908–1974) $\Rightarrow$ G. Wittstock (1939) $\Rightarrow$ E. Behrends (1946) $\Rightarrow$
{P. Harmand (1953), D. Werner (1955), W. Werner (1958), H. Pfitzner (1959)}

Kiel:
Hermann König (1949) wrote a well-known book on *Eigenvalue Distribution of Compact Operators* (1986) and made important contributions to the local theory.
C. Schütt (1949) was a Ph.D. student of K. Floret.
A conference on *Geometric Aspects of Fourier and Functional Analysis* (1998) was organized by H. König and D. Müller.

Oldenburg:
K. Floret (1941–2002), a pupil of J. Wloka (1929) and strongly influenced by G. Köthe, changed from locally convex linear spaces to Banach spaces. In collaboration with his pupil A. Defant (1953) he wrote a standard treatise on *Tensor Norms and Operator Ideals* (1993).
The meetings on the East Frisian Islands (Spiekeroog, Wangerooge) had a good tradition.

Paderborn:
The only genuine representative of Banach space theory is W. Lusky (1948), a pupil of B. Fuchssteiner (1941).
Conferences on *Functional Analysis* (1976, 1979, 1983) were organized by K.D. Bierstedt and B. Fuchssteiner; see [BIER$^{\cup}_{1}$].

Saarbrücken:
E. Albrecht (1944) and J. Eschmeier (1956) work on spectral theory of operators in Banach spaces.

In addition to the preceding list, I mention some colleagues who concentrated on Banach space theory only for a short period.

G. Neubauer (1930–2003) received his Ph.D. under G. Köthefrom the University of Heidelberg. Among other things, he investigated the behavior of unbounded Fredholm operators under perturbation.

B. Gramsch (1938) was a formal Ph.D. student of H.G. Tillmann. His thesis [1967] dealt with the collection of all closed operator ideals in $\mathfrak{L}(H)$, where H is a Hilbert space of arbitrary dimension. He also contributed to the theory of Fredholm operators, but finally switched to pseudodifferential operators. Among his students were E. Albrecht (1944) and W. Kaballo (1952).

K. Höllig (1953), a pupil of K. Scherer, estimated approximation numbers of embedding maps between Sobolev spaces.

8.2.8.8 Since the 1980s, semi-groups of linear and non-linear operators have become a major subject at several places in West Germany:

Essen: W. Rueß (1944), a pupil of D. Pallaschke,
Karlsruhe: L. Weis (1950), a pupil of E. Schock,
Tübingen: R. Nagel (1940) and F. Räbiger (1958), pupils of H.H. Schaefer,
Ulm: W. Arendt (1950), a pupil of H.H. Schaefer.

I stress that there is a remarkable interplay between the theory of semi-groups and the geometry of Banach spaces; see 5.3.2.11.

8.2.8.9 Das Mathematische Forschungsinstitut Oberwolfach, founded in 1944, became a famous meeting place for mathematicians from all over the world. At the beginning, many conferences on functional analysis took place there. Later on, specialized meetings on *Geometrie der Banachräume* were organized by Behrends, Fuchssteiner, König, Lindenstrauss, Pełczyński, and Tomczak-Jaegermann (1980, 1981, 1986, 1991, 1996, and 2003).

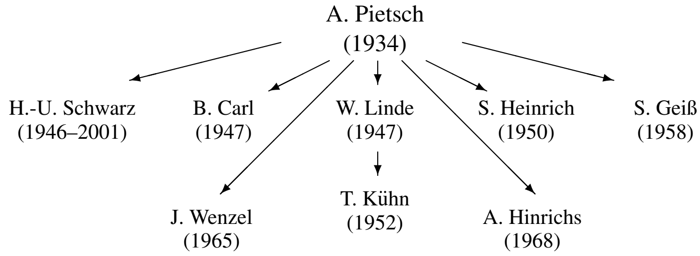
8.2.8.10 Due to the scientific isolation of East Germany, the development of mathematics in the German Democratic Republic was quite specific. Most of the leading researchers were self-taught, while the others received their education in the Soviet Union. A political credo said that

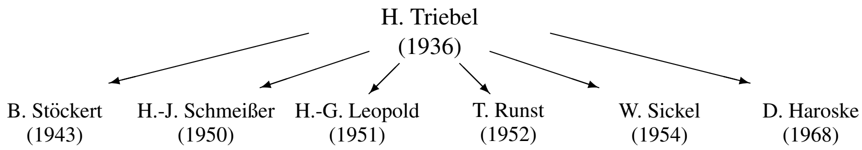
„Von der Sowjetunion lernen, heißt siegen lernen!“
(*Learning from the Soviet Union means learning to win*)

By the way, concerning mathematics, this strategy was not too bad.

8.2.8.11 In 1965, the author of this text was appointed professor in Jena. Since then, the university of this city has become a good place for Banach spaces. While A. Pietsch and his pupils were interested in the abstract theory of *operator ideals*, his contemporary H. Triebel founded a well-known *school of function spaces*. The books [CARL$^+$], [EDM$^+$], [LINDE], [PIE$_1$],...,[PIE$_4$], [PIE$^+$], [SCHM$^+$], [SCHZ], [TRI$_1$],...,[TRI$_7$] have been well-thought-of by the mathematical community.

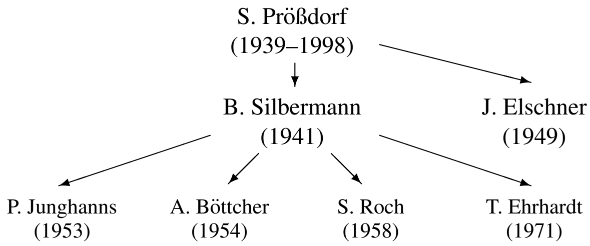
From 1965 to 2000, Jena produced about 70 Ph.D.s in functional analysis. The most successful members of this offspring are presented in the following genealogical trees:





8.2.8.12 Conferences on *Operator Algebras, Ideals, and Their Applications in Theoretical Physics* were organized by H. Baumgärtel, G. Laßner, A. Pietsch, and A. Uhlmann (physicist) in Leipzig (1977, 1983); see [BAUM$^{\cup}_{1}$], [BAUM$^{\cup}_{2}$]. One third of these meetings had been devoted to Banach space theory. More specialized were the *East German Polish Seminars* in Oberhof (1976) and Georgenthal (1979, 1981, 1984, 1986, 1989) and their Polish counterparts; see 8.2.1.16.

8.2.8.13 A powerful *school of singular integral equations* developed at the Technical University of Chemnitz (then called Karl-Marx-Stadt):



S. Prößdorf, a Ph.D. student of S.G. Mikhlin (1908–1990) at the University of Leningrad, worked in Karl-Marx-Stadt from 1968 to 1975; see [ELS$^{\cup}$, pp. 1–26]. Afterward he went to Berlin, together, with his pupil J. Elschner. From this point on, B. Silbermann (who wrote his masters thesis at the University of Moscow) became the leader. The output of Chemnitz contains a remarkable number of monographs: [BÖT^{+}], [MICH^{+}], [PRÖ], [PRÖ$^{+}$], etc.

8.2.8.14 Due to the unpleasant political situation, several functional analysts emigrated from the East to the West, among them my teacher H.H. Schaefer in 1956 as well as my pupils B. Carl and H.-U. Schwarz in 1989. H. Langer (1935), a close collaborator of M.G. Kreĭn, left Dresden in 1989; see [DIJK$_1^{\cup}$, pp. 1–22]. Much more seriously, K. Gröger (1936), who worked on non-linear operator equations, was sentenced in 1960 because of *"Spionage und staatsgefährdender Hetze"* (spying and subversive agitation); see Schmidt/Ziegler [1997•]. He had to stay in prison for five years. "Generously," the communists allowed him to translate some Russian books into German. Gröger also edited an extended version of the *Ljusternik–Sobolev* [LYUS$^+$].

8.2.8.15 In spite of the Iron Curtain, mathematicians from West and East tried to maintain contact. I thankfully remember the help of many friends, who sent me reprints and books. Only by the mid 1980s could some light at the end of the tunnel be seen. My first visit to Oberwolfach became possible in spring 1984. In March 1990, the FIRST AND UNIQUE *German German Seminar on Functional Analysis* took place in Georgenthal (GDR). Subsequently, a UNITED GERMAN counterpart was organized in Essen (1991); see [BIER$_2^{\cup}$].

8.2.8.16

The example of a Banach space shows
that being of second category
is not always a discriminating property.

This is a good place to finish this subsection. Because of personal emotions, I will not write about the time after the so-called "German Unification." Unfortunately, the present credo is only a slight modification of that quoted on p. 633: now the NEW Federal States (East) "learn" from the OLD ones (West). Typically, scientific work done in the former German Democratic Republic was classified as communist activity (staatsnahe Tätigkeit). This is why (old) professors

"... who came in from the cold"

have been treated so badly; see the title of [CARRÉ].

„Geschichte ist immer die Geschichte der Sieger!"
(*History is always the history of the winners*)

8.3 Short biographies of some famous mathematicians

We live in a world that is based solely on money: show masters and pop stars, actors and models, racing drivers and tennis players have become the modern heroes. However, there is more.

8.3.1 Frigyes Riesz

8.3.1.1 Frigyes (Friedrich, Frédéric) **Riesz**, the son of a Jewish physician, was born in Győr (Hungary) on January 22, 1880. After studies of engineering at the Polytechnikum Zürich, he converted to mathematics and received his Ph.D. in 1902 from the

University of Budapest. His Hungarian teachers were G. König (1849–1913) and J. Kürschák (1864–1933). During several visits in Göttingen, Riesz was strongly influenced by D. Hilbert and H. Minkowski.

In 1912, Riesz became a professor in Kolozsvár. When Transylvania was ceded to Romania, he intermediately stayed in Budapest (1918–1920). Most important was his activity in Szeged (1920–1946). Here, in collaboration with A. Haar (1885–1933), he founded the Bolyai Institute and the journal *Acta Scientiarum Mathematicarum*. Without being able to leave his apartment, Riesz survived the Nazi occupation from March to October 1944 thanks to the help of good friends. In 1946, he moved to the University of Budapest, where he taught until spring 1955. Riesz died on February 28, 1956.

Frigyes had a younger brother, Marcel (1886–1969), who was professor at the University of Lund (Sweden). There is a famous Riesz–Riesz theorem; see 6.7.12.17.

8.3.1.2 Without any doubt, we may refer to Riesz as the nestor of Banach space theory. Here is a selection of his functional analytic achievements:

- the Fischer–Riesz theorem: 1.5.2, Fischer [1907] and Riesz [1907a],
- representation of functionals on $C[a,b]$; see 2.2.9 and [1911],
- contributions to the moment problem: 2.3.1 and [1909a], [1911], [RIE],
- the creation of the classes L_p; see 1.1.4 and [1909a],
- spectral theory of compact operators: 2.6.4.3 and [1918],
- boundary behavior of analytic functions: 6.7.12.2 and [1923],
- the impetus for the study of Banach lattices: 4.1.9 and [1928],
- the mean ergodic theorem in L_p spaces: 5.3.5.4 and [1938].

In 1913, Riesz published the monograph *Les systèmes d'équations linéaires à une infinité d'inconnues* [RIE] in which the theory of l_p spaces is developed. This seminal treatise also contains the original form of many spectral theoretic results: analyticity of the resolvent and the operational calculus.

Together with B. Szőkefalvi-Nagy, he wrote the best-seller

Leçons d'analyse fonctionnelle (1952),

which was translated into German, English, Russian, Japanese, and Chinese.

Riesz had only a few, but distinguished, pupils,

Szeged: T. Radó (1895–1965), B. Szőkefalvi-Nagy (1913–1998), A. Rényi (1921–1970),
Budapest: A. Császár (1922), J. Aczél (1924).

8.3.1.3 The preceding presentation is based on the following sources:

Rogosinski [1956•], Halmos [1981•], Kreyszig [1990•]. The reader is also referred to a lively article of Hersh/John-Steiner [1993•, pp. 19–22].

obituary: Acta Sci. Math. (Szeged) **7** (1956), 1–3.

His *Œuvres complètes* [$\mathrm{RIE}_1^{\bowtie}$] were published in 1960.

8.3.2 Eduard Helly

8.3.2.1 Eduard **Helly** was born in Vienna on June 1, 1884. In 1902, he enrolled at the University of Vienna. His mathematical teachers were G. von Eschrich (1849–1935), F. Mertens (1840–1927) and W. Wirtinger (1865–1945). For a thesis *Beiträge zur Theorie der Fredholm'schen Integralgleichung* he received his Ph.D. in 1907, with the degree summa cum laude. Afterward, Helly spent one year in Göttingen. From 1910 to the outbreak of World War I, he taught at a Gymnasium. Helly was called to the army in 1915 and soon wounded at the Austrian–Russian front. Kept as a prisoner of war in Siberia, he was able to return to Vienna only in November 1920. As early as spring 1921, Helly submitted his Habilitationsschrift *Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten*. The referee, professor H. Hahn (1879–1934), was certainly inspired by reading this fundamental work. Unfortunately, despite his talent, no academic position was offered to Helly. Since being a *Privatdozent* did not guarantee any salary, he became employed in a bank (Bodenkreditanstalt, 1921–1929) and, subsequently, in a life insurance company (Phönix, 1930–1938). After the Nazi occupation, people of Jewish origin were no longer safe in Vienna. Thus Helly and his family emigrated to the USA in September 1938. Until 1943, he worked as an underpaid lecturer at several colleges. A better offer from the Illinois Institute of Technology (Chicago) came too late. Helly taught there only for a few months and died on November 28, 1943, in Chicago.

8.3.2.2 Mainly due to social and political circumstances, Helly published only five mathematical papers. Nevertheless, his results have become landmarks:

- Auswahlprinzip: 3.4.2.4 and [1912],
- Momentenproblem: 2.3.1 and [1912], [1921],
- Normierte lineare Folgenräume: 1.8.1 and [1921],
- Dualität von Abstandsfunktionen: 1.8.1 and [1921],
- Helly's theorem: 2.3.7 and [1923].

In the first period of his research, Helly worked in the concrete (Banach) space $C[a,b]$. Anticipating Riesz, he could have said: *Die in der Arbeit gemachte Einschränkung auf stetige Funktionen ist nicht von Belang;* see 1.9.2. But, he did not. Probably, the time was not ripe. Of course, one may speculate what would have happened without World War I.

8.3.2.3 The previous presentation is based on two articles of Butzer et al. [1980$^\bullet$], [1984$^\bullet$]; see also Pinl [1974$^\bullet$, p. 192]. The history of the Hahn–Banach theorem is described in Section 2.3; see Hochstadt [1980$^\bullet$].

8.3.3 Stefan Banach

8.3.3.1 Stefan **Banach** was born in Kraków on March 30, 1892. His parents, Stefan Greczek and Katarzyna Banach, were not married. It is said that Banach spent some years of his childhood under the wings of his grandmother in Ostrowsko, the birthplace of his father. However, he was mainly brought up in Kraków by a foster-mother, Franziszka Płowa.

After the matura in 1910, Banach enrolled at the Faculty of Engineering of the Polytechnical Institute in Lwów. Due to the outbreak of World War I, he was not able to finish his studies.

A well-known story spread among mathematicians says that sometime in 1916, Steinhaus (then assistant at the University of Lwów) walked through a park in Kraków. Suddenly, he overheard the words *Lebesgue integral*; the youngsters who were discussing this unusual matter were Stefan Banach and Otton Nikodym; see [BAN$^{\bowtie}$, Vol. I, p. 13]. This encounter was the beginning of a lifelong collaboration and the big bang of the famous Lwów school. Steinhaus always claimed that Banach was his *greatest mathematical discovery*.

In 1920, Banach became an assistant of A. Łomnicki (1881–1941) at the Lwów Polytechnical Institute. This was his first paid academic job. In June of the same year, he submitted his thesis *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales* at the Jan Kazimierz University. His habilitation followed in April 1922, and subsequently, in July 1922, he was appointed professor extraordinarius.

Banach stayed in Lwów throughout the rest of his life. During the Soviet occupation from September 1939 to June 1941 he was made dean of the Physical-Mathematical Faculty and head of the Department of Mathematical Analysis. Most horrible was the period from June 1941 to July 1944. Banach survived the Nazi pogrom thanks to the fact that he had an identity card as an employee of the Rudolf Weigl Bacteriological Institute. However, his job was embarrassing. He had to carry lice in a box that was placed on the back of his hands. These lice were used for producing anti-typhoid vaccines; additional information can be found in Alexander's review [2004$^{\bullet}$, p. 224].

After the end of the war, Banach immediately resumed his academic duties. However, all of his plans for the future were stopped by a serious illness, lung cancer. Banach died in Lwów on August 31, 1945. Pictures of Banach's grave can be found in an article of Ciesielski [1988$^{\bullet}$].

8.3.3.2 A mysterious story about the origin of Stefan Banach reads as follows:

In Tel Aviv, V. Milman [2003$^{\bullet}$, стр. 292–295] met an old lady whose maiden name was Banach. She told him that her grandmother (Netl Banach married to a cousin, by the name of Moshe Banach) had a younger brother who became a famous mathematician in Lwów. Since the boy did not like to live according to the orthodox Jewish tradition, he went away at the age of about fifteen and converted to catholicism. As a consequence, nobody from his former family was allowed to remember this black sheep, and therefore no further information is available.

As an antithesis, I add a quotation from [KAŁ•, p. 4]:

In 1902, after finishing the elementary school, Banach, then 10 years of age, entered the first grade at Cracow's Henryk Sienkiewicz Gymnasium Number 4.

If the life of a famous person contains blank space, then there is room for all kinds of speculation.

