Hilbert Space Methods Used in a First Course in Quantum Mechanics:

Green's Functions in Quantum Theory

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Outline

- Perturbation Theory in Quantum Mechanics.
- Permutation Symmetry of a System of Identical Particles. Bosons and Fermions.
- Second Quantization for Bosons.
- Green's Functions for an Ideal Gas of Free Particles.
- The Adiabatic Hypothesis. Wick's Chronological *T*-Operator.
- Perturbation Series for Green's Functions.
- Wick's Theorem.
- Feynman Diagrams.
- Dyson's Equation and Renormalization.

Perturbation Theory in Quantum Mechanics

Part 1. The Stationary Case. No Time Dependence

Finding approximate solution of the Stationary Schrödinger's Equation: $H\Psi = E\Psi$.

Let $H = H_0 + \lambda V$, λ is a small parameter. Assume the eigenvalue problem with H_0 is solved, $H_0\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}$. We are trying to find approximate solutions of perturbed eigenvalue problem: $\left(H_0 + \lambda V\right)\psi = E\psi$. Its matrix version is:

$$\psi = \sum_m c_m \psi_m^{(0)}$$

$$\left(H_0 + \lambda V\right)\sum_m c_m \psi_m^{(0)} = E\sum_m c_m \psi_m^{(0)}, \qquad V\psi_m^{(0)} = \sum_m V_{km}\psi_k^{(0)}, \quad V_{km} = \left\langle \psi_k^{(0)} \middle| V \middle| \psi_m^{(0)} \right\rangle$$

$$\left(E - E_k^{(0)}\right)c_k = \sum_m V_{km} c_m$$

$$E = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \ldots, \qquad c_m = c_m^{(0)} + \lambda c_m^{(1)} + \lambda^2 c_m^{(2)} + \ldots$$

$$E_n^{(1)} = V_{nn} = \left\langle \psi_n^{(0)} \middle| V \middle| \psi_n^{(0)} \right\rangle, \qquad c_k^{(1)} = \frac{\left\langle \psi_k^{(0)} \middle| V \middle| \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}}, \ (k \neq n) \qquad \psi_n^{(1)} = \sum_{k \neq n} \frac{\left\langle \psi_k^{(0)} \middle| V \middle| \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} \psi_k^{(0)}$$

It is true if $\left|V_{kn}\right| << \left|E_n^{(1)} - E_k^{(1)}\right|$

Perturbation Theory in Quantum Mechanics

Part 2. The Time-Dependent Case.

Finding approximate solution of the Time-Dependent Schrödinger's Equation: $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$

with $H = H_0 + \lambda V$, λ being a small parameter. Assume the eigenvalue problem with H_0 is solved, $H_0\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}$. We are trying to find approximate solutions of perturbed eigenvalue problem: $i\hbar\frac{\partial\Psi}{\partial t} = \left(H_0 + \lambda V\right)\Psi$. Its matrix version is:

$$\Psi(t) = \sum_m c_m(t)\Psi_m^{(0)}$$

$$\left(H_0 + \lambda V\right)\sum_m c_m\psi_m^{(0)} = E\sum_m c_m\psi_m^{(0)},$$

$$i\hbar\frac{\partial c_k}{\partial t} = \sum_m V_{km}(t)c_m \qquad V_{km}(t) = \left\langle\Psi_k^{(0)}\middle|V(t)\middle|\Psi_m^{(0)}\right\rangle = V_{km}e^{i\omega_{km}t}, \quad \omega_{km} = \frac{E_k^{(0)} - E_m^{(0)}}{\hbar}$$

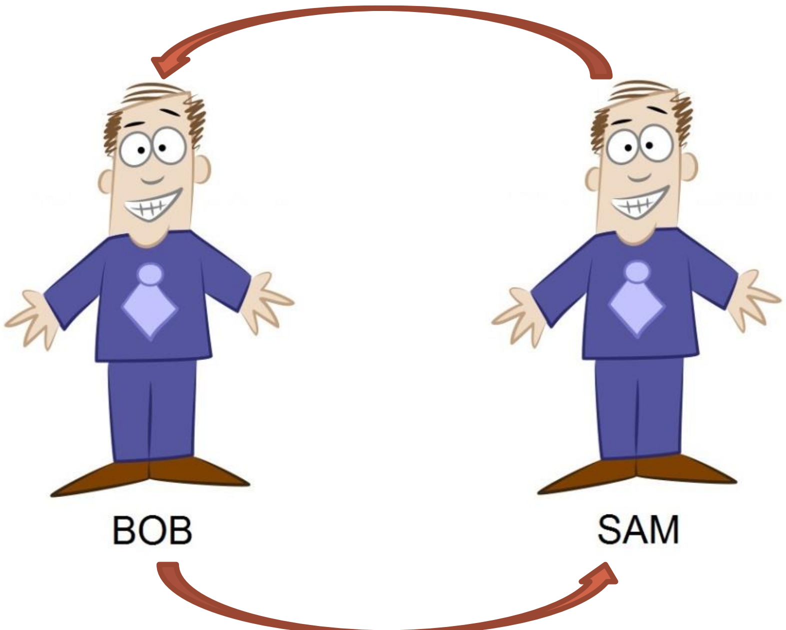
$$c_m(t) = c_m^{(0)}(t) + \lambda c_m^{(1)}(t) + \lambda^2 c_m^{(2)}(t) + \ldots$$

