From Circuit Networks to Modern-Day Quantum Information Theory: Kirchhoff's Matrix Tree Theorem

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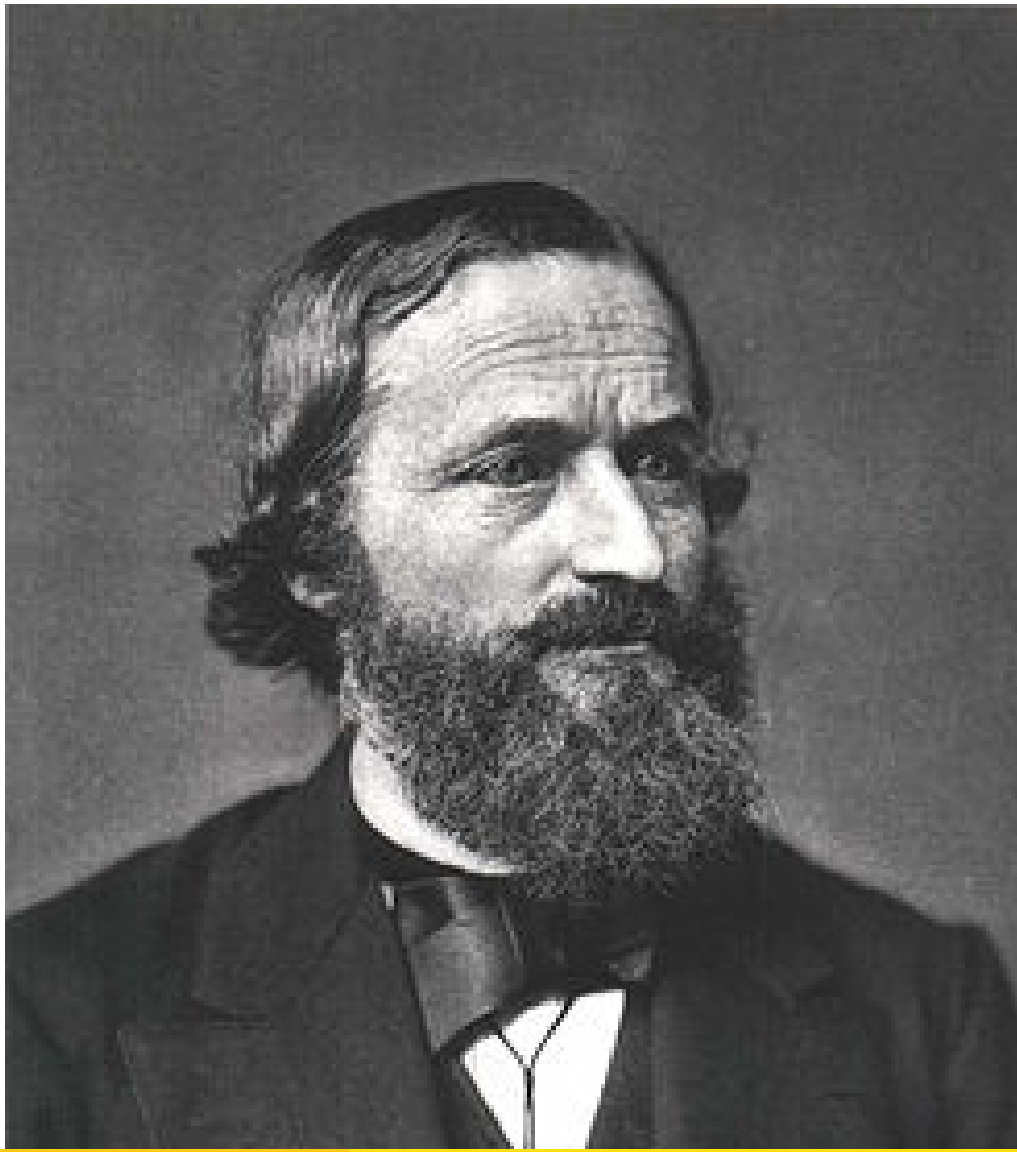
Introduction

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to objects of reality?" - Albert Einstein (1921)