8.3.3.3 A short glance at the number of citations listed in the bibliography of this text shows that Banach is the most quoted author. Since it is impossible to give a full account of his achievements, I present only some highlights:

- the concept of a complete normed linear space, which includes the rediscovery of the axioms of a linear space; see 1.1.1 and [1922] as well as Section 1.2.
- his version of the principle of uniform boundedness: 2.4.5 and [1922] as well as Banach/Steinhaus [1927],
- the bounded inverse theorem, the closed graph theorem, and the open mapping theorem: Section 2.5 and [1929, Part II],
- the Banach–Stone theorem: 4.5.5 and [BAN],
- the Hahn–Banach theorem: 2.3.5 and [1929, Part I],
- the existence of translation invariant additive measures on the line and in the plane: 2.3.5 and [1923],
- the Banach–Tarski paradox: 7.5.21 and Banach/Tarski [1924],
- Banach limits: 6.9.11.6 and [1923],
- the isometric embedding of separable Banach spaces into $C[0,1]$; see 4.9.1.7 and [BAN],
- the first example of a non-complemented subspace: 4.9.1.10 and Banach/ Mazur [1933],

Most importantly, Banach's book

Théorie des opérations linéaires (1932)

gave the impetus for creating a new mathematical discipline: *la théorie des espaces du type* (B).

Curiously enough, the best-known result of Banach seems to be his fixed point theorem; see 2.1.12.

8.3.3.4 There is a lively biography of Banach: [KAŁ•]. Further information can be found in [KUR•, pp. 153–158] and [ULAM•]. I also refer to the articles of Ulam [1946•], Steinhaus [1948•, 1960•], and Köthe [1989•], as well as to Alexander [2004•].
His *Œuvres* [BAN$^{\bowtie}$] were published in 1967/79.

Banach's impact to the foundation of the *Polish school of functional analysis* is described in Subsection 8.2.1.

8.3.4 John von Neumann

8.3.4.1 János **Neumann** was born in Budapest (Hungary) on December 28, 1903. In 1913, his father, a rich Jewish banker, received a minor title of nobility; see Halmos [1973$^\bullet$, p. 382]. This is why the young János became "Neumann **von** Margitta."

John von Neumann, as he called himself later, was a *Wunderkind*. Under the supervision of L. Fejér (1880–1959), he wrote a thesis on ordinal numbers and earned his Ph.D. from Budapest University in 1926. Some months later, the ETH Zürich awarded to him a diploma in chemical engineering. In Switzerland, H. Weyl (1885–1955) and G. Pólya (1887–1985) were his teachers. Next, von Neumann spent some time at Göttingen under the wings of the old Hilbert. Then he went to Berlin, submitted his Habilitationsschrift *Der axiomatische Aufbau der Mengenlehre* and received the title Privatdozent. Von Neumann's stay in Berlin (1927–1929) turned out to be a decisive step, since E. Schmidt (1876–1959) introduced him to spectral theory. The upshot was the famous monograph *Mathematische Grundlagen der Quantenmechanik*. One year at Hamburg followed. In 1930, von Neumann accepted a visiting lectureship at Princeton, and after 1933 he was at the newly founded Institute for Advanced Study until his untimely death on February 8, 1957.

8.3.4.2 Certainly, von Neumann cannot be considered as a member of the sect of Banach space people. Nevertheless, many of his fundamental results had a large impact on our theory:

- axiomatic set theory,
- spectral theory of self-adjoint operators: [1927], [1930a], [1930b], [vNEU],
- the first mean ergodic theorem: 5.3.5.4, and [1932b],
- the existence of a Haar measure: 4.6.5 and [1934],
- the concept of a locally convex linear space: 3.3.1.3 and [1935],
- metrization of matrix-spaces: 4.10.1.5 and [1937],
- rings of operators: Subsection 4.10.4, [1930b], [1940] [1949] and [1936, 1937, 1940, 1943, in collaboration with Murray],
- cross-spaces of linear transformations: 4.10.1.5, 5.7.2.3, 6.3.1.6 and [1946, 1948, in collaboration with Schatten],
- preliminary versions (unpublished) of the lifting theorem 5.1.3.7, the Maharam theorem 4.8.2.7, and the invariant subspace theorem 5.2.4.2.

8.3.4.3 There is an extensive literature on J. von Neumann that, in particular, covers his achievements in computer science. Volume **64** of Bull. Amer. Math. Soc. is devoted to his life and work. For further information the reader may consult the following sources:

[vNEU$^{\bowtie}$], [GLI$^{\cup}$], [LEG$^{\cup}$, pp. 11–35], [MACR$^\bullet$], [ULAM$^\bullet$],

Ulam [1958$^\bullet$], Smithies [1959$^\bullet$], Halmos [1958$^\bullet$, 1973$^\bullet$], Pinl [1969$^\bullet$, p. 187].

His *Collected Works* [vNEU$^{\bowtie}$] were published in 1961/63.

8.3.5 Mark Grigorievich Kreĭn

8.3.5.1 Марк Григорьевич **Крейн** was born in Kiev (Ukraine) on April 3, 1907. At the age of 17, he moved to Odessa; Kreĭn loved that city very much and stayed there most of his life. Without having finished regular studies, he became a research student of N.G. Chebotarev (1894–1947) and defended his thesis in 1929. In view of his fundamental results, he received the degree д-р. физ.-мат. наук from Moscow State University (1938) without submitting a dissertation. Kreĭn was a professor at the Odessa University only from 1930 to 1941. Because of World War II, he worked in Kuĭbyshev from 1941 to 1944. Returning to Odessa, Kreĭn was dismissed from the university due to anti-Semitism. From 1944 to 1952, he held a part-time position at the Mathematical Institute of the Ukrainian Academy of Science in Kiev. However, Kreĭn was dismissed from there as well. Thus he had to teach at colleges of lesser quality:

Одесский институт инженеров морского флота, 1944–1954,
(Odessa Marine Engineering Institute)

Одесский инженерно-строительнй институт, from 1954 on.
(Odessa Civil Engineering Institute)

Despite his miserable working conditions, Kreĭn was the supervisor of more than 40 Ph.D.s. In 1982, he was awarded the prestigious Wolf Prize from Israel. His life clearly shows that good mathematics can be done under bad political circumstances.

Deeply depressed by the death of his wife and his only grandson, Kreĭn died on October 17, 1989, in Odessa.

8.3.5.2 Concerning Banach space theory, Kreĭn was certainly the most important of all prominent Soviet functional analysts. Among others, he made fundamental contributions to the following fields:

- moment problem: [AKH$^+$],
- Banach spaces with a cone: Kreĭn/Rutman [1948],
- the characterization of abstract M-spaces: 4.8.4.4 and [1940, in collaboration with his younger brother] ,
- the Kreĭn–Shmulyan theorem: 3.6.7 and Kreĭn/Shmulyan [1940],
- the Kreĭn–Milman theorem: 5.4.1.2 and Kreĭn/Milman [1940],
- non-self-adjoint operators in Hilbert spaces; see [GOH$_3^+$],
- Wiener–Hopf and Toeplitz operators: Subsection 6.9.9.

Kreĭn's course on functional analysis, delivered in 1936/37, was the starting point of the famous *Odessa functional analytic school*. Here is a list of the first generation:

V.R. Gantmakher (1909–1942)), sister of F.R. Gantmakher,

M.A. Naĭmark (1909–1978),

A.P. Artemenko (1909–1944),

D.P. Milman (1913–1982),

M.S. Brodskiĭ (1913–1989),
V.L. Shmulyan (1914–1944),
I.M. Glazman (1916–1968); see [$\text{GOH}_4^{\cup}$],
M.A. Rutman (1917),
M.S. Livsic (1917); see [$\text{ALP}_1^{\cup}$].

During his activity in Kiev, Kreĭn supervised M.A. Krasnoselskiĭ (1920–1997), Yu.M. Berezanskiĭ (1925), as well as his younger brother, Selim Gregorievich (1917–1999).

The seminal monograph *Introduction into the Theory of Nonselfadjoint Operators in Hilbert Space*, written in collaboration with I.C. Gohberg, gave the decisive impetus for developing corresponding concepts in the setting of Banach spaces.

Without any doubt, one can say that Gohberg's *operator theory school* in Kishinev was formed under the essential influence of the grandmaster from Odessa: Mark Grigorievich Kreĭn.

8.3.5.3 Here is a list of articles:

50th birthday: Uspekhi Mat. Nauk 13, # 3 (1958), 213–224,
60th birthday: Uspekhi Mat. Nauk 23, # 3 (1968), 197–214,
70th birthday: Uspekhi Mat. Nauk 33, # 3 (1978), 197–203,
80th birthday: Uspekhi Mat. Nauk 42, # 4 (1987), 201–206.
Obituaries:
Notices Amer. Math. Soc. 37 (1990), 284–285,
Функционал. Анализ и его Прилож. 24, # 2 (1990), 1–2,
Украин. Мат. Журнал 42 (1990), 576,
[$\text{ADAM}_1^{\cup}$, pp. 5–8], [$\text{ADAM}_2^{\cup}$, p. ix], Gohberg [1998$^\bullet$].

Volume 46 of the Ukrainian Mathematical Journal is for the most part dedicated to the work of M.G. Kreĭn. In particular, I refer to the articles of Levin [1994$^\bullet$] and Nudelman [1994$^\bullet$], [2000$^\bullet$]. Furthermore, there are the *Proceedings of the Mark Kreĭn International Conference* [$\text{ADAM}_1^{\cup}$], [$\text{ADAM}_2^{\cup}$].

I hope that there will be *Collected* or *Selected Works* soon.

8.3.6 Alexandre Grothendieck

8.3.6.1 Alexandre (Schurik) **Grothendieck** was born in Berlin on March 28, 1928. His childhood and youth was overshadowed by German fascism. Grothendieck's father, *Alexander Shapiro* (1890) alias Tanaroff, was a Russian anarchist of Jewish origin, who became involved in several uprisings during 1905 and 1919. He was incarcerated in prisons and concentration camps for about 10 years. Shapiro also participated in the October Revolution. But, not being a Bolshevik, he left the Soviet Union around 1921. According to Scharlau [2004$^\bullet$], there were two different Shapiros, and Grothendieck's father was not the "famous" one described by Cartier

[1998•, Engl. transl. p. 390]. Anyway, when living in Berlin, Shapiro met *Johanna* (*Hanka*) *Grothendieck* (1900–1957), born in Hamburg. These were Grothendieck's parents. When Hitler came to power, the unmarried couple escaped to France, and the young Alexandre was given to a family in Hamburg; the name of his foster-father was Wilhelm Heydorn. In 1939, Grothendieck moved to France, where he lived with his mother in the Cévennes Mountains, a southern part of the Massif Central. For some time they were detained in the Camp Riencros near Mende. Then Hanka was moved to the Camp de Gurs (Basses, Pyrénées), while Alexandre had the opportunity to finish his secondary education at the Collège de Cévenol (Chambon-sur-Lignon), which, founded in 1938, offered shelter and security to children enduring discrimination during World War II.

Grothendieck's father was picked up by the (French?) police and sent via Vernet to the concentration camp Auschwitz, where he died in 1942.

After studies at the University of Montpellier, Grothendieck spent the academic year 1948/49 as "auditeur libre" at the École Normale Supérieure. In 1949, he went to Nancy and worked there under Dieudonné and Schwartz. Their paper *La dualité dans les espaces* $(\mathscr{F})$ *and* $(\mathscr{L}\mathscr{F})$ had a great impact on Grothendieck's early work. His thesis

Produits tensoriels topologiques et espaces nucléaires

HANKA GROTHENDIECK in Verehrung und Dankbarkeit gewidmet

was submitted in 1953. Afterward he visited the Universities of São Paulo (1953/54) and Kansas (1955). In Brazil, Grothendieck gave a course on *Espaces vectoriels topologiques* [GRO_2]. Most importantly, in this period he wrote the

Résumé [1956b, Trabalho recibido pela Sociedade de Matemática de São Paulo em Junho de 1954],

his outstanding contribution to Banach space theory. About 1955, Grothendieck changed to homological algebra and algebraic geometry.

From 1959 to 1970 he was a professor at the newly founded Institut des Hautes Études Scientifiques (Bures-sur-Yvette). In the next three years, Grothendieck acted as a visiting professor at the Collège de France and the University of Paris-Sud (Orsay). Finally, he accepted a chair in Montpellier, where he retired in 1988.

8.3.6.2 Beginning in 1970, Grothendieck developed a strong interest in political, ecological, philosophical and religious problems. He converted to Buddhism and now lives in the countryside (Mormoiron near Carpentras, South of France).

His views can be found in a long reflection *Récoltes et Semailles*, [GRO•, *Reaping and Sowing*, *Ernten und Säen*], that–in his own words–*became a portrait of the morals of the mathematical world* [of his time].

The following quotation from a letter [1989•] to the Swedish Royal Academy of Sciences in which he refused the Crafoord Prize provides a rough impression:

Meanwhile, the ethics of the scientific community (*at least among mathematicians*) *have declined to the point that outright theft among colleagues* (*especially at the expense of those who are in no position to defend themselves*) *has nearly become the general rule, and is in any case tolerated by all, even in the most obvious and iniquitous cases.*

...

I do not doubt that before the end of the century, totally unforeseen events will completely change our notions about "science" and its goals and the spirit in which scientific work is done.

Right! But, I am afraid that it became a change for the worse.

8.3.6.3 At the International Congress of Mathematicians in Moscow (1966), Alexandre **Grothendieck** was awarded the Fields Medal, mainly for his contributions to homological algebra and algebraic geometry. However, taking into account only his fundamental results in functional analysis would have yielded a sufficient reason. Here is the relevant passage from Dieudonné's laudation, [1966•, p. 24] (original version on p. 682):

So far, I have said nothing about the first memoirs of Grothendieck about topological vector spaces (*1950–1955*)*, partly because they are well known and more and more used in functional analysis. This is, in particular, true for the theory of nuclear spaces which "explains" the phenomena one encounters in the theory of distributions. I have had personally the privilege to be present, in those days, of the birth of the talent of this extraordinary "beginner" who at the age of 20 years was already a master; and, 10 years later, I am still convinced that the work of Grothendieck of this period can be compared with that of Banach, the one which has had the greatest influence on this part of mathematics.*

8.3.6.4 At the age of twenty-two, Grothendieck presented his first notes to the Comptes Rendus. From 1949 to 1955, he wrote 25 papers on locally convex and Banach spaces. It is no exaggeration to say that his ideas laid the foundation for the work of a whole generation of functional analysts.

Grothendieck's main achievements in Banach space theory include

- deep results about weakly compact operators, the Dunford–Pettis property, and related concepts: 4.8.5.3 and [1953],
- fundamental contributions to the theory of vector measures: 5.1.6.1 and [1953],
- an abstract theory of determinants: 6.5.2.12 and [1956a],
- introducing local techniques: Introduction to Section 6.1 and [1956b],
- enriching the theory of tensor products through his 14 natural norms: 6.3.11.6 and [1956b],
- stressing the significance of approximation properties and giving various equivalent reformulations: Subsection 5.7.4 and [GRO_1],
- pointing out the use of Banach spaces in the theory of (nuclear) locally convex linear spaces: quotation on p. xxii, 6.3.21.6 and [1956b].

8.3.6.5 The presentation above is based on an article of Cartier [1998•], the foreword of *The Grothendieck Festschrift* [CART$^{\cup}$], some pages of the autobiography of Schwartz [SCHW•, Engl. transl. pp. 282–286], as well as investigations of a journalist, Lisker [LISK•]. Very recently, Jackson [2004•] wrote a comprehensive article about the life of Grothendieck. Of special interest are Scharlau's *Materialien zu einer Biographie von Alexander Grothendieck*, which are available only via the Internet; see Scharlau [2004•].

Unfortunately, there are several contradictory statements. For example, a Russian dictionary [AAA_{06}, стр. 163], Lisker [LISK•, Part III, p. 18], and Dath [2002•] (in a well-known German newspaper) claim that the last name "Grothendieck" comes from his foster-mother. This is certainly false. Though his German first name was *Alexander*, all French publications appeared under the name *Alexandre* Grothendieck.

8.3.7 Nicolas Bourbaki

8.3.7.1 Nicolas **Bourbaki** is the pseudonym of a group of mathematicians whose names are kept top secret (at least in principle). The idea of its founding came up in December 1934 and was realized in July 1935 at a meeting in Besse-en-Chandesse (France). The original goal was to write a modern substitute of Goursat's *Traité d'Analyse*. Soon it turned out that this project had to be enlarged significantly. The final result Bourbaki's

Éléments de mathématique.

The first volume [BOU_{1a}] appeared in 1939. The working style of the Bourbaki team guaranteed *the control of the specialist by the nonspecialists*; see Borel [1998•, p. 376].

This is how Bourbaki identifies himself, [1948, footnote † on p. 221]:

Professor N. Bourbaki, formerly of the Royal Poldavian Academy, now residing in Nancy, France, is the author of a comprehensive treatise of modern mathematics ... .

In a later publication [1949], he moved his residence to *Nancago*. Moreover, Boas [1986•] once received a letter from *Bourbaki's ashram in the Himalayas* (secluded dwelling of a religious community and its guru):

You miserable worm, how dare you say that I do not exist?

A vehement reaction of J.R. Kline (at this time, secretary of the American Mathematical Society) can be found in an article of Lorch [1989•, p. 149]:

Now, really, these French are going too far. They have already given us a dozen independent proofs that Nicolas Bourbaki is a flesh and blood human being. He writes papers, sends telegrams, has birthdays, suffers from colds, sends greetings. And now they want us to take part in their canard. They want him to become a member of the American Mathematical Society. My answer is "No."

8.3.7.2 Almost all initiators were former students of the *École Normale Supérieure*:

Szolem **Mandelbrojt** (1899–1983),
*Jean **Delsarte** (1903–1968),
*Henri **Cartan** (1904),
René **de Possel** (1905–1974),
*André **Weil** (1906–1998),
*Jean **Dieudonné** (1906–1992),
*Claude **Chevalley** (1909–1984).

Weil was the spiritus rector, and those marked by * are considered as the real founding members.