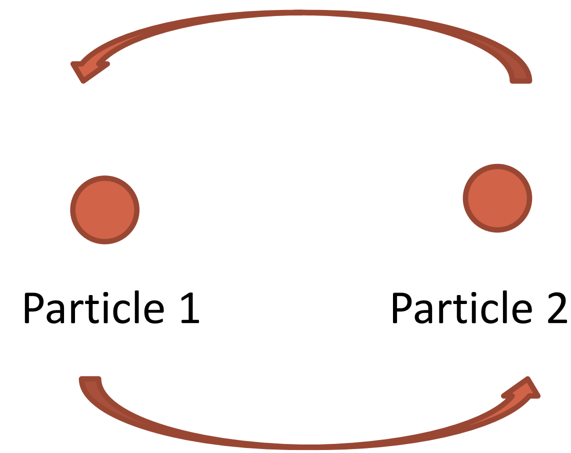
First-order correction: $i\hbar\frac{\partial c_k^{(1)}}{\partial t} = V_{km}(t)$ $\qquad c_k^{(1)}(t) = -\frac{i}{\hbar}\int V_{kn}(t)dt = -\frac{i}{\hbar}\int V_{kn}e^{i\omega_{kn}t}dt$

It is true if $\left|V_{kn}\right| << \left|E_n^{(1)} - E_k^{(1)}\right|$

Permutation Symmetry of a System of Identical Particles. Bosons and Fermions.

Permutation Symmetry of Identical Twins.





$$P(1,2)\Psi(r_1,r_2) = e^{i\alpha}\Psi(r_2,r_1)$$

$$P(1,2)P(1,2)\Psi(r_1,r_2) = e^{2i\alpha}\Psi(r_1,r_2) = \Psi(r_1,r_2)$$

$$e^{2i\alpha} = 1 \quad \Rightarrow \quad \left(e^{i\alpha}\right)^2 = 1 \quad \Rightarrow \quad e^{i\alpha} = \pm\sqrt{1}$$

$$e^{i\alpha} = \pm 1 \quad (\text{meaning} \;\; \alpha = 0 \;\; \text{or} \;\; \alpha = \pm\pi)$$

$$P(1,2)\Psi(r_1,r_2) = \Psi(r_2,r_1) \quad \text{or} \quad P(1,2)\Psi(r_1,r_2) = -\Psi(r_2,r_1)$$

The Bose-Einstein case

The Fermi-Dirac case

Permutation Symmetry of a System of Identical Particles. Non-Interacting Particles.

Bosons (phonons, plasmons, magnons, etc.):

$$H = H(r_1) + H(r_2)$$

$$\Psi_{mn}(r_1, r_2) = \psi_m(r_1)\psi_n(r_2)$$

Fermions (electrons, nucleons, mesons, etc.):

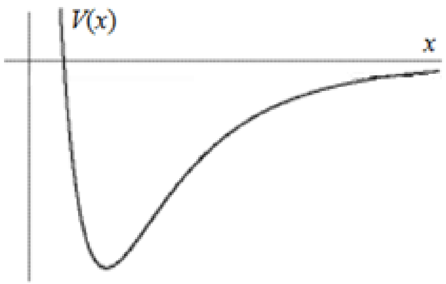
$$H = H(r_1) + H(r_2)$$

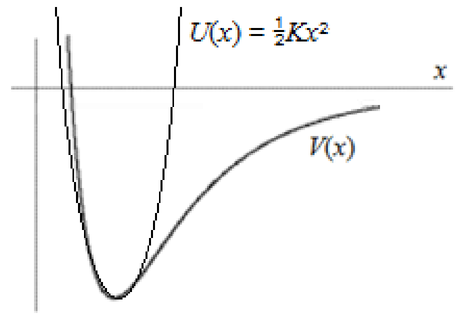
$$\Psi_{mn}(r_1, r_2) = \frac{1}{\sqrt{2}}\left[\psi_m(r_1)\psi_n(r_2) - \psi_m(r_2)\psi_n(r_1)\right]$$

$$\Psi_{mn}(r_1, r_2) = \frac{1}{\sqrt{2}}\begin{vmatrix} \psi_m(r_1) & \psi_m(r_2) \\ \psi_n(r_1) & \psi_n(r_2) \end{vmatrix}$$

Second Quantization for Bosons

- The Harmonic Approximation: Modeling the true potential energy, $V(x)$, with the parabola, $U(x) = \frac{1}{2}Kx^2$.





