Historical Approaches to the Existence and Nature of Abstract Objects in Mathematics

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What Will We Do Today?

Today we will discuss and, possibly, argue about a number of philosophical positions on abstracta held by various logicians, philosophers and mathematicians from the time of the ancient Greeks to today.

I will show a slide or two, set up an idea, and then we will chat and/or argue about it.

At a certain point we will run out of slides or time or patience.

Be aware that virtually every slide in this talk is the subject of entire university courses and practically every word carries nuances explored by philosophers with details quite beyond the person giving this talk.

Our goal today is to describe a millennia-long discussion, not answer major questions.

Generalities

"Meaning" in all human languages is founded in sensation or belief. If you look up the definition of a word in a dictionary it is given in terms of other words.

Meaning of some of these words—ultimate root words—is determined by introspection or pointing out the window.

When I believe I understand the utterances of another I am relying on my repeated and varied experiences with the words and past interactions with humans (LLM anyone?) and a belief that what is going on inside the head of "other" is closely related to the mental state of "me."

We cannot prove facts about our ability to recognize and manipulate finitude in the form of finite lists and sets.

We know what we mean by finite but what is it exactly? This ability forms part of our cognitive apparatus and, perhaps, a big part of the substance of cognition itself.

Though there are several precise definitions of the word "finite" in mathematics these have meaning only due to this aspect of our thought processes.

Logicians refer to various aspects of our thinking process that give meaning to mathematics as metamathematics. There MUST BE metamathematical aspects to ANY mathematical argument.

The Ancients

The word "ontology" refers to the objects presumed to exist in a subject area. "Epistemology" on the other hand describes the means by which subject-experts acquire knowledge about these objects.

Plato (circa 427-347 BCE) believed in the literal existence of abstract objects, universals, in a separate world of forms.

This is called Platonism.

Aristotle (384-322 BCE) believed that abstract objects are real things but must be tied to objects in this world, with aspects apprehended by humans through sensory experience. So an abstraction such as "roundness" exists wherever it is instantiated within a round object; the same abstraction is present in every round object.

This is called Aristotelian realism.

Victor and I Argue About This

These ancient issues continue to cause heated disputation to this day, even in this very room.

Consider the group consisting of the real numbers $(\mathbb{R}, +)$ with addition and the group of positive real numbers $(\mathbb{R}^{\text{pos}}, \cdot)$ with multiplication.

The identity in $(\mathbb{R}, +)$ is 0 and the inverse for x is $-x$.

The identity in $(\mathbb{R}^{\text{pos}}, \cdot)$ is 1 and the inverse for x is $1/x$.

The function $f\colon \mathbb{R} \to \mathbb{R}^{\text{pos}}$ defined by $f(x) = e^x$ satisfies $f(0) = 1$ and $f(-x) = 1/f(x)$ and generally $f(x + y) = f(x) \cdot f(y)$. This function is one-to-one, onto and is a group isomorphism.

Victor claims these are simply two instances of the same group, which exists independent of any particular instantiation. I claim they are different but isomorphic. Shall we have a vote?

Euclid of Alexandria circa 300 BCE reputedly assembled and organized and improved the work of previous mathematicians in The Elements. Earlier mathematicians who probably contributed included Pythagoras circa 570-495 BCE, Hippocrates of Chios 470-410 BCE and Eudoxus of Cnidus circa 408-355 BCE. It is the most successful textbook ever written, having been used continuously in one form or another for over 2000 years as the primary text for mathematical instruction in Europe and the Islamic countries.

Euclid's system posited an ontology consisting of points, line segments, circles and other geometrical objects assumed to exist in some sense and an epistemology based on deductions founded on Aristotelian logic based on the syllogism and constructions based on axioms or "postulates."

The Phrasing of Axioms in The Elements

Below we have a translation of a couple of the definitions about numbers from Book VII of Euclid's Elements with a few comments in parentheses.

