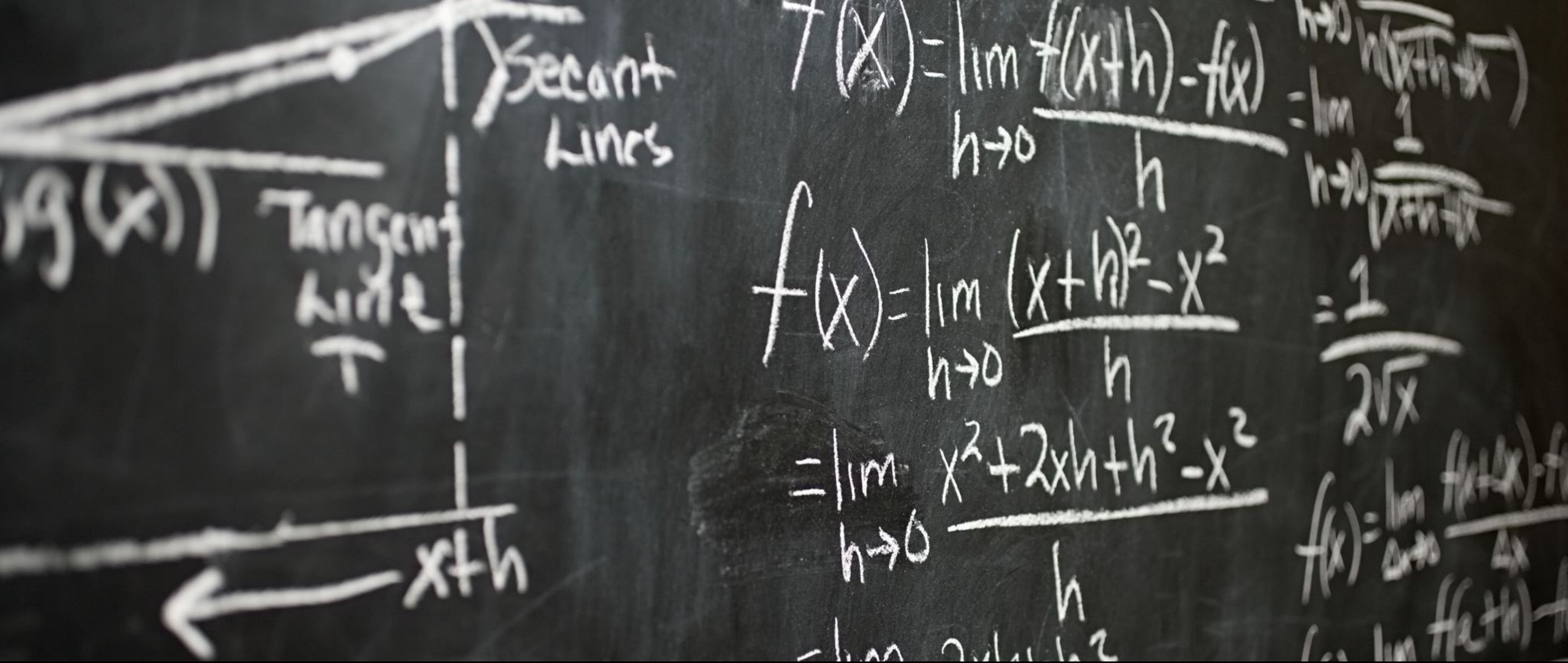

Advanced Integration Techniques Not Normally Taught In Calculus

By Liam Anderson

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$$e^{ix}$$

A Quick Proof for e^{ix} Using Series

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{(i\theta)} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \mp \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \mp \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

You can quickly Prove Euler's Formula by expanding e^{x} as a series and inputting $x = i\theta$, an important note is after simplifying group all the imaginary terms and factor out the i.

Properties of $e^{ix} = \cos x + i \sin x$

Remembering the multi angle formulas:

$$e^{i2x} = \cos 2x + i \sin 2x$$

$$(e^{ix})^2 = \cos 2x + i \sin 2x$$

$$(e^{ix})^2 = (\cos x + i \sin x)^2$$

$$e^{i2x} = cos^2 x + i\, 2\sin x \cos x - sin^2 x$$

$$e^{i2x} = (cos^2 x - sin^2 x) + i\,(2\sin x \cos x)$$

$$(cos^2 x - sin^2 x) + i\,(2\sin x \cos x) = \cos 2x + i \sin 2x$$

$$\boxed{Re\left[e^{i2x}\right] = cos^2 x - sin^2 x = \cos 2x}$$

$$\boxed{Im\left[e^{i2x}\right] = 2 \sin x \cos x = \sin 2x}$$

1

1 1

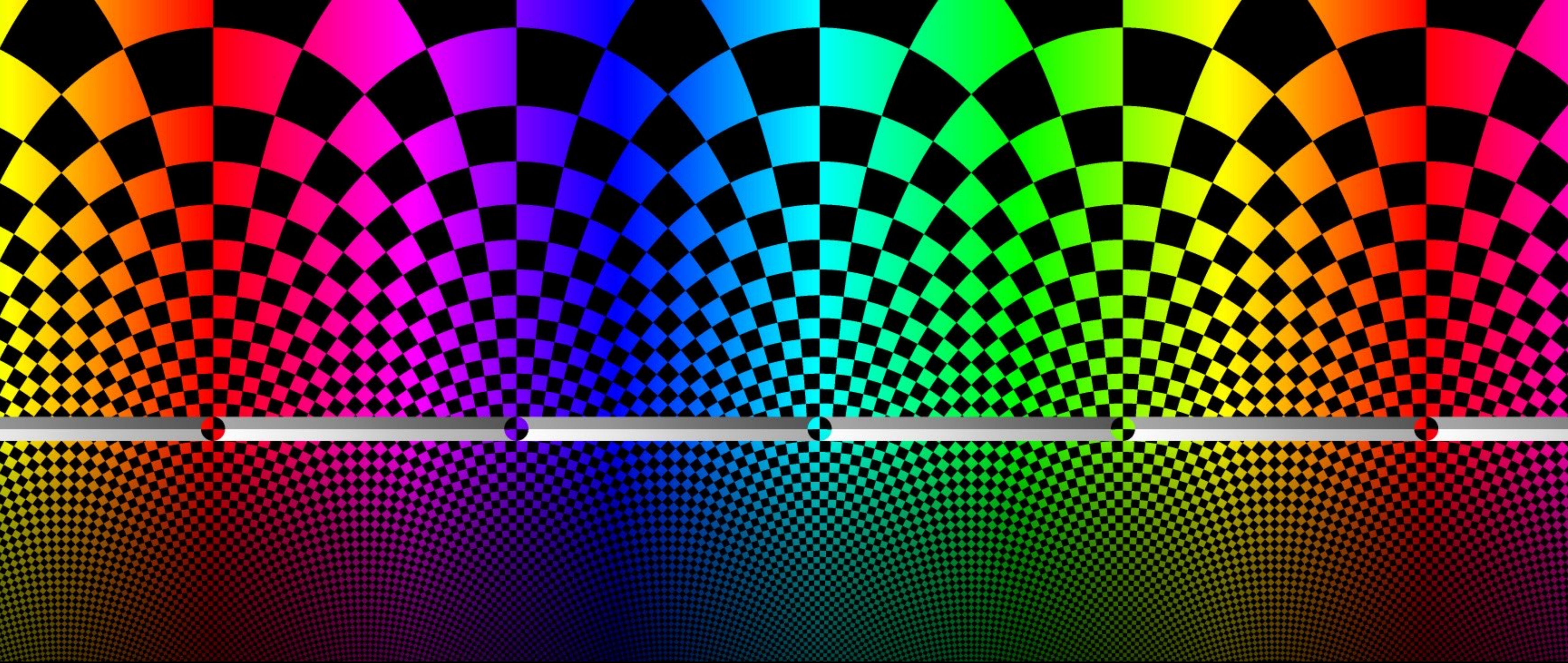
1 2 1

There are many different uses of Euler's formula, one of which can be argued to be included in Pre-calculus with multi-angle formulas.

We use the properties of exponents to factor the 2 as Euler's formula squared. We then use "Not Pascals Triangle" to determine the desired Coefficients from expanding terms.

After doing the algebra we come to one of the first important things that I briefly highlighted in the previous slide, with imaginary numbers we can as I like to think of it "Call Upon" the different components, either the real or imaginary components.

Here we see that the real part of this complex exponential is the double angle form of cosine, the imaginary part on the other hand is the double angle formula of sine.



Complexifying The Integral

Complexifying The Integral

$$\int e^{-x} \cos 5x \, dx$$

$$Re\left[\int e^{-x}\left\{e^{i5x}\right\} dx\right]$$

$$Re\left[\int e^{-x+i5x} \, dx\right]$$

$$Re\left[\int e^{x(-1+i5)} \, dx\right]$$

$$Re\left[\frac{1}{-1+i5} e^{x(-1+i5)} + C\right]$$

$$Re\left[\frac{-1-i5}{26} e^{x(-1+i5)} + C\right]$$

$$Re\left[\frac{-1-i5}{26} e^{-x} e^{i5x} + C\right]$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$\frac{1}{-1+i5}\left(\frac{-1-i5}{-1-i5}\right) = \frac{-1-i5}{26}$$

Using both the Idea of "Calling Upon" the Real or Imaginary parts. As well as Euler's Formula we can drastically simplify some integrals where we would have to use I.B.P, sometimes even twice, and turn them simply into a single exponential integral.

A very important and often overlooked step is to simplify the complex fraction from integration, which can really be problematic solving later.

Complexifying The Integral Cont.

We then want to break up the exponential into its sines and cosines using Euler's Formula so we can call upon our real or imaginary term we are looking for.

$$Re\left[\frac{-1-i5}{26}e^{-x}e^{i5x}+C\right]$$

$$Re\left[\frac{-1-i5}{26}e^{-x}(\cos 5x+i\sin 5x)+C\right]$$

$$Re\left[\frac{-e^{-x}-i5e^{-x}}{26}(\cos 5x+i\sin 5x)+C\right]$$

$$Re\left[\frac{-e^{-x}\cos 5x-ie^{-x}\sin 5x}{26}+\frac{-i5e^{-x}\cos(5x)+5e^{-x}\sin 5x}{26}+C\right]$$

Complexifying the Integral Cont. Cont.

$$Re\left[\frac{-e^{-x}\cos 5x - ie^{-x}\sin 5x}{26} + \frac{-i5e^{-x}\cos 5x + 5e^{-x}\sin 5x}{26} + C\right]$$

$$Re\left[\frac{-e^{-x}\cos 5x + 5e^{-x}\sin 5x}{26} + i\,\frac{-e^{-x}\sin 5x - 5e^{-x}\cos 5x}{26} + C\right]$$

$$\boxed{\frac{-e^{-x}\cos 5x + 5e^{-x}\sin 5x}{26} + \mathrm{C} = \int e^{-x}\cos 5x\,dx}$$

A very important and crucial step is to separate the real and imaginary terms because we are only looking for the real term, the constant of integration can be used for both real and imaginary component answers (can you possibly think of why?)

We then after taking only the real part of this answer arrive at the answer to this integral.

Using some intuition of how complex numbers and taking real and imaginary terms work can you also tell me from the work shown the answer for the integral $\int e^{-x}\sin 5x\,dx$?

Introduction to Integral Transforms (Laplace Transform)

The Laplace Transform

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

$$\int_0^{\infty} e^{-5x} \cos 2x \, dx$$

$$\int_0^{\infty} e^{-3x} \ln^2 x \, \mathrm{dx}$$

The Laplace Transform which is often found in Diff EQ classes can also be used to solve integrals, mainly by looking at the integrals form and looking at the table.