$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2, \qquad \omega = \sqrt{\frac{K}{m}}, \quad K = m\omega^2$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]$$

- Introducing operators a and $a^\dagger$.

$$a = \frac{1}{\sqrt{2m\hbar\omega}}\left(ip + m\omega x\right), \qquad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(-ip + m\omega x\right)$$

As $p = -i\hbar\frac{d}{dx}$, we have

$$a = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x + \hbar\frac{d}{dx}\right), \qquad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)$$

Second Quantization for Bosons

Since $\left[\frac{\partial}{\partial x}, x\right] = \frac{\partial}{\partial x}x - x\frac{\partial}{\partial x} = 1$, we have $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$.

Then
$$aa^\dagger = \left(\frac{1}{\sqrt{2m\hbar\omega}}\right)^2 (-ip + m\omega x)(ip + m\omega x)$$

$$= \frac{1}{2m\hbar\omega}\left\{p^2 + im\omega(xp - px) + (m\omega x)^2\right\}$$

$$a^\dagger a = \left(\frac{1}{\sqrt{2m\hbar\omega}}\right)^2 (ip + m\omega x)(-ip + m\omega x)$$

$$= \frac{1}{2m\hbar\omega}\left\{p^2 - im\omega(xp - px) + (m\omega x)^2\right\}$$

Therefore,
$$aa^\dagger + a^\dagger a = \frac{1}{2m\hbar\omega}\left\{p^2 + im\omega(xp - px) + (m\omega x)^2 + p^2 - im\omega(xp - px) + (m\omega x)^2\right\}$$

$$= \frac{1}{m\hbar\omega}\left\{p^2 + (m\omega x)^2\right\} \quad \Rightarrow \quad p^2 + (m\omega x)^2 = m\hbar\omega\left(aa^\dagger + a^\dagger a\right)$$

As $aa^\dagger - a^\dagger a = 1$, we have $aa^\dagger = 1 + a^\dagger a$ and, therefore,

$$p^2 + (m\omega x)^2 = m\hbar\omega\left(2a^\dagger a + 1\right) = 2m\hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

Second Quantization for Bosons

Plugging $p^2 + (m\omega x)^2 = 2m\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ into $H = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]$, we come to $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$

- **Theorem 1:** **Let $\psi(x)$ be eigenfunction of $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ with the eigenvalue E.**

  Then $\Psi(x) = a^\dagger\psi(x)$ is another eigenfunction of H with the eigenvalue $E + \hbar\omega$

- **Proof:** $H\,\Psi = H\,a^\dagger\psi = \hbar\omega(a^\dagger a + \frac{1}{2})a^\dagger\psi = \hbar\omega(a^\dagger a a^\dagger + \frac{1}{2}a^\dagger)\psi$

  Plugging $aa^\dagger = 1 + a^\dagger a$, we have

$$H\,\Psi = \hbar\omega[a^\dagger(1 + a^\dagger a) + \tfrac{1}{2}a^\dagger]\psi = a^\dagger\hbar\omega[a^\dagger a + \tfrac{1}{2} + 1]\psi$$
$$= a^\dagger[\hbar\omega(a^\dagger a + \tfrac{1}{2}) + \hbar\omega]\psi = a^\dagger[H + \hbar\omega]\psi$$
$$= a^\dagger[E + \hbar\omega]\psi = [E + \hbar\omega]a^\dagger\psi = [E + \hbar\omega]\Psi$$

Second Quantization for Bosons

- **Theorem 2:** **Let $\psi(x)$ be eigenfunction of $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ with the eigenvalue E.**

  Then $\Psi(x) = a\psi(x)$ is another eigenfunction of H with the eigenvalue $E - \hbar\omega$

- **Proof:** $H\,\Psi = Ha\psi = \hbar\omega(a^\dagger a + \frac{1}{2})a\psi = \hbar\omega(a^\dagger aa + \frac{1}{2}a)\psi$

  Plugging $a^\dagger a = aa^\dagger - 1$, we have

$$H\,\Psi = \hbar\omega[(aa^\dagger - 1)a + \tfrac{1}{2}a]\psi = a\hbar\omega[a^\dagger a + \tfrac{1}{2} - 1]\psi$$
$$= a\,[\hbar\omega(a^\dagger a + \tfrac{1}{2}) - \hbar\omega]\psi = a\,[H - \hbar\omega]\psi$$
$$= a\,[E - \hbar\omega]\psi = [E - \hbar\omega]a\psi = [E + \hbar\omega]\Psi$$