History

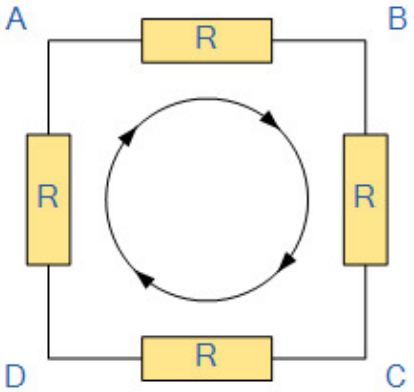
Gustav Kirchhoff

1. 1845 at 21 years old Kirchhoff published his first paper which included what would become known as Kirchhoff's Circuit Laws
2. 1847 publishes second paper which included Matrix Tree Theorem (MTT)
3. Debate whether Kirchhoff explicitly proved the theorem or implied MTT and proved a similar theorem



Kirchhoff's Circuit Laws

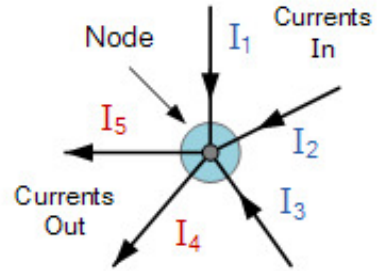
The sum of all the Voltage Drops around the loop is equal to Zero



$$V_{AB} + V_{BC} + V_{CD} + V_{DA} = 0$$

- **Conservation of Energy**

Currents Entering the Node Equals Currents Leaving the Node



$$I_1 + I_2 + I_3 + (-I_4 + -I_5) = 0$$

- **Conservation of Charge**

Definitions

Degree of a Vertex

Given a vertex $v \in \mathcal{V}(G)$, the degree of v is the number of edges $e \in \mathcal{E}(G)$ that are connected to v, denoted $deg(v_i)$.

Spanning Tree

An acyclic connected subgraph of G.

Degree Matrix

Let $\{v_1, v_2, v_3, ..., v_n\} \in \mathcal{V}(G)$, the *degree matrix* is defined to be the $\mathbf{n} \times \mathbf{n}$ matrix where $D(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} \deg(v_i) & \text{if } v_i = v_j \in \mathcal{V}(G) \\ 0 & \text{if } v_i \neq v_j. \end{cases}$$

Further Definitions

Adjacency Matrix

A graph G with vertex set $\mathcal{V}(G) = \{v_1, v_2, v_3, ..., v_n\}$ and edge set $\mathcal{E}(G) = \{e_1, e_2, e_3, ..., e_m\}$ can be described as a $\mathbf{n} \times \mathbf{n}$ adjacency matrix, denoted $A(G) = [a_{ij}]$, where

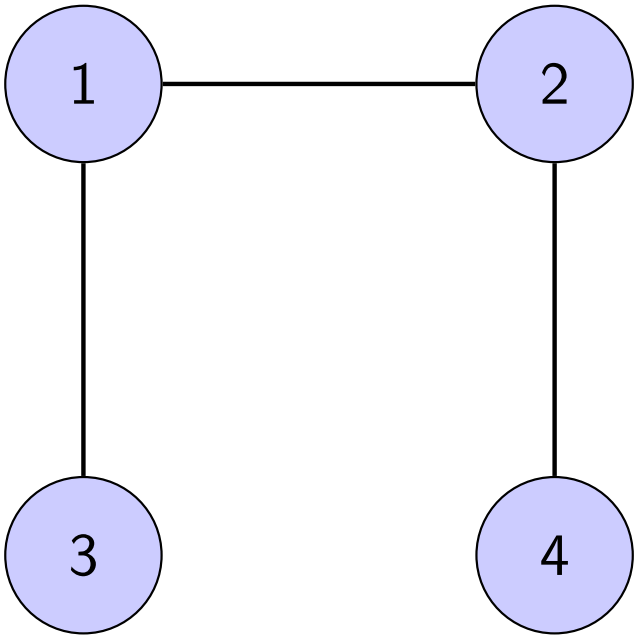
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in \mathcal{E}(G) \\ 0 & \text{if } v_i v_j \notin \mathcal{E}(G). \end{cases}$$

