Norms, the Dual, Continuity

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The Evaluation Map

We suppose V is a vector space and V^* is its dual.

V^* is, itself, a vector space so it too has a dual, $V^{**} = (V^*)^*$.

There is an obvious collection of members of V^{**}, namely *evaluation of a functional at members of* V.

Define the **evaluation map** $\mathbf{E}\colon \mathbf{V} \to (\mathbf{V}^*)^*$ by

$$E(x)(f) = f(x) \quad \text{for each } x \in V \text{ and } f \in V^*.$$

The Evaluation Map

E is linear, and also one-to-one:

$$\text{that is, } E(x) = E(y) \text{ exactly when } x = y.$$

So if E is onto then it is invertible and an isomorphism.

In that case, $(V^*)^*$ can be identified with (i.e. it *is*) V.

If V is finite dimensional, V and V^* have the same dimension. And it follows that $(V^*)^*$ has the same dimension as does V. So E must be onto and therefore an isomorphism.

The infinite dimensional case is much more delicate and we will consider the extent to which we can recover this important identification in some form later.

Convexity

A nonempty subset S of a real vector space V is called **convex** if

$$tu + (1-t)v \in S \quad \forall t \in [0,1] \text{ and } u, v \in S.$$

In other words, all points on the line segment connecting u and v are in S whenever u and v are in S.

If V is any real vector space we say that a function $P\colon X \to \mathbb{R}$ is **convex** provided

$$P(tu + (1-t)v) \le tP(u) + (1-t)P(v) \quad \forall t \in [0,1].$$

Convexity

Geometrically, and in case $V = \mathbb{R}$, this means that the graph of a convex function always lies on or beneath the straight line connecting any two points on the graph. For this reason convex functions are also called **sublinear.**

So the region above the graph of such a function is a convex subset of $\mathbb{R}^2$.

Any seminorm is convex: a seminorm is the most common source of convex functions.

The Hahn-Banach Theorem

Theorem

The Hahn-Banach Theorem

If $Y \overset{\text{Real Vector Subspace}}{\subset} X \overset{\text{Real Vector Space}}{}$ *and* $P\colon X \to \mathbb{R}$ *is convex*

and $\Lambda \in Y^*_{\mathbb{R}}$ *satisfies* $\Lambda \le P|_Y$

then $\exists \Psi \in X^*_{\mathbb{R}}$ *with* $\Lambda = \Psi|_Y$ *and* $\Psi \le P$.

Proof.

If $w \in X - Y$ and α, β are positive and $u, v \in Y$

$$\begin{aligned}\beta\,\Lambda u + \alpha\,\Lambda v &= (\alpha+\beta)\,\Lambda\left(\frac{\beta}{\alpha+\beta}u + \frac{\alpha}{\alpha+\beta}v\right)\\ &\le (\alpha+\beta)\,P\left(\frac{\beta}{\alpha+\beta}(u-\alpha w) + \frac{\alpha}{\alpha+\beta}(v+\beta w)\right)\\ &\le \beta\,P(u-\alpha w) + \alpha\,P(v+\beta w).\end{aligned}$$

$$\text{So} \quad \frac{1}{\alpha}\,[\,\Lambda u - P(u - \alpha\,w)\,] \;\le\; \frac{1}{\beta}\,[\,P(v+\beta\,w) - \Lambda v\,].$$

Proof (Cont.)

The left side does not depend on v or β, while the right is independent of α and u. So there is a real number a with

$$\sup_{\substack{u\in Y\\ \alpha>0}} \frac{1}{\alpha}\left[\,\Lambda u - P(u-\alpha w)\,\right] \le a \le \inf_{\substack{v\in Y\\ \beta>0}} \frac{1}{\beta}\left[\,P(v+\beta w) - \Lambda v\,\right].$$

Define $\Theta\colon Y\bigoplus\mathbb{R}w \to \mathbb{R}$ by $\Theta(v+rw) = \Lambda v + ra$ for each $r\in\mathbb{R}$ and $v\in Y$.