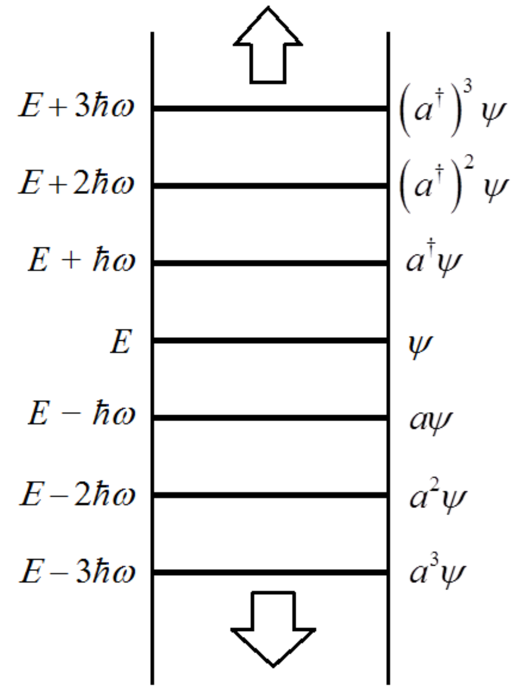
- **How deep can we go down in energy?**

Total energy: $E = \dfrac{p^2}{2m} + V(x) \geq \dfrac{p^2}{2m} + V_{\min} \geq V_{\min}$

So, there is a minimal energy, E_0, the system can have.

The corresponding state, ψ_0, is called **ground state**: $a\psi_0 = 0$.

$$a = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x + \hbar\frac{d}{dx}\right) \quad \Rightarrow \quad \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x + \hbar\frac{d}{dx}\right)\psi_0 = 0$$



The "ladder" of equidistant energy levels of a harmonic oscillator

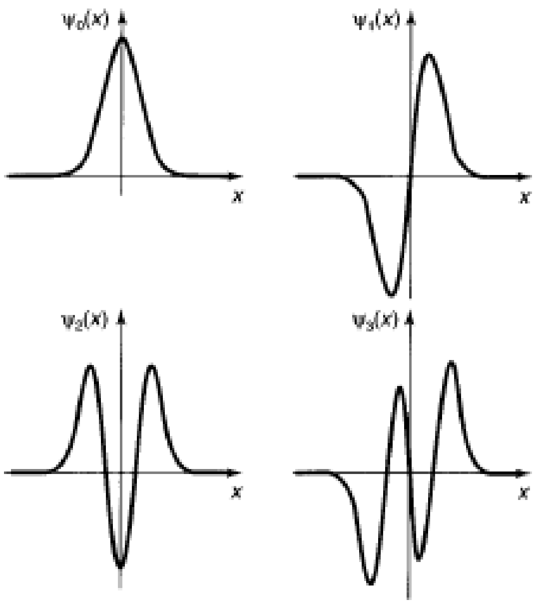
Second Quantization for Bosons

- **Ground state:** We come to separable differential equation, $m\omega x\psi_0 - \hbar\frac{d\psi_0}{dx} = 0$

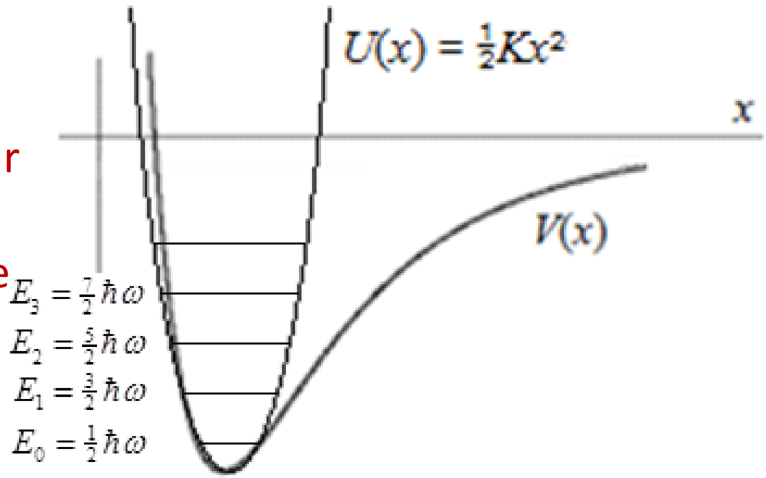
Its solution is $\psi_0(x) = N_0 e^{-\alpha x^2}$ with $\alpha = \frac{m\omega}{2\hbar}$ and $N_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

- **Excited states:**

$$\psi_1 = a^\dagger\psi_0 = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)Ne^{-\alpha x^2} = N_1 x e^{-\alpha x^2} \text{ with } N_1 = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$



The first four stationary states of the harmonic oscillator



$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ $\quad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ $\quad\Rightarrow\quad n = a^\dagger a$

- ***n*** **the number of *"elementary excitations"***

The Heisenberg representation

- **Time dependence:**

$$i\hbar\frac{\partial\Psi}{\partial t}=\mathbf{H}\Psi(x,t)\quad\Rightarrow\quad\Psi(x,t)=\psi(x)e^{-iEt/\hbar}=e^{-i\mathbf{H}t/\hbar}\psi(x)$$

$$\bar{A}=\int\psi^*(x)e^{i\mathbf{H}t/\hbar}A(x)e^{-i\mathbf{H}t/\hbar}\psi(x)dx$$