LAPLACE TRANSFORMS 183

Table of Special Laplace Transforms

	$f(s)$	$F(t)$
33.25.	$\frac{1}{s}$	1
33.26.	$\frac{1}{s^2}$	t
33.27.	$\frac{1}{s^n} \quad n = 1,2,3,\ldots$	$\frac{t^{n-1}}{(n-1)!}, \quad 0! = 1$
33.28.	$\frac{1}{s^n} \quad n > 0$	$\frac{t^{n-1}}{\Gamma(n)}$
33.29.	$\frac{1}{s-a}$	e^{at}
33.30.	$\frac{1}{(s-a)^n} \quad n = 1,2,3,\ldots$	$\frac{t^{n-1}e^{at}}{(n-1)!}, \quad 0! = 1$
33.31.	$\frac{1}{(s-a)^n} \quad n > 0$	$\frac{t^{n-1}e^{at}}{\Gamma(n)}$
33.32.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
33.33.	$\frac{s}{s^2+a^2}$	$\cos at$
33.34.	$\frac{1}{(s-b)^2+a^2}$	$\frac{e^{bt}\sin at}{a}$
33.35.	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
33.36.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
33.37.	$\frac{s}{s^2-a^2}$	$\cosh at$
33.38.	$\frac{1}{(s-b)^2-a^2}$	$\frac{e^{bt}\sinh at}{a}$

A list of Laplace Transforms from *SCHAUM'S® outlines Mathematical Handbook of Formulas and Tables by Murray R. Spiegel, PhD, Seymour Lipschutz, PhD and John Liu, PhD*

The Laplace Transform Examples

$$\int_0^\infty e^{-5x} \cos 2x \, dx \qquad \begin{matrix} s = 5 \\ a = 2 \end{matrix}$$

33.33.	$\frac{s}{s^2 + a^2}$	$\cos at$

Directly From the Table of Laplace Transforms
from *SCHAUM'S® outlines*

$$\frac{s}{s^2 + a^2} = \int_0^\infty e^{-5x} \cos 2x \, dx \,, s = 5, a = 2$$

Using the Laplace Transform to solve integrals is very simple, Identify the form, look at the table and plug in the numbers.

$$\frac{5}{5^2 + \ 2^2} = \int_0^\infty e^{-5x} \cos 2x \, dx$$

$$\boxed{\frac{5}{29} = \int_0^\infty e^{-5x} \cos 2x \, dx}$$

The Laplace Transform Examples Cont.

$$\int_0^\infty e^{-3x}\ln^2 x\,\mathrm{dx} \qquad s = 3$$

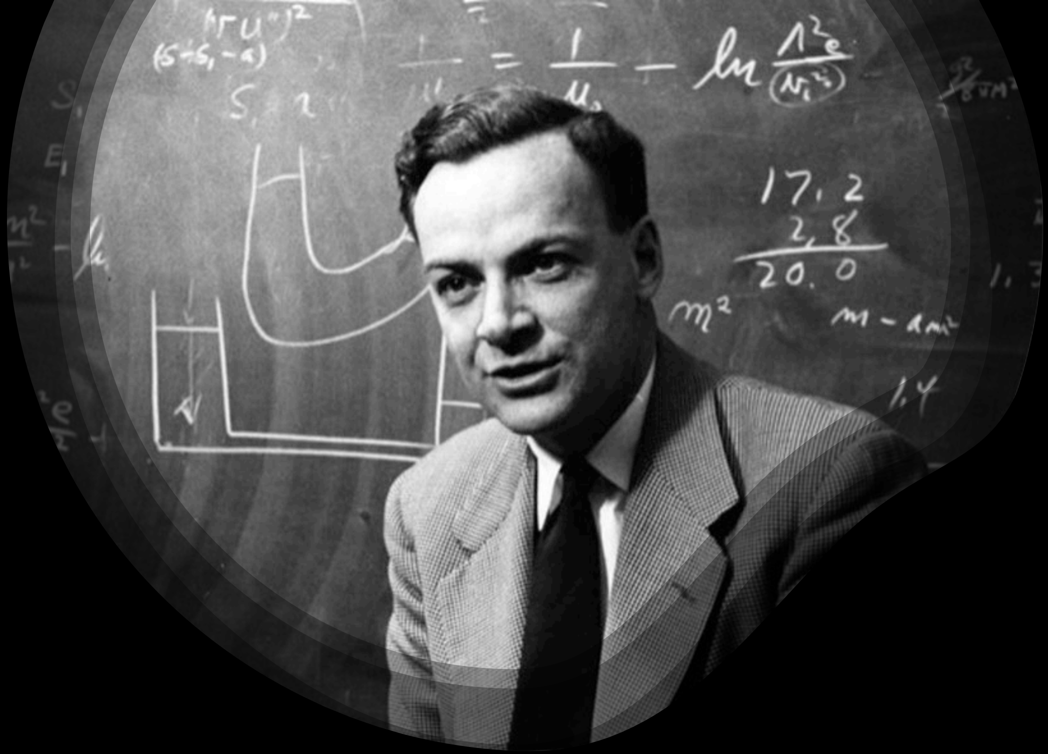
33.119.	$\dfrac{\pi^2}{6s} + \dfrac{(\gamma + \ln s)^2}{s}$ γ = Euler's constant = .5772156 …	$\ln^2 t$

$$\frac{\pi^2}{6s} + \frac{(\gamma + \ln s)^2}{s} = \int_0^\infty e^{-3x}\ln^2 x\,\mathrm{dx}\,, \mathrm{s} = 3$$

$$\boxed{\frac{\pi^2}{18} + \frac{(\gamma + \ln 3)^2}{3} = \int_0^\infty e^{-3x}\ln^2 x\,\mathrm{dx}}$$

Directly From the Table of Laplace Transforms from *SCHAUM'S® outlines*

Doesn't matter how scary the integral looks, if its in the form of a Laplace transform, identify the form, plug in and simplify.



