# The Banach-Tarski Paradox

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## 1 Overview

The **Banach-Tarski Paradox** (BT) (1) states that one can cut a solid ball into a small finite number of pieces, and reassemble those pieces into two balls, each identical in size to the original.

This result clearly does violence to our ordinary understanding of volume. The volume of the original ball is NOT equal to the sum of its pieces. This means that the pieces don't have well defined volumes. This fact is closely related to the existence of non-measurable sets (2). In both constructions the Axiom of Choice will play a starring role.

In fact, the essential construction in BT does not work with solid balls, but rather with spheres. We will show that the unit sphere in $\mathbb{R}^3(S^2)$ can be decomposed into two identical spheres. Repeating this for each sphere of radius $0 < r \leq 1$, and building the ball from all the spheres, we have the BT Paradox, except for the single missing point at the center.

The decomposition of the sphere is based on the Word/Rotation-Splitting Theorem, explained in (3). As was the case there and in other relevant results, notions of countability and uncountability will be extensively used. (4) provides the background.

## References

[1] Anders Kaseorg, "The Banach-Tarski Paradox"

[2] Steve Ziskind, "A non-Measurable Set"

[3] Steve Ziskind, "Free Groups and Rotations"

[4] Steve Ziskind, "Cardinality Basics"

## 2 Construction

As proved in a prior memo (3), there is a group of $\mathbb{R}^3$ rotations that have the structure of the free group on 2 elements. Denote this set of rotations by F. In addition, when we double F by creating $G = F \times \{1, -1\}$, then the Splitting Theorem shows that F and G are equidecomposable.

**<u>Lemma:</u>** Let D denote the set of points on $S^2$ that are fixed by some non-trivial member of F. Then $S^2$ is equidecomposable with $S^2 \setminus D$.

**<u>Proof:</u>** Note that every rotation in F fixes the 2 points on $S^2$ that intersect the axis of rotation. Because F is countable, D is a countable set. Thus there is an axis through the origin that does not pass through any point in D. Fix that axis.

Consider all pairs of points in D, the set of points of the form $(d_1, d_2)$, a countable set. For any rotation $\rho$ about our fixed axis, there cannot be more than a countable set of values of n that satisfy $\rho^n d_1 = d_2$. (Generally there will be none.) This means that are rotation angles for which $\rho^n d_1 = d_2$ never happens. Pick one.

With this choice of $\rho$, the sets $D, \rho D, \rho^2 D, \rho^3 D, \ldots$ are pairwise disjoint. If we now let their union be

$$A = \bigcup_{n=0}^{\infty} \rho^n D$$

then $\rho A = A \setminus D$. As in the $\mathbb{R}$ to $\mathbb{I}$ proof (4), this will give us a 1-1 mapping between $S^2$ and $S^2 \setminus D$.

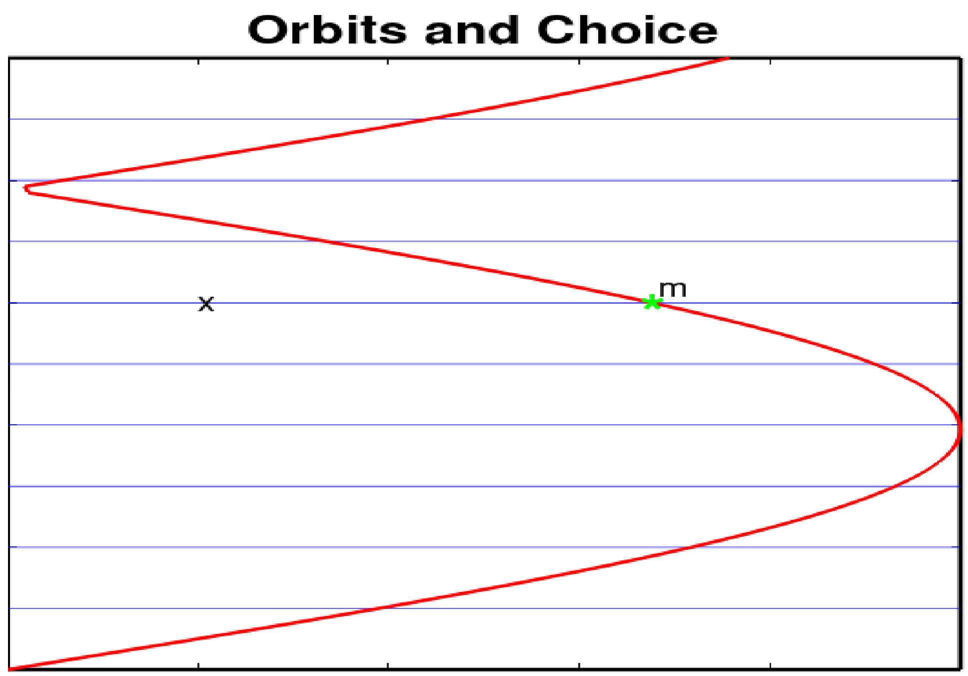
QED

Next we will partition $S^2 \setminus D$ into two copies of itself, using the rotations in F. The last lemma was needed to make the images in the next theorem disjoint. We will also make use of the Axiom of Choice.

**Theorem:** $S^2 \setminus D$ can be partitioned into 2 disjoint sets, each equidecomposable with $S^2 \setminus D$.

**Proof:** Distinct rotations in F will send any x in $S^2 \setminus D$ to distinct points, because if $f_1 x = f_2 x$ then x is a fixed point of $f_1 f_2^{-1}$. Define the equivalence relation $x \sim y$ when $y \in Fx$. (It is an equivalence because F is a group.). We call each equivalence class an **orbit** because the points in a given class can be moved from one to another by the action of some $f \in F$.

Using the Axiom of Choice, form the set M by choosing one element of each equivalence class. Now we know that every element of $S^2 \setminus D$ can be uniquely written as $fm \in FM$. That is, each orbit consists of Fm, for some unique $m \in M$.




In this figure, the blue lines are the orbits and the red curve $M$ is the choice function. Every point, $x$, is in the orbit of some choice of a point $m \in M$.

Now apply the splitting theorem: F can be split into 2 equivalent copies of F, via the functions $\phi_1 : F_1 \to F$ and $\phi_2 : F_2 \to F$. This then means that the sets $F_1 M$ and $F_2 M$ partition FM $= S^2 \setminus D$ into disjoint sets, for if they overlapped then that common value would have two distinct representations as $fm$, violating the uniquiness of representation. More explicitly, if $f_1m_1 = f_2m_2$, then $m_1$ and $m_2$ would be in the same orbit.

On the other hand, each of these disjoint images of M is G-equidecomposable to the image of M under all of F, which is $S^2 \setminus D$.This means that we have formed two equivalent copies of FM, namely $F_1M$ and $F_2M$. Repeating this on each layer of the ball we have duplicated it.

QED