- A unit is that by virtue of which each of the things that exist is called one. (So a unit, or monad, must first be selected among the things "that exist" to be used in subsequent definitions.)
- A number is a multitude composed of units. (The monad itself is technically not a number. Euclid's numbers start with the dyad, 2.)

Locke, Hume, Kant

Enlightenment philosophers such as John Locke (1632-1704) and David Hume (1711-1776) discarded the notion of the reality of abstract objects entirely. Locke proposed that humans began life as a tabula rasa (blank slate) and all subsequent thought was determined by experience and patterns of experience, a stance that brings him among the empiricist philosophers.

Hume also felt that our capacity for understanding and reason is based on perceived patterns in personal experience alone.

Hume did believe that logical deductions from assumed axioms were valid, presuming the axioms to be non-contradictory. But an axiom bound to a pattern of sensory impressions in the world could not be seen as surely true. Statements about the world derived by logical maneuvers on such axioms must remain uncertain.

If it is about the world, and you didn't literally see it, there must be doubt. This brought him to reject the legitimacy of causal and inductive reasoning, putting him squarely among the most skeptical philosophers.

Immanuel Kant (1724-1804), a German philosopher and pillar of the European Enlightenment, rejected the "tabula rasa" viewpoint of Locke. And while he credited Hume with waking him from his "dogmatic slumber" he built a philosophical system in which our knowledge about time, Euclidean geometry and number is innate and central.

Kant called innate knowledge synthetic a priori, both independent of experience and contingent upon the nature of the world.

	Analytic: truth determined by definitions alone	**Synthetic**: truth not determined by definitions alone
a priori: can be seen to be true by reason alone	all analytic true statements are a priori true	truth not determined by definitions alone but possibly by definitions plus features of the reasoning mechanism
a posteriori: truth contingent on experience and observation	Most philosophers find these conditions to be contradictory	all a posteriori true statements are synthetic true

Table 1 Combinations of some properties of statements: how they can be true, and how they may become known to be true.

Example

One such concept is "substance." This is the idea of a permanence, an underlying substratum. Experiences are not a chaotic sequence of unrelated sensations. We perceive changes, but changes in what? Substance is the stable thing, against which changes are measured and interpreted.

This idea is not contained in the raw sensations themselves. We contribute that idea, by our nature.

Example

There are other "truths" Kant classified as synthetic a priori, such as the principle of causality: that every experience has a cause.

He claimed that a Humean observer could never acquire by sensory impressions alone the concept of causality. An awareness of regular association, yes, but the idea of causality comes from our nature as beings with agency. We act, things happen. When we see a billiard ball strike another and the second moves we ascribe that motion to the action of the first.

Nowhere in the pure sensations of the event is the idea of causality hidden. That comes, according to Kant, from our cognitive apparatus: synthetic but a priori.

Example

Human thought itself depends on various synthetic a priori features of our understanding; they reflect the nature of our thought processes. Time, for example, is built into the very nature of thought.

We "arrive" at an understanding. We "come to understand" an idea. We "follow" an argument. Without time there is no arriving. There is no "cause and effect." And without change over time there can be no awareness of substance.

Synthetic a priori knowledge forms the boundary, even the elements, of cognition. We simply cannot conceive of anything without it. Though there may be objects in the world for which all synthetic a priori notions are irrelevant, they could not be important to us: humans can never think about or know of them, according to Kant.

Time, geometry and number, including arithmetic, are on his list of synthetic a priori knowledge types. Logical deductions based on these could be secure.

Kant was enormously influential. In the century after the second edition of his Critique of Pure Reason was published in 1787 philosophers, both proponents and opponents, had to contend with his ideas. To this day he is regarded as among the most important philosophers.

The Kantian Diaspora

Mathematicians interested in foundational matters have to address the issues raised by Plato, Aristotle and the Enlightenment philosophers.

When you prove a mathematical fact what is it you have actually done? What is the ontological status of the things you are talking about? In what sense is a group or a topological space a real thing?

We describe next a few of the positions that developed in response to this question.