The Leibniz Integral Rule (*Surely You're Joking, Mr. Anderson!*)

Foundation of Multivariable Functions and Double Integration (Featuring Desmos)

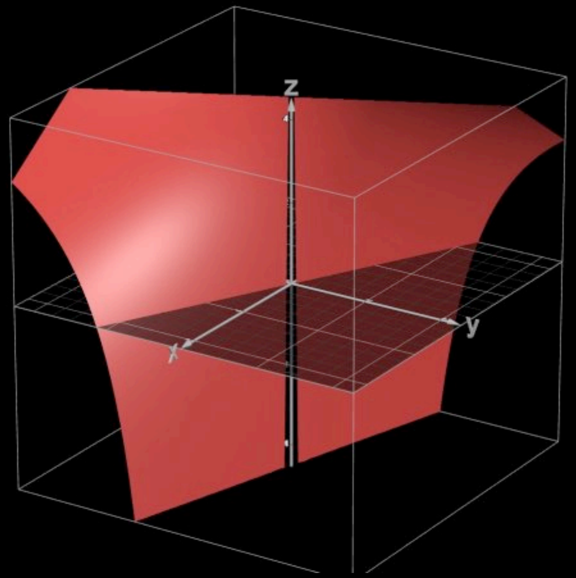
$$z = f(x,y) = 3x + 6$$

$$z = g(x,y) = \frac{3x}{y} + 6$$

$$z = g(x,1) = \frac{3x}{(1)} + 6 = 3x + 6$$

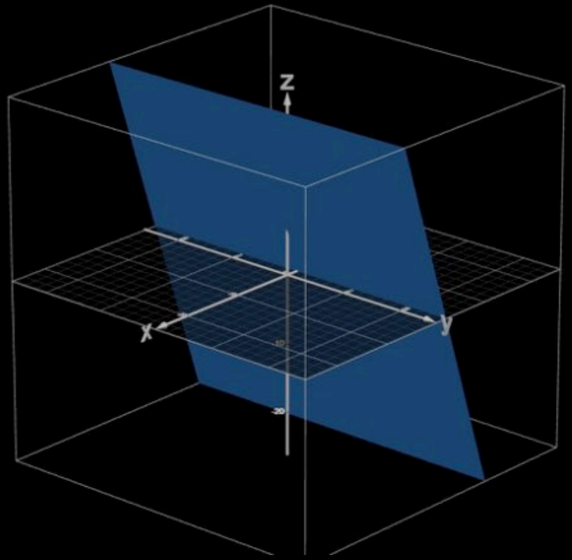
$$\boxed{f(x,y) = g(x,1)}$$

Even though it might not seem important or might feel like review, understanding multivariable functions is the key to the Leibnitz Rule.