Considering the cases of r positive, negative or zero separately, the definition of a yields

$$\Theta(v+rw) = \Lambda v + ra \le \Lambda v + P(v+rw) - \Lambda v = P(v+rw).$$

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Proof (Cont.)

So Λ can be extended one dimension at a time while preserving its relationship with P.

Let S be the set of all linear extensions of Λ to subspaces of X which are dominated by P on their domain.

Partially order this set of extensions by $\Theta\le\Psi$ if Ψ is an extension of Θ.

Chains in S have upper bounds in S and we invoke Zorn's lemma and assert that there is a maximal member Ψ of S.

The domain of Ψ is X, else it could be extended by one dimension, contradicting maximality. □

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The Hahn-Banach Theorem: Complex Version

Corollary

The Hahn-Banach Theorem

If $\underset{\text{Subspace}}{Y^{\text{Complex Vector}}} \subset \underset{\text{Space}}{X^{\text{Complex Vector}}}$ *and* $P\colon X\to\mathbb{R}$ *satisfies*

$P(\alpha v+\beta u) \le |\alpha|P(v) + |\beta|P(u)$ *if* $u, v\in X$ *and* $|\alpha|+|\beta| = 1$

and if $\Lambda\in Y^*_{\mathbb{C}}$ *satisfies* $|\Lambda|\le P|_Y$

then $\exists\Psi\in X^*_{\mathbb{C}}$ *with* $\Lambda = \Psi|_Y$ *and* $|\Psi|\le P$.

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Norms

If you want to do calculus in your space you must have limits, and the easiest way to talk about limits in a vector space is through the explicit notion of distance provided through a norm.

If V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, a seminorm on V is a function

$$\|\cdot\|\colon V\to[0,\infty)$$

with the property that for any number k and vectors v and w

$$\|kv\| = |k|\,\|v\| \quad \text{and} \quad \|\,v+w\,\| \le \|v\| + \|w\|$$

The seminorm is called **homogeneous** by virtue of the first line. The second of these is called the **triangle inequality**.

Seminorms

The triangle inequality can be tweaked slightly to produce a lower limit for the norm of a sum too.

$$|\,\|v\| - \|w\|\,| \leq \|\, v + w \,\| \leq \|v\| + \|w\|.$$

A **seminormed linear space**, abbreviated **SNLS**, is a real or complex vector space endowed with a seminorm.

A seminorm satisfies

$$\|\, \alpha v + \beta u \,\| \leq |\alpha|\, \|\, v \,\| + |\beta|\, \|\, u \,\| \quad \text{if } u, v \in X \text{ and } \alpha, \beta \in \mathbb{F}.$$

So a seminorm is an example (the most important example) of a sublinear function as found in the statement of the Hahn-Banach Theorem.

Norms

If you add the condition

$$\|v\| = 0 \text{ when and only when } v = 0$$

the seminorm is called a norm.

If $G\colon V \to F$ is any linear functional, the map $|G|\colon V \to [0, \infty)$ given as $|G|(v) = |G(v)|$ is a seminorm, and a common source of them too. This seminorm can never be a norm unless V has dimension 1.

A **normed linear space**, abbreviated **NLS**, is a real or complex vector space endowed with a norm.

Metric from a Norm

The **distance** between vectors v and w in a SNLS V is defined by

$$d(x, y) = \|v - w\|.$$

The distance notion is a pseudometric on V, and is a particularly nice one, having the properties

$$d(x + z, y + z) = d(x, y) \;\forall x, y, z \in V \text{ (\textbf{translation invariance})}$$

$$d(a\,x, a\,y) = |a|\, d(x, y) \;\forall x, y \in V \text{ and } a \in \mathbb{F} \text{ (\textbf{homogeneity})}$$

not required of a general metric or pseudometric.

Whenever a notion of distance is used in an SNLS it is this pseudometric which will be intended.

This pseudometric is a metric exactly when the seminorm is a norm.