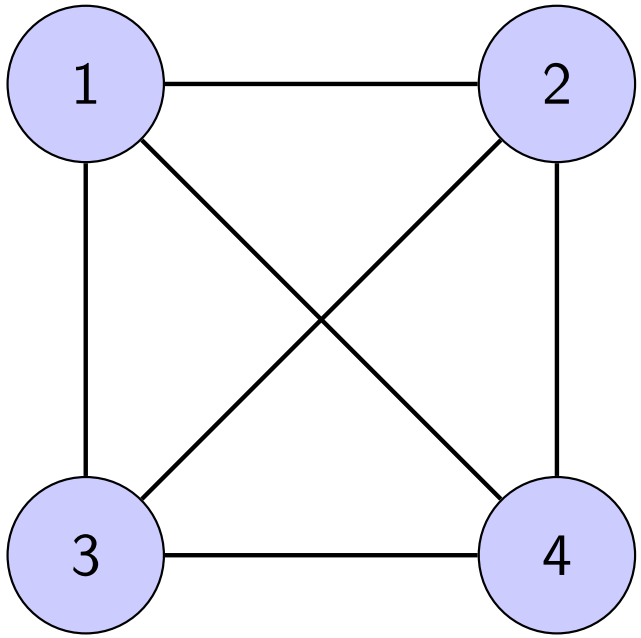
Incidence Matrix

Let G be a simple, connected graph with n vertices and m edges. The incidence matrix hereby denoted $\mathcal{I}$, is the $\mathbf{n} \times \mathbf{m}$ matrix $\mathcal{I} = [i_{ij}]$, where

$$i_{ij} = \begin{cases} 1 & \text{if } v_i \textit{ is incident to } e_j, \\ 0 & \text{if } v_i \textit{ is not incident to } e_j. \end{cases}$$

Spanning Tree Example





Methods From Matrix Algebra Used to Count Spanning Trees

Let G be a connected simple graph with vertex set $\{v_1, v_2, v_3, ..., v_n\} \in \mathcal{V}(G)$, and edge set $\{e_1, e_2, e_3, ..., e_m\} \in \mathcal{E}(G)$, where $\mathcal{L}$ is the $n \times n$ matrix found by subtracting the adjacency matrix from the degree matrix. Let $\mathcal{L}^*$ denote the $(n-1) \times (n-1)$ Laplacian matrix of G, with the i^{th} row and j^{th} column removed, then

$$det(\mathcal{L}^*) = \tau,$$

where τ is the number of spanning trees in a graph.

Kirchhoff's Matrix Tree Theorem Corollary

Let $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ be all the non-zero eigenvalues of the Laplacian matrix of a graph, $\mathcal{L}$. The distinct number of spanning trees in a graph, τ, can be computed as

$$\tau(G) = \frac{\lambda_1 * \lambda_2 * ... * \lambda_{n-1}}{n}$$

Lemma 1: Let A be an $n \times n$ matrix where the rows and columns sum to zero, and A^* be the matrix obtained by removing the last row and column of A. Then the coefficient of x in $det(A - xI)$ is equal to $-n * det(A^*)$.
Lemma 2: Let A be an $n \times n$ symmetric matrix. The product of the n eigenvalues of A is equal to the determinant of A.
Lemma 3: Let A be an $n \times n$ symmetric matrix. If the sum of each row of A equals s, then s is an eigenvalue of A.

Proof.

I will begin by observing that by **Lemma 3**, $\mathcal{L}$ has an eigenvalue of $\lambda_n = 0$ as the columns and rows sum to 0.

When finding

$$det(\mathcal{L} - xI) = (\lambda_1 - x)(\lambda_2 - x)...(\lambda_{n-1} - x)(\lambda_n - x).$$