$$g(x,y) = \frac{3x}{y} + 6$$

As a function in 3d Space

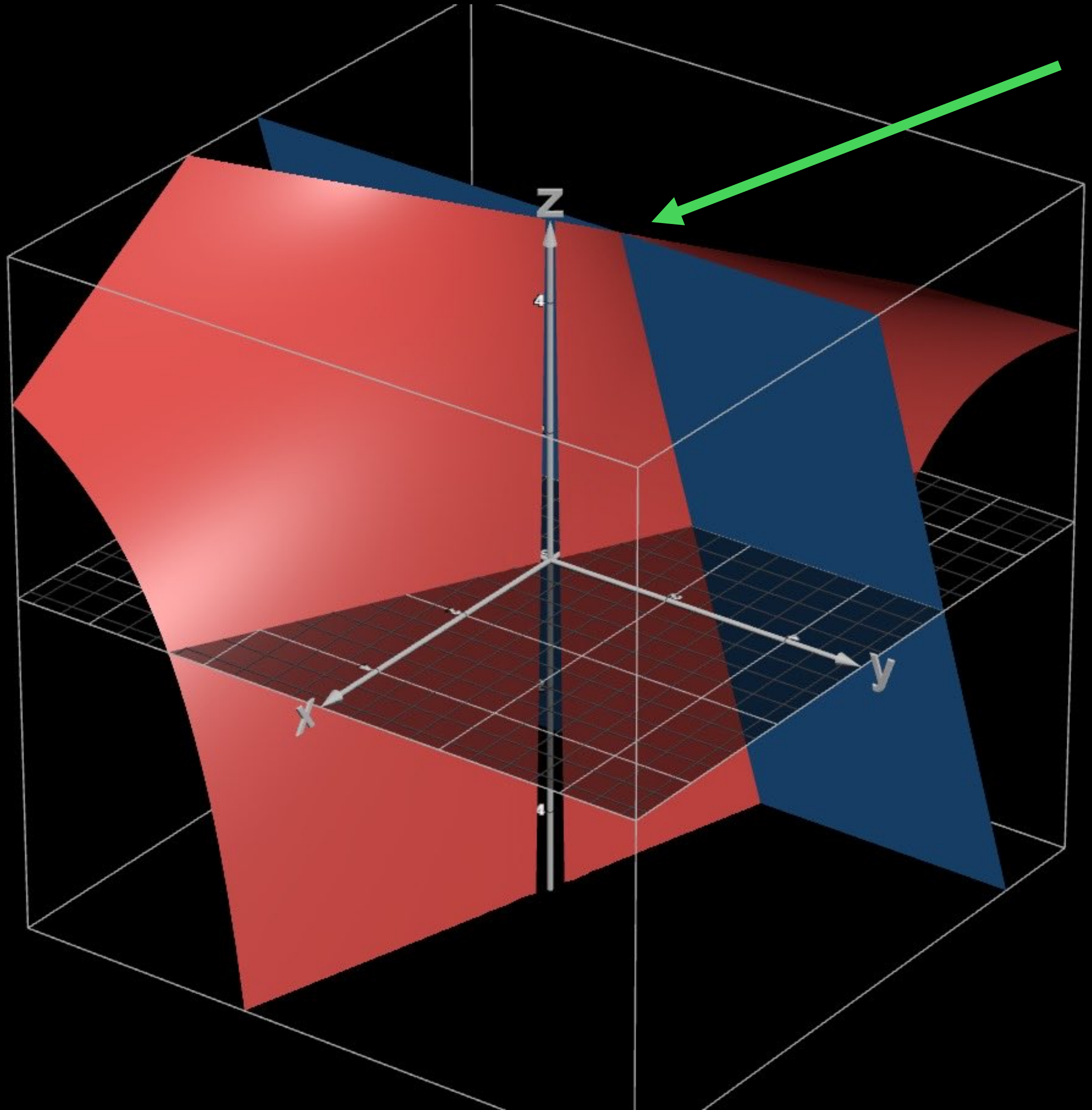


$$f(x,y) = 3x + 6$$

The function doesn't depend on y so each coordinate just maps out a plane

By looking at the multivariable functions we find a case of similarity and find an equality for a specific value of y. Which is an important piece of knowledge to using L.I.R correctly, The name of the game here is patterns!

Foundations of Multivariable Functions cont.



Both $f(x, y)$ and $g(x, y)$ mapped in 3D space together Note the intersection line!

Foundation of Double Integration

$$\int_0^2 2x\,dx = 4 = I$$

$$\int_0^2 \int_{-x}^{x} 1\,dy\,dx$$

$$\int_0^2 \left[\int_{-x}^{x} 1dy\right] dx \qquad \int_{-x}^{x} dy = x - (-x) = 2x$$

$$\int_0^2 2x\,dx = I$$

$$\boxed{\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx = I}$$

Double Integration is also another important thing for understanding and using L.I.R

Double Integration looks and seems complicated, but you use the same rules as partial differentiation but reversed! And work inside out.

How I view double integration is that with each integral you “collapse” a variable. two variables collapses to one variable then collapses to a value I.

Foundation of Double Integration Cont.

$$\int_a^b f(x)dx = I$$

$$\boxed{\int_a^b f(x,t)dx = I(t)}$$

You can see even this idea of collapsing works with single variable integrals by just looking at the F.T.C as shown to the left

Using the idea of double integration and collapsing variables we arrive at a key statement, the integral with respect to a single variable of a multivariable function turns into a function of the remaining variables.

Can you think intuitively why this is?

What is L.I.R and what is the Parallel of L.I.R to Multivariable functions

The Leibniz Integral Rule

$$\frac{d}{dt}\left[\int_a^b f(x,t)dx\right] = \int_a^b \frac{\partial}{\partial t}[f(x,t)]dx$$

What is the L.I.R? Cont.

$$\int_a^b f(x,t)dx = I(t)$$

$$\frac{d}{dt}\left[\int_a^b f(x,t)dx\right] = \frac{d}{dt}[I(t)]$$

$$\int_a^b \frac{\partial}{\partial t}[f(x,t)]dx = I'(t)$$

$$\int \left[\int_a^b \frac{\partial}{\partial t}[f(x,t)]dx\right] dt = I(t)$$

$$\boxed{\int \left[\int_a^b \frac{\partial}{\partial t}[f(x,t)]dx\right] dt = \int_a^b f(x,t)dx}$$

Using the formula, we arrived at a few slides ago, we are going to add our understanding of L.I.R to generalize what we are going to do in future slides and examples.

We are going to take a function of one variable and make it a function of two variables and use L.I.R to solve the integral.

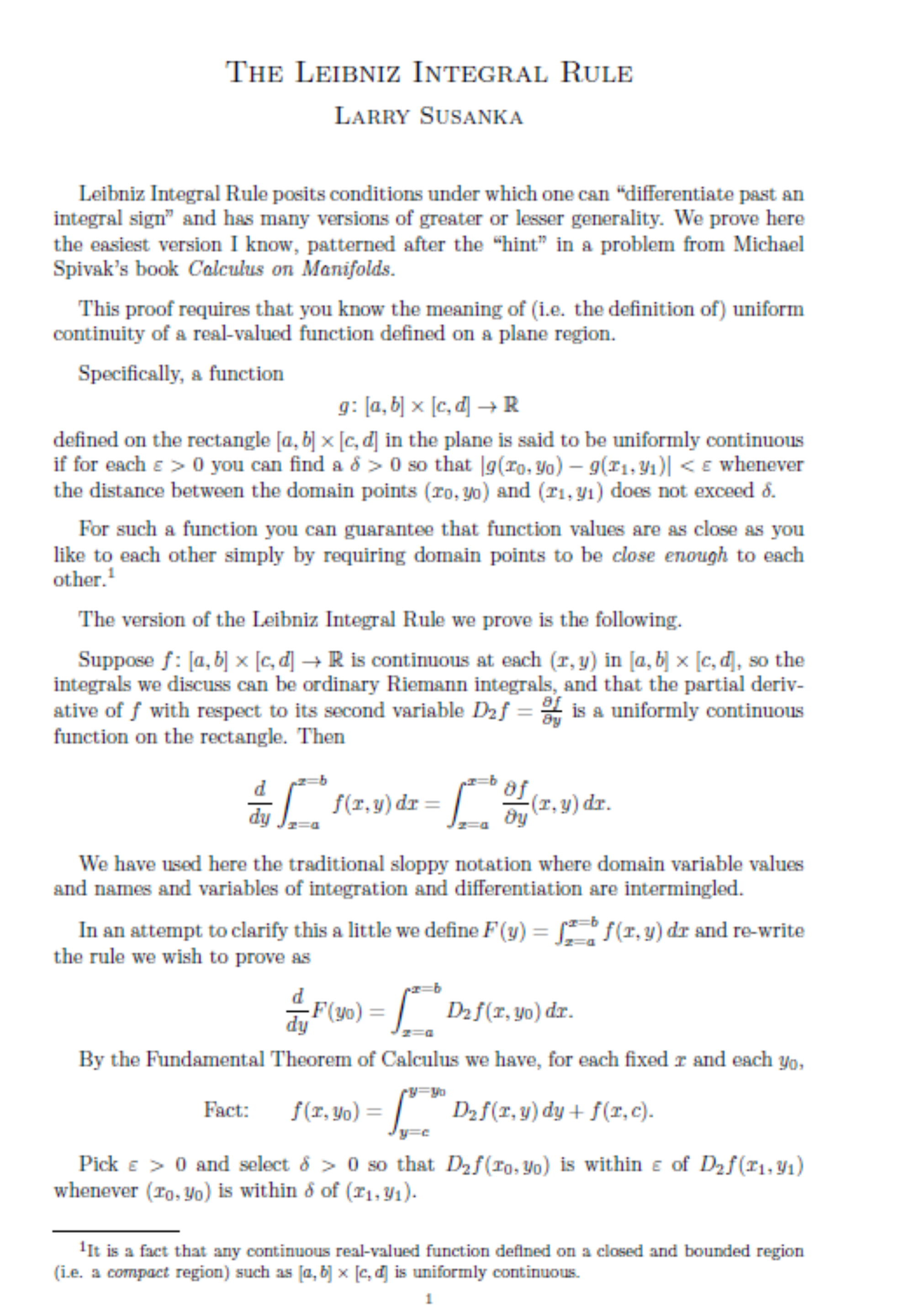
The Leibniz Integral Rule has a few names, Differentiation Under the Integral Sign and more infamously "Feynman's Trick".

