The product of primes $\leq$ n

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1 The Theorem

It was known to Euclid that there are infinitely many primes, but the ancient proof gives no insight into their rarity. We will show here that they must be somewhat rare by placing an upper bound on the product of the primes less than or equal to n. In particular, we will prove that

$$\prod_{p \leq n} p < 4^n \tag{1}$$

This table shows the values of both sides of (1), for small values of n.

n	$p \leq n$	4^n	$\prod p$
2	2	16	2
3	$2, 3$	64	6
4	$2, 3$	256	6
5	$2, 3, 5$	1024	30
6	$2, 3, 5$	4096	30
7	$2, 3, 5, 7$	16384	210
8	$2, 3, 5, 7$	65536	210

Several things should be noted from this table. First, equation (1) is certainly true for these small values of n. Second, the left hand side of (1), i.e. the product of the primes, only grows when n reaches the next prime. Thus, for example, the last column will stay at 210 until n=11, and then it will grow to a value of $210 \times 11 = 2310$. This means, among other things, that the theorem only needs to be proved for odd values of n, because once it is true for an odd value, the right side of (1) for the next (even) number will grow even larger while the left side won't change at all.

The fact that the product grows in occasional large jumps says something about the rarity of primes. As the table is extended to larger and larger values of n, extremely large primes will be folded into the product, causing enormous jumps in the value of the product (i.e. the left side of (1)). The fact that it never catches up to the steadily increasing power of 4 indicates that these jumps rarely happen. It is a tortoise and hare phenomenon: the right side of (1), the tortoise, increases steadily, while the left side, the hare, takes an occasional huge jump, but mostly just snoozes. The hare never catches up.

2 The Proof

The proof begins with an unusual use of mathematical induction. Recalling that we only need to concern ourselves with odd values of n, we split n in "half", even plus odd separated by 1. To make the proof easier to digest, we will deal with a specific "large" odd n. Let's consider $n = 723$. Split $723 = 361 + 362$, and split the product in (1) in half:

$$\prod_{p \leq 723} p = \left(\prod_{p \leq 362} p \right) \left(\prod_{p > 362} p \right) \tag{2}$$

By induction, we can presume that the first product on the right of (2) is less than 4^{362}, an odd use of induction because we usually rely on the immediately previous integer, but clearly acceptable. Equation (2) becomes

$$\prod_{p \leq 723} p < 4^{362} \left(\prod_{p > 362} p \right) \tag{3}$$

We need a bound on the product of the primes in the upper half of our range of n, and get it by looking at binomial coefficients. We note that

$$\binom{723}{362} = \binom{723}{361} = \frac{723!}{362!361!} = \frac{363 \times 364 \times \cdots \times 723}{1 \times 2 \times \cdots \times 361}$$

This last fraction will, of course, reduce to an integer when the denominator is simplified away, but any prime in the numerator will be untouched by the process, because each is larger than any factor in the denominator, and cannot be cancelled by any combination of factors because it is prime. This means that the second factor in the right side of (3) is less than this binomial coefficient. This observation is the heart of the proof.

We now know that

$$\prod_{p \leq 723} p < 4^{362} \binom{723}{362} \tag{4}$$

Next notice that

$$\binom{723}{362} + \binom{723}{361} < \sum_{k=0}^{k=723} \binom{723}{k} = (1+1)^{723} = 2^{723}$$

using the binomial theorem. Because the two terms are equal, each is less than half of 2^{723}. (This is the only place where we exploit the oddness of n.) Thus each is less than $2^{722} = 4^{361}$. Putting this into equation (4) we have