Convergence

If a sequence $v_0, v_1, v_2, \ldots$ converges in a SNLS using this pseudometric we say that the sequence **converges in seminorm (or norm).**

Sometimes this is also called **strong convergence**, particularly when we have a norm from an inner product.

(There is a weaker concept of convergence which is also useful there.)

In case there might be confusion about the type of convergence involved, we might indicate intent by

$$v_i \xrightarrow[\text{strong}]{} w.$$

Null Vectors for a Seminorm

If V is an SNLS the set $\mathcal{N} = \{ x \in V \mid \|x\| = 0 \}$ is a vector subspace of V, sometimes called the set of **null vectors** for the seminorm (not to be confused with vectors from the nullspace of a linear transformation.)

In an SNLS a sequence can converge to more than one point.

In fact, if $v_i \to w$ then

$$v_i \to x \text{ exactly when } x = w + y \text{ for some } y \in \mathcal{N}.$$

This implies that $\mathcal{N}$ is a closed set.

The seminorm is a norm exactly when $\{ 0 \}$ is closed.

Completeness

Completeness is a very important property for us. Normed linear spaces which are complete are called **Banach spaces.**

$\mathbb{R}$ and $\mathbb{C}$ are themselves Banach spaces, a critical fact that is used often and assumed without discussion in most first-year calculus classes.

If S is a vector subspace of V then $\overline{S}$ is also a SNLS space, a subspace of V.

If V is Banach, so is $\overline{S}$.

Continuity

For V and W both SNLSs, a function $F\colon V \to W$ is continuous at $p \in V$ exactly when

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ so that if } \| p - v \| < \delta \text{ then } \| F(p) - F(v) \| < \varepsilon.$$

In general, δ will depend on both ε and p, but for linear functions it does not. Using linearity, one shows that continuity at *any point*, that is for just one v, implies continuity at *every point*, and the δ chosen for a particular ε does not depend on *which* point.

Linear functions between two SNLSs
which are continuous at a point are uniformly continuous.

Continuity

In fact if you can find $\delta_1 > 0$ for the single value $\varepsilon = 1$ and $p = 0$ then F is continuous.

That is because for any $\varepsilon > 0$ we have

$$\|v\| < \delta_1 \varepsilon \ \Rightarrow\ \frac{\|v\|}{\varepsilon} < \delta_1 \Rightarrow \left\| F\left(\frac{v}{\varepsilon}\right) \right\| < 1 \Rightarrow \frac{1}{\varepsilon}\ \|F(v)\| < 1$$

and so

$$\|F(v)\| < \varepsilon.$$

Linearity then allows us to translate the argument away from the origin.

Continuity

A useful equivalent condition, again using linearity of F, is that if $F^{-1}(B)$ is open for even *one open ball* B then $F^{-1}(B)$ will be open for *every* open subset B contained in W and therefore F will be continuous.

One implication: for continuous linear F, the set $\mathcal{K}er(F)$ is a closed subspace of V.

Boundedness

For $F^{\text{Linear}}: V^{\text{SNLS}} \to W^{\text{SNLS}}$ we define $\|\mathbf{F}\|$ by

$$\|F\| = \sup\{\, \|F(x)\| \mid x \in S_1(0)\,\}.$$

If $\|F\|$ is finite we say F itself is **bounded**, and it turns out that $\|F\|$ is bounded exactly when F is continuous. This is important enough that we enshrine the result in:

A linear functions between two SNLSs
is continuous exactly when it is bounded.

Operator Norm

$\mathbf{B}(\mathbf{V}, \mathbf{W})$ is defined to be the set of bounded linear functions from V to W.

The number $\|F\|$ given for each $F \in B(V, W)$ defines a norm on $B(V, W)$, generally called the **operator norm**.

If the range space W is a Banach space, so is $B(V, W)$.

If there are multiple norms floating around, we might distinguish this one by the notation $\|\cdot\|_{op}$.