"That book also showed how to differentiate parameters under the integral sign—it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again."

– Richard Feynman from his book *"Surely You're Joking, Mr. Feynman!" pg. 86 and 87*

A Quick Proof Of The Leibniz Rule By Larry Susanka

Big thank you to Larry Susanka for letting me use his proof in my presentation today!



THE LEIBNIZ INTEGRAL RULE

LARRY SUSANKA

Leibniz Integral Rule posits conditions under which one can "differentiate past an integral sign" and has many versions of greater or lesser generality. We prove here the easiest version I know, patterned after the "hint" in a problem from Michael Spivak's book *Calculus on Manifolds*.

This proof requires that you know the meaning of (i.e. the definition of) uniform continuity of a real-valued function defined on a plane region.

Specifically, a function

$$g\colon [a,b]\times[c,d]\to\mathbb{R}$$

defined on the rectangle $[a,b]\times[c,d]$ in the plane is said to be uniformly continuous if for each $\varepsilon>0$ you can find a $\delta>0$ so that $|g(x_0,y_0)-g(x_1,y_1)|<\varepsilon$ whenever the distance between the domain points (x_0,y_0) and (x_1,y_1) does not exceed δ.

For such a function you can guarantee that function values are as close as you like to each other simply by requiring domain points to be *close enough* to each other.[1]

The version of the Leibniz Integral Rule we prove is the following.

Suppose $f\colon [a,b]\times[c,d]\to\mathbb{R}$ is continuous at each (x,y) in $[a,b]\times[c,d]$, so the integrals we discuss can be ordinary Riemann integrals, and that the partial derivative of f with respect to its second variable $D_2f=\frac{\partial f}{\partial y}$ is a uniformly continuous function on the rectangle. Then

$$\frac{d}{dy}\int_{x=a}^{x=b} f(x,y)\,dx=\int_{x=a}^{x=b}\frac{\partial f}{\partial y}(x,y)\,dx.$$

We have used here the traditional sloppy notation where domain variable values and names and variables of integration and differentiation are intermingled.

In an attempt to clarify this a little we define $F(y)=\int_{x=a}^{x=b} f(x,y)\,dx$ and re-write the rule we wish to prove as

$$\frac{d}{dy}F(y_0)=\int_{x=a}^{x=b} D_2f(x,y_0)\,dx.$$

By the Fundamental Theorem of Calculus we have, for each fixed x and each y_0,

$$\text{Fact:}\qquad f(x,y_0)=\int_{y=c}^{y=y_0} D_2f(x,y)\,dy+f(x,c).$$

Pick $\varepsilon>0$ and select $\delta>0$ so that $D_2f(x_0,y_0)$ is within ε of $D_2f(x_1,y_1)$ whenever (x_0,y_0) is within δ of (x_1,y_1).

[1]It is a fact that any continuous real-valued function defined on a closed and bounded region (i.e. a *compact* region) such as $[a,b]\times[c,d]$ is uniformly continuous.

Again, big thanks to Larry Susanka for his proof!

An approximation to the derivative $\frac{d}{dy}F(y_0)$ is given by

$$\frac{F(y_0+\delta)-F(y_0)}{\delta} = \frac{1}{\delta}\int_{x=a}^{x=b} f(x,y_0+\delta)-f(x,y_0)\,dx$$
$$= \frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=c}^{y=y_0+\delta} D_2f(x,y)\,dy + f(x,c) - \int_{y=c}^{y=y_0} D_2f(x,y)\,dy - f(x,c)\right]dx$$
$$= \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_0+\delta} D_2f(x,y)\,dy\,dx.$$

Since (x,y_0) is within δ of $(x,y_0+\delta)$ for each x we have the approximation to $\frac{d}{dy}F(y_0)$ trapped between the values

$$\frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=y_0}^{y=y_0+\delta} D_2f(x,y_0)-\varepsilon\,dy\right]dx$$
$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_0+\delta} D_2f(x,y)\,dy\,dx$$
$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\left[\int_{y=y_0}^{y=y_0+\delta} D_2f(x,y_0)+\varepsilon\,dy\right]dx.$$

The left and right inner integrands don't depend on y and can be integrated as constants with respect to y on the interval $[y_0, y_0+\delta]$.

