Cardinality of the Irrationals

S. Ziskind

February 25, 2026

Because the set of rationals ($\mathbb{Q}$) is countable, and the set of reals ($\mathbb{R}$) is uncountable, it is "obvious" the the set of irrationals ($\mathbb{I}$) has the same cardinality as $\mathbb{R}$. But many obvious facts regarding the cardinality of infinite sets are not necessarily true, so a proof is needed to ensure that a 1-to-1 mapping exists between $\mathbb{R}$ and $\mathbb{I}$. Here is one.

Let λ be your favorite irrational number. Consider the sets

$$\mathbb{Q}, \mathbb{Q} + \lambda, \mathbb{Q} + 2\lambda, \mathbb{Q} + 3\lambda, \ldots$$

The irrationality of λ guarantees that these sets are pairwise disjoint. Define their union as

$$A = \bigcup_{k=0}^{\infty} (\mathbb{Q} + k\lambda)$$

Note that A is countable, being the countable union of countable sets, and that $A + \lambda = A \setminus \mathbb{Q}$. Now we can define the 1-1 mapping from $\mathbb{R}$ to $\mathbb{I}$ as:

$$f(x) = \begin{cases} x + \lambda & \text{if } x \in A \\ x & \text{else} \end{cases}$$