Basics of Countability

Steve Ziskind

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- Such a function is said to establish a one-to-one correspondence between the elements of A and B.

Examples

- Example 1: let $A = [0, 1, 2, 3, \ldots]$ and $\mathbb{N} = [1, 2, 3, \ldots]$. The function f(x)=x+1 from A to $\mathbb{N}$ shows that card(A)=card($\mathbb{N}$). The function f mechanizes a shift operation, a method we can/will use repeatedly.

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- An explicit mapping function from $\mathbb{Z}$ to $\mathbb{N}$ is:

$$f(x) = \begin{cases} 1 + 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

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- To send A to B in a 1-1 way we use the shift operator on a particular subset of A, and use the identity function on the rest of A. Specifically, define

$$f(x) = \begin{cases} 1/(n+1) & \text{if } x = 1/n \\ x & \text{else} \end{cases}$$

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- Example 4: let A be the solid unit ball in $\mathbb{R}^3$ and let B be A with the center point removed. We could imitate the method of Example 3 by taking a ray from the center to the surface sphere, using the shift technique on it to remove the single endpoint, and sending everything else to itself.

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- Define our 1-1 function act as rotation by ρ on that set of points, and as the identity everywhere else. The origin disappears.

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- Within each sum, order from 1/n to n/1, with the numerator increasing and the denominator decreasing. Thus, the sequence begins: 1/1; 1/2, 2/1; 1/3, 2/2, 3/1; 1/4, 2/3, 3/2, 4/1; 1/5, ...

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$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
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- The idea behind this array shows that: A countable union of countable/finite sets is countable.

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- Each x can be expressed as an infinite decimal:

  $x_1 = 0.x_{11}x_{12}x_{13}x_{14}\dots$
  $x_2 = 0.x_{21}x_{22}x_{23}x_{24}\dots$
  $x_3 = 0.x_{31}x_{32}x_{33}x_{34}\dots$
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  $\ldots$
- Choose a sequence of digits: $a_1 \neq x_{11}, a_2 \neq x_{22}, a_3 \neq x_{33}\ldots$

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- Because $\mathbb{Q}$ and $\mathbb{R}$ have different cardinalities, they cannot be the same set. This proves the existence of irrational numbers.

Example

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- Let λ be your favorite irrational number. Consider the sets

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- Now define $f : \mathbb{R} \to \mathbb{I}$ as

$$f(x) = \begin{cases} x+\lambda & \text{if } x \in A \\ x & \text{else} \end{cases}$$