$$= (\lambda_1 - x)(\lambda_2 - x)...(\lambda_{n-1} - x)(-x)$$

By **Lemma 1** we get the negative coefficient of $-x$ is equal to

$$n * det(\mathcal{L}^*) = \lambda_1\lambda_2...\lambda_{n-1}.$$

By dividing through we arrive at

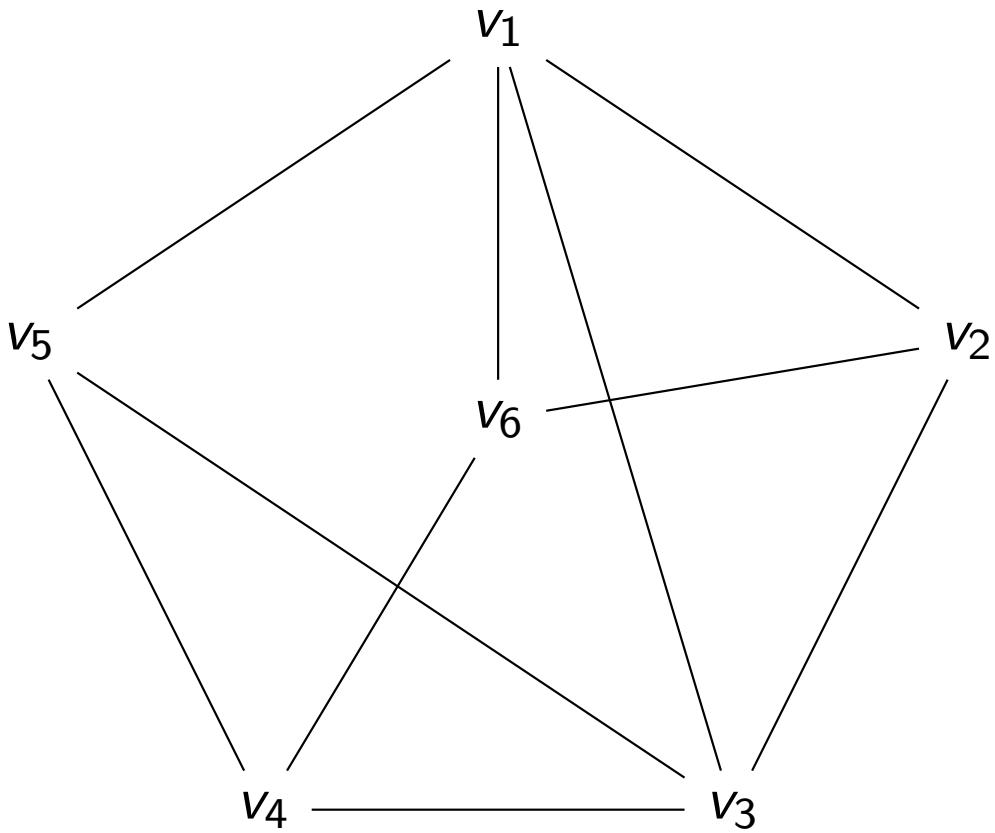
$$det(\mathcal{L}^*) = \frac{1}{n}\lambda_1\lambda_2...\lambda_{n-1}.$$

The Matrix Tree Theorem gives us $det(\mathcal{L}^*) = \tau$

$$\therefore \ \tau = \frac{\lambda_1 * \lambda_2 * ... * \lambda_{n-1}}{n}.$$



Base Case Graph



First, we must form the degree matrix and adjacency matrix.

The degree matrix would be constructed as:

$$\mathbf{D}_{6\times 6} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The adjacency matrix:

$$\mathbf{A}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Forming the Laplacian

We will define the Laplacian matrix,$\mathcal{L}$, by taking the difference of the degree matrix and the adjacency matrix.

$$\mathcal{L} = (\mathbf{D} - \mathbf{A}) = \begin{bmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & -1 & 0 & 3 \end{bmatrix}.$$

Please note that all the columns and rows sum to 0.

We will delete its 1^{st} row and 1^{st} column making this an $n-1 \times n-1$ matrix, and we will call this $\mathcal{L}^*$. Please note, we could have equivalently removed any of the six rows or columns

$$\mathcal{L}^* = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & -1 & 0 & 3 \end{bmatrix}.$$

Put simply, by finding the determinant of $\mathcal{L}^*$ will give us the distinct number of spanning trees of the graph. Using MATLAB, the value returned for the number of spanning trees was 130.

MATLAB Code:

```
>> Lstar = L(2:6,2:6);
>> det(Lstar)
ans =
130.0000
```

But, let's dig into *why* this works.