- **The Schrödinger representation**

$$\Psi(x,t)=\psi(x)e^{-iEt/\hbar}=e^{-i\mathbf{H}t/\hbar}\psi(x)$$

$$A=A(x)$$

The Heisenberg representation

$$\Psi(x)=\psi(x),\qquad U(t)=e^{-i\mathbf{H}t/\hbar},$$

$$A(x,t)=e^{i\mathbf{H}t/\hbar}A(x)e^{-i\mathbf{H}t/\hbar}=U^\dagger(t)A(x)U(t)$$

$$\frac{\partial}{\partial t}A(x,t)=\left[A(x,t),H\right]=A(x,t)H-HA(x,t)$$

- **Wave function of a system of several non-interacting oscillators:**

$$\Psi(x_1,x_2,x_3)=\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)=|n_1,n_2,n_3\rangle$$

Occupation-number representation

- **Ground state:** $|0, 0, 0, \ldots, 0\rangle = |\mathbf{0}\rangle$

Vacuum state

Interaction representation

So far, $H(x) = H_0(x) + \lambda V(x)$ **. What if** $H(x,t) = H_0(x) + \lambda V(x,t)$ **?**

Assume we know solutions of the unperturbed eigenvalue problem,

$$H_0 \Psi_n^{(0)}(x,t) = E_n^{(0)} \Psi_n^{(0)}(x,t), \qquad \Psi_n^{(0)}(x,t) = e^{-(i/\hbar)E_n^{(0)}t} \psi_n^{(0)}(x) .$$

We want to solve the time-dependent Schrödinger equation for the perturbed case,

$$\frac{\partial \Psi(x,t)}{\partial t} = -\frac{i}{h} H(x,t) \Psi(x,t)$$

Expanding the perturbed wave function $\Psi(x,t)$ **in terms of the orthonormal basis set of unperturbed states,**

$$\Psi(x,t) = \sum_n c_n(t) \Psi_n^{(0)}(x,t) = \sum_n c_n(t) e^{-(i/\hbar)E_n^{(0)}t} \psi_n^{(0)}(x)$$

$$= \sum_n c_n(t) e^{-(i/\hbar)H_0 t} \psi_n^{(0)}(x) = e^{-(i/\hbar)H_0 t} \sum_n c_n(t) \psi_n^{(0)}(x)$$

Consider the function

$$\tilde{\Psi}(x,t) = e^{(i/\hbar)H_0 t} \Psi(x,t) = e^{(i/\hbar)H_0 t} e^{-(i/\hbar)H_0 t} \sum_n c_n(t) \psi_n^{(0)}(x) = \sum_n c_n(t) \psi_n^{(0)}(x)$$

This change is achieved using the operator of unitary transformation $U = e^{(i/\hbar)H_0 t}$

Respectively, for operators,

$$\tilde{A}(x,t) = U A(x) U^{-1} = e^{(i/\hbar)H_0 t} A(x) e^{-(i/\hbar)H_0 t}$$

This is the interaction reperesentation

Interaction representation. Part 2.

Consider time dependence of $\tilde{\Psi}(x,t) = e^{(i/\hbar)H_0 t}\Psi(x,t)$. **Its time derivative is**

$$\frac{\partial\tilde{\Psi}(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[e^{(i/\hbar)H_0 t}\right]\Psi(x,t) + e^{(i/\hbar)H_0 t}\frac{\partial}{\partial t}\left[\Psi(x,t)\right]$$

$$= e^{(i/\hbar)H_0 t}\left(\frac{i}{\hbar}H_0\right)\Psi(x,t) + e^{(i/\hbar)H_0 t}\left[-\frac{i}{h}H\Psi(x,t)\right]$$

$$= \frac{i}{h}e^{(i/\hbar)H_0 t}\left(H_0 - H\right)\Psi(x,t)$$

$$= e^{(i/\hbar)H_0 t}\left[-\frac{i}{h}\lambda V(x,t)\right]\Psi(x,t)$$

$$= e^{(i/\hbar)H_0 t}\left[-\frac{i}{h}\lambda V(x,t)\right]e^{-(i/\hbar)H_0 t}e^{(i/\hbar)H_0 t}\Psi(x,t)$$

Thus, $\frac{\partial\tilde{\Psi}(x,t)}{\partial t} = -\frac{i\lambda}{h}\tilde{V}(x,t)\tilde{\Psi}(x,t)$ **. Converting it into an integral equation, we have:**

$$\tilde{\Psi}(t) = \tilde{\Psi}_n^{(0)}(-\infty) - \frac{\lambda i}{\hbar}\int_{-\infty}^{t}\tilde{V}(x,\ t_1)\tilde{\Psi}(t_1)\,dt_1$$

The Adiabatic Hypothesis.
Wick's Chronological *T*-Operator. Part 1.