By the end of 1935, the Bourbaki team had coopted Charles **Ehresmann** (1905–1979) and the geophysicist Jean **Coulomb** (1904–1999).

8.3.7.3 Based on the opinion that younger mathematicians are better mathematicians, the Bourbakists agreed on the rule that members should retire at the age of 50. In this way, a permanent self-renewing process was ensured.

According to [WEIL•, p. 169], Laurent **Schwartz** (1915–2002) became a member of Bourbaki in 1940. To the best of my knowledge, in the subsequent generations, functional analysis (in the widest sense) was represented by

Roger **Godement** (1921),
Jaques **Dixmier** (1924),
Alexandre **Grothendieck** (1928),
François **Bruhat** (1929),
Pierre **Cartier** (1932),
Adrien **Douady** (1935),
Alain **Connes** (1947).

8.3.7.4 Next, I describe the state of the art in the mid 1930s.

Taking Hilbert's *Grundlagen der Geometrie* (1899) as a pattern, several authors had used the axiomatic method in their monographs that were devoted to special mathematical disciplines:

Hausdorff	:	*Grundlagen der Mengenlehre,* (concerning Hausdorff spaces)	1914,
van der Waerden	:	*Moderne Algebra,*	1931,
Stone	:	*Linear Transformations in Hilbert Space,*	1932,
Banach	:	*Théorie des opérations linéaires,*	1932.

Now, in the opinion of Bourbaki, the time was ripe to do the same for mathematics as a whole. In order to arrange the material, one had to find a general scheme. In contrast to the classical division into arithmetic, algebra, geometry, analysis, ... , the concept of *structure* came into play. In the words of Dieudonné [1982•, p. 619]:

... *what really matters is not the nature of the objects under consideration, but their mutual relations,* ... .

Bourbaki's point of view was outlined in [1948], where he proposed to distinguish

algebraic structures, order structures and topological structures

as well as combinations thereof. For example, a Banach space is an algebraic-topological structure. Taking a little bit more of algebra, we get Banach algebras, whereas adding some order yields Banach lattices.

However, nothing is perfect: there was no proper place for *tribes* or *σ-rings* (in the awful Anglo-Saxon terminology; see Dieudonné [1982•, p. 623]). As a consequence, probability theory did not fit into Bourbaki's program and was therefore dropped.

Bourbakism was not and is not loved by all colleagues. However, during the 1950s and 1960s, it had a significant influence on the development of mathematics. Concerning Banach space theory, the *Espaces vectoriels topologiques* has been of greatest importance.

At present, there is a discussion about *The continuing silence of Bourbaki*, and Cartan said in an interview:

"What Bourbaki had to do is done now";

see Senechal [1998•, title] and Jackson [1999•, p. 785], respectively.

8.3.7.5 Besides the *Éléments de mathématique*, Bourbaki published only a few papers. The following are relevant for this text:

- *Sur les espaces de Banach* [1938],
- *L'architecture des mathématiques* [1948],
- *Foundations of mathematics for the working mathematician* [1949],
- *Sur certains espaces vectoriels topologiques* [1950a],
- *Sur le théorème de Zorn* [1950b].

It was Bourbaki's Note [1938], which in conjunction with Alaoglu's announcement [1938], provided *a mathematical tempest in the teapot*; see [EWI$^{\cup}$, p. 99] and 3.4.2.6. Indeed, one may be surprised why great scientists are vehemently fighting for priority in a case in which the same result was–for sure–obtained independently. I also refer to the Dieudonné–Halmos controversy concerning the "right approach" to the theory of measure and integration; see Section 4.7.

8.3.7.6 Bourbaki's standpoint concerning the presentation of a mathematical theory was described by Cartan [1959•, Engl. transl. p. 178] (original German version on p. 682):

There are never any historical references in the text itself, for Bourbaki never allowed the slightest deviation from the logical organization of the work.

Exceptionally, a few theorems are labeled by names: Hahn–Banach, Krein–Milman. This intentional anonymity also reflects the collective spirit of Bourbaki. Hence historians are not able to locate the source of those results or proofs that appeared for the first time in the *Éléments de mathématique*; and this happens quite often.

The lack just described is compensated by the *Notes historiques*. These notes have a very specific flavor that can be explained by the following quotation from Dieudonné [1982•, p. 620]:

... *mathematics is the opposite of democracy. History shows that the really seminal ideas are due to a small number of first-rate mathematicians.*

Therefore Bourbaki's historical comments are concerned with the main streams of the development, and single results are considered as minor pieces of the puzzle of mathematics. Ironically, Bourbaki shifted the ground in his own case: the quarrel with Alaoglu.

The *Notes historiques*, which are spread over the whole work of Bourbaki, have been collected in a special volume [BOU•]. Reading this text is a must for every mathematician.

8.3.7.7 The history of Bourbaki is described in the autobiography [WEIL•, Chap. V]. In what follows, I give some further references.

members of Bourbaki: Borel [1998•], Cartan [1959•], Dieudonné [1970•, 1982•], Schwartz [1949•], [SCHW•, Engl. transl.: pp. 146–165],

Interviews with members of Bourbaki: Guedj [1985•], Jackson [1999•], Senechal [1998•],

miscellaneous: [CHOU•], [MASH•], Beaulieu [1993•, 1994•], Boas [1986•], Choquet [1962•], Köthe [1955•].

8.4 Banach space theory at the ICMs

For their work (which includes Banach space theory), **Fields Medals** were awarded to

Laurent Schwartz,	Cambridge, Mass.	1950,
Alexandre Grothendieck,	Moscow	1966,
Jean Bourgain,	Zürich	1994,
Timothy Gowers,	Berlin	1998.

This list could be extended by

Charles Fefferman (Helsinki 1978) and Alain Connes (Warsaw 1983).

Invited addresses:

Oslo 1936:

S. Banach,

Die Theorie der Operationen und ihre Bedeutung für die Analysis.

Moscow 1966:

B.S. Mityagin, A. Pełczyński,

Nuclear operators and approximative dimension.

NICE 1970:

D.A. Edwards,
Compact convex sets.

J. Lindenstrauss,
The geometric theory of the classical Banach spaces.

J. Peetre,
Interpolation functors and Banach couples.

VANCOUVER 1974:

Z. Ciesielski,
Bases and approximation by splines.

P. Enflo,
Recent results on general Banach spaces.

B. Maurey,
Quelques problèmes de factorisation d'operateurs linéaries.

HELSINKI 1978:

$\emptyset$

WARSAW 1983:

A. Pełczyński, (`Plenary Address`)
Structural theory of Banach spaces
and its interplay with analysis and probability.

J. Bourgain,
New Banach space properties of spaces of analytic functions.

T. Figiel,
Local theory of Banach spaces and some operator ideals.

B. Kashin,
Некоторые результаты об оценках поперечников,
(*Some results on estimates of diameters*).

G. Pisier,
Finite rank projections on Banach spaces
and a conjecture of Grothendieck.

BERKELEY 1986:

J. Bourgain,
Geometry of Banach spaces and harmonic analysis.

E.G. Effros,
Advances in quantized functional analysis.

E.D. Gluskin,
Вероятность в геометрии банаховых пространств,
(*Probability in geometry of Banach spaces*).

V.D. Milman,
The concentration phenomenon and
linear structure of finite-dimensional normed spaces.

KYOTO 1990:

M. Talagrand,
Some isoperimetric inequalities and their applications.

ZÜRICH 1994:

T. Gowers,
Recent results in the theory of infinite-dimensional Banach spaces.

E. Odell, T. Schlumprecht
Distortion and stabilized structure in Banach spaces;
New geometric phenomena for Banach and Hilbert spaces.

BERLIN 1998:

G. Pisier, (`Plenary Address`)
Operator spaces and similarity problems.

V.D. Milman,
Randomness and pattern in convex geometric analysis.

N. Tomczak-Jaegermann,
From finite- to infinite-dimensional phenomena
in geometric functional analysis on local and asymptotic level.

BEIJING 2002:

S. Alesker,
Algebraic structures on valuations, their properties and applications.

R. Latała,
On some inequalities for Gaussian measures.

8.5 The Banach space archive

The idea to have a *Banach Space Newsletter* goes back to P. Casazza. However, due to the use of computers, this project was never implemented in paper form. Instead, D. Alspach created an electronic *Bulletin Board* in August 1989, located at Oklahoma State University under the address

`http://www.math.okstate.edu/~alspach/banach`

In April 1998, the *Banach Space Archive* was added to the *Los Alamos Archive* and can be approached via

`banach@math.okstate.edu`

At present, the attached *Address Book* includes the names of about 400 subscribers.

More information about *The history of the Banach space archive and implications for electronic archives of publications* may be found in an article of Alspach that is available only via the Internet.

8.6 Banach space mathematicians

This section contains biographical data about mathematicians who have contributed to Banach space theory and related fields. It was my intention to provide a picture of the different generations that were involved. Though I spent a large amount of time, my attempt is necessarily incomplete and I apologize for any omissions.
Readers who are interested to know the year in which the Ph.D. (or some equivalent academic degree) was awarded should consult *The Mathematics Genealogy Project*, [www_1]. The heuristic formula

$$\text{year of birth} + 24 \leq \text{year of Ph.D.} \leq \text{year of birth} + 30$$

yields a good approximation. In a few instances, this inequality was applied to guess the (unknown) year of birth. Such uncertain data are marked by $\sim$.

ARGENTINA
Santaló, Luis A. (1911–2001)

AUSTRALIA
Dodds, Peter (1942)
Edwards, Robert E. (1926–2000)
Robertson, Alexander P. (1925–1995)
Robertson, Wendy (1927)
Sims, Brailey (1947)
Yost, David (1955)

AUSTRIA
Cigler, Johann (1937)
Cooper, James B. (1944)
Fischer, Ernst S. (1875–1954)
Hahn, Hans (1879–1934)
Helly, Eduard (1884–1943)
Langer, Heinz (1935)
Losert, Viktor (1952)
Michor, Peter (1949)
Müller, Paul F.X. (1960)
Racher, Gerhard (1952)
Radon, Johann (1887–1956)
Schachermayer, Walter (1950)
Schmetterer, Leopold K. (1919–2004)
Schmuckenschläger, Michael (1960)
Stegall, Charles P. (1943)
Tauber, Alfred (1866–1942)
Vietoris, Leopold F. (1891–2002)

BELGIUM
Bourgain, Jean (1954)
Daubechies, Ingrid (1954)
Delbaen, Freddy (1946)
Finet, Catherine (1956)
Garnir, Henri G. (1921–1985)
Lumer, Gunter (Günter) (1929–2005)
Schmets, Jean (1940)
Waelbroeck, Lucien (1929)
Wilde, Marc De (1940)

BRAZIL
Araujo, Aloisio (1946)
Barroso, Jorge A. (1928)
Matos, Mario C. (1939)
Mujica, Jorge (1946)
Nachbin, Leopoldo (1922–1993)
Prolla, Joao B. (1935)
Zapata, Guido I. (1940)

BULGARIA
Alexandrov, Georgi A. (1950)
Karadzhov, Georgi E. (1946)
Kutzarova, Denka N. (1956)
Maleev, Rumen P. (1943)
Troyanski, Stanimir L. (1944)

CANADA
Atkinson, Frederick V. (1916–2003)
Borwein, Jonathan M. (1951)
Boyd, David W. (1941)
Davis, Chandler (1926)
Fournier, John J.F. (1942)
Ghoussoub, Nassif (1953)
Koosis, Paul J. (1929)
Lancaster, Peter (1929)
Lowig (Löwig), Henry F. J. (1904–1995)
Radjavi, Heydar (1935)

Rosenthal, Peter M. (1941)
Swaminathan, Srinivasan (1926)
Tomczak-Jaegermann, Nicole (1945)
Whitfield, John H.M. (1939)
Zizler, Václav (1943)

China

Bu, Shanquan (1963)
Chang, Shih-Hsun (1900–1985)
Chen, Shutao (1950)
Cheng, Lixin (1959)
Kwan, Chao-Chih (1919–1982)
Ren, Zhong-Dao (1938)
Tseng, Yuan-Yung (1903–1994)
Wang, Tingfu (1933–2000)
Wang, Yuwen (1950)
Wu, Congxin (1935)
Xia, Dao-Xing (1930)

Czechoslovakia

Čech, Eduard (1893–1960)
Dobrakov, Ivan (1940–1997)
Fabian, Marián (1949)
Frolík, Zdeněk (1933–1989)
Fučik, Svatopluk (1944–1979)
Habala, Petr (1968)
Hájek, Petr (1968)
John, Kamil (1942)
John, Oldřich (1940)
Katětov, Miroslav (1918–1995)
Kolomý, Josef (1934–1993)
Kufner, Alois (1934)
Kurzweil, Jaroslav (1926)
Löwner, Karel (1893–1968)
Nečas, Jindřich V. (1929–2002)
Pelant, Jan (1950)
Preiss, David (1947)
Pták, Vlastimil (1925–1999)
Zizler, Václav (1943)

Denmark

Bonnesen, Tommy (1873–1935)
Fenchel, Werner (1905–1988)
Haagerup, Uffe V. (1949)
Hoffmann-Jørgensen, Jørgen (1942)
Jensen, Johan L.W.V. (1859–1925)
Jessen, Børge (1907–1993)
Laursen, Kjeld B. (1942)
Nielsen, Niels J. (1943)
Pedersen, Gert K. (1940–2004)
Poulsen, Ebbe Thue (1931)

Finland

Apiola, Heikki (1942)
Astala, Kari (1953)
Geiß, Stefan (1958)
Ketonen, Timo (1950)
Taskinen, Jari (1966)
Tylli, Hans-Olav (1958)
Vala, Klaus (1930–2000)

France

Badrikian, Albert (1933–1994)
Baire, René (1874–1932)
Beauzamy, Bernard (1949)
Becker, Richard (1947)
Bloch, André (1893–1948)
Borel, Émile (1871–1956)
Bretagnolle, Jean P. (1937)
Brézis, Haïm (1944)
Brunel, Antoine (1920–2003)
Cam, Lucien Le (1924–2000)
Cartan, Henri (1904)
Cartier, Pierre (1932)
Chevallier, Claude (1909–1984)
Chevet, Simone (1941)
Choquet, Gustave (1915)
Connes, Alain (1947)
Dacunha-Castelle, Didier (1937)
Delsarte, Jean (1903–1968)
Deville, Robert (1959)
Dieudonné, Jean (1906–1992)
Dixmier, Jacques (1924)
Doeblin, Wolfgang (1915–1940)
Ferenczi, Valentin (1970)
Fernique, Xavier (1934)
Fortet, Robert (1912–1998)
Fréchet, Maurice (1878–1973)
Gâteaux, René (1889–1914)
Godefroy, Gilles (1953)
Godement, Roger (1921)
Grisvard, Pierre (1940)
Grothendieck, Alexandre (1928)
Guerre-Delabrière, Sylvie (1953)
Hadamard, Jacques (1865–1963)
Kahane, Jean-Pierre (1926)
Krivine, Jean-Louis (1939)
Lancien, Gilles (1964)
Lapresté, Jean-Thierry (1949)
Lebesgue, Henri (1875–1941)
Ledoux, Michel (1958)
Levy, Mireille ($\sim$1955)

Lévy, Paul (1886–1971)
Lions, Jacques-Louis (1928–2001)
Malgrange, Bernard (1928)
Mantelbrojt, Szolem (1893–1983)
Maurey, Bernard (1948)
Meyer, Paul-André (1934–2003)
Meyer, Yves (1939)
Montel, Paul (1876–1975)
Mourier, Edith (1920)
Neveu, Jacques (1932)
Pajor, Alain (1949)
Pfitzner, Hermann (1959)
Pisier, Gilles (1950)
Raynaud, Yves (1952)
Salem, Raphaël (1898–1963)
Saphar, Pierre D. (1934)
Schwartz, Laurent (1915–2002)
Stern, Jacques (1949)
Talagrand, Michel (1952)
Xu, Quanhua (1961)
Weil, André (1906–1998)

GERMANY

Albrecht, Ernst (1944)
Arendt, Wolfgang (1950)
Bauer, Heinz (1928–2002)
Baumgärtel, Hellmut (1934)
Behrends, Ehrhard (1946)
Bernstein, Felix (1878–1956)
Bierstedt, Klaus D. (1945)
Böttcher, Albrecht (1954)
Bois-Reymond, Paul du (1831–1889)
Braunß, Hans-Andreas (1959)
Butzer, Paul L. (1928)
Cantor, Georg (1845–1918)
Carathéodory, Constantin (1873–1950)
Carl, Bernd (1947)
Defant, Andreas (1953)
Defant, Martin (1955)
Ehrhardt, Torsten (1971)
Elschner, Johannes (1949)
Eschmeier, Jörg (1956)
Floret, Klaus (1941–2002)
Freudenthal, Hans (1905–1990)
Frobenius, F. Georg (1849–1917)
Fuchssteiner, Benno (1941)
Geiß, Stefan (1958)
Gramsch, Bernhard (1938)
Hamburger, Hans L. (1889–1956)
Hankel, Hermann (1839–1873)
Harmand, Peter (1953)
Hausdorff, Felix (1868–1942)
Heinrich, Stephan (1950)
Hellinger, Ernst (1883–1950)
Hensgen, Wolfgang (1957)
Heuser, Harro (1927)
Hilbert, David (1862–1943)
Hinrichs, Aike (1968)
Hölder, L. Otto (1859–1937)
Höllig, Klaus (1953)
Hopf, Eberhardt (1902–1983)
Hopf, Heinrich (Heinz) (1894–1971)
John, Fritz (1910–1994)
Junek, Heinz (1944)
Junge, Marius (1962)
Junghanns, Peter (1953)
Kaballo, Winfried (1952)
Keller, Ott-Heinrich (1906–1990)
König, Heinz (1929)
König, Hermann (1949)
Köthe, Gottfried (1905–1989)
Kühn, Thomas (1952)
Kürsten, Klaus-Detlef (1950)
Landau, Edmund (1877–1938)
Langer, Heinz (1935)
Laßner, Gerd (1940–2005)
Leopold, Hans-Gerd (1951)
Linde, Werner (1947)
Lipschitz, Rudolf O.S. (1832–1903)
Löwig (Lowig), Heinrich F.J. (1904–1995)
Lubitz, Claus (1953)
Lusky, Wolfgang (1948)
Meise, Reinhold (1945)
Meyer-Nieberg, Peter (1944)
Minkowski, Hermann (1864–1909)
Müntz, Chaim (Herman) (1884–1956)
Nagel, Rainer (1940)
Neubauer, Gerhard (1930–2003)
Neumann, Michael M. (1950)
Noether, Fritz E.A. (1884–1941)
Pallaschke, Diethard (1940)
Pfitzner, Hermann (1959)
Pietsch, Albrecht (1934)
Prößdorf, Siegfried (1939–1998)
Räbiger, Frank (1958)
Rademacher, Hans (1892–1969)
Rellich, Franz (1906–1955)
Roch, Steffen (1958)
Rueß, Wolfgang (1944)
Runst, Thomas (1952)
Schaefer, Helmut H. (1925–2005)

