Hardy & Wright : $\sum \frac{1}{p}$

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The are a great many proofs that the sum of the reciprocals of the prime numbers is a divergent series. Here is an entirely elementary (but clever) one from Hardy and Wright's famous book on number theory.

We begin by defining a special function, $N(x, j)$, as the number of values of $n \leq x$, which can be constructed as a product using only the first j prime numbers, $\{2, 3, 5, \ldots, p_j\}$. Described alternatively, it is the number of values of n which are <u>not</u> divisible by any primes beyond the first j.

To make this definition clear, let us compute $N(30, 4)$. The first 4 primes are {2,3,5,7}. Using these, sometimes multiple times, we can construct this set of numbers up to 30: {1,2,3,4,5,6,7,8,9,10,12,14,15,16,18,20,21,24,25,27,28,30}. The other numbers, {11,13,17,19,22,23,26,29}, cannot be built without the use of primes beyond the first 4. Thus we find that $N(30, 4) = 22$, while $30 - N(30, 4) = 8$.

We need to obtain estimates on the sizes of $N(x, j)$ and of $x - N(x, j)$, via 2 different methods.

To begin, factor any natural number $n \leq x$ as a perfect square times a square-free number. For example,

$$123480 = 2^3 \times 3^2 \times 5 \times 7^3 = (2 \times 3 \times 7)^2 \times (2^1 \times 3^0 \times 5^1 \times 7^1)$$

In general, the part of such an n that is squared cannot exceed $\sqrt{x}$, and the number of possible values of the square-free number, given that each of the j primes will have an exponent of either 0 or 1, is at most 2^j. Combining these facts, we conclude that

$$N(x, j) \leq 2^j \sqrt{x}$$

Now we consider the numbers below x that weren't counted. They weren't counted because they were divisible by one of $\{p_{j+1}, p_{j+2}, \ldots\}$. The number divisible by p_{j+1} cannot exceed $\frac{x}{p_{j+1}}$. (Actually it is the greatest integer in this quotient.). Similarly for the rest of the larger primes. Summing over all primes beyond the j^{th} we conclude

$$x - N(x, j) < \sum_{k=j+1}^{\infty} \frac{x}{p_k}$$

Combining these two equations we find that

$$x < 2^j \sqrt{x} + \sum_{k=j+1}^{\infty} \frac{x}{p_k} = 2^j \sqrt{x} + x \sum_{k=j+1}^{\infty} \frac{1}{p_k}$$

Suppose now that $\sum \frac{1}{p}$ were convergent. There would then be some particular value of j for which the infinite tail end would sum to less than 1/2. For that j we could then conclude that

$$x < 2^j \sqrt{x} + \frac{x}{2}$$

for all x, and this can be rearranged (subtract $x/2$, multiply by 2, divide by $\sqrt{x}$, then square) to force $x \leq 2^{2j+2}$, which violates the fact that it should hold for <u>all</u> x. Thus the sum of the reciprocals of the primes must diverge.