The Adiabatic Hypothesis: Assume $\lambda V(x)$ **turns on very slowly from** $V(x, -\infty) = 0$ to $V(x, t)$ **at a finite time** t**. This means** $V(x, t) = e^{\varepsilon t} V(x)$ **with** $\varepsilon = +0$**. Then** $H(x, t) = H_0(x) + \lambda V(x, t)$.

$$\tilde{\Psi}(t) = \tilde{\Psi}_n^{(0)}(-\infty) - \frac{\lambda i}{\hbar} \int_{-\infty}^{t} \tilde{V}(x, t_1) \tilde{\Psi}(t_1) dt_1$$

Picard's Method of Successive Approximations:

$$\tilde{\Psi}_n(t) \approx \tilde{\Psi}_n^{(0)}(-\infty), \quad \tilde{\Psi}(t) \approx \tilde{\Psi}_n^{(0)}(-\infty) - \frac{\lambda i}{\hbar} \int_{-\infty}^{t} \tilde{V}(x, t_1) \tilde{\Psi}(t_1) dt_1$$

$$\tilde{\Psi}_n(t) = \tilde{\Psi}_n^{(0)}(-\infty) - \frac{\lambda i}{\hbar} \int_{-\infty}^{t} \tilde{V}(x, t_1) \left[\tilde{\Psi}_n^{(0)}(-\infty) - \frac{\lambda i}{\hbar} \int_{-\infty}^{t_1} \tilde{V}(x, t_2) \tilde{\Psi}(t_2) dt_2 \right] dt_1$$

$$\tilde{\Psi}_n(t) = \tilde{\Psi}_n^{(0)}(-\infty) + \tilde{\Psi}_n^{(1)}(t) + \tilde{\Psi}_n^{(2)}(t) + \tilde{\Psi}_n^{(3)}(t) + \ldots = S(t, -\infty) \tilde{\Psi}_n^{(0)}(-\infty)$$

$$\tilde{\Psi}_n^{(k)}(t) = \left(-\frac{\lambda i}{\hbar} \right)^k \left[\int_{-\infty}^{t} dt_1 \tilde{V}(x, t_1) \int_{-\infty}^{t_1} dt_2 \tilde{V}(x, t_2) \int_{-\infty}^{t_2} dt_3 \tilde{V}(x, t_3) \ldots \int_{-\infty}^{t_{k-1}} dt_k \tilde{V}(x, t_k) \right] \tilde{\Psi}_n^{(0)}(-\infty)$$

The Adiabatic Hypothesis.
Wick's Chronological *T*-Operator. Part 2.

The evolution operator, $\tilde{\Psi}_n(t) = S(t,-\infty)\tilde{\Psi}_n^{(0)}(-\infty)$ **, is:**

$S(t,-\infty) =$

$$1-\frac{\lambda i}{\hbar}\int_{-\infty}^{t}\tilde{V}(x,\ t_1)dt_1+\left(-\frac{\lambda i}{\hbar}\right)^2\int_{-\infty}^{t}dt_1\tilde{V}(x,\ t_1)\int_{-\infty}^{t_1}dt_2\tilde{V}(x,\ t_2)+\left(-\frac{\lambda i}{\hbar}\right)^3\int_{-\infty}^{t}dt_1\tilde{V}(x,\ t_1)\int_{-\infty}^{t_1}dt_2\tilde{V}(x,\ t_2)\int_{-\infty}^{t_2}dt_3\tilde{V}(x,\ t_3)+\ldots$$

Here $t > t_1 > t_2 > t_3 > \ldots$ **. The connected integrals could be simplified if** $\tilde{V}(x,\ t) = e^{\frac{i}{\hbar}H_0 t}V(x)e^{-\frac{i}{\hbar}H_0 t}$ **would be just a** ***c*-number, not an operator. At different times, the commutator is NOT zero:**

$$\left[\tilde{V}(x,\ t_1),\tilde{V}(x,\ t_2)\right]=\tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2)-\tilde{V}(x,\ t_2)\tilde{V}(x,\ t_1)$$

$$=e^{\frac{i}{\hbar}H_0 t_1}V(x)e^{-\frac{i}{\hbar}H_0 t_1}e^{\frac{i}{\hbar}H_0 t_2}V(x)e^{-\frac{i}{\hbar}H_0 t_2}-e^{\frac{i}{\hbar}H_0 t_2}V(x)e^{-\frac{i}{\hbar}H_0 t_2}e^{\frac{i}{\hbar}H_0 t_1}V(x)e^{-\frac{i}{\hbar}H_0 t_1}$$

$$=e^{\frac{i}{\hbar}H_0 t_1}V(x)e^{-\frac{i}{\hbar}H_0(t_1-t_2)}V(x)e^{-\frac{i}{\hbar}H_0 t_2}-e^{\frac{i}{\hbar}H_0 t_2}V(x)e^{\frac{i}{\hbar}H_0(t_1-t_2)}V(x)e^{-\frac{i}{\hbar}H_0 t_1}\neq 0$$

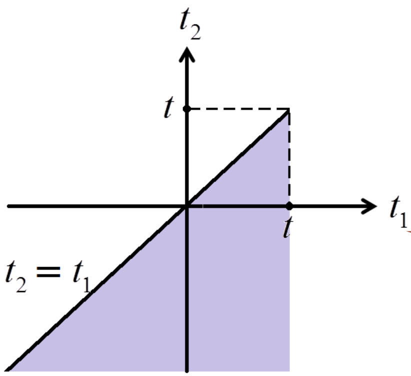
The Adiabatic Hypothesis. Wick's Chronological *T*-Operator. Part 3.