Schlumprecht, Thomas B. (1954)
Schmeißer, Hans-Jürgen (1950)
Schmidt, Erhardt (1876–1959)
Schock, Eberhard (1939)
Schoenflies, Arthur Moritz (1853–1928)
Schröder, Ernst (1841–1902)
Schur, Issai (1875–1941)
Schütt, Carsten (1949)
Schütt, Ingo (1955)
Schwarz, Hans-Ulrich (1946–2001)
Sickel, Winfried (1954)
Silbermann, Bernd (1941)
Steinitz, Ernst (1871–1928)
Stephani, Irmtraud (1935)
Stöckert, Bernd (1943)
Teichmüller, Oswald (1913–1943)
Tillmann, Heinz-Günther (1924)
Toeplitz, Otto (1881–1940)
Triebel, Hans (1936)
Unger, Heinz (1914)
Vogt, Dietmar (1941)
Wecken, Franz (1912)
Weidmann, Joachim (1939)
Weierstrass, Karl (1815–1897)
Weis, Lutz (1950)
Wenzel, Jörg (1965)
Werner, Dirk (1955)
Werner, Elisabeth (1958)
Werner, Wend (1958)
Weyl, Hermann (1885–1955)
Wittstock, Gerd (1939)
Wloka, Joseph (1929)
Zeller, Karl (1924–2006)
Zermelo, Ernst F.F. (1871–1953)
Zorn, Max A. (1906–1993)

GREECE

Androulakis, George (1967)
Argyros, Spiros A. (1950)
Carathéodory, Constantin (1873–1950)
Deliyanni, Irene (1964)
Farmaki, Vasiliki (1946)
Giannopoulos, Apostolos (1963)
Kappos, Demetrios A. (1904–1985)
Mercourakis, Sophocles (1956)
Negrepontis, Stelios (1939)
Pantelidis, Georgios (1936)
Papadopoulou, Souzanna (1945)

HUNGARY

Fejér, Lipót (1880–1959)
Haar, Alfred (1885–1933)
Horváth, John (1924)
Kürschák, Josef A. (1864–1933)
Neumann, János (John) von (1903–1957)
Riesz, Frigyes (1880–1956)
Riesz, Marcel (1886–1969)
Szőkefalvi-Nagy, Bela (1913–1998)
Wintner, Aurel F. (1903–1958)

INDIA

Chandrasekharan, Komaravolu (1920)
Gupta, Manjul (1950)
Kamthan, P.K. (1938–1990)
Namboodiri, M.N. Narayana (1953)
Parthasarathy, Kalyanapuram. R. (1938)
Rao, C. Radhakrishna (1920)
Rao, Taduri S.S.R.K. (1954)
Roy, Ashoke K. (1939)
Sunder, Viakalathur S. (1952)
Vaidyanathaswamy, R. (1894–1960)
Varadarajan, Veeravalli S. (1937)
Varadhan, S.R. Srinivasa (1940)

IRELAND

Boland, Philip J. (1944)
Dineen, Seán (1944)
Murphy, Gerard J. (1948)
Ryan, Raymond A. (1953)
Smyth, Malcolm Roger F. (1946)
West, Trevor T. (1938)

ISRAEL

Agmon, Shmuel (1921)
Aharoni, Israel (1945)
Alesker, Semyon (1972)
Altshuler, Zvi (1940)
Amir, Dan (1933)
Arazy, Jonathan (1942)
Artstein, Shiri (1978)
Benyamini, Yoav (1943)
Brudnyi, Yuri (1934)
Cwikel, Michael (1948)
Dar, Sean (1970)
Dvoretzky, Arieh (1916)
Dym, Harry (1938)
Feldman, Israel (1933)
Foguel, Shaul R. (1931)
Fonf, Vladimir (1949)
Gluskin, Efim (Haim) (1953)
Gohberg, Israel C. (1928)
Gordon, Yehoram (1940)

Gorelik, Evgeni (1947–1993)
Katznelson, Itzhak (1934)
Klartag, Boaz (1978)
Krupnik, Naum (1932)
Lazar, Aldo Joram (1936)
Lerer, Leonid (Ariel) (1943)
Lindenstrauss, Joram (1936)
Litvak, Alexander E. (1969)
Livsic, Moshe (1917)
Lyubich, Yuri (1931)
Maiorov, Vitali (1946)
Markus, Alexander (1932)
Matsaev, Vladimir (1937)
Milman, David (1913–1982)
Milman, Vitali (1939)
Noar, Assaf (1975)
Olevskii, Alexander (1939)
Pazy, Amnon (1936)
Pinkus, Allan (1946)
Reisner, Shlomo (1943)
Rudelson, Mark (1965)
Saphar, Pierre D. (1934)
Schechtman, Gideon (1947)
Schur, Issai (1875–1941)
Solomyak, Michael (1931)
Sternfeld, Yaki (1944–2001)
Szankowski, Andrzej (1945)
Toeplitz, Otto (1881–1940)
Tsolomitis, Antony (1967)
Tsirelson, Boris (1950)
Tzafriri, Lior (1936)
Wagner, Roy (1973)
Wolfson, Haim (1951)
Zippin, Morry (1939)

ITALY

Aiena, Pietro ( )
Altomare, Francesco (1951)
Arzelà, Cesare (1847–1912)
Ascoli, Giulio (1843–1896)
Ascoli, Guido (1887–1957)
Cacciopoli, Renato (1904–1959)
Gagliardo, Emilio (1930)
Metafune, Giorgio (1962)
Moscatelli, V. Bruno (1945)
Papini, Pier L. (1943)
Peano, Guiseppe (1858–1939)
Pincherle, Salvatore (1853–1936)
Terenzi, Paolo (1940)
Volterra, Vito (1860–1940)

JAPAN

Amemiya, Ichiro (1923–1995)
Ando, Tsuyoshi (1932)
Aoki, Tosio ($\sim$1915–1989)
Hida, Takeyuki (1927)
Kakutani, Shizuo (1911–2004)
Kato, Mikio (1948)
Kato, Tosio (1917–1999)
Matsuda, Minoru (1946)
Miyazaki, Ken-ichi (1930–2001)
Nagumo, Mitio (1905)
Nakano, Hidegorô (1909–1974)
Sakai, Shôichirô (1928)
Shiga, Kôji (1930)
Shimogaki, Tetsuya (1932–1971)
Takahashi, Yasuji (1944)
Takesaki, Masamichi (1933)
Tomita, Minuro (1924)
Uchiyama, Akihito (1948–1997)
Yosida, Kôsaku (1909–1990)

NETHERLANDS

Dulst, Dick van (1938)
Freudenthal, Hans (1905–1990)
Kuiper, Niclaas H. (1920–1994)
Kaashoek, Marinus A. (1937)
Lekkerkerker, C. Gerrit (1922–1999)
Luxemburg, Wilhelmus A.J. (1929)
Monna, Antonie F. (1909–1995)
Pagter, de Ben (1953)
Zaanen, Adrian C. (1913–2003)

NORWAY

Alfsen, Erik M. (1930)
Lima, Åsvald (1942)
Olsen, Gunnar (1946)

POLAND

Alexiewicz, Andrzej (1917)
Auerbach, Herman (1901–1942)
Banach, Stefan (1892–1945)
Bergman(n), Stefan (1895–1977)
Bessaga, Czesław (1932)
Birnbaum, Z. Wilhelm (1903–2000)
Bogdan(owicz), Witold M. (1933)
Borzyszkowski, Andrzej (1955)
Ciesielski, Zbigniew (1934)
Domański, Paweł (1959)
Drewnowski, Lech (1944)
Eidelheit, Maks (1911–1943)
Figiel, Tadeusz (1948)
Goebel, Kazimierz (1940)

Hudzik, Henryk (1945)
Kac, Marek (1909–1984)
Kaczmarz, Stefan (1895–1939)
Kamińska, Anna (1950)
Komorowski, Ryszard (1955–2003)
Kwapień, Stanisław (1942)
Labuda, Iwo (1945)
Latała, Rafał (1971)
Leżański, Tadeusz (1923)
Maligranda, Lech (1953)
Mankiewicz, Piotr (1943)
Marczewski, Edward (1907–1976)
Mastyło, Mieczysław (1956)
Mazur, Stanisław (1905–1981)
Müntz, Chaim (Herman) (1884–1956)
Musielak, Julian (1928)
Nikodym, Otton M. (1887–1974)
Oleszkiewicz, Krzysztof (1972)
Orlicz, Władysław (1903–1990)
Pełczyński, Aleksander (1932)
Przeworska-Rolewicz, Danuta (1931)
Rolewicz, Stefan (1932)
Saks, Stanisław (1897–1942)
Schauder, Juliusz (1899–1943)
Schreier, Józef (1909–1942)
Semadeni, Zbigniew (1934)
Sikorski, Roman (1920–1983)
Słowikowski, Wojciech (1932)
Steinhaus, Hugo D. (1887–1972)
Sternbach, Ludwik (1905–1943)
Szankowski, Andrzej (1945)
Szarek, Stanisław (1952)
Szlenk, Wiesław (1935–1995)
Szpilrajn = Marczewski see 8.2.1.10
Tomczak-Jaegermann, Nicole (1945)
Toruńczyk, Hendryk (1945)
Ulam, Stanisław (1909–1984)
Waszak, Aleksander (1936)
Wojciechowski, Michał (1966)
Wojtaszczyk, Przemysław (1948)
Wolniewicz, Tomasz M. (1955)
Woyczyński, Wojbor A. (1943)
Wozniakowski, Krzysztof (1964)
Żelazko, Wiesław (1933)
Zemánek, Jaroslav (1946)
Zygmund, Antoni S. (1900–1992)

Romania

Apostol, Constantin (1936–1987)
Călugăreanu, Gheorghe (1902–1976)
Colojoară, Ion (1930)
Cristescu, Romulus (1928)
Dinculeanu, Nicolae (1925)
Foiaş, Ciprian I. (1933)
Ghika, Alexandru (1902–1964)
Lalesco, Trajan (1882–1929)
Nicolescu, Miron (1903–1975)
Niculescu, Constantin P. (1947)
Onicescu, Oktav (1892–1983)
Popa, Nicolae (1943)
Singer, Ivan (1929)
Vasilescu, Florian-Horia (1941)
Voiculescu, Dan V. (1949)
Vuza, Dan (1955)
Zsidó, Laszlo (1946)

Russia see Soviet Union (former)

South Africa

Eldik, Peter van (1945)
Engelbrecht, Johann (1946)
Fourie, Jan H. (1951)
Grobler, Jacobus J. (1943)
Raubenheimer, Heinrich (1952)
Ströh, Anton (1965)
Swart, Johan (1948)

Soviet Union (former) СССР

Abramovich, Yuriĭ A. (1945–2003)
Абрамович, Юрий А.

Akhiezer, Naum I. (1901–1980)
Ахиезер, Наум И.

Akilov, Gleb A. (1921–1986)
Акилов, Глеб П.

Alexandrov, Alexander D. (1912–1999)
Александров, Александр Д.

Alexandrov, Pavel S. (1896–1982)
Александров, Павел С.

Allakhverdiev, Dzhalal E. (1929)
Аллахвердиев, Джалал Э.

Arkhangelskiĭ, Alexander V. (1938)
Архангельский, Александр В.

Artemenko, Alexander P. (1909–1944)
Артеменко, Александр П.

Bari, Nina K. (1903–1961)
Бари, Нина К.

Berezanskiĭ, Yuriĭ M. (1925)
Березанский, Юрий М.

Bernstein, Sergeĭ N. (1880–1968)
Бернштейн, Сергей Н.

Besov, Oleg V. (1933)
Бесов, Олег В.

Birman, Mikhail Sh. (1928)
Бирман, Михаил Ш.

Bochkarev, Sergeĭ V. (1941)
Бочкарев, Сергей В.

Brodskiĭ, Mikhail S. (1913–1989)
Бродский, Михаил С.

Brudnyĭ, Yuriĭ A. (1934)
Брудный, Юрий А.

Bukhvalov, Alexander V. (1948)
Бухвалов, Александр В.

Chintchine see Khintchine

Chobanyan, Sergeĭ A. (1942)
Чобанян, Сергей А.

Daugavet, Igor K. (1932)
Даугавет, Игорь К.

Dzhrbashyan, Mkhutar M. (1918)
Джрбашян, Мхутар М.

Egorov, Dimitriĭ F. (1869–1931)
Егоров, Дмитрий Ф.

Feldman, Israel A. (1933)
Фельдман, Израиль А.

Fichtenholz see Fikhtengolts

Fikhtengolts, Grigoriĭ M. (1888–1959)
Фихтенгольц, Григорий М.

Fomin, Sergeĭ V. (1917–1975)
Фомин, Сергей В.

Fonf, Vladimir P. (1949)
Фонф, Владимир П.

Gantmakher, Vera R. (1909–1942)
Гантмахер, Вера Р.

Gavurin, Mark K. (1911–1992)
Гавурин, Марк К.

Gelfand, Israel M. (1913)
Гельфанд, Израиль М.

Gikhman, Iosif I. (1918)
Гихман, Иосиф И.

Glazman, Israel M. (1916–1968)
Глазман, Израиль М.

Gluskin, Efim (Haim) D. (1953)
Глускин, Ефим Д.

Gohberg, Israel C. (1928)
Гохберг, Израиль Ц.

Gorelik, Evgeniĭ M. (1947–1993)
Горелик, Евгений М.

Govurin or Gowurin see Gavurin

Grinblum, Maximilian M. (1903–1951)
Гринблюм, Максимилиан М.

Gulko, Sergeĭ P. (1950)
Гулько, Сергей П.

Gurariĭ, Vladimir I. (1935–2005)
Гурарий, Владимир И.

Gurevich, Lev A. (1910–1993)
Гуревич, Лев А.

Havin see Khavin

Helemskiĭ see Khelemskiĭ

Henkin see Khenkin

Ibragimov, Ildar A. (1932)
Ибрагимов, Ильдар А.

Ismagilov, Rais. S. (1938)
Исмагилов, Раис. С.

Jerbashian see Dzherbashyan

Kadets, Mikhail I. (1923)
Кадец, Михаил И.

Kadets, Vladimir M. (1960)
Кадец, Владимир М.

Kantorovich, Leonid V. (1912–1986)
Канторович, Леонид В.

Kashin, Boris S. (1951)
Кашин, Борис С.

Keldysh, Mstislav V. (1911–1978)
Келдыш, Мстислав В.

Khavin, Viktor P. (1933)
Хавин, Виктор П.

Khelemskiĭ, Alexander Ya. (1943)
Хелемский, Александр Я.

Khenkin, Gennadiĭ M. (1942)
Хенкин, Геннадий М.

Khintchine, Alexander Ya. (1894–1959)
Хинчин, Александр Я.

Kislyakov, Sergeĭ V. (1950)
Кисляков, Сергей В.

Koldobskiĭ, Alexander L. (1955)
Колдобский, Александр Л.

Kolmogorov, Andreĭ N. (1903–1987)
Колмогоров, Андрей Н.

Kondrashov, Vladimir I. (1919)
Кондрашов, Владимир И.

Korotkov, Vitaliĭ B. (1936)
Коротков, Виталий Б.

Kostyuchenko, Anatoliĭ G. (1931)
Костюченко, Анатолий Г.

Krasnoselskiĭ, Mark A. (1920–1997)
Красносельский, Марк А.

Kreĭn, Mark G. (1907–1989)
Крейн, Марк Г.

Kreĭn, Selim G. (1917–1999)
Крейн, Селим Г.

Kruklyak, Natan Ya. (1949)
Кругляк, Натан Я.

Krupnik, Naum Ya. (1932)
Крупник, Наум Я.

Kudryavtsev, Lev D. (1923)
Кудрявцев, Лев Д.

Lerer, Leonid (Arieh) (1943)
Лерер, Леонид Е.

Levin, Boris Ya. (1906–1993)
Левин, Борис Я.

Lewin see Levin

Lidskiĭ, Viktor B. (1924)
Лидский, Виктор Б.

Linnik, Yuriĭ V. (1915–1972)
Линник, Юрий В.

Livsic, Mikhail (Moshe) (1917)
Лившиц, Михаил С.

Lizorkin, Petr I. (1922–1993)
Лизоркин, Петр И.

Ljusternik see Lyusternik

Lomonosov, Victor I. (1946)
Ломоносов, Виктор И.

Lorentz, Georg G. (1910–2006)
Лоренц, Георгий Р.

Lozanovskiĭ, Grigoriĭ Ya. (1937–1976)
Лозановский, Григорий Я.

Luzin (Lusin) , Nikolaĭ N. (1883–1950)
Лузин, Николай Н.

Lyantse, Vladislav E. (1920)
Лянце, Владислав Э.

Lyubich, Yuriĭ I. (1931)
Любич, Юрий И.

Lyusternik, Lazar A. (1899–1981)
Люстерник, Лазарь А.

