Hilbert Space for Quantum Mechanics
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Opening Remarks

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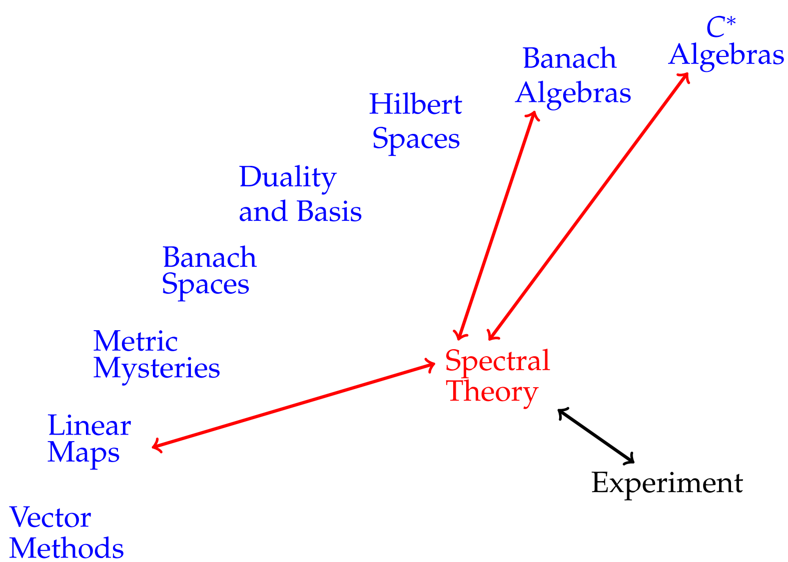
The Arc of Quantum Wisdom



Convergence of Sequences of Operators

If V and W are normed spaces, and T_n is a sequence of members of $\mathcal{L}(V, W)$ we say that T_n converges to a continuous linear map S uniformly, or **in operator norm**, if $\| S - T_n \|$ converges to 0.

We say that T_n converges to S strongly, or in the **strong operator topology**, if $\| S(v) - T_n(v) \|$ converges to 0 for each $v \in V$.

We say that T_n converges to S weakly, or in the **weak operator topology**, if $| \phi(S(v)) - \phi(T_n(v)) |$ converges to 0 for each $v \in V$ and each $\phi \in W'$.

Because of the opportunity to represent functionals using the inner product in a Hilbert Space, if W is Hilbert this is equivalent to

$$\langle S(v) - T_n(v) , w \rangle \to 0 \quad \text{for all } v \in V \text{ and } w \in W.$$

Convergence of Sequences of Operators 2

uniform op.	strong op.	weak op.
$\| S - T_n \| \to 0$	$\| S(v) - T_n(v) \| \to 0$	$\mid \phi(S(v)) - \phi(T_n(v)) \mid \to 0$
one norm	one norm for each v	one norm for each v and ϕ.

These modes of convergence are progressively weaker, left to right. Uniform operator convergence is ideal.

Strong operator convergence is often all you can get, and sufficient for the application.

Weak operator convergence is sufficient to deduce that a (not-necessarily-continuous) linear operator exists that matches the limit values. With other conditions, this can often be promoted to one of the stronger forms of convergence.

The Schrödinger Equation

- To solve $i \frac{d}{dt}\psi(t) = H\psi(t)$ with $\psi(0) = \psi_0$ we need to calculate

$$e^{-itH} = I - itH + \frac{(itH)^2}{2} - \frac{(itH)^3}{6} + \cdots$$

  but here we have many new issues.
- What does it mean for a series to converge when H is an operator on an infinite dimensional function space?
- What if H is unbounded? Can one make any sense at all out of such a sum in that case?
- And how does the crucial self-adjointness condition come into all this?

What We Need To Do Now I

We need to discuss the following topics in some (not necessarily this) order:

- Recall the definition of adjoint of an operator.
- Recall the meaning of normal, self-adjoint, positive, isometric, unitary and projection pperators and their immediately pertinent qualities.
- Projection-valued measures and a Stieltjes-type integral for complex valued functions with domain in $\mathbb{R}$ against such measures.
- Discuss the meaning of densely defined operators and symmetric operators, and the possibility of extension.
- Define closed operators (such as the derivative operator with appropriate norm) and their properties.

What We Need To Do Now II

- Define Compact Operators (such as Fredholm operators) and examine some of their properties.
- Define Banach and C^* algebras and give examples of the three main types: Operator, Function and Convolution Algebras.
- State the Schauder fixed point theorem, Picard iteration and the Contraction Mapping Theorem.
- The spectral theorem for compact operators with application to Integral Equations of Fredholm type.
- The spectral theorem and the functional calculus for Banach and C^* algebras.