The Continuous Dual

The collection of bounded linear functionals on an NLS V is a particularly important Banach space (with operator norm.)

It is called the **continuous dual** of V, denoted

$$V' = B(V, \mathbb{F}).$$

The continuous dual is a subspace of the algebraic dual,

$$V^* = Hom_{\mathbb{F}}(V, \mathbb{F}).$$

Evaluation is into V''

If V is an NLS, then V' is Banach. It too has a an algebraic dual $(V')^*$ and continuous dual $V'' \subset (V')^*$.

Recall the evaluation map

$$E\colon V \to (V^*)^* \text{ given by } E(x)(f) = f(x).$$

Every member of $(V^*)^*$ produces a member of $(V')^*$ by restriction and $\widetilde{E}(x)$ defined to be $E(x)|_{V'}$ is such a member.

And if $f \in V'$ and $\|f\| = 1$ then

$$\left|\widetilde{E}(x)(f)\right| = |f(x)| \le \|f\| \, \|x\| = \|x\|.$$

So each $\widetilde{E}(x)$ is bounded as a function from Banach space V' with operator norm to $\mathbb{F}$. In other words,

$$\widetilde{E}\colon V \to V''.$$

Hahn-Banach and the Continuous Dual

Suppose x is nonzero in NLS V. The linear transformation Λ defined on $\mathbb{F}x$ by $\Lambda(a\,x) = a\,\|x\|$ satisfies the condition of the Hahn-Banach Theorem (or its Corollary) where P is the norm on V. In this one dimensional case we have equality:

$$|\,\Lambda(a\,x)\,| = |a|\,\|x\| = \|a\,x\|.$$

So Λ can be extended to linear Ψ defined on all of V and for which

$$|\Psi(v)| \le \|v\| \quad \text{for all } \ v \in V.$$

This means that the operator norm $\|\Psi\|$ cannot exceed 1.

But $\Psi(x/\|x\|) = \Lambda(x/\|x\|) = 1$, So $\|\Psi\| = 1$.

To recap, for each $x \in V$ there is a functional Ψ in V' with

$$\|\Psi\| = 1 \quad \text{and} \quad \Psi x = \|x\|.$$

Reflexivity

But by the result of the previous slide there is a functional $\Psi \in V'$ with operator norm 1 for which $|\Psi(x)| = \|x\|$. This means that $\left|\widetilde{E}(x)\right|$ actually attains its maximum value, $\|x\|$, on the members of V' with operator norm 1.

Even more, this means that

$$\widetilde{E}\colon V \to V'' \quad \text{is an isometry.}$$

So the image of $\widetilde{E}$ with operator norm and V itself with its norm are not only isomorphic as vector spaces but are interchangeable in any calculation involving norms as well.

Spaces for which $\widetilde{E}$ is onto V'' are very important, and are called **reflexive**. We will have occasion to refer to this property later. Hilbert spaces are reflexive.

Overview

The Baire category theorem implies that no complete infinite dimensional space can have a countable basis. However, if we can't have a countable *basis*, we can do almost as well with a Schauder basis, defined below, which uses concepts of limit and continuity provided by a norm to get most of what a true basis provides in the finite dimensional setting.

The ideas to follow make sense in more general settings but we will confine consideration here to Banach spaces.

Schauder Basis

A **Schauder basis** for Banach X is a countable *ordered* set of vectors $v_0, v_1, \ldots$ for which every member x of X can be written in a *unique* way as

$$x = \sum_{n=0}^{\infty} a^n(x)\, v_n$$

for $a^n(x) \in \mathbb{F}$. The uniqueness refers to the values of the **coordinate functionals** a^n, which are therefore linear, and the convergence of the sequence of partial sums is in norm:

$$\sum_{n=0}^{k} a^n(x)\, v_n \underset{\text{strong}}{\longrightarrow} x \quad \forall x \in X.$$

The Coordinate Functionals

When referring to a sequence of vectors in vector space X the notation $\mathbf{v} = (\mathbf{v_n}) \subset X$ will be used. Thus, for Schauder basis as above we have the paired sequences

$$\mathbf{v} \subset X \quad \text{with unique associated coordinate functionals} \quad \mathbf{a} \subset X^*.$$

Any Banach space with a Schauder basis is separable, so there are Banach spaces without Schauder bases. In fact, there are Banach spaces for which no infinite dimensional subspace has a Schauder basis. Still, Banach spaces that have these bases are common in practice.