Maĭorov, Vitaliĭ E. (1946)
Майоров, Виталий Е.

Makarov, Boris M. (1932)
Макаров, Борис М.

Makovoz, Yuly (1937)
Маковоз, Юлий

Markov, Andreĭ A. (junior) (1903–1979)
Марков, Андрей А. (млад.)

Markus, Alexander S. (1932)
Маркус, Александр С.

Matsaev, Vladimir I. (1937)
Мацаев, Владимир И.

Mazya, Vladimir G. (1937)
Мазья, Владимир Г.

Menshov, Dmitriĭ E. (1892–1988)
Меньшов, Дмитрий Е.

Michlin Mikhlin

Mikhlin, Solomon G. (1908–1990)
Михлин, Соломон Г.

Milman, David P. (1913–1982)
Мильман, Давид П.

Milman, Vitaliĭ D. (1939)
Мильман, Виталий Д.

Milyutin, Alexeĭ A. (1925–2001)
Милютин, Алексей А.

Minlos, Robert A. (1931)
Минлос, Роберт А.

Mityagin, Boris S. (1937)
Митягин, Борис С.

Müntz, Herman M. (1884–1956)
Мюнц, Герман М.

Naĭmark, Mark A. (1909–1978)
Наймарк, Марк А.

Neumark see Naĭmark

Nikishin, Evgeniĭ M. (1945–1986)
Никишин, Евгений М.

Nikolskiĭ, Nikolaĭ K. (1940)
Никольский, Николай К.

Nikolskiĭ, Sergeĭ M. (1905)
Никольский, Сергей М.

Nudelman, Adolf A. (1931)
Нудельман, Адольф А.

Oja nee Martin, Eve (1948)
Оя, Эве Ф.

Olevskiĭ, Alexander M. (1939)
Олевский, Александр М.

Ostrovskiĭ, Mikhail I. (1960)
Островский, Михаил И.

Ovchinnikov, Vladimir I. (1947)
Овчинников, Владимир И.

Peller, Vladimir V. (1955)
Пеллер, Владимир В.

Petunin, Yuriĭ I. (1937)
Петунин, Юрий И.

Pinsker, Aron G. (1905–1985)
Пинскер, Арон Г.

Ples(s)ner, Abraham I. (1900–1961)
Плеснер, Абрам И.

Plichko, Anatoliĭ N. (1949)
Пличко, Анатолий Н.

Pontryagin, Lev S. (1908–1988)
Понтрягин, Лев С.

Potapov, Vladimir P. (1914–1980)
Потапов, Владимир П.

Prokhorov, Yuriĭ V. (1929)
Прохоров, Юрий В.

Raĭkov, Dmitriĭ A. (1905–1980)
Райков, Дмитрий А.

Reĭnov, Oleg I. (1951)
Рейнов, Олег И.

Rokhlin, Vladimir A. (1919–1984)
Рохлин, Владимир А.

Rutitskiĭ, Yakov B. (1922)
Рутицкий, Яков Б.

Rutman, Moiseĭ A. (1917)
Рутман, Моисей А.

Sazonov, Vyacheslav V. (1935)
Сазонов, Вячеслав В.

Schilow see Shilov

Schnirelman see Shnirelman

Semenov, Evgeni M. (1940)
Семенов, Евгений М.

Shevchyk, Vitaliĭ V. (1949)
Шевчик, Виталий В.

Shilov, Georgiĭ E. (1917–1975)
Шилов, Георгий Е.

Shmulyan, Vitold L. (1914–1944)
Шмульян, Витольд Л.

Shnirelman, Lev G. (1905–1938)
Шнирельман, Лев Г.

Shvarts, Albert S. (1934)
Шварц, Альберт С.

Sirvint, Yuriĭ F. (1913–1942)
Сирвинт, Юрий Ф.

Skorokhod, Anatoliĭ V. (1930)
Скороход, Анатолий В.

Šmulian see Shmulyan

Sobolev, Vladimir I. (1913–1995)
Соболев, Владимир И.

Sobolev, Sergeĭ L. (1908–1989)
Соболев, Сергей Л.

Solomyak, Michail Z. (1931)
Соломяк, Михаил З.

Stechkin, Sergeĭ B. (1920–1995)
Стечкин, Сергей Б.

Sudakov, Vladimir N. (1934)
Судаков, Владимир Н.

Tamarkin, Jacob D. (1888–1945)
Тамаркин, Яков Д.

Tarieladze, Vazha I. (1949)
Тариеладзе, Важа И.

Tikhomirov, Vladimir M. (1934)
Тихомиров, Владимир М.

Tikhonov, Andreĭ N. (1906–1993)
Тихонов, Андрей Н.

Timan, Alexander F. (1920–1988)
Тиман, Александр Ф.

Tsirelson, Boris S. (1950)
Цирельсон, Борис С.

Tychonoff see Tikhonov

Ulyanov, Peter L. (1928)
Ульянов, Петр Л.

Vakhaniya, Nikolaĭ N. (1930)
Вахания, Николай Н.

Vershik, Anatoliĭ M. (1933)
Вершик, Анатолий М.

Vilenkin, Naum Ya. (1920–1991)
Виленкин, Наум Я.

Vinogradov, Stanislav A. (1941–1997)
Виноградов, Станислав А.

Vitushkin, Anatoliĭ G. (1931–2004)
Витушкин, Анатолий Г.

Vladimirov, Denis A. (1929–1994)
Владимиров, Денис А.

Vulikh, Boris Z. (1913–1978)
Вулих, Борис З.

Wilenkin see Vilenkin

Wladimirow see Vladimirov

Wulich see Vulikh

Yudin, Abraham I. (1919–1941)
Юдин, Абрам И.

SPAIN

Arias de Reyna, Juan (1947)
Bastero, Jesús (1950)
Blasco, Oscar (1959)
Bombal, Fernando (1944)
Cascales, Bernardo (1958)
Castillo, Jesús M.F. (1959)
Cobos, Fernando (1956)
Freniche, Fancisco J. (1957)
García-Cuerva, José (1949)
González, Manuel (1957)
Guerra, Pedro J. (1951)
Hernández, Francisco L. (1953)
Martín-Peinador, Elena (1948)
Moltó, Anibal (1952)
Montesinos, Vincente (1952)
Onieva, Victor (1938–1988)
Orihuela, José (1958)
Payá, Rafael A. (1955)
Plans, Antonio (1922–1998)
Reyes, Andrés (1951–1983)
Rodés, Alvaro (1953)
Rodríguez-Palacios, Ángel (1947)
Rodríguez-Salinas, Baltasar (1925)
Rubio de Francia, José L. (1949–1988)
Santaló, Luis A. (1911–2001)
Torrea, José L. (1955)
Valdivia, Manuel (1928)

SWEDEN

Asplund, Edgar (1931–1974)
Bergh, Jöran (1941)
Beurling, Arne (1905–1986)
Carleman, T. G. Torsten (1892–1949)
Carleson, Lennart A. E. (1928)
Enflo, Per (1944)
Fredholm, Ivar (1866–1927)
Gårding, Lars (1919)
Hanner, Olov (1922)
Holmstedt, Tord (1938)
Hörmander, Lars V. (1931)
Jawerth, Björn (1952)
Josefson, Bengt (1947)
Kaijser, Sten (1940)
Koch, N. F. Helge von (1870–1924)
Löfström, Jörgen (1941)
Müntz, Chaim (Herman) (1884–1956)
Nilsson, Per (1955)
Nordlander, Göte (1930)
Peetre, Jaak (1935)
Persson, Arne (1933)
Persson, Lars-Erik (1944)
Rådstrøm, Hans (1919–1970)
Ribe, Martin (1945)
Riesz, Marcel (1886–1969)
Sparr, Gunnar (1942)
Thorin, G. Olof (1912)

SWITZERLAND

Chandrasekharan, Komaravolu (1920)
Chatterji, Sristhi D. (1935)
Delbaen, Freddy (1946)
Jarchow, Hans (1941)
Mascioni, Vania D. (1962)
Matter, Urs (1955)

UKRAINE see Soviet Union (former)

UNITED KINGDOM

Aldous, David J. (1952)
Allan, Graham R. (1936)
Ball, Keith M. (1960)
Bollobás, Béla (1943)

Bonsall, Frank F. (1920)
Borwein, Jonathan M. (1951)
Carne, Thomas K. (1953)
Chapman, Sydney (1888–1970)
Cooper, J. Lionel B. (1915–1979)
Davie, Alexander M. (1946)
Dilworth, Stephen J. (1959)
Dixon, Alfred C. (1865–1936)
Dowson, Henry R. (1939)
Duncan, John (1938)
Edmunds, David E. (1931)
Edwards, David A. (∼1929)
Edwards, Robert E. (1926–2000)
Fremlin, David H. (1942)
Garling, David J.H. (1937)
Gillespie, T. Alastair (1945)
Gowers, W. Timothy (1963)
Hardy, G. Harold (1877–1947)
Haydon, Richard G. (1947)
Jameson, Graham J.O. (1942)
Kalton, Nigel J. (1946)
Littlewood, John E. (1885–1977)
Mercer, James (1883–1932)
Montgomery-Smith, Stephen J. (1963)
Paley, Raymond E.A.C. (1907–1933)
Partington, Jonathan R. (1955)
Pitt, Sir Harry R. (1914)
Power, Stephen C. (1951)
Read, Charles J. (1958)
Ringrose, John R. (1932)
Robertson, Alexander P. (1925–1995)
Rogers, Claude A. (1920)
Ruston, Anthony F. (1920–2005)
Smithies, Frank (1912–2002)
Titchmarsh, Edward C. (1899–1963)
White, Michael C. (1963)
Young, William H. (1862–1942)

UNITED STATES OF AMERICA

Alaoglu, Leonid (1914–1981)
Aldous, David J. (1952)
Aliprantis, Charalambos D. (1946)
Alspach, Dale E. (1950)
Androulakis, George (1967)
Arens, Richard F. (1919–2000)
Arias, Alvaro (1961)
Aron, Richard M. (1944)
Aronszajn, Nachman (1907–1980)
Axler, Sheldon J. (1949)
Bade, William G. (1924)
Barnes, Bruce A. (1938)
Bartle, Robert G. (1927–2003)
Beck, Anatol (1930)
Bellenot, Steven F. (1948)
Bennett, Albert A. (1888–1971)
Bennett, Colin (1946)
Bennett, Grahame (1945)
Bellow, Alexandra see Ionescu Tulcea
Bercovici, Hari (1955)
Bergman, Stefan (1895–1977)
Bernau, Simon J. (1937)
Birkhoff, Garrett (1911–1996)
Birkhoff, George D. (1884–1944)
Birnbaum, Z. William (1903–2000)
Bishop, Errett A. (1928–1983)
Bliss, Gilbert A. (1876–1951)
Bochner, Solomon (1899–1982)
Bogdan(owicz), Victor M. (1933)
Bohnenblust, Henri F. (1906–2000)
Bonic, Robert A. (∼1933)
Bourgain, Jean (1954)
Bourgin, David G. (1900–1984)
Bourgin, Richard D. (1943)
Brace, John W. (1926)
Branges de Boureia, Louis de (1932)
Brown, H. Arlen (1926)
Burkholder, Donald L. (1927)
Burkinshaw, Owen (1942)
Calderón, Alberto P. (1920–1998)
Calkin, John W. (1909–1964)
Cam, Lucien Le (1924–2000)
Caradus, Selwyn R. (1935)
Carothers, Neal L. (1952)
Casazza, Peter G. (1944)
Cima, Joseph A. (1933)
Clarkson, James A. (1906–1970)
Coburn, Lewis A. (1940)
Cohen, Joel S. (1941)
Coifman, Ronald R. (∼1938)
Comfort, W. Wistar (1933)
Corson, Harry H. (1931)
Cowen, Carl. C. (1945)
Cudia, Dennis F.R. (∼1935)
Daubechies, Ingrid (1954)
Davis, Burgess J. (∼1941)
Davis, William J. (1940)
Day, Mahlon M. (1913–1992)
Dean, David W. (1931)
Diestel, Joseph (1943)
Dilworth, Stephen J. (1959)
Dor, Leonard E. (∼1948)

Douglas, Ronald G. (1938)
Dowling, Patrick N. (1958)
Dubinsky, Edward L. (1935)
Dudley, Richard M. (1938)
Dunford, Nelson (1906–1986)
Duren, Peter L. (1935)
Eberlein, William F. (1917–1986)
Edgar, Gerald A. (1949)
Effros, Edward G. (1935)
Elton, John H. (∼1951)
Enflo, Per (1944)
Fan, Ky (1914)
Fefferman, Charles L. (1949)
Fialkow, Lawrence A. (1946)
Fleming, Richard J. (1938)
Frampton, John M. (∼1938)
Frazier, Michael (1956)
Foiaş, Ciprian I. (1933)
Gamelin, Theodore W. (1939)
Garnett, John B. (1940)
Gelbaum, Bernard R. (1922)
Getoor, Ronald K. (1929)
Giesy, Daniel P. (∼1937)
Gilbert, John E. (∼1936)
Girardi, Maria (1963)
Goldberg, Seymour (1928)
Goldstine, Herman H. (1913–2004)
Goodman, Victor W. (1943)
Goodner, Dwight B. (1913–1995)
Graves, Lawrence M. (1896–1973)
Gross, Leonhard (1931)
Grünbaum, Branko (1929)
Gupta, Chaitan P. (1939)
Hagler, James N. (1945)
Halberg, Charles J.A. (1921)
Halmos, Paul R. (1916–2006)
Henson, C. Ward (1940)
Herz, Carl S. (1930)
Hewitt, Edwin (1920–1999)
Highberg, Ivar E. (1911)
Hildebrandt, Theophil H. (1888–1980)
Hill, George W. (1838–1914)
Hille, Einar (1894–1980)
Hoffman, Kenneth (1930)
Holub, James R. (1942)
Horn, Alfred (1918–2001)
Horváth, John (1924)
Huff, Robert E. (1942)
Hutton, Charlene V. (1946)
Hyers, Donald H. (1913–1997)
Ionescu Tulcea, Alexandra (1935)
Ionescu Tulcea, Cassius (1923)
Jackson, Dunham (1888–1946)
James, Robert C. (1918–2003)
Jamison, James E. (1943)
Jerison, Meyer (1922)
John, Fritz (1910–1994)
Johnson, William B. (1944)
Joichi, James T. (∼1932)
Junge, Marius (1962)
Kadison, Richard V. (1925)
Kakutani, Shizuo (1911–2004)
Kalton, Nigel J. (1946)
Kamińska, Anna (1950)
Kaplansky, Irving (1917–2006)
Karlin, Samuel (1924)
Kato, Tosio (1917–1999)
Kelley, John L. (1916–1999)
Klee, Victor L. (1925)
Koosis, Paul J. (1929)
Korenblum, Boris I. (1923)
Kuo, Hui-Hsiung (1941)
Lacey, H. Elton (1937)
Lamson, Kenneth W. (1885–1960)
Lay, David C. (1941)
Leih, Thomas (∼1946)
Lennard, Christopher J. (1957)
Lewis, Daniel R. (1944)
Lin, Bor-Luh (1935)
Lin, Pei-Kee (1952)
Loewner (Löwner), Charles (1893–1968)
Lomonosov, Victor I. (1946)
Lorch, Edward R. (1907–1990)
Lorentz, Georg G. (1910–2006)
Lotz, Heinrich P. (1934)
Lovaglia, Anthony R. (1923)
Luecking, Daniel H. (1950)
MacCluer, Barbara D. (1953)
Mackey, George W. (1916–2006)
Maharam (Stone), Dorothy (1917)
Maynard, Hugh B. (1943)
McArthur, Charles W. (1921–2001)
McCarthy, Charles A. (1936)
McShane, Edward J. (1904–1989)
Megginson, Robert E. (1948)
Michal, Aristotle D. (1899–1953)
Montgomery-Smith, Stephen J. (1963)
Moore, Eliakim H. (1862–1932)
Morris, Peter D. (∼1939)
Morrison, Terry J. (1947)

Murray, Francis J. (1911–1996)
Nachbin, Leopoldo (1922–1993)
Namioka, Isaac (1928)
Neumann, John (János) von (1903–1957)
Neumann, Michael M. (1950)
Newman, Charles M. (1946)
Nikodym, Otton M. (1887–1974)
Nordgren, Eric A. (1933)
Odell, Edward (1947)
Paulsen, Vern I. (1951)
Pearcy, Carl M. (1935)
Peck, N. Tenney (1937–1996)
Peressini, Anthony L. (1934)
Pettis, Billy J. (1913–1979)
Pfaffenberger, Williams E. (1943)
Phelps, Robert R. (1926)
Phillips, Ralph S. (1913–1998)
Ramanujan, Melapalayam S. (1931)
Randrianantoanina, Beata (1963)
Randrianantoanina, Narcisse J. (1960)
Retherford, J. Ronald (1937)
Rickart, Charles E. (1913–2002)
Rieffel, Marc A. (1937)
Roberts, James W. (∼1943)
Rochberg, Richard H. (1943)
Rosenthal, Haskell P. (1940)
Royden, Halsey (1928–1993)
Ruan, Zhong-Jin (∼1960)
Rubel, Lee A. (1928–1995)
Ruckle, William H. (1936)
Rudin, Walter (1921)
Russo, Bernard (1939)
Saab, Elias (1946)
Saab, Paulette (1952)
Sagher, Yoram (1939)
Sakai, Shôichirô (1928)
Salinas, Norberto (1940–2005)
Sarason, Donald E. (1933)
Schäffer, Juan Jorge (1930)
Schatten, Robert (1911–1977)
Schlumprecht, Thomas B. (1954)
Schwartz, Jacob T. (1930)
Segal, Irving E. (1918–1998)
Shapiro, Joel H. (1940)
Sharpley, Robert C. (1946)
Shields, Allen L. (1927–1989)
Simons, Stephen (1938)
Smith, Kennan T. (1926–2000)
Smith, Mark A. (1947)
Sobczyk, Andrew (1915–1981)
Starbird, Thomas W. (∼1949)
Stegall, Charles P. (1943)
Stein, Elias M. (1931)
Stone, Marshall H. (1903–1989)
Sucheston, Louis (1926)
Sundaresan, Kondagunta (1939)
Szarek, Stanisław (1952)
Tamarkin, Jacob D. (1888–1945)
Taylor, Angus E. (1911–1999)
Thorp, Edward O. (1932)
Tonge, Andrew M. (1948)
Tukey, John W. (1915–2000)
Turett, J. Barry (1949)
Uhl, J. Jerry (1940)
Ulam, Stanisław M. (1909–1984)
Varadarajan, Veeravalli S. (1937)
Varadhan, S.R. Srinivasa (1940)
Walsh, Joseph L. (1895–1973)
Weiss, Guido L. (1928)
Wells, James H. (1932)
Werner, Elisabeth (1958)
Whitley, Robert J. (1937)
Widom, Harold (1932)
Wiener, Norbert (1894–1964)
Wilansky, Albert (1921)
Woyczyński, Wojbor A. (1943)
Yood, Bertram (1917–2004)
Zhu, Kehe H. (1961)
Zorn, Max A. (1906–1993)
Zygmund, Antoni S. (1900–1992)

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Cotlar, Mischa (1913)
Giné, Evarist (1944)

8.7 Anniversary volumes and articles, obituaries

From the historical point of view, Banach space theory is a relatively young discipline. This explain why the following collection contains only little biographical material about Banach space specialists.