After cancelling δ and integrating ε with respect to x on $[a,b]$ we have

$$\int_{x=a}^{x=b} D_2f(x,y_0)\,dx - \varepsilon(b-a)$$
$$\le \frac{1}{\delta}\int_{x=a}^{x=b}\int_{y=y_0}^{y=y_0+\delta} D_2f(x,y)\,dy\,dx$$
$$\le \int_{x=a}^{x=b} D_2f(x,y_0)\,dx + \varepsilon(b-a).$$

Taking the limit as $\delta \to 0$ produces the estimate

$$\int_{x=a}^{x=b} D_2f(x,y_0)\,dx - \varepsilon(b-a) \le \frac{d}{dy}F(y_0) \le \int_{x=a}^{x=b} D_2f(x,y_0)\,dx + \varepsilon(b-a).$$

But ε can be chosen to be arbitrarily small, and the result follows.

It is worth remarking that far less stringent requirements than uniform continuity on the partial derivative suffice for this theorem to hold, but we will leave that for other treatments.

Basic Example of L.I.R

$$\int_0^2 2x\,dx = I$$

$$\int_0^2 ax\,dx = I(\mathrm{a}) \qquad \int_0^2 (0)x\,dx = I(0) = 0$$

$$\frac{d}{da}\left[\int_0^2 ax\,dx\right] = I'(a) \qquad I(2) = I$$

$$\int_0^2 \frac{\partial}{\partial a}[ax]dx = I'(a)$$

$$\int_0^2 x\,dx = I'(a)$$

To start using L.I.R we must start basic.
We start by either substituting or creating a term in the function to make the function multivariable, in this case a. if we substituted, we keep a note that the integral converges to the value we specified. (remember $f(x,y) = g(x,1)$?)

Because of reasons that will become apparent later we also need to find a value of a that will help us and collapse the integral to a value we can use. Most of the time we try to collapse to zero or an infinity. Think of this as a boundary or initial condition in Diff EQ.

After making the function multivariable we differentiate with respect to the "dummy" variable a.

Basic Example of L.I.R Cont.

$$\int_0^2 2x\,dx = I$$

$$\int_0^2 (0)x\,dx = I(0) = 0$$

$$\int_0^2 x\,dx = I'(a)$$

$$I(2) = I$$

$$2 = I'(a)$$

$$\int 2\,da = I(a)$$

$$2a + C = I(a) \qquad I(0) = 0$$

$$\boxed{C = 0}$$

$$\boxed{2a = I(a)}$$

$$\boxed{I(2) = I = 2(2) = 4}$$

After keeping some important notes for later we then see we can easily evaluate the integral shown.

After evaluating we run into a problem, we have the derivative of what we want, we must integrate back up. Remember double integration?

After integrating back up we have another issue C, this is where the artificial condition we made becomes useful, we then solve for C using our condition made at the beginning of the integral.

After we solve that C is zero, we finally have our equivalent function to the original integral that's only in terms of a. by substituting back the value of a we find the integral converges to four.

Introduction In Use With Non-Elementary Integrals

The Logarithmic Integral:

$$\int_0^1 \frac{x^3 - 1}{\ln(x)}\,dx$$

The Dirichlet Integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

The Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

By using L.I.R we can solve complicated and non-elementary integrals with relative ease and critical thinking!

The Fun Mystery Integral:

? ? ? ? ? ?

The Logarithmic Integral

$$\int_0^1 \frac{x^3-1}{\ln(x)}\,dx = I$$

The first step of L.I.R is to pick how to add in the parameter a, there are so many different cases we can include the parameter.

But in the end, there's only one that works the best and requires some prediction and foresight, the best tip I can give is think about what you want to cancel out, ask yourself how can I cancel it out?

Many different Combinations for I(a) you can choose:

$$\int_0^1 \frac{x^3-1}{\ln(ax)}\,dx = I(a) \qquad \int_0^1 \frac{ax^3-1}{\ln(x)}\,dx = I(a)$$

$$\int_0^1 \frac{x^3-a}{\ln(x)}\,dx = I(a) \qquad \int_0^1 \frac{ax^3-1}{\ln(ax)}\,dx = I(a)$$

$$\boxed{\int_0^1 \frac{x^a-1}{\ln(x)}\,dx = I(a)}$$

The Logarithmic Integral Cont.

$$\int_0^1 \frac{x^a - 1}{\ln x}\,dx = I(a)$$

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln x}\,dx = \int_0^1 0\,dx = 0$$

After determining the form of the integral we are going to use for L.I.R we then set our artificial condition.

$$\frac{d}{da}\left[\int_0^1 \frac{x^a - 1}{\ln x}\,dx\right] = I'(a)$$

$$\int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a - 1}{\ln x}\right] da = I'(a)$$

$$\frac{d}{dt}[x^t] = x^t \ln x$$

We see that $\frac{d}{dt}[x^t] = x^t \ln x$ gives exactly what we want, which is to cancel out $\ln x$.

$$\int_0^1 x^a\,dx = I'(a)$$

$$\frac{x^{a+1}}{a+1}\Big|_0^1 = I'(a)$$

$$\boxed{\frac{1}{a+1} = I'(a)}$$

The integral then just turns into a standard power rule integral.

Logarithmic Integral Cont. Cont.

$$\int_0^1 \frac{x^3 - 1}{\ln(x)}\,dx = I \qquad I(0) = \int_0^1 \frac{x^0 - 1}{\ln x}\,dx = \int_0^1 0\,dx = 0 \qquad a = 3$$

$$\frac{1}{a+1} = I'(a)$$

$$\int \frac{1}{a+1}\,da = I(a)$$

$$\boxed{\ln(a+1) + C = I(a)}$$

$$\ln(1) + C = 0$$

$$\boxed{C = 0}$$

$$\boxed{\ln(a+1) = I(a) = \int_0^1 \frac{x^a - 1}{\ln x}\,dx}$$

$$\boxed{I(3) = \ln(4) = I}$$

EXTRA NOTE!
Notice how our resulting answer $\ln(a+1) = \int_0^1 \frac{x^{a-1}}{\ln x}\,dx$ is a function family? We have answers to every integral in this form! We can solve any value of a we choose, be it 1 or 1,000,000.