Uniqueness of coefficients implies that 0 is not among the vectors in a Schauder basis, and in fact the vectors in a Schauder basis must constitute a linearly independent list of vectors.

Continuity of the Coordinate Functionals

If $(\mathbf{v_n})$ is a Schauder basis, so is $(\mathbf{v_n}/\|\mathbf{v_n}\|)$ and the latter Schauder basis is called **normalized.**

Normalized or not, the members of the sequence of linear functionals $(\mathbf{a^n})$ are all continuous. In fact, (though it takes a bit of work to prove) their norms satisfy

$$1 \leq \|a^n\|\,\|v_n\| \leq K \quad \forall n \in \mathbb{N}$$

for a positive constant K that will vary with the basis.

So for Schauder basis $\mathbf{v} \subset X$ we actually have $\mathbf{a} \subset X'$, not just $\mathbf{a} \subset X^*$.

Unconditional and Bounded Bases

A Schauder basis is called **unconditional** if, for any $x \in X$ the series obtained by any permutation of the terms in the series representation for x in this Schauder basis also converges to x.

A Schauder basis is called **bounded** if there are positive constants A and B for which

$$A \leq \|v_n\| \leq B \quad \forall n \in \mathbb{N}.$$

Dual Schauder Bases

Since each vector in a Schauder basis $\mathbf{v}$ can be conceived of as (that is, *it is*) a member of X'' the possibility arises that $\mathbf{a}$ could be a Schauder basis for the Banach space $\overline{Span(\mathbf{a})}$ with operator norm, with coordinate functionals $\mathbf{v}$.

This is, in fact, the case. Even more, we have the following.

The Dual Basis Theorem

Suppose $\mathbf{v} \overset{\text{Schauder}}{\underset{\text{Basis}}{}} \subset X^{\text{Banach}}$ with coordinate functionals $\mathbf{a} \subset X'$.
Then $\mathbf{a} \overset{\text{Schauder}}{\underset{\text{Basis}}{}} \subset \overline{Span(\mathbf{a})}$ with coordinate functionals $\mathbf{v}$.
Moreover, if $\mathbf{v}$ is unconditional or bounded, so is $\mathbf{a}$.

The most interesting case, of course, is when X is reflexive. Then $X' = \overline{Span(\mathbf{a})}$ and so we have a correspondence between Schauder bases with their coefficient sequences for X and those for X'.

The Banach Adjoint

Suppose $F \in B\left(V^{\text{Banach}}, W^{\text{Banach}}\right)$.

Define for each $w' \in W'$ the member $F^*(w') \in V^*$ given by

$$F^*(w')(v) = w'(F(v)).$$

Note that $\|F^*(w')(v)\| \le \|w'\| \, \|F\| \, \|v\|$.

So, in fact

$$\|F^*(w')\| \le \|w'\| \, \|F\|$$

and this means $F^*(w')$ is actually in V', not just V^*.

Looking again we see that $F^* \in B(W', V')$ and $\|F^*\| \le \|F\|$.

Application of the Hahn-Banach theorem proves more: that $\|F^*\| = \|F\|$.

It is also a fact that if F is an isometry, so is F^*.

The Banach Adjoint

The map $F^* \colon W' \to V'$ is called the **Banach adjoint** of F and the **Banach adjoint operator**

$$^* : B(V, W) \to B\left(W', V'\right) \quad \text{is an isometry.}$$

Finally, if V and W are reflexive, we note that $F^{**} = F$, and the adjoint operator is an isomorphism.