Akhieser, N.I.	Uspekhi Mat. Nauk **6**, # 2 (1951), 191–194; **16**, # 4 (1961), 223–234; **26**, # 6 (1971), 257–261; **36**, # 4 (1981), 183–184.
Akilov, G.P.	Uspekhi Mat. Nauk **43**, # 1 (1988), 181–182.
Alexandrov, A.D.	Uspekhi Mat. Nauk **54**, # 5 (1999), 143–146.
Alexandrov, P.S.	Uspekhi Mat. Nauk **39**, # 5 (1984), 3–6 (Yu.M. Smirnov); **52**, # 5 (1997), 5–8 (S.M. Nikolskiĭ); Mitteil. Deutsch. Math.-Verein. 1996, # 3 , 17–21 (G. Frei, U. Stammbach);
Ando, T.	[FUR$^{\cup}$, pp. 1–4].
Apostol, C.	[GOH$_3^{\cup}$, pp. 1–6].
Arzelà, C.	Atti Accad. Naz. Lincei Rend., Cl. Sci. Fis. Mat. Natur. (5), **21** (1912), 879–884; Periodico di Matematiche (3^a), **10** (1913), 45–48; [ARZ$^{\bowtie}$].
Ascoli, Giulio	Periodico di Matematiche (1), **17** (1901), 144–151.
Ascoli, Guido	Boll. Unione Mat. Ital. (3) **12** (1957), 347–350 (F.G. Tricomi).
Asplund, E.	[AAA$_{20}$, pp. i–iv] (E.T. Poulsen).
Auerbach, H.	Ann. Soc. Math. Pol. Ser. II, Wiadom Mat. **29** (1992), 225–231.
Badrikian, A.	Ann. Math. Blaise Pascal, Numéro Spécial 1996, 1–11; Gazette Math. # 65 (1995), 89–92.
Baire, R.	Enseig. Math. **31** (1932), 5–13 (R.-L. Buhl, G. Bouligand); C.R. Acad. Sci. Paris **195** (1932), 86–88; [BAI$^{\bowtie}$].
Banach, S.	see 8.3.3.4.
Bari, N.	Uspekhi Mat. Nauk **17**, # 1 (1962), 120–133.
Bartle, R.G.	Notices Amer. Math. Soc. **51** (2004), 239–240 (J.E. Kister, D.R. Sherbert).
Bauer, H.	Jahresber. Deutsch. Math.-Verein. **105**, # 2 (2003), 63–78; [BAU$^{\bowtie}$].
Bergman(n), S.	Applicable Analysis **8** (1979), 195–199 (M.M. Schiffer, H. Samelson); Ann. Polon. Math. **39** (1981), 5–9 (M.M. Schiffer); Pinl [1969$^{\bullet}$, p. 174].
Bernstein, F.	Frewer [1981$^{\bullet}$].
Bernstein, S.N.	Uspekhi Mat. Nauk **24**, # 3 (1969), 211–218; **38**, # 3 (1983), 191–203; [BERN$^{\bowtie}$].
Beurling, A	Math. Intelligencer **15**, # 3 (1993), 28–31 (B. Kjellberg); **15** (1993), # 3, 32–33 (J. Wermer); [BEUR$^{\bowtie}$].
Birkhoff, Garrett	[ALB$^{\cup}$, pp. 3–15]; [BIRK$_1^{\bowtie}$]; Modern Logic **7** (1997), 81–82; J. Approx. Theory **95** (1998), 1–4 (R.S. Varga).

Birkhoff, George	J. London Math. Soc. **20** (1945), 121–128 (E.T. Whittaker); Trans. Amer. Math. Soc. **60** (1946), 1–2 (R.E. Langer); Bull. Amer. Math. Soc. **52** (1946), 357–391 (M. Morse); [BIRK$_2^{\bowtie}$].
Birman, M.Sh.	Uspekhi Mat. Nauk **43**, # 3 (1988), 201–202.
Birnbaum, W.	Math. Intelligencer **23**, # 2 (2001), 36–46 (W. Woyczyński).
Bishop, E.	[BISH$^{\bowtie}$].
Bliss, G.A.	Bliss [1952$^{\bullet}$].
Bochner, S.	Pinl [1972$^{\bullet}$, p. 200]; [BOCH$^{\bowtie}$].
Bois-Reymond, P. du	Math. Ann. **35** (1890), 458–469 (H. Weber).
Bonnesen, T.	Mat. Tidsskirift B (1936), 16–24 (J. Mollerup).
Borel, É.	C.R. Acad. Sci. Paris **242**:A (1956), 845–850; Enseig. Math. (2) **11** (1965), 1–98 (M. Fréchet); [BOREL$^{\bowtie}$].
Borsuk, K.	[BORS$^{\bowtie}$].
Bourbaki, N.	see 8.3.7.7.
Bourgain, J.	Proc. ICM 1994, Zürich, Vol. I, 3–5 (L. Caffarelli); Notices Amer. Math. Soc. **41** (1994), 1103–1105 (J. Lindenstrauss); Gazette Math. # 63 (1995), 2–15.
Bretagnolle, J.	[AAA$_{17}$, p. 803].
Cacciopoli, R.	Ann. Math. Pura Appl. **47** (1959), v–viii.
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Călugăreanu, G.	[CĂL$^{\bowtie}$].
Cam, L. Le	Ann. Statist. **30** (2002), 617–630 (G.L. Yang).
Cantor, G.	Jahresber. Deutsch. Math.-Verein. **31** (1922), 97–106 (A. Schoenflies); **39** (1930), 189–266 (A. Fraenkel); [DAU$^{\bullet}$], [PUR$^{+\bullet}$], [CAN$^{\bowtie}$].
Carathéodory, C.	Jahresber. Deutsch. Math.-Verein. **55** (1952), 39–51 (O. Perron); [GEO$^{\bullet}$], [CARA$^{\bowtie}$].
Carleman, T.	Acta Math. **82** (1950), i–iii (F. Carlson); [CARL$^{\bowtie}$].
Cartan, H.	The 100th issue of Gazette des Mathématiciens is devoted to Cartan's 100th birthday; see also European Math. Soc. Newsletter **53**, 20–21; [CART$^{\bowtie}$].
Chapman, S.	Bull. London Math. Soc. **3** (1971), 221–250 (V.C.A. Ferraro).
Čech, E.	Czechoslovak. Math. J. **43** (1993), 567–575 (B. Balcar et al.); Math. Intelligencer **16** (1994), # 4, 48–49 (J.J. Gray); [KAT$^{\cup}$], [ČECH$^{\bowtie}$].
Cooper, J.L.B.	Bull. London Math. Soc. **13** (1981), 429–450 (D.E. Edmunds).
Cotlar, M.	As. Mat. Venezolana, Bol. **1** (1994), 75–83 (E. L. de Sá, L. Recht).
Courant, R.	Math. Phys. Semester-Ber. **15** (1968), 1–2 (H. Benke); Uspekhi Mat. Nauk **30**, # 4 (1975), 205–226 (P.S. Aleksandrov, O.A. Oleĭnik), Pinl [1971$^{\bullet}$, p. 171], [REID$_2^{\bullet}$].

Dacunha-Castelle, D.	[AAA$_{17}$, p. 805–806].
Day, M.M.	[BARTL$^{\cup}$, pp. ix–x].
Dieudonné, J.	Gazette Math. # 55 (1993), 3–4 (H. Cartan); [DUG$^{\bullet}$]; [DIEU$^{\bowtie}$].
Dixon, A.C.	J. London Math. Soc. **12** (1937), 145–155 (E.T. Wittaker).
Doeblin, W.	[COHN$^{\cup}$, pp. 1–13].
Dunford, N.	Notices Amer. Math. Soc. **28** (1981), 509–510; **34** (1987), 287.
Dym, H.	[ALP$^{\cup}_{2}$, pp. 1–17].
Enflo, P.	[SAXE, pp. 122–124].
Fan, Ky	[LIN$^{\cup}_{4}$, pp. iii–v].
Fejér, L.	Mat. Lapok **1** (1950), 160–170; **11** (1960), 8–18; J. London Math. Soc. **36** (1961), 501–506 (G. Polya); [FEJ$^{\bowtie}$].
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Floret, K.	Bierstedt [2005$^{\bullet}$].
Foiaş, C.	Indiana Univ. Math. J. **42** (1993), 677–679 (H. Bercovici, R. Temam); see also **50** (2001), i–iv.
Fomin, S.V.	Uspekhi Mat. Nauk **31**, # 4 (1976), 199–212.
Fréchet, M.	C.R. Acad. Sci. Paris **277** (1973), 73–76 (M.S. Mandelbrojt); Taylor [1982/87$^{\bullet}$].
Fredholm, I.	Acta Math. **54** (1930), i–xvi (N. Zeilon); [FRED$^{\bowtie}$].
Freudenthal, H.	Notices Amer. Math. Soc. **38** (1991), 113–114; Pinl [1969$^{\bullet}$, p. 178].
Frobenius, G.	[FRO$^{\bowtie}$].
Frolík, Z.	Czechoslovak Math. J. **40** (1990), 697–702 (J. Pelant, P. Simon).
Garnir, H.G.	Bull. London Math. Soc. **19** (1987), 609–622 (P.L. Butzer, J. Vaillant).
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Glazman, I.M.	[GOH$^{\cup}_{4}$, pp. 1–5].
Gohberg, I.C.	[DYM$^{\cup}$], [DIJK$^{\cup}_{2}$], Gohberg [1989$^{\bullet}$].
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Hahn, H.	Monatsh. Math. **41** (1934), 221–238 (K. Mayrhofer); Fund. Math. **24** (1934), 317–320 (K. Menger); Heuser [1995$^\bullet$]; [HAHN$^\bowtie$].
Halmos, P.R.	[HAL$^\bullet$], [EWI$^\cup$], [ALB$^\cup$, pp. 119–132]; [HAL$^\bowtie$].
Hamburger, H.L.	J. London Math. Soc. **1933** (1958), 377–383 (M.E. Grimshaw); Pinl [1972$^\bullet$, p. 182].
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Hardy, G.H.	J. London Math. Soc. **25** (1950), 81–101 (E.C. Titchmarsh); 107–109 (F. Smithies); [HAR$^\bullet$]; [HAR$^\bowtie$].
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Hellinger, E.	Pinl [1969$^\bullet$, p. 214]; [BRAN$^\cup$, pp. 1–41] (J. Rovnyak); [SCHW$^\cup$, pp. 106–108].
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Hewitt, E.	Results Math. **37** (2000), 199–203 (W. Schempp); Hewitt [1990$^\bullet$]; `http://at.yorku.ca/t/o/p/c/82.htm` (K.A. Ross).
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John, F.	Notices Amer. Math. Soc. **42** (1995), 256–257; Comm. Pure Appl. Math. **51** (1998), 971–989 (S. Hilde- brandt); Pinl [1971$^\bullet$, p. 176],[JOHN$^\bowtie$].
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Kantorovich, L.V.	Uspekhi Mat. Nauk **17**, # 4 (1962), 201–215; **27**, # 3 (1972), 221–227; **37**, # 3 (1982), 201–209; **42**, # 2 (1987), 177–182; Notices Amer. Math. Soc. **33** (1986), 503–504; [LEIF$^\cup$, pp. 1–7] (V.L. Makarov, S.L. Sobolev); Kantorovich [1987$^\bullet$]; [KAN$^\bowtie$].
Karlin, S.	[AND$^\cup_2$, pp. xi–xii].

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Kreĭn, S.G.	Uspekhi Mat. Nauk **33**, # 2 (1978), 217–224; **42**, # 5 (1987), 223–224; **55**, # 2 (2000), 125–126.
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Langer, H.	[DIJK$_1^{\cup}$, pp. 1–22].
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Lévy, P.	Ann. Probab. **1** (1973), 1–18 (M. Loéve); Bull. London Math. Soc. **7** (1975), 300–320 (S.J. Taylor); [LEVY$^{\bowtie}$].
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Livšic, M.S.	[GOH$_2^{\cup}$]; [ALP$_1^{\cup}$].

Lizorkin, P.I.	Uspekhi Mat. Nauk **48**, # 1 (1993), 205–207; **49**, # 3 (1994), 177–179.
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Löwner, K.	(Loewner, C.), Netuka [1993$^\bullet$], Pinl [1974$^\bullet$, p. 175]; [LOEW$^{\bowtie}$].
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Lyusternik, L.A.	Uspekhi Mat. Nauk **5**, # 2 (1950), 234–236; **15**, # 2 (1960), 215–230; **25**, # 4 (1970), 3–10; **35**, # 6 (1980), 3–10; **37**, # 1 (1982), 125–126.
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Marcinkiewicz, J.	Banach Center Publ. **56** (2002), 31–35 (K. Dabrowski, E. Hensz-Chadzyńska); [KUR$^\bullet$, pp. 83–86]; [MAR$_1^{\bowtie}$].
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Nagumo, M.	Funkcial. Ekvac. **9** (1966), vii–viii; [NAG$^{\bowtie}$].
Naĭmark, M.A.	Uspekhi Mat. Nauk **15**, # 2 (1960), 233–236; **35**, # 4 (1980), 135–140.
Neumann, J. von	see 8.3.4.3.
Nikishin, E.M.	Uspekhi Mat. Nauk **51**, # 2 (1996), 181–182.
Nikodym, O.M.	Ann. Soc. Math. Pol. Ser. II, Wiadom. Mat. **25** (1983), 75–88; Math. Intelligencer **12**, # 2 (1990), 27–31 (W. Szymanski).
Nikolskiĭ, S.M.	Uspekhi Mat. Nauk **11**, # 2 (1956), 237–240; **20**, # 5 (1965), 275–287; **30**, # 4 (1975), 271–280; **40**, # 5 (1985), 268–277; **50**, # 6 (1995), 223–228.
Noether, F.	Integral Equations Operator Theory **8** (1985), 573–576; **13** (1990), 303–305 (G.E. Noether); Pinl [1969$^{\bullet}$, p. 203]; Schriftenr. Geschichte Natur. Tech. Med. (Leipzig) **28** (1991), 33–41 (K.-H. Schlote); [REN$^{\bullet}$, pp. 245–255].
Orlicz, W.	[CIE$^{\cup}$, pp. 13–19], [HUD$^{\cup}$, pp. 23–29] (L. Maligranda, W. Wnuk); Notices Amer. Math. Soc. **38** (1991), 21; [ORL$^{\bowtie}$].
Paley, R.E.A.C.	Bull. Amer. Math. Soc. **39** (1933), 476 (N. Wiener); J. London Math. Soc. **9** (1934), 76–80 (G.H. Hardy).
Peano, G.	[PEA$^{\bowtie}$], [KEN$^{\bullet}$].
Pearcy, C.	[BER$^{\cup}$, pp. ix–xi].
Peetre, J.	[CWI$^{\cup}$, pp. 1–37].
Pełczyński, A.	Studia Math. **159** (2003), front matter.
Pettis, B.J.	Contemp. Math. **2** (1980), vii–ix.
Phillips, R.S.	Phillips [1984$^{\bullet}$], [1994$^{\bullet}$].
Pincherle, S.	Boll. Un. Mat. Ital. **15** (1936), 149–152 (L. Berzolari); [PIN$^{\bowtie}$].
Pinsker, A.G.	Uspekhi Mat. Nauk **21**, # 6 (1966), 169–170; **41**, # 2 (1986), 177–178.
Plans, A.	[MAR$^{\cup}$, pp. 1–10] (E. Martín-Peinador).
Plemelj, J.	Matematička Biblioteka (Beograd) **25** (1963), 3–10; Obzornik za Matematiko in Fiziko **16**, # 4 (1969), 146–148; Uspekhi Mat. Nauk **28**, # 6 (1973), 223–226.
Ples(s)ner, A.I.	Uspekhi Mat. Nauk **16**, # 1 (1961), 212–218; Math. Intelligencer **14**, # 3 (1992), 31–36 (D. Gaier); Pinl [1969$^{\bullet}$, p. 223].
Pólya, G.	Notices Amer. Math. Soc. **32** (1985), 599.
Prößdorf, S.	[ELS$^{\cup}$, pp. 1–26].
Pták, V.	Czechoslovak Math. J. **35** (1985), 662–670; **46** (1996), 337–349; **50** (2000), 445–447 (M. Fiedler, V. Müller).
Rademacher, H.	Pinl[1969$^{\bullet}$, p. 205]; [RADE$^{\bowtie}$].
Radon, J.	Monatsh. Math. **62** (1958), 189–199 (P. Funk); [RADO$^{\bowtie}$].

