Curvature: Euler and The Boeing Company

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1 Discussion

In 2000, while doing engineering work at Boeing, a colleague showed me this equation:

$$K(\theta) = cos^2(\theta)K(N) + sin^2(\theta)K(E) \tag{1}$$

where the 3 different values of K correspond to curvatures of a surface in various directions. We got this from a subcontractor. He asked me if this was correct, or if K should really be radius of curvature instead of curvature. I had never seen this equation before, but took it home and figured it out over a weekend: it was correct.

Several obvious questions arise. What is this equation and how is it proved? Why would Boeing engineers have any interest in it? How is it applied in the special case relevant to Boeing?

2 Curvature: $y = f(x)$

The curvature at a point on the graph of y=f(x) is the rate at which the direction of the tangent line changes, as a function of arc length. If φ is the angle the tangent makes with the x-axis, then $tan(\varphi) = y'$. Differentiating again we obtain $sec^2(\varphi) * d\varphi/dx = y''$. But $sec^2 = 1 + tan^2 = 1 + (y')^2$, so that

$$d\varphi/dx = \frac{y''}{1 + (y')^2}$$

On the other hand, if s denotes arc length along the graph, then $ds/dx = \sqrt{1 + (y')^2}$. Taking the quotient of these two, we have the curvature, denoted K, as

$$K = d\varphi/ds = \frac{y''}{[1 + (y')^2]^{3/2}} \tag{2}$$

When the first derivative is zero, K is just the second derivative. If the graph in question is the circle of radius=r, centered at the origin, then we start with $x^2 + y^2 = r^2$. Differentiating twice we have $1 + yy'' + (y')^2 = 0$. The first derivative is 0 when $y = \pm r$, and at that point the equation becomes $y'' = \pm 1/r$. For this reason the reciprocal of $|K|$ is called the radius of curvature, and denoted ρ.

3 The Ellipse

We now consider curvature of an ellipse, defined by $b^2x^2 + a^2y^2 = a^2b^2$, where a and b indicate the x and y intercepts of the graph. We take $a > b$ and call them the semi-major and semi-minor axes. Taking the gradient of the LHS we get the vector $(2b^2x, 2a^2y)$, which says that λ, the angle the normal to the graph makes with the x-axis satisfies $tan(\lambda) = \frac{a^2}{b^2}\frac{y}{x}$, and derivative y' is the negative reciprocal. The angle λ is known as the geodetic latitude.

Taking another derivative, we find that $b^2 + a^2[yy'' + (y')^2] = 0$. The first derivative is 0 when y=b and x=0, and that means $K = -b/a^2$ and $\rho = a^2/b > a$. Similarly at the point (a,0) the value of $\rho = b^2/a < b < a$. This makes sense visually, because an ellipse is a flattened version of the circle of radius a, so the radius is greater than a on the y axis and less than a on the x axis.

At a general point (x,y) on the graph we now calculate K using two different methods. The first simply uses equation 1 and our just computed values of first and second derivative:

$$y' = -\frac{x}{y}\frac{b^2}{a^2} \quad \text{and} \quad y'' = -\frac{b^2}{a^2}\left[\frac{y - xy'}{y^2}\right]$$

The resulting formula for K is an algebraic mess of no particular interest, but will be used to numerically check our second method.

The second method, which will be used when we analyze surfaces, rotates the ellipse so that the point of interest now has its first derivative equal to zero, so that K is simply the second derivative. Figure 1 below shows the geometry.

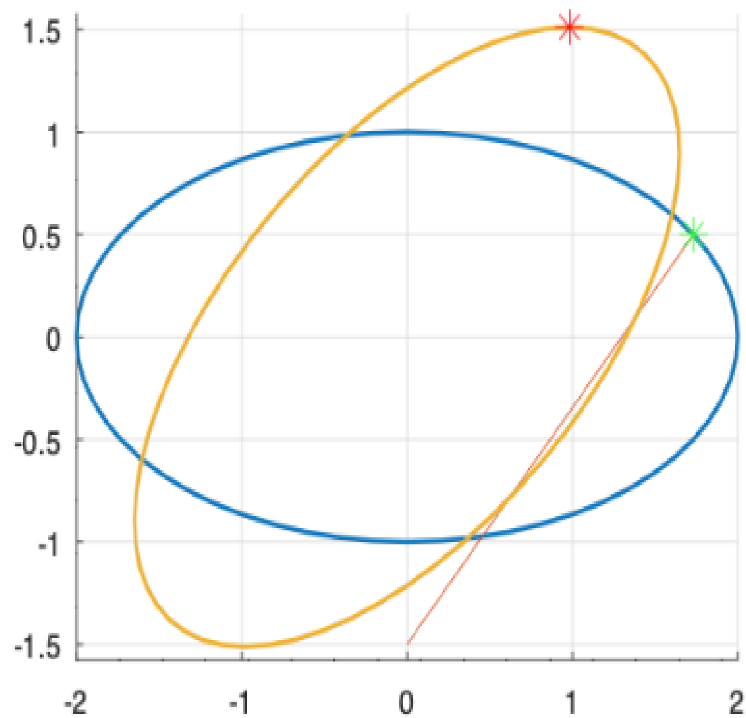

Figure 1: Original and Rotated Ellipse

The required rotation is by the compliment of λ. Denoting this angle by $\theta = \pi/2 - \lambda$, we change from (x,y) coordinates to (u,v) coordinates using