Consider the 2nd-order term.

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) = \int_{-\infty}^{t} dt_2 \int_{t_2}^{t} dt_1 \tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) = \int_{-\infty}^{t} dt_1 \int_{t_2}^{t} dt_2 \tilde{V}(x,\ t_2)\tilde{V}(x,\ t_1)$$

Changing the order of integration

Renaming $t_1 \rightleftarrows t_2$



Operators are chronologically ordered

Thus,

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) = \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) + \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{t_2}^{t} dt_2 \tilde{V}(x,\ t_2)\tilde{V}(x,\ t_1)$$

Here $t_1 \geq t_2$

Here $t_2 \geq t_1$

The Adiabatic Hypothesis.
Wick's Chronological *T*-Operator. Part 4.

Thus, the second-order term can be written in the following form:

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) = \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 T\tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2) + \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{t1}^{t} dt_2 T\tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2)$$

$$= \frac{1}{2}\int_{-\infty}^{t} dt_1 T\tilde{V}(x,\ t_1)\left[\int_{-\infty}^{t_1} dt_2 \tilde{V}(x,\ t_2) + \int_{t1}^{t} dt_2 \tilde{V}(x,\ t_2)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 T\tilde{V}(x,\ t_1)\tilde{V}(x,\ t_2)$$

$$= \frac{1}{2}\left[\int_{-\infty}^{t} T\tilde{V}(x,\ t_1)dt_1\right]^2$$

Gian Carlo Wick
1909 - 1992

Definition of *T*-ordering:
$TA(t_1)B(t_2) = A(t_1)B(t_2)$ if $t_1 > t_2$
$TA(t_1)B(t_2) = B(t_2)A(t_1)$ if $t_1 < t_2$

Similar *T*-ordering can be done in all other terms. Then

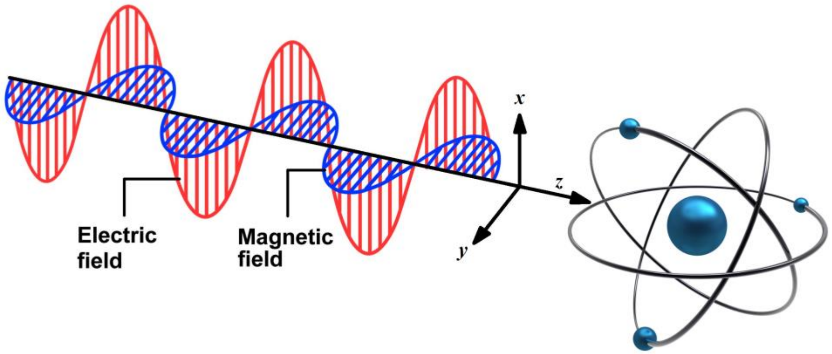
$$S(t,-\infty) = T\left\{1 + \left(-\frac{\lambda i}{\hbar}\right)\int_{-\infty}^{t} \tilde{V}(x,\ t_1)dt_1 + \frac{1}{2}\left(-\frac{\lambda i}{\hbar}\right)^2\left[\int_{-\infty}^{t} \tilde{V}(x,\ t_1)dt_1\right]^2 + \frac{1}{3!}\left(-\frac{\lambda i}{\hbar}\right)^3\left[\int_{-\infty}^{t} \tilde{V}(x,\ t_1)dt_1\right]^3 + \dots\right\}$$

$$= T\exp\left[-\frac{\lambda i}{\hbar}\int_{-\infty}^{t} \tilde{V}(x,\ t_1)dt_1\right]$$

In general, $S(t_2,t_1) = T\exp\left[-\frac{\lambda i}{\hbar}\int_{t_1}^{t_2} \tilde{V}(x,\ t_1)dt_1\right], \quad t_2 \geq t_1$

Also, $S(t_2,\ t_1)S(t_1,\ t_2) = 1$. **Then** $S(t_2,\ t_1) = S^{-1}(t_1,\ t_2)$.
Besides, $S(\infty,\ t)S(t,\ -\infty) = S(\infty,\ -\infty)$. **Therefore,** $S(t,\ -\infty) = S^{-1}(\infty,\ t)\ S(\infty,\ -\infty)$.

