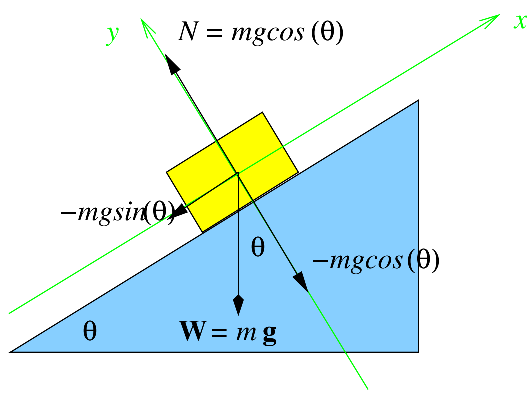
How To Do Physics

February 14, 2013

Let's start with the canonical Newton's 2nd Law problem:

1 Mass on an Inclined Plane

A block is sliding down a frictionless plane that is inclined an angle θ to the horizontal. Find the acceleration of the block.



<u>x-direction:</u>

$$\begin{aligned}\sum F_x &= ma_x \\ -mg\sin\theta &= ma_x\end{aligned}$$

$$a_x = -g\sin\theta \tag{1}$$

This acceleration is manifestly constant, so if we wanted the velocity of the block at some time t, we would use

$$\begin{aligned}v(t) &= v_0 + a_x t \\ &= v_0 - gt\sin\theta\end{aligned}$$

and if we wanted its position,

$$\begin{aligned} x(t) &= x_0 + v_0 t + \frac{1}{2} a_x t^2 \\ &= x_0 + v_0 t - \frac{1}{2} \left(g \sin \theta \right) t^2 \end{aligned}$$

However, let's think a bit more carefully (i.e., mathematically) about what just happened. First of all, Newton's 2nd Law reads

$$\frac{1}{m} F = \frac{d^2 x}{dt^2}$$

The fact that we write the second derivative of x of course implies that, in general, $x = x(t)$. Thus I can write Eq. 1 as

$$\frac{d^2 x}{dt^2} = -g \sin \theta \tag{2}$$

To find the velocity of the block, let's assume it starts from rest at the point shown in the diagram (where I put my origin), and has moved a distance $\Delta x = x - x_0$ after some amount of time t. Integrating Eq. 2 with respect to t gives

$$\begin{aligned} \int \frac{d^2 x}{dt^2} dt &= -\int g \sin \theta dt \\ v(t) \equiv \frac{dx}{dt} &= -gt \sin \theta + v_0 \end{aligned}$$

where v_0 is our constant of integration.

However, we are told that the initial velocity is zero. This means

$$v\left(t = 0s\right) = v_0 - \left(g \sin \theta \right) \left(0s \right) \equiv 0 \frac{m}{s}$$

which gives $v_0 = 0\frac{m}{s}$; we thus interpret this integration constant to be the initial velocity of the block.

And of course, a second integration gives

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_x t^2$$

Applying the condition that $x = 0m$ when $t = 0s$ gives

$$x(0) = x_0 - \frac{1}{2} \left(g \sin \theta \right) \left(0s \right)^2 \equiv 0m$$

This integration constant $x_0 = 0m$ is thus seen to be the initial position, and so our function $x(t)$, in this case, reads

$$x(t) = -\frac{1}{2} (g \sin \theta) \, t^2$$

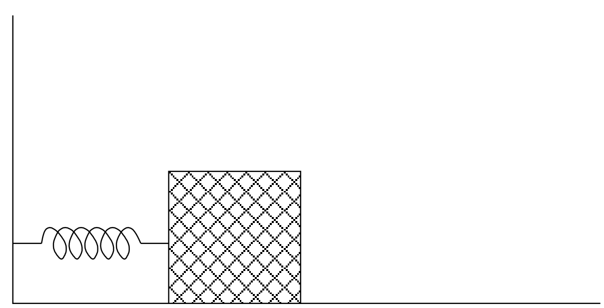
Note carefully that a different set of initial conditions would result in a different specific function.

1.1 Summary

What we have done, of course, is solve a second-order ODE (Eq. 2), got the general solution which has two adjustable parameters (v_0 and x_0), then obtained our specific solution (specific to the problem at hand) by requiring that the general solution so obtained match the physical requirements of the problem–i.e., we imposed our boundary conditions.