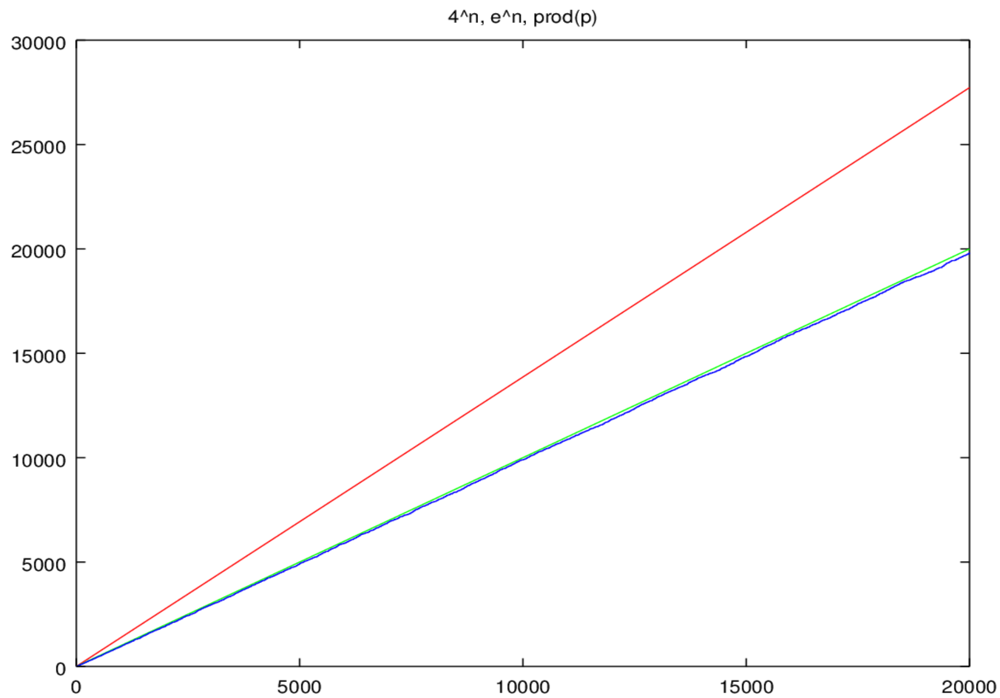
$$\prod_{p \leq 723} p < 4^{362} \times 4^{361} = 4^{723}$$

Our proof is complete.

3 A Proposed Extension (No Proof)

Now that we know that the product of the primes below n grows slower than 4^n it is natural to ask: just how rapidly **does** it grow?

Using software (Mathematica or Matlab) that has built-in functions involving primes, a numerical investigation of this question was made. Rather than looking at the product of the primes below n, we examined the logarithm of the product, which is the sum of the logarithm of the primes. This table shows a few values, and the accompanying graph shows the logarithmic behavior of 4^n, of e^n, and the product of the primes.



n	$\sum \log(p)$
10	5
100	83
1000	956
10000	9896
100000	99685
1000000	998484

Clearly we are seeing that $\sum \log p$ for the primes below n looks very much like n. This means that the product of the primes below n looks a lot like e^n. While we have no proof we strongly suspect that $\sum \log p \sim n$.

To make sense of this we turned to the Prime Number Theorem. This deep theorem, one of the landmark achievements of mathematics, states that the number of primes less than n is "asymptotic" to $n/\log(n)$, which means that their ratio approaches 1 for large values of n. Proving this result requires methods far outside the scope of this discussion, but we can use it to see why the product of primes assertion is reasonable.

Given the Prime Number Theorem we can calculate the density of primes near a given value of n by the derivative of $n/\log(n)$. This derivative is $\dfrac{\log(n)-1}{\log^2(n)} \approx \dfrac{1}{\log(n)}$. Now we can estimate the sum of the prime logarithms by adding the logarithm of *every* number, but weighted by the probability that it is prime, which is the density of primes at that value. The result is

$$\sum_{k=1}^{n} \log(k) * \frac{1}{\log(k)} = n$$

This is what we wanted.

Naturally we cannot claim that the above argument is a solid proof - it is really just proofiness. Nevertheless it seems like a reasonable heuristic justification for a numerically observed pattern. It remains an open challenge for someone to turn this idea, using the Prime Number Theorem, of course, into a true formal proof.