Raĭkov, D.A.	Mat. v Shkole # 6 (1980), 63.
Rainwater, J.	Phelps [2002$^\bullet$].
Rao, C.R.	Linear Algebra Appl. **237/238** (1996), 1–40 (S. Puntanen, G.P.H. Styan).
Rellich, F.	Math. Ann. **133** (1957), 185–190 (R. Courant).
Rickart, C.E.	Yale Bull. & Calendar **30**, # 29, May 10, 2002.
Riesz, F.	see 8.3.1.3.
Riesz, M.	Acta Math. **124** (1970), i–xi (L. Gårding); [RIE$_2^{\bowtie}$].
Robertson, A.P.	`http://www.austms.org.au/Publ/Gazette/Oct95/robertson.html` (W.R. Bloom, P. Boyce).
Rodríguez-Salinas , B.	Rev. Mat. Univ. Complutense Madrid **9** (1996), 23–72 (J. Horváth).
Rubio de Francia, J.L.	Collect. Math. **38** (1987), 3–15 (J. García-Cuerva).
Rudin, W.	[RUD$^\bullet$].
Saks, S.	Math. Intelligencer **9**, # 1 (1987), 36–41 (A. Zygmund); 41–43 (P. Wojtaszczyk); [KUR$^\bullet$, pp. 86–88].
Schauder, J.	[SCHAU$^{\bowtie}$, pp. 9–10] (W. Orlicz); [KUR$^\bullet$, p. 86],
Schmidt, E.	Jahresber. Deutsch. Math.-Verein. **69** (1968), 209–224 (H. Rohrbach); **72** (1970), 3–17 (A. Dinghas); [HILB$^+$]
Schröder, E.	Jahresber. Deutsch. Math.-Verein. **12** (1903), 249–265 (J. Lüroth).
Schur, I.	[SCHU$^{\bowtie}$, Vol. I, pp. v–xiv] (A. Brauer); Pinl [1969$^\bullet$, p. 195]; [JOS$^\cup$].
Schwartz, J.T.	Notices Amer. Math. Soc. **28** (1981), 510–511; Rota [1996$^\bullet$].
Schwartz, L.	[SCHW$^\bullet$]; Schwartz [1981$^\bullet$]; Maurey [1985$^\bullet$].
Segal, I.E.	Notices Amer. Math. Soc. **46** (1999), 659–668.
Semenov, E.M.	Uspekhi Mat. Nauk **56**, # 6 (2001), 171–175.
Shields, A.L.	Math. Intelligencer **12**, # 2 (1990), 3–22.
Shilov, G.E.	Uspekhi Mat. Nauk **31**, # 1 (1976), 217–228.
Shmulyan, V.L.	Uspeki Mat. Nauk **20**, # 2 (1965), 131–133 (M.G. Kreĭn); 135–147 (D.A. Raĭkov).
Shnirelman, V.L.	Uspeki Mat. Nauk (old series) **6** (1939), 3–8.
Sikorski, R.	Demonstratio Math. **18** (1985), 1–8 (M. Mączyński, T. Traczyk).
Silbermann, B.	[BÖT$^\cup$, pp. 1–12] (A. Böttcher).
Smithies, F.	Bull. London Math. Soc. **36** (2004), 841–853 (J. Ringrose).
Sobolev, S.L.	Uspekhi Mat. Nauk **14**, # 3 (1959), 203–214; **23**, # 5 (1968), 177–186; **34**, # 1 (1979), 3–15; **43**, # 5 (1988), 3–16; **44**, # 4 (1989), 5; Notices Amer. Math. Soc. **36** (1989), 853.
Solomyak, M.Z.	Uspekhi Mat. Nauk **46**, # 4 (1991), 183–184.
Stechkin, S.B.	Uspekhi Mat. Nauk **51**, # 6 (1996), 3–10; 11–20 (P.L. Ulyanov).

Steinhaus, H.	Amer. Math. Monthly **81** (1974), 572–581 (M. Kac); [KUR$^{\bullet}$, pp. 173–179]; [STEIN$^{\bowtie}$].
Stone, M.H.	Notices Amer. Math. Soc. **36** (1989), 221–223.
Szegő, G.	Notices Amer. Math. Soc. **32** (1985), 590–591; Math. Intelligencer **18**, # 3 (1996), 10–22 (R. Askey, P. Nevai); [SZE$^{\bowtie}$].
Szlenk, W.	Ann. Soc. Math. Pol. Ser. II, Wiadom Mat. **33** (1997), 204–209.
Szőkefalvi-Nagy, B.	[KÉR$^{\cup}$, pp. xi–xxxviii].
Tamarkin, J.D.	Bull. Amer. Math. Soc. **53** (1947), 440–457 (E. Hille); Историко-Мат. Исслед., 2nd Ser., **1** (1996), 100–108 (N.S. Ermolaeva).
Taylor, A.E.	Taylor [1984$^{\bullet}$].
Thorin, O.E.	[CWI$^{\cup}$, pp. 41–45].
Timan, A.F.	Uspekhi Mat. Nauk **44**, # 3 (1989), 145–146.
Tikhonov, A.N.	Uspekhi Mat. Nauk **11**, # 6 (1956), 235–245; **22**, # 2 (1967), 133–188; **31**, # 6 (1976), 3–12; **42**, # 3 (1987), 1–12; Ж. Вычисл. Мат. и Мат. Физ. **34** (1994), 318–319; Differ. Equations **32** (1996), 1297–1299.
Titchmarsh, E.C.	[1964$^{\bullet}$] .
Toeplitz, O.	Jahresber. Deutsch. Math.-Verein. **66** (1963), 1–16 (H. Benke, G. Köthe); [GOH$^{\cup}_{1}$, pp. 545–556, 575–584] (G. Köthe); Pinl [1969$^{\bullet}$, p. 201].
Tukey, J.W.	[TUK$^{\bowtie}$, Vol. I].
Ulam, S.M.	[ALB$^{\cup}$, pp. 353–361]; [ULAM$^{\bullet}$]; [ULAM$^{\bowtie}$].
Ulm, H.	Contemp. Math. **87** (1989), 1–10 (R. Göbel).
Urysohn, P.S.	Fund. Math. **7** (1925), 138–140 (P.S. Alexandrov); [URY$^{\bowtie}$].
Vaidyanathaswamy, R.	J. Indian Math. Soc. **64** (1997), 1–13 (A. Narasinga Rao, V. Ganapati Iyer); 15–18 (V.K. Balachandran).
Valdivia, M.	[BIER$^{\cup}_{3}$, pp. 1–55]; [BIER$^{\cup}_{4}$, pp. 1–17].
Vershik, A.M.	Uspekhi Mat. Nauk **49**, # 3, (1994), 195–204.
Vietoris, L.	Mitteil. Deutsch. Math.-Verein. 2000, # 3 , 44–46 (G. Helmberg); Notices Amer. Math. Soc. **49** (2002), 1232–1236 (H. Reitberger).
Vilenkin, N.Ya.	Mat. v Shkole # 6 (1980), 63–64; # 1 (1992), 77.
Vinogradov, S.A.	[HAV$^{\cup}$, pp. 1–22].
Vitushkin, A.G.	Uspekhi Mat. Nauk **57**, # 1 (2002), 179–184; Mat. Zametki **76** (2004), 323; Engl. transl.: Math. Notes **76** (2004), 305.
Volterra, V.	J. London Math. Soc. **16** (1941), 131–139 (E.T. Whittaker); Fichera [1994$^{\bullet}$]; [VOL$^{\bowtie}$].
Vulikh, B.Z.	Uspekhi Mat. Nauk **18**, # 6 (1963), 242–243; **34**, # 4 (1979), 133–138.

Walsh, J.L.	Bull. Amer. Math. Soc. **81** (1975), 45–65 (M. Marden); Birkhoff [1977$^\bullet$, pp. 59–60].
Weierstrass, K.	Jahresber. Deutsch. Math.-Verein. **6** (1889), 27–44; **24**, (1915), 416–438 (E. Lampe); **96** (1994), 56–75 (R. Bölling); [WEI$^{\bowtie}$].
Weil, A.	Notices Amer. Math. Soc. **46** (1999), 440–447 (A. Borel et al.), 633–636 (H. Cartan); Mitteil. Deutsch. Math.-Verein. 1999, # 3, 7–12 (P. Cartier); [WEIL$^\bullet$]; [WEIL$^{\bowtie}$].
Weyl, H.	J. London Math. Soc. **33** (1978), 500–511 (M.H.A. Newman); Enseig. Math. (2) **3** (1957), 157–187 (C. Chevalley, A. Weil) Pinl [1971$^\bullet$, p. 185], [WEYL$^{\bowtie}$].
Widom, H.	[BAS$^\cup$, pp. ix–xix].
Wiener, N.	Bull. Amer. Math. Soc. **72** (1966); Notices Amer. Math. Soc. **42** (1995), 430–438 (D. Jerison, D. Stroock); 664–669 (V. Mandrekar); [MASA$^\bullet$]; [WIE$^\bullet$]; [WIE$^{\bowtie}$].
Wintner, A.	J. London Math. Soc. **37** (1962), 483–503 (P. Hartman).
Yosida, K.	[YOS$^{\bowtie}$].
Young, W.H.	J. London Math. Soc. **17** (1942), 218–237 (G.H. Hardy); [YOU$^{\bowtie}$].
Zaanen, A.C.	[BUT$^\cup$, pp. 41–45] (J. Korevaar); [HUIJ$^\cup$, p. 1–14]; Nieuw Archief Wiskunde (5) **5** (2004), 21–25 (W. Luxemburg, M. Kaashoek, B. de Pagter).
Zorn, M.	Notices Amer. Math. Soc. **40** (1993), 640; Pinl [1972$^\bullet$, p. 157].
Zygmund, A.	Notices Amer. Math. Soc. **39** (1992), 848–849; Studia Math. **103** (1992), 119–121; J. Approx. Theory **75** (1993), 1–7 (G.G. Lorentz); Coifman/Strichartz [1989$^\bullet$]; [ZYG$^{\bowtie}$].

Chronology

1902 Dixon's theory of equations in infinitely many variables,
Lebesgue's thesis and his integral,

1903 Fredholm determinants,

1906 closed unit ball of l_2, complete continuity, Hilbert–Schmidt matrices,
Fréchet's thesis,

1907 Fischer–Riesz theorem, s(ingular)-numbers,

1908 Hilbert space, Gram–Schmidt process,

1909 function spaces L_p, Schur's inequality,

1910 Haar functions, Faber functions,

1911 Riesz representation theorem: $C[a,b]^*$, Fréchet derivatives,

1912 Jackson–Bernstein theorem,

1913 *Les systèmes d'équations linéaires à une infinité d'inconnues* [RIE],
sequence spaces l_p,

1914 *Grundzüge der Mengenlehre* [HAUS$_1$],
Gâteaux derivatives, Hardy classes,

1918 theory of compact operators,

1920 *Banach's thesis*,

1921 Schur property of l_1, Helly's lemma, index of an operator,

1922 uniform boundedness principle, Bergman spaces, Wiener measure,

1923 Walsh functions, Khintchine's inequality,
H_p viewed as a class of functions,

1924 bicompactness,

1926 Marcel Riesz convexity theorem,

1927 *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten* [HEL$^+$],
reflexivity, Schauder bases, Sidon sets,

1928 linear lattices, Franklin system,

1929 Hahn–Banach theorem, dual operators, bounded inverse theorem,

1930 abstract Hilbert spaces, weak topology of Hilbert spaces,
rings of operators ($W^\star$-algebras),
$T \in \mathfrak{K} \Leftrightarrow T^* \in \mathfrak{K}$, open mapping theorem (Satz von der Gebietsinvarianz),
Tychonoff's theorem, Radon–Nikodym theorem, $\Lambda(2)$-sets,

1931 $L_1^* = L_\infty$, Wiener–Hopf operators,

1932 *Théorie des opérations linéaires* [BAN], closed graph theorem,
Linear Transformations in Hilbert Space [STONE],
Mathematische Grundlagen der Quantenmechanik [vNEU],
every separable Banach space is isometric to a subspace of $C[0,1]$,
Banach–Mazur distance, normal solvability, ε-entropy, Orlicz spaces,

1933 *Grundbegriffe der Wahrscheinlichkeitsrechnung* [KOL],
cylindrical sets, Bochner integral, Orlicz property,
every separable Banach space is isometric to a quotient of l_1,

1934 *Theorie der konvexen Körper* [BONN$^+$], *Inequalities* [HARD$^+$],
topological linear spaces, Lipschitz–Hölder spaces,

1935 *Trigonometric Series* [ZYG],
separation of convex sets, locally convex linear spaces,
Birkhoff integral, Zorn's lemma,
Kolmogorov widths, Borel measures on Banach spaces,

1936 strictly convex (rotund) norms, uniformly convex norms,
lineare metrische Ringe, rings of operators,
Stone's representation of Boolean algebras,
weak derivatives (in the sense of Sobolev),

1937 *Theory of the Integral* [SAKS],
non-complemented subspaces, Banach lattices, ultrafilters,
Weierstrass–Stone approximation theorem, Banach–Stone theorem,
Stone–Čech compactification, traces on $W^\star$-algebras,

1938 Goldstine's theorem, weak topology of Banach spaces, Alaoglu's theorem,
locally bounded spaces, complex Hahn–Banach theorem,
Dunford–Gelfand integral, Pettis integral,
Radon–Nikodym property (classical version),
ergodic theorems for abstract operators,
Sobolev spaces, embedding theorems,

1939 smooth (differentiable) norms, measure algebras,

1940 *Lattice Theory* [BIRK], Banach lattices,
Topologie générale [BOU$_{3a}$],
regularly convex sets,

1941 Kreĭn–Milman theorem, abstract L- and M-spaces,
operator ideals on Hilbert space, Calkin's theorem,
normierte Ringe, commutative Banach algebras,

1942 absolute (now unconditional) bases, Maharam's theorem,

1943 operational (or functional) calculus, spectral mapping theorem,
$B^\star$-algebras,

1946 *Коммутативные нормированные кольца* [GEL$^+$],
trace class operators, splines,

1947 *Algèbre linéaire* [BOU$_{2a}$],
Eberlein–Shmulyan theorem,

1948 *Functional Analysis and Semi-Groups* [HIL],
Algèbre multilinéaire [BOU$_{2b}$],
Schatten–von Neumann operators, John's theorem,
$C[0,1]$ has no unconditional basis, Gelfand–Naĭmark–Segal representation,
weighted Bergman spaces,

1949 Weyl inequalities, Santaló inequality,

1950 *Measure Theory* [HAL_1], *Théorie des distributions* [SCHW_1],
Функциональный анализ в полуупорядоченных пространствах [KAN_2^+],
Modulared Semi-Ordered Linear Spaces [NAK],
A Theory of Cross-Spaces [SCHA_1],
Некоторые применения функционального анализа в математической физике [SOB],
Dvoretzky–Rogers theorem, extension property,
Lorentz spaces, modular spaces, H_p viewed as a Banach space,

1951 *Элементы функционального анализа* [LYUS^+],
Fredholm operators, determinant theory for abstract operators,
Nikolskiĭ spaces, disk algebra,
a non-reflexive Banach space isomorphic with its second conjugate space,
completeness of root vectors for certain operators in Hilbert spaces,

1952 *Leçons d'analyse fonctionnelle* [RIE^+],

1953 *Espaces vectoriels topologiques* [BOU_{5a}], *Linear Analysis* [ZAA_1],
Grothendieck's thesis [GRO_1], nuclear, integral and 1-summing operators,
nuclear locally convex linear spaces, Dunford–Pettis property,
first versions of the law of large numbers and the central limit theorem
for vector-valued random variables,

1954 *Элементы теории функций и функционального анализа* [KOL^+],
invariant subspaces of compact operators, operators with a Riesz theory,
$B^\star$-algebras = $C^\star$-algebras,

1955 *General Topology* [KEL],
lifting property,

1956 *Нормированные кольца* [NAI],
Grothendieck's *Résumé*, Hilbertian operators, 2-summing operators,
Grothendieck's theorem and his 14 natural tensor norms,
Choquet's representation, characterization of $W^\star$-algebras,

1957 *Algèbres de von Neumann* [DIX_1],
Nehari's theorem for Hankel matrices,
tight or Radon measures on completely regular topological spaces,

1958 *Linear Operators I* [DUN_1^+], *Normed Linear Spaces* [DAY],
Functional Analysis [TAY], *Integral Equations* [SMI],
Выпуклые Функции и пространства Орлича [KRA^+],
Bessaga–Pełczyński c_0-theorem and selection principle,
spectral measures, spectral operators, scalar type operators,
strictly singular operators, Slobodetskiĭ spaces, Minlos–Sazonov theorem,

1959 *Функциональный анализ в нормированных пространствах* [KAN_1^+],
separable Banach spaces are locally uniformly convexifiable,
trace formula for operators in Hilbert spaces,
Corson compact spaces,

1960 *Rings of Continuous Functions* [GIL^+], *Banach Algebras* [RIC],
Topologische lineare Räume I [KÖT_1],
Norm Ideals of Completely Continuous Operators [SCHA_2],
Dvoretzky's theorem, projection constants,
vector-valued martingales, $\Lambda(p)$-sets,

1961 *Обобщенные функции* IV [GEL_4^+],
all Banach spaces are subreflexive,
Besov spaces, Bessel potential spaces, *BMO*,
complex interpolation method,

1962 *Banach Spaces of Analytic Functions* [HOF], *Spectral Theory* [LOR],
B-convexity,

1963 *Linear Operators II* [DUN_2^+], *Linear Topological Spaces* [KEL^+],
A Theory of Interpolation of Normed Spaces [PEE_1],
principle of related operators, approximation numbers, strongly exposed points,
decomposable operators, generalized scalar operators, spectral distributions,
real interpolation methods, J-functional, K-functional, Daugavet equation,