Quantum Transitions Due to a Periodic Perturbation



Let, say, a linearly polarized electromagnetic wave with $E_x(t) = E_0 e^{i\Omega t}$ is incident on a quantum system, say, an atom. If $\mathbf{d}$ is the electric dipole moment of the quantum system, the respective coupling is the dot product $\mathbf{E}\cdot\mathbf{d} = E_0 d_x e^{i\Omega t}$. It generates quantum transitions from the ground state, Ψ_0, to excited states Ψ_n. This is observed as absorption of the electromagnetic wave by the quantum system. Intensity of the absorption is proportional to

$$I(\Omega) = \sum_n \left|\langle \Psi_n | d_x | \Psi_0 \rangle\right|^2 \delta(\Omega - \omega_{0n}) \quad \text{with} \quad \omega_{0n} = \frac{E_n - E_0}{\hbar}, \quad \delta(\omega) = \lim_{\gamma \to 0}\left(\frac{1}{\pi}\frac{\gamma}{\omega^2 + \gamma^2}\right)$$

Plugging $\delta(\Omega - \omega_{0n}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\Omega - \omega_{0n})t} dt$, it can be transformed in the following way:

$$I(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_n \left|\langle \Psi_0 | d_x | \Psi_n \rangle\right|^2 e^{i(\Omega - \omega_{0n})t} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Omega t} \sum_n \left|\langle \Psi_0 | d_x | \Psi_n \rangle\right|^2 e^{-i\omega_{0n}t} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Omega t} I(t)\, dt$$

Here $I(t)$ is The Fourier transform of the absorption intensity $I(\Omega)$,

$$I(t) = \sum_n \left|\langle \Psi_0 | d_x | \Psi_n \rangle\right|^2 e^{-i\omega_{0n}t} = \sum_n \langle \Psi_0 | d_x | \Psi_n \rangle \langle \Psi_n | d_x | \Psi_0 \rangle e^{-(i/\hbar)(E_n - E_0)t}$$

$$= \sum_n \langle \Psi_0 | e^{(i/\hbar)E_0 t} d_x e^{-(i/\hbar)E_n t} | \Psi_n \rangle \langle \Psi_n | d_x | \Psi_0 \rangle = \sum_n \langle \Psi_0 | e^{(i/\hbar)Ht} d_x e^{-(i/\hbar)Ht} | \Psi_n \rangle \langle \Psi_n | d_x | \Psi_0 \rangle$$

Correlation Functions and Green's Functions

$$I(t)=\sum_n\langle\Psi_0|e^{(i/\hbar)Ht}d_x e^{-(i/\hbar)Ht}|\Psi_n\rangle\langle\Psi_n|d_x|\Psi_0\rangle=\langle\Psi_0|\underbrace{e^{(i/\hbar)Ht}d_x e^{-(i/\hbar)Ht}}\Big(\underbrace{\sum_n|\Psi_n\rangle\langle\Psi_n|}\Big)d_x|\Psi_0\rangle$$

This is Heisenberg time dependence, $d_x(t)$

This is just **1**

Thus,

$I(t) = \langle\Psi_0|d_x(t)d_x(0)|\Psi_0\rangle$

Equivalently,

$I(t) = \langle 0|d_x(t)d_x(0)|0\rangle$

In general, for any two operators A and C, we have

$I_{AC}(t_1 - t_2) = \langle A(t_2)|C(t_1)\rangle = \langle 0|A(t_2)C(t_1)|0\rangle$ ← This is the so-called "correlation function"

The T-ordered (chronological) Green's function is: $D_{AC}(t_1 - t_2) = (i/\hbar)\langle 0|TA(t_2)C(t_1)|0\rangle$.

Changing to the interaction representation, $\hat{A}(t_2) = S^{-1}(-\infty, t_2)\, A(t_2)S(-\infty, t_2)$

And $\hat{C}(t_2) = S^{-1}(-\infty, t_1)C(t_1)S(-\infty, t_1)$, **we come to**

$$D_{AC}(t_1 - t_2) = (i/\hbar)\langle 0|T\, S^{-1}(-\infty, t_2)\hat{A}(t_2)\, S(-\infty, t_2)S^{-1}(-\infty, t_1)\, \hat{C}(t_1)S(-\infty, t_1)|0\rangle$$
$$= (i/\hbar)\langle 0|S^{-1}[T\hat{A}\,(t_2)\hat{C}(t_1)]S|0\rangle$$

Here $S = S(-\infty, \infty)$. **At the same time,** $S(-\infty, \infty)|0\rangle = e^{i\alpha}|0\rangle$. **Then**

$\langle 0|S^{-1} = \langle 0|e^{-i\alpha} \rightarrow \langle 0|S(-\infty, \infty)|0\rangle = e^{i\alpha}$. **Then** $e^{-i\alpha} = \langle 0|S(-\infty, \infty)|0\rangle^{-1}$. **Therefore,**

$\langle 0|S^{-1} = \langle 0|S(-\infty, \infty)|0\rangle^{-1}\langle 0|$.

Thus,

$$D_{AC}(t_2 - t_1) = \frac{\langle 0|T\hat{A}(t_2)\hat{C}(t_1)S|0\rangle}{\langle 0|S|0\rangle}$$