2 Mass on A Spring

Our next system is a block of mass m on a frictionless horizontal surface, attached to a horizontal spring that is in turn attached to a perfectly rigid wall, as shown in the diagram below:



This system will obey Hooke's law:

$$F(x) = -\kappa x \tag{3}$$

Now, from Newton's second law,

$$F = ma = m \frac{d^2 x}{dt^2} \tag{4}$$

Substituting this into Equation 3 and dividing both sides by m gives

$$\frac{d^2 x}{dt^2} = -\frac{\kappa}{m} x \tag{5}$$

This is an ordinary differential equation (ODE) for $x = x(t)$; its solution is

$$x(t) = A \sin\left(\sqrt{\frac{\kappa}{m}}t\right) + B \cos\left(\sqrt{\frac{\kappa}{m}}t\right) \tag{6}$$

or, defining $\omega \equiv \sqrt{\frac{\kappa}{m}}$,

$$x(t) = A \sin(\omega t) + B \cos(\omega t) \tag{7}$$

Applying initial/boundary conditions is the same here; we have two adjustable parameters (A and B) whose values are determined by applying specific initial position and initial velocity values. For example, let $x(0) = x_{max}$ and $v(0) = 0$; then

$$\begin{aligned} x(0) &= x_{max} = A \sin[\omega(0)] + B \cos[\omega(0)] \\ x_{max} &= B \end{aligned}$$

This leaves A still undetermined; for that, we use the velocity function:

$$\begin{aligned} v(t) &= \frac{d}{dt}x(t) \\ v(t) &= \omega A \cos(\omega t) - \omega B \sin(\omega t) \end{aligned}$$

Hence

$$\begin{aligned} v(0) &= 0 = \omega A \cos[\omega(0)] - \omega B \sin[\omega(0)] \\ 0 &= \omega A \\ A &= 0 \end{aligned}$$

and thus our specific solution is

$$x(t) = x_{max} \cos(\omega t)$$

3 The Wave Equation

Suppose you have a very long string held taut by a tension T (Fig. 1).

In equilibrium it coincides with the x-axis, but if you shake it (thus providing a restoring force), a **wave** $y(x,t)$ will propagate down the string. I will state without proof that the net transverse force on the piece of string shown is

$$F = T\frac{\partial^2 y}{\partial x^2}\Delta x$$

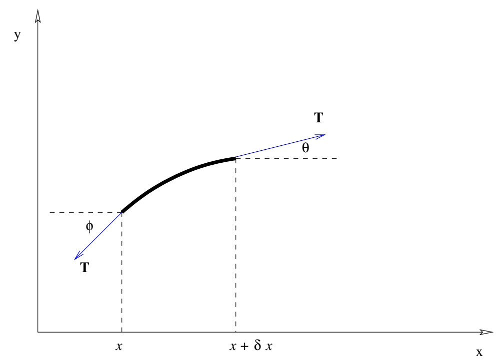


Figure 1: Tension on a vibrating string

If the mass per unit length is μ, Newton's second law gives

$$F = \mu\,(\Delta x)\,\frac{\partial^2 y}{\partial t^2}$$

Since F is equal to both of these, they must be equal to each other:

$$T\frac{\partial^2 y}{\partial x^2}\Delta x = \mu\,(\Delta x)\,\frac{\partial^2 y}{\partial t^2}$$

and therefore

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Notice that the quantity $\frac{\mu}{T}$ has units of $\frac{s^2}{m^2}$, which is the inverse of the units of velocity squared. Thus, we interpret its inverse, $\sqrt{\frac{T}{\mu}}$ to be the velocity of the wave as it moves down the string:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{8}$$

This equation is called the **wave equation**. It is a partial differential equation that describes the space and time behavior of the wave as it propagates down the string. It turns out that this same equation can be derived for *any* system that undergoes simple harmonic motion.

4 Fundamental Starting Point

A crucial idea in all of these examples is the role played by Newton's 2nd Law. It is the starting point of every problem, because we make it a *postulate* that the force, and the acceleration it causes, are related in this way; that the acceleration is linearly proportional to the force, inversely proportional to the mass, and that Newton's 2nd Law is a second-order ODE whose solution $x(t)$–in every case–has a *physical* interpretation (the position of the object).

It is also worth pointing out that the function $x(t)$ *predicts* where the particle will be (with, be it noted, 100% certainty); but to have confidence in the validity of N2L, you have to *check* it–you have to actually go into the lab and make a measurement of the object's position.

4.1 A Reminder

Recall that a (conservative) force[1] is related to a potential energy function by

$$\vec{F} = -\left(\frac{\partial V}{\partial x}\hat{e}_x + \frac{\partial V}{\partial y}\hat{e}_y + \frac{\partial V}{\partial z}\hat{e}_z\right)$$

or simply

$$\vec{F} = -\nabla V$$

where $V = V(x, y, z)$. And of course, F and V can be expressed in cylindrical or spherical coordinates, if desired. Thus, Newton's 2nd Law reads

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \frac{1}{m}\vec{F} &= \vec{a} \\ -\frac{1}{m}\nabla V &= \frac{d^2\vec{r}}{dt^2} \end{aligned}$$

[1] And conservative forces are the only kind we deal with in quantum mechanics.