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \text{i.e.} \quad u = cx - sy \quad \text{and} \quad v = sx + cy \tag{3}$$

where we denote $c = cos(\theta)$ and $s = sin(\theta)$.

We invert the transformation to find that $x = cu + sv$ and $y = -su + cv$. Replacing these expressions for x and y in the original ellipse equation yields an equation in u and v, and now our point of interest occurs where dv/du is zero. The calculation of the second derivative, which is somewhat of a mess, provides a direct K value. The expression for the first derivative is:

$$\frac{dv}{du} = \frac{(b^2c^2 + a^2s^2)u - (a^2 - b^2)scv}{-(b^2s^2 + a^2v^2)v + (a^2 - b^2)scu}$$

The second derivative is even worse (but the calculation is routine). Our two methods have been coded in Octave and compared numerically: agreement.

4 Curvature: $z = f(x, y)$

We want to extend our analysis of curvature to surfaces of the form $z = f(x, y)$. It will be assumed that such surfaces are smooth, with no creases or corners, so f has continuous partial derivatives of all orders. At any point the surface will have a normal direction to it, and we will reorient the surface, if needed, so that the normal points along the z-axis, and both x and y partials are zero. Each plane containing the z-axis will determine a curve when it intersects the surface, and we wish to determine the curvature of that resulting curve. As noted above, this is just the second derivative of the curve. We need a formula for this.

The derivative of a function in a given direction is known as the directional derivative. If the direction is defined by the unit column vector U, then the directional derivative is $\nabla(f) \cdot U$, where $\nabla(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ is the vector of partial derivatives. In order to take a second derivative we repeat the operation and have $\nabla(\nabla(f) \cdot U) \cdot U$. Remembering that a dot product of 2 column vectors is just the transpose of the first times the second, we can express it as

$$K = U^T \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} U \tag{4}$$

Because the mixed partials in the matrix are equal, the matrix is symmetric. An important fact about about any real n-by-n symmetric matrix, M, is that there must be a set of n unit length vectors (eigenvectors) and associated real numbers (eigenvalues) with the property that $Mv = \lambda v$ for each of the n (v, λ) pairs. Furthermore, these n eigenvectors are pairwise perpendicular.. This wonderful fact from linear algebra is called the Spectral Theorem.

While the proof of the general theorem takes a bit of doing, in this small 2-by-2 case we can show it directly. Suppose the symmetric matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

We need to show that for some value of λ the matrix

$$\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a non-trivial real solution. This happens when the determinant of the matrix, $(a-\lambda)(c-\lambda)-b^2=0$. Solving the quadratic, we need to know the discriminant is positive. In fact it equals $(a-c)^2+4b^2 \geq 0$, so the roots are real and the eigenvalue/eigenvector pair exists.

Furthermore, distinct eigenvalues have perpendicular eigenvectors. To see this suppose that V and W are associated with distinct values λ and μ. Then, remembering that a dot product is just the transpose of the first times the second, we have

$$\begin{aligned} (\lambda-\mu)V\cdot W = \lambda V\cdot W - \mu V\cdot W = MV\cdot W - V\cdot \mu W \\ = V^T M^T W - V^T MW = V^T MW - V^T MW = 0 \end{aligned}$$

Because $(\lambda-\mu)$ is not zero, $V\cdot W$ must be.

So now suppose that we want the curvature in the direction that makes an angle of θ with V. The unit vector in that direction is $U = cos(\theta)V + sin(\theta)W$. Plugging that into equation (2) (and calling the matrix M), we have a curvature of:

$$K_\theta = U\cdot MU = (cos(\theta)V + sin(\theta)W)\cdot M(cos(\theta)V + sin(\theta)W) \tag{5}$$

$$= cos^2(\theta)V\cdot MV + sin^2(\theta)W\cdot MW = cos^2(\theta)K_V + sin^2(\theta)K_W \tag{6}$$

In other words: **the curvature in a direction intermediate between the perpendicular eigenvalue directions is the weighted sum of the curvatures in those directions, where the weights are the squares of the cos and sin of the angle to the principle curvatures.** The curvatures in the direction of V and W are called the principal curvatures. This result is a classic formula of Euler.

