The Banach-Shröder-Bernstein Theorem

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1 Overview

The **Banach-Shröder-Bernstein Theorem** says that if there are two sets, A and B, each of which each has the same cardinality as a subset of the other, then A and B have the same cardinality.

Basically we are told that neither is larger than the other, and need to show that they are the same size.

This theorem has many proofs. We present the one in (1) .

References

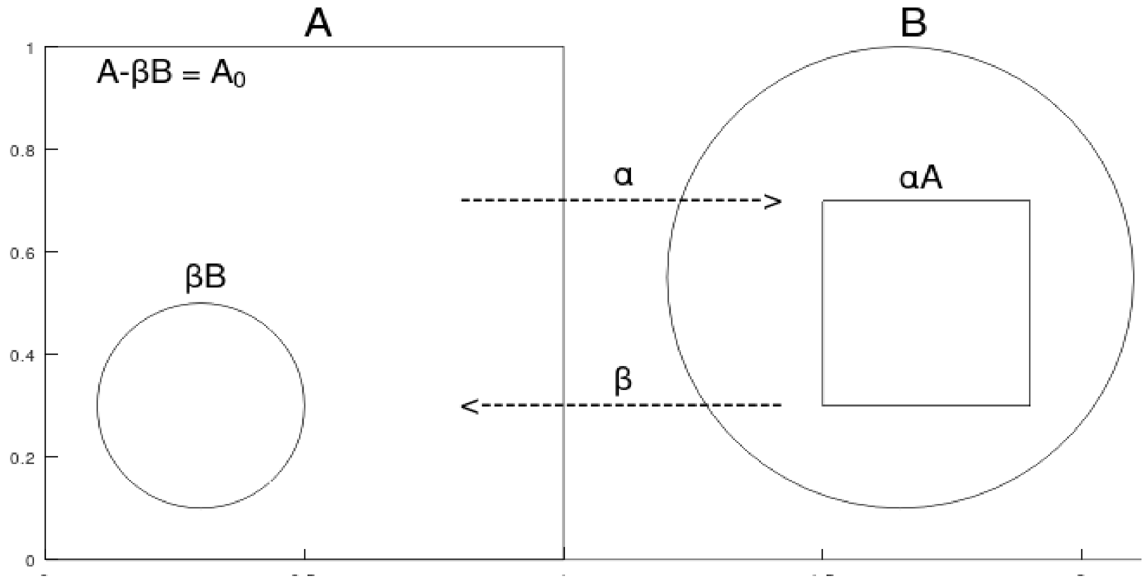
[1] Anders Kaseorg, "The Banach-Tarski Paradox"

2 Proof

By assumption there exist two 1-1 functions: $\alpha : A \to \alpha A \subseteq B$ and $\beta : B \to \beta B \subseteq A$. We cannot assume that either function is onto A or onto B. From these two given functions we need to construct a new improved 1-1 function that goes from A to B.

Denote the points in A that are not in βB as $A_0 = A \setminus \beta B$. Next define $A_1 = \beta\alpha A_0, A_2 = \beta\alpha A_1, A_3 = \beta\alpha A_3, \ldots$. Let

$$A_\infty = \bigcup_{n=0}^{\infty} (\beta\alpha)^n A_0$$



and let its complement be $\bar{A} = A \setminus A_\infty$. Define $B_\infty = \alpha A_\infty$ and $\bar{B} = B \setminus B_\infty$

Because $A_\infty = A_0 \cup \beta\alpha A_\infty$, it follows that

$$\bar{A} = (A \setminus A_0) \cap (A \setminus \beta\alpha A_\infty) = \beta B \setminus \beta\alpha A_\infty = \beta B \setminus \beta B_\infty = \beta\bar{B}$$

This means that everything in A is either in A_∞ or is the image of something in B under the action of β. We can now define our sought after 1-1 mapping from A to B by:

$$f(x) = \begin{cases} \alpha x & \text{if } x \in A_\infty \\ \beta^{-1}x & \text{if } x \in \bar{A} \end{cases}$$

3 An Example

Let $A = [0, 1]$ and $B = [0, 1)$. We choose $\alpha(x) = x/2$ and $\beta(x) = x$. This makes $\alpha A = [0, 1/2]$ and $\beta B = [0, 1)$. Now we have $A_0 = A \setminus \beta B = \{1\}$, $A_1 = \{1/2\}$, $A_2 = \{1/4\}, \ldots$

Overall, $A_\infty = \{1, 1/2, 1/4, 1/8, \ldots\}$, and $\bar{A} = [0, 1] \setminus A_\infty$.
$B_\infty = \alpha A_\infty = \{1/2, 1/4, 1/8, \ldots\}$ and
$\bar{B} = [0, 1) \setminus \{1/2, 1/4, 1/8, \ldots\} = [0, 1] \setminus \{1, 1/2, 1/4, \ldots\} = \beta\bar{A}$

$$f(x) = \begin{cases} x/2 & \text{if } x \in A_\infty \\ x & \text{if } x \notin A_\infty \end{cases}$$

4 A Question

Let us say that A is **cardinally larger** than B if: 1) they do not have the same cardinality, and 2) A has a proper subset that does have the same cardinality as B.

It seems obvious that if two sets do not have the same cardinality then one is cardinally larger than the other. Nevertheless, I have never seen this stated, and do not possess a proof.

Is it true?