Cantorian Platonists

Georg Cantor and Kurt Gödel and many sympathizers.

These folks took the position that infinite sets are real but abstract objects in a Platonic sense. Following Georg Cantor's revelations (1874-1884) about the structure and properties of infinite sets many people found his work to be so compelling that they felt that these entities must be literally real.

There was a quasi-religious (in Gödel's case not so "quasi") cast to much of the writing about this at the time.

Nominalism

This is a position that contrasts with Platonists and Aristotelian realists of various types. It takes the stance that mathematical abstract nouns are useful fictions, incredibly helpful in organizing our thinking but not literally real, similar to a fictional characters in a novel versus an actual human.

A persistent element in critique of Platonism is the epistemological issue of how humans can gain access to information about these slippery fish, "things" that exist out of space and time and which are not directly causally connected to something one can touch.

The tension between types of realists and types of nominalists is a recurring theme.

The Logicists

Richard Dedekind, Bertrand Russell and Gottlob Frege

Natural numbers and their arithmetic, and ultimately most of mathematics, can be reduced entirely to a few logical principals.

Russell and Whitehead's Principia Mathematica in three volumes published from 1910-1913 was the logicist bible.

Formalists

David Hilbert, Paul Bernays and other sympathizers including John von Neuman.

They took the position that mathematics was nothing more than the manipulation of symbols according to rules and did not produce objective truth about objects in some abstract universe apart from the human mind.

They distinguished between the syntax of symbol manipulation and the semantics of meaning and interpretation. A symbolic framework does not need to correspond to "truth" in some larger sense to be practical and useful: only consistent.

Practices of modern day model theorists with their distinction between syntax and semantics descend from this tradition but depart from that in considering their "intended models" to be real in some sense.

Intuitionists

After 1907: L. E. J. Brouwer, Arent Heyting, Hermann Weyl
Before 1907: Henri Poincaré, Leopold Kronecker

Mathematics is a purely mental process and nothing mathematical is independent of the mental processes of the practitioner.

But practicing intuitionists (particularly after 1907) take this further and reject the law of the excluded middle and the whole idea of "completed" infinite objects. Many versions forbid all impredicative definitions.

This leads to constructive mathematics: the idea that unless you can produce an explicit algorithm to build the object under study to completion you are really not talking about anything at all.

To them most of mathematics is an elaborate hallucination.

Suppose $A \cup B = \{2\}$. Is "$2 \in A$ or $2 \in B$" a tautology?

Classic: Yes. Intuitionistic: No.

Impredicative definition: A set is defined by a condition that references a whole of which it is a part.

"the set of members of the NY Knicks team who are not the three tallest members"

No problem.

"the set of natural numbers that cannot be explicitly identified using less than 1000 words of ordinary English"

Contradictory.

Realists (Other than Platonists)

Many varieties. For instance

Naturalists, Aristotelian Realists, like my favorite living philosopher Penelope Maddy. For Maddy, mathematics is about the patterns inherent in the physical world. I recommend her A Plea for Natural Philosophy and Other Essays.

Structural realists like Stewart Shapiro who make the case that mathematics is about structural relations between objects in the world, and not about the objects themselves.

... And Then There Is Tarski

Alfred Tarski (1901-1983) in three papers The Concept of Truth in Formalized Languages (1933) and The Establishment of Scientific Semantics (1936) and On the Concept of Logical Consequence (1936) revolutionized the concepts of truth and consequence in mathematics and logic generally, and essentially invented model theory in its modern form.

These papers and much of his other work from the period 1923-1938 are assembled in the book Logic, Semantics, Metamathematics.

Tarski was notoriously cagey in his philosophical beliefs, describing himself simply as a working mathematician and logician.

However the practice of model theory inherently assumes a semantic metatheory in which the axioms acquire meaning and are not just strings of symbols, and an intended model, the thing in the world a model is intended to emulate.

The way model theorists talk about their models seems to entail a Platonist ontology, but perhaps Tarski himself was too busy doing mathematics to worry about that.