1964 Khintchine–Kahane inequality, Kolmogorov numbers,
$L_p[0,1]$ and $L_q[0,1]$ with $p \neq q$ and $\max\{p,2\} > 2$ fail to be uniformly homeomorphic,
algebraic properties of Toeplitz operators,

1965 *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве* [GOH_3^+],
Nukleare lokalkonvexe Räume [PIE_1],
Gelfand widths, strictly cosingular operators,
the linear group of the Hilbert space is contractible,

1966 *Vector Measures* [DINC], *Unbounded Linear Operators* [GOL_1],
Perturbation Theory for Linear Operators [KATO],
Approximation of Functions [LORZ], *Lectures on Choquet's Theorem* [PHE_1],
absolutely p-summing operators,
Calderón couples, dentable sets, denting points,
completeness of root vectors for certain operators in Banach spaces,

1967 *Semi-Groups of Operators and Approximation* [BUT_1^+],
all infinite-dimensional separable Banach spaces are homeomorphic,

1968 *Theory of Generalized Spectral Operators* [COL^+],
Some Random Series of Functions [KAH], *Equations in Linear Spaces* [PRZ^+],
$\mathscr{L}_p$-spaces, weakly compactly generated Banach spaces, Eberlein compact sets,
Radon–Nikodym property (modern version), Asplund spaces,
operator ideals on Banach spaces, spectral capacity, composition operators,

1969 *Uniform Algebras* [GAM], *Topics in the Theory of Lifting* [ION],
Intégration sur les espaces topologiques séparés [BOU_{6d}],
Приближение функций многих переменных и теория вложения [NIK],
principle of local reflexivity,
radonifying operators, p-nuclear and p-integral operators, trace duality,
vector-valued Fourier transform, Fourier type,

1970 *Theory of H^p Spaces* [DUR], *Bases in Banach Spaces I* [SIN_1], solution of the complemented subspace problem, Gelfand numbers,

1971 *Linear Operators III* [DUN_3^+], *Riesz Spaces I* [LUX^+],
Banach Spaces of Continuous Functions [SEMA],
Compact Non-Self-Adjoint Operators [RIN],
C^*-Algebras and W^*-Algebras [SAKAI],
weakly compactly generated Banach spaces are locally uniformly convexifiable,

1972 *Theorie der Operatorenideale* [PIE_2],
Enflo's space without the approximation property,
finite representability, ultraproducts, superreflexivity, finite tree property,
subquadratic Gaussian and Rademacher average,
stable type and cotype, Kwapień's theorem,
p-factorable operators, local unconditional structure,
M-ideals,

1973 Rademacher type and cotype, Lomonosov's theorem,
approximation property $\not\Rightarrow$ bounded approximation property,

1974 *Calkin Algebras and Algebras of Operators on Banach Spaces* [CAR^+],
The Isometric Theory of Classical Banach Spaces [LAC],
Banach Lattices and Positive Operators [SCHAE],
Rosenthal's l_1-theorem, Kadets–Snobar theorem,
Tsirelson's space not containing l_p's or c_0,
UMD spaces, Gordon–Lewis property, Bloch spaces,

1975 *Geometry of Banach Spaces – Selected Topics* [DIE_1],
superproperties, spreading models, Radon–Nikodym property of operators,
three space properties, negative solution of the three space problem for Hilbert spaces,
Haar (martingale) type and cotype,
more examples of $\mathscr{L}_p$-spaces, *VMO*,

1976 *Geometry of Spheres in Normed Spaces* [SCHÄ],
Interpolation Spaces [BERG^+], *New Thoughts on Besov Spaces* [PEE_3],
Maurey–Pisier theorem, Krivine's theorem, B-convexity = K-convexity,
Enflo announced his negative solution of the invariant subspace problem,
uniformly convex and uniformly convexifiable operators, Ribe's theorem,
analytic Radon–Nikodym property,
Lizorkin–Triebel spaces, spline bases of Besov spaces,

1977 *Vector Measures* [DIE_2^+], *Classical Banach Spaces I* [LIND_1^+],
Banach Spaces of Analytic Functions and Absolutely Summing Operators [PEŁ],
Hilbert numbers,

1978 *Spectral Theory of Linear Operators* [DOW],
Reflexive and Superreflexive Banach Spaces [DUL_1],
Интерполяция линейных операторов [KREIN^+],
Operator Ideals [PIE_3],
Interpolation Theory, Function Spaces, Differential Operators [TRI_1],
spaces satisfying Grothendieck's theorem,
characterization of Sidon sets by Banach space properties,

1979 *Classical Banach Spaces II* [LIND_2^+], *Topological Vector Spaces II* [KÖT_2],
Weyl inequalities in Banach spaces, eigenvalues of nuclear operators,
computing 2-summing norms with few vectors,
twisted sums, Kalton–Peck spaces,

1980 *Lectures on H_p Spaces* [KOO],
eigenvalues of p-summing operators, Weyl numbers, volume ratio,
trace formula for operators in Banach spaces,
Peller's theorem for Hankel operators,

1981 *Geometry and Probability in Banach Spaces* [SCHW$_3$],
Bases in Banach Spaces II [SIN$_2$],
stable Banach spaces, diameter of the Minkowski compactum,
$\mathfrak{L}(H)$ does not have the approximation property,
traces on operator ideals, ζ-convexity,
non-commutative Hahn–Banach theorem, weakly countably determined spaces,

1982 *Introduction to Banach Spaces and their Geometry* [BEAU$_2$],
Lipschitz homeomorphy to l_p or $L_p[0,1]$ with $1<p<\infty$ implies linear homeomorphy,

1983 *Orlicz Spaces and Modular Spaces* [MUS], *Riesz Spaces II* [ZAA$_2$],
Pisier's space: $X_{\text{Pis}}\,\widetilde{\otimes}_\pi X_{\text{Pis}} = X_{\text{Pis}}\,\widetilde{\otimes}_\varepsilon X_{\text{Pis}}$, $\overline{\mathfrak{F}}(X_{\text{Pis}}) = \mathfrak{N}(X_{\text{Pis}})$,
vector-valued Hilbert transform, Bourgain–Burkholder theorem: $UMD = HT$,

1984 *Modèles étalés des espaces de Banach* [BEAU$^+$],
Sequences and Series in Banach Spaces [DIE$_2$],
The Method of Orbits in Interpolation Theory [OVCH],
Banach Lattices and Operators [SCHZ],
wavelets,

1985 *Пространства С.Л. Соболева* [MAZ],
n-Widths in Approximation Theory [PINK],
quotient of subspace theorem, cotype 2 dichotomy for $C^\Lambda(\mathbb{T})$,

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Probability in Banach Spaces [LED$^+$], *Banach Lattices* [MEY-N],
Banach Spaces for Analysts [WOJ$_1$],
Schlumprecht's space, which is arbitrarily distortable, K-divisibility theorem,

1992 *Classical Sequences in Banach Spaces* [GUER],

1993 *Tensor Norms and Operator Ideals* [DEF$^+$],
Smoothness and Renormings in Banach Spaces [DEV$^+$],
M-Ideals in Banach Spaces and Banach Algebras [HARM$^+$],
Gowers–Maurey space, which provides a negative solution of the
unconditional basic sequence problem, Banach spaces with few operators,
hereditarily indecomposable spaces, l_2 is arbitrarily distortable,

1994 Gowers's space not containing c_0, l_1 or any infinite-dimensional reflexive subspace, negative solution of the hyperplane problem, Rosenthal's c_0-theorem,

1995 *Absolutely Summing Operators* [DIE_1^+],
dichotomy of Komorowski/Tomczak-Jaegermann, asymptotic l_p spaces,

1996 Gowers's dichotomy, solution of the homogeneous subspace problem,
negative solution of the Schroeder–Bernstein problem,
additivity of the spectral trace, self-dual operator Hilbert spaces,

1997 *Three-Space Problems in Banach Space Theory* [CAST^+],
non-classical prime spaces,

1998 *Orthonormal Systems and Banach Space Geometry* [PIE^+],

1999

2000 *Geometric Nonlinear Functional Analysis I* [BENY_1^+],
An Introduction to Local Spectral Theory [LAU^+],
Operator Spaces [EFF^+],
Lipschitz homeomorphy to c_0 implies linear homeomorphy,

Recent Banach space textbooks (of different levels) are [HAB^+, 1996], [MEG, 1998], [FAB^+, 2001], [MOR, 2001], [RYAN, 2002], [SAXE, 2002], [LIN, 2004], [CARO, 2005], [LI^+, 2005], [ALB^+, 2006].

2001/2003
Handbook of the Geometry of Banach Spaces I–II [$\mathrm{JOHN}_1^\cup$, $\mathrm{JOHN}_2^\cup$].

Original Quotations

In order to preserve the historical flavor, most quotations were given in the original language. Concerning mathematical statements, this should not cause any trouble. On the other hand, quotations of general nature are harder to grasp. Therefore I have sometimes presented an English translation. The original versions follow.

p. xv, Grimms Märchen (Sneewittchen und die sieben Zwerge):
Spieglein, Spieglein an der Wand,
wer ist die Schönste im ganzen Land?

p. xx, [BOU$_{5b}$, p. 173] or [BOU$^{\bullet}$, p. 273]:
La publication du traité de Banach sur les ≪Opérations linéaires≫ marque, pourrait-on dire, le début de l'âge adulte pour la théorie des espaces normés.

p. xxi, [BOU$_{5b}$, p. 174] or [BOU$^{\bullet}$, p. 273, modified version]:
Mais, malgré un grand nombre de recherches entreprises depuis 20 ans sur les espaces de Banach, peu de progrès importants ont été réalisés dans les problèmes laissés ouverts par ce dernier; d'autre part – si l'on excepte la théorie de algèbres de Banach et ses applications à l'analyse harmonique –, l'absence presque totale de nouvelles applications de la théorie aux grands problèmes de l'Analyse classique a quelque peu déçu les espoirs fondés sur elle.

p. xxi, [PIE$_1$, Vorwort]:
Abgesehen von einigen Ausnahmen lassen sich die in der Analysis auftretenden lokalkonvexen Räume in zwei Klassen einteilen. Das sind einmal die normierten Räume, die zum klassischen Bestand der Funktionalanalysis gehören und deren Theorie im wesentlichen als abgeschlossen angesehen werden kann. Die zweite Klasse besteht aus den sogenannten nuklearen lokalkonvexen Räumen, die im Jahre 1951 von A. GROTHENDIECK *eingeführt wurden. Beide Klassen haben einen trivialen Durchschnitt, denn es zeigt sich, daß nur die endlichdimensionalen lokalkonvexen Räume gleichzeitig normierbar und nuklear sind.*

p. xxii, Grothendieck [1956b, p. 1]:
En effet, presque toutes les questions de la théorie des produites tensoriels topologiques d'espaces localement convexes généraux, y compris la théorie des espaces nucléaires, se ramènent en réalité à des questions sur les espaces de Banach.

p. 55, [BOU$_{5b}$, pp. 173–174] or [BOU$^{\bullet}$, p. 273]:
La publication du traité de Banach sur les ≪Opérations linéaires≫ marque, pourrait-on dire, le début de l'âge adulte pour la théorie des espaces normés. Tous les résultats

dont nous venons de parler, ainsi que beaucoup d'autres, se trouvent exposés dans ce volume, de façon encore un peu désordonnée, mais accompagnés de multiples exemples frappants tirés de domaines variés de l'Analyse. ... De fait, l'ouvrage eut un succès considérable, et un de ses effets les plus immédiats fut l'adoption quasi-universelle du langage et des notations utilisés par Banach.

p. 55, Heuser [1986$^\bullet$, p. 660]:

Banach findet nie aus den reellen (B)-Räumen heraus, ungeachtet der Rieszschen und Wienerschen Fingerzeige, und läßt so ohne Not ein fruchtbares Feld unbeackert liegen.

p. 55, [HAUS$_2$, Vorwort]:

Es wird vielleicht bedauert werden, daß ich zu weiterer Raumersparnis in der Punktmengenlehre den topologischen Standpunkt, durch den sich die erste Auflage anscheinend viele Freunde erworben hat, aufgegeben und mich auf die einfachere Theorie der metrischen Räume beschränkt habe.

p. 117, Bauer, in: [RADO$^{\bowtie}$, Vol. I, pp. 30–31]:

Das Studium der abstrakten Maße hatte über viele Jahre hinweg die enge Kopplung der von Radon untersuchten Maße mit der Topologie und damit mit der Geometrie des $\mathbb{R}^n$ in Vergessenheit geraten lassen. Die Rückbesinnung auf die Bedeutung dieser Kopplung war es dann, welche Anfang der 40er Jahre das Interesse für die Ideen Radons im Zusammenhang mit dem Studium von Maßen auf zunächst lokal-kompakten und polnischen Räumen, später dann weitgehend allgemeinen topologischen Räumen, neu belebte; see, for example, [BOU$_{6d}$].

Die Vertreter der ‚abstrakten' und der ‚konkreten' [read: topological] *Richtung standen sich eine Zeitlang nahezu kriegerisch gegenüber.*

p. 118, [BOU$_{6b}$, p. 123] or [BOU$^\bullet$, pp. 286–287, modified version]:

Mais c'est avec [CARA] *aussi que la notion d'intégrale cède le pas pour la première fois à celle de mesure, qui avait été chez Lebesgue un moyenne technique auxiliaire. Depuis lors, les auteurs qui ont traité l'intégration se sont partagés entre ces deux points de vue, non sans entrer dans de débats qui ont fait couler beaucoup d'encre sinon beaucoup de sang.*

p. 545, Milman [2003$^\bullet$, pp. 297–298]:

Таким образом, 20-летний период между началом 70-х и 90-ми годами в деятельности, связанной с теорией бесконечномерных пространств Банаха, оказался ненужным для получения результатов, на которые были направлены все главные усилия этой теории. Вместе с тем в это время были получены несколько замечательных результатов.

p. 566, [HERM$^\bullet$, Vol. II, p. 318]:

Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui n'ont point de dérivées.

p. 566, [BOU_{1b}, p. 86] or [BOU$^\bullet$, p. 27]:
Avec les recherches de Riemann lui-même sur l'integration, et plus encore avec les exemples de courbes sans tangente, construits par Bolzano et Weierstrass, c'est toute la pathologie des mathématiques qui commençait. Depuis un siècle, nous avons vu tant de monstres de cette espèce que nous sommes un peu blasés, et qu'il faut accumuler les caractères tératologiques les plus biscornus pour arriver encore à nous étonner.

p. 567, [PER]; see also Kahane [1998$^\bullet$, p. 126]:
C'est un cas où il est vraiment naturel de penser à ces fonctions continues sans dérivées que les mathématiciens ont imaginées, et que l'on regardait à tort comme de simples curiosités mathématiques, puisque l'expérience peut les suggérer.

p. 589, Brecht *Dreigroschenoper*,
Final verses of the ballad of Mack the Knife:
Denn die einen sind im Dunkeln
Und die andern sind im Licht.
Und man siehet die im Lichte
Die im Dunkeln sieht man nicht.

p. 620, [GARN$^+$, Part I, Introduction]:
Nous n'avons usé que des modes de raisonnement constructifs universellement admis en analyse, basés sur l'usage d'une langue correcte, de la logique traditionelle, de la théorie naïve des ensembles, des propriétés des nombres entiers et de l'axiome du choix dénombrable.

Nous nous sommes refusés à invoquer l'axiome du choix non dénombrable ou ses formes équivalentes (axiomes de Zermelo, Zorn, ...) jadis si controversés et aujourd'hui souvent utilisés sans conviction ou par routine.

p. 644, Dieudonné [1966$^\bullet$, p. 24]:
Enfin, je n'ai rien dit des premiers mémoires de Grothendieck sur les espaces vectoriels topologiques (1950–1955), en partie parce qu'ils sont fort connus et de plus en plus utilisés en Analyse Fonctionnelle, notamment la théorie des espaces nucléaires, qui ≪explique≫ les phénomènes rencontrés dans la théorie des distributions. J'ai eu personnellement le privilège d'assister de près, à cette époque, à l'éclosion du talent de cet extraordinaire ≪débutant≫ qui à 20 ans était déjà un maître; et, avec 10 ans de recul, je considère toujours que l'œuvre de Grothendieck de cette période reste, avec celle de Banach, celle qui a le plus fortement marqué cette partie des mathématiques.

p. 647, Cartan [1959$^\bullet$, p. 14]:
Im eigentlichen Text des Werkes gibt es niemals historische Hinweise. Denn Bourbaki duldet nicht die geringste Abweichung von logischen Aufbau.

Bibliography

The bibliography is divided into the following parts:

Textbooks and monographs . . . . pp. 684–709

referred to by several capitals from the author's name: [BAN]
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Historical and biographical books . . . . pp. 709–712

quoted as above but additionally marked by the superscript $^{\bullet}$:
[KALU$^{\bullet}$], [CART$^{+\bullet}$].

Collected and selected works . . . . pp. 712–716

items are indicated by the symbol $^{\bowtie}$: [BAN$^{\bowtie}$].

Collections (proceedings, memorial volumes) . . . . pp. 716–722

referred to by several capitals from the name of the first editor and
marked by the superscript $^{\cup}$: [GOH$^{\cup}_{1}$].

Seminars . . . . pp. 722–723

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items are referred as follows: [AAA$_1$], . . . ; [aaa$_1$], . . . ; [www$_1$],. . . .

Mathematical papers . . . . pp. 725–821

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Banach [1923], Banach/Steinhaus [1927].

Historical and biographical papers . . . . pp. 822–829

quoted as above but additionally marked by the superscript $^{\bullet}$:
Ulam [1946$^{\bullet}$], Butzer/Nessel/Stark [1984$^{\bullet}$].

Coauthors . . . . pp. 830–833

Labeled by the name of the first author, joint works are listed only once. In order to compensate for this injustice, I have appended an index of coauthors.

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