We can equivalently express $\mathcal{L}$ as the product of its incidence matrix and the incidence transpose. Denoted mathematically as

$$\mathcal{L} = \mathcal{I}\mathcal{I}^T.$$

The incidence matrix is given an arbitrary but fixed direction for each edge,

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The matrix $\mathcal{M}$ is the $(n-1) \times m$ matrix comprised of the columns of $\mathcal{I}$ with the first row removed,

$$\mathcal{M} = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Well, $\mathcal{L}^* = \mathcal{M}\mathcal{M}^T$, and we will look to the Cauchy-Binet formula to understand how we can find the the determinant of a non-square matrix.

Cauchy-Binet formula

Let A be a $n \times m$ matrix and B be an $m \times n$ matrix. Then,

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A(S)) \det(B(S)),$$

where $S = (s_1, s_2, ..., s_n), 1 \leq s_1 \leq s_2 < ... < s_n \leq m$, runs through all such multi-indices, $A(S)$ denotes the matrix formed from A using columns S (in that order), and $B(S)$ denotes the matrix formed using rows S of B in that order.
Also, an important corollary for our purpose of looking at Kirchhoff's Matrix Tree Theorem,

$$det(AA^T) = \sum_S (detA(S))^2.$$

In our case, we are finding $det(\mathcal{L}^*) = \sum_{S \in \binom{[m]}{n-1}} det(\mathcal{M}[S]\mathcal{M}[S]^T)$, and S ranges across the subset of $(n-1)$, where $(n-1)$ refers to the original graph with n vertices. Which brings us back to the definition of a spanning tree because to have a spanning tree on a graph with n vertices *requires* $(n-1)$ edges.

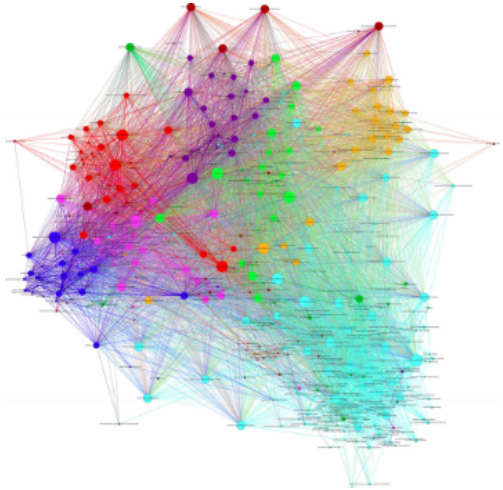
Further, it can be shown that a spanning tree is created by S if and only of the determinant is ± 1.

Using MATLAB, the eigenvalues were computed as

$$\lambda_1 = 0, \lambda_2 = 2.2679, \lambda_3 = 3, \lambda_4 = 4, \lambda_5 = 5, \lambda_6 = 5.7321.$$

```
>>eigenval = eig(L);
>> tau = 1;
for k = 2:6
tau = tau * eigenval(k);
end
tau/k
ans =
130.0000
```

- Mixed states in Quantum Mechanics
- Measuring the "complexity" of a network
- Study of spectral graph theory



- Feynman integrals

Application to Quantum Information Theory

"By reinterpreting the Kirchhoff's Matrix-Tree Theorem in the context of quantum information theory, we give an exact formula to count spanning trees based on the notion of quantum relative entropy...We show that the number of spanning trees is proportional to the distinguishability/distance between a certain density matrix associated to the graph in context and the maximally mixed state, i.e., the state with maximum von Neumann entropy, or, equivalently, maximum amount of classical uncertainty." (Giovannetti, Severini 1)

Conclusion

- Although Kirchhoff may not have been seeking an efficient way to count spanning trees, his findings can be applied to various network type problems in many different branches of mathematics.
- Only a grasp of introductory linear algebra skills are needed to apply Kirchhoff's Matrix Tree theorem.
- Kirchhoff's Matrix Tree theorem allows you to quantify the *complexity* of a network when case inspection would prove difficult to impossible.
- Spanning trees and Kirchhoff's Matrix Tree Theorem have interesting applications in quantum information theory.

Thank You

A big thank you to all my past professors–you have undoubtedly played an integral role (pun intended) in my mathematical development. A special thank you to my professors Jen Townsend and Larry Susanka for being such great resources of insight and advice leading up to this presentation.

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