We integrate both sides with respect to a to give the desired function that equates to our original integral.

We then use our artificial condition to find C.

We then evaluate with the value of a we replaced thus solving the integral.

You might already be starting to see the name of the game here.

Insert the parameter.
Set an artificial condition.
Use L.I.R.
Arrive at an indefinite integral.
Find C.
Substitute and solve.

The Dirichlet Integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

The Dirichlet Integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = I$$

$$2\int_{0}^{\infty} \frac{\sin x}{x} dx = I$$

$$2\int_{0}^{\infty} e^{-ax}\frac{\sin x}{x} dx = I(\mathrm{a}) \qquad I(0) = I \qquad \lim_{a\to\infty} I(a) = 0$$

$$-2\int_{0}^{\infty} e^{-ax} \sin x \, dx = I'(a)$$

We see something that is strikingly similar to the beginning of the presentation. You have a choice.

Complexify The Integral or Laplace Transform?

Since $\frac{\sin x}{x}$ is an even function we can split the interval up, it also takes care of the issue of discontinuity at x=0

We now see the next important step, adding the parameter a, which isn't a substitution in this case. How does this work you might ask; we still have an artificial value we can set for convergence to our function which is all that matters, in this case when a is zero we get our original integral. We also use as our artificial value, the limit as a approaches infinity, which drags the entire function to zero.

After taking the derivative with respect to a we get an integral that's very familiar and we can solve easily.

The Dirichlet Integral

$$-2\int_0^\infty e^{-ax}\sin x\,dx = I'(a) \qquad I(0) = I \qquad \lim_{a\to\infty} I(a) = 0$$

$$-2\left(\frac{1}{1+a^2}\right) = \mathrm{I}'(\mathrm{a})$$

$$-2\int \frac{1}{1+a^2}\,da = I(a)$$

$$\boxed{-2\tan^{-1} a + C = I(a)}$$

$$\boxed{C = \pi}$$

$$-2\tan^{-1} 0 + \pi = I(0)$$

$$\boxed{I = \pi}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = I = \pi}$$

After Either using Complexifying the Integral or the Laplace Transform we see that our integral we must do will result in the arc tangent function.

We are then able to use our artificial value to find C which in this case is π.

We then can then solve this integral with ease and thus prove The Dirichlet Integral is equal to π.

The Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

The Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$2\int_{0}^{\infty} e^{-x^2}\,dx = I$$

$$2\int_{0}^{\infty} e^{-a^2x^2}e^{-a^2}\,dx = I(a)$$

$$2\int_{0}^{\infty} e^{-a^2(1+x^2)}dx = I(a)$$

We need to think outside the box, we need to "make" the target integral

$$\boxed{2\int_{0}^{\infty} \frac{e^{-a^2(1+x^2)}}{1+x^2}dx = F(a)}$$

We now have arrived at the most fun and important for Differentiating Under The Integral Sign The Gaussian Integral, this integral is important for probability distributions and is the main reason the L.I.R is Taught in higher level physics.

This Integral is very interesting, and requires outside the box thinking, instead of starting with an integral we differentiate, we want our **<u>result</u>** to be in the form of the integral we have made **<u>after</u>** we differentiate with respect to a.

The Gaussian Integral Cont.

$$2\int_0^\infty \frac{e^{-a^2(1+x^2)}}{1+x^2}dx = F(a)$$

$$F(0) = 2\int_0^\infty \frac{1}{1+x^2}dx = \pi$$

$$F(\infty) = \lim_{a\to\infty} 2\int_0^\infty \frac{e^{-a^2(1+x^2)}}{1+x^2}dx = 0$$

$$2\int_0^\infty -2ae^{-a^2(1+x^2)}dx = f(a)$$

$$-4e^{-a^2}\int_0^\infty ae^{-a^2x^2}dx = f(a)$$

$$-4e^{-a^2}\int_0^\infty ae^{-(ax)^2}dx = f(a)$$

$$-2e^{-a^2}2\int_0^\infty e^{-u^2}du = f(a)$$

$$u = ax$$
$$du = adx$$

$$-2e^{-a^2}I = f(a)$$

After taking the derivative with respect to A of $F(a)$ we factor out as much as we can in terms of a but strategically leave an a inside the Integrand so we can make a U-Substitution.

After making the U-Substitution we see that the integral is just out original value we are trying to find.

The Gaussian Integral Cont. Cont.

$$-2e^{-a^2}I = f(a)$$

$$\int_0^b f(a)da = -2I\int_0^b e^{-a^2}da$$

Fundamental Theorem of Calculus

$$F(b) - F(0) = -I\left[2\int_0^b e^{-a^2}da\right]$$

$$\lim_{b\to\infty} 2\int_0^b e^{-a^2}da = I$$

$$0 - \pi = I[I]$$

$$-\pi = -I^2$$

$$\boxed{\sqrt{\pi} = I}$$

$$F(\infty) = \lim_{a\to\infty} 2\int_0^\infty \frac{e^{-a^2(1+x^2)}}{1+x^2}dx = 0$$

$$\boxed{F(\infty) = 0}$$

$$F(0) = 2\int_0^\infty \frac{1}{1+x^2}dx = \pi$$

$$\boxed{F(0) = \pi}$$

We Are now able to Integrate with respect to a to get back our original function (note I is a number!) from 0 to b.

We then use the Fundamental Theorem of Calculus.

Note that since we used bounds, we don’t have a constant of integration but instead we are still able to use these values to our advantage, we take the limit as b goes to ∞. Because the integral is equilent to I we get $-\pi = -I^2$ thus proving the gaussian integral is the square root of π

Challenge Question! Using L.I.R Can you Prove this Equality?

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}$$