5 The Oblate Spheroid

Now that we have Euler's basic formula for curvature of a surface we will apply it to the Earth. The model of the Earth (WGS-84) is an oblate spheroid, which is basically a squashed sphere, or a rotated ellipse. The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{7}$$

where a = 6,378,137 meters is the equatorial radius, and b is the polar radius. The difference between these is described by $f = \frac{a-b}{a}$, the flattening factor, which is roughly 1/298. (Note: for Saturn, which spins quite fast and is much flatter, $f = 1/10$).

As noted above, the angle that the normal to the surface makes with equatorial plane is the geodetic latitude. The angle that the line line from a point to the center of the earth makes with the equatorial plane is the geocentric latitude. As an amusing homework calculation, you might want to verify that for the Earth, whose flattening factor is small, the largest difference between these latitudes is roughly $f = 3.3$ mrad., and it occurs at the mid-latitudes (i.e. Seattle).

To apply Euler's formula to the Earth, we will emulate our procedure of rotating the spheroid so that the point of interest is maximal, making its first derivatives zero. Then we must compute the matrix of second derivatives.

As a simplification, note that the curvatures depend on latitude, but not longitude. Thus we can set y = 0, putting the point on the prime meridian. As in the ellipse calculation, we will rotate the spheroid about the y-axis by an angle of $\theta = \pi/2 - \lambda$, where the geodetic latitude λ satisfies $tan(\lambda) = (a^2 z)/(b^2 x)$.

As before, both partials of w with respect to u and v are zero. After much algebra, the second partials satisfy:

$$\frac{\partial^2 w}{\partial u^2} = \frac{-(b^2c^2 + a^2s^2)}{w(b^2s^2 + a^2c^2) - (a^2 - b^2)scu} \tag{8}$$

$$\frac{\partial^2 w}{\partial v^2} = \frac{-b^2}{w(b^2s^2 + a^2c^2) - (a^2 - b^2)scu} \tag{9}$$

$$\frac{\partial^2 w}{\partial u \partial v}\left[w(b^2s^2 + a^2c^2) - (a^2 - b^2)scu\right] = 0 \tag{10}$$

Because the non-zero multiple of the mixed partial is zero, the mixed partials must be zero, so our matrix is a diagonal matrix. This means that the directions of the extreme curvatures run North-South and East-West. Our original equation is now proved.

6 Inertial Navigation

In 2000 I was the lead engineer for the navigation system on the F-22, which is how I encountered Euler's formula. We got it from Litton, our subcontractor for the inertial instruments.

Inertial navigation calculates the velocity and position of an object from sensed acceleration. This acceleration is derived from outputs of accelerometers and gyroscopes. GPS has taken over much, but not all, of this activity. Starting from an initial position, output from a triad of accelerometers is combined to a single acceleration vector, integrated to obtain velocity, and then integrated again to obtain position. The gyroscopes supply attitude information so that the accelerometer outputs can be stabilized to a common frame, typically North/East/Down.

Accelerometers are basically (very sophisticated) pendulums, and their deviation from a neutral position indicate acceleration in the given direction. However, there is an important exception: gravity will pull an accelerometer vertically. So as not to misinterpret this as arising from motion, the direction and strength of gravity must be carefully known, and its effect removed from the instrument output. Furthermore, as an aircraft flies across the Earth, the sense of vertical changes. To account for this change, the sense of level must rotate at a rate proportional to the velocity times the curvature. It is for this reason that the value of curvature in the direction of travel must be known with high precision. Hence the use of Euler's curvature formula developed above.

6.1 The Vertical Channel

It is an interesting fact that a pure inertial navigator, without external aiding, has an unstable vertical channel. To see what happens, suppose that an inertial box (i.e 3 accels and 3 gyros) is sitting in the basement of a building on a table, connected to a computer that integrates its outputs. Suppose further that the box stays on the table, never moving, but that its altitude is initialized to be on the roof, a slight error. Let the actual distance to the center of the Earth be R, but that it is initialized as $R + y$, with y small.

The actual force acting via gravity is $\frac{GMm}{R^2}$, according to Newton's law of gravity, where G is the universal gravitational constant, and M and m are the masses of the Earth and the pendulum. But because of the initialization error, the computer expects a force of $\frac{GMm}{(R+y)^2}$, a smaller value because of the greater distance to the Earth. The computer subtracts from the reported force what it thinks is due to gravity, and interprets the difference as acceleration of motion. Thus

$$F = ma = \frac{GMm}{R^2} - \frac{GMm}{(R+y)^2} = \frac{GMm}{R^4}(2Ry + y^2) \approx \frac{2GMmy}{R^3} \qquad (11)$$

But the value of $\frac{GM}{R^2}$ is just $g = 32.17 feet/sec^2 = 9.8 meters/sec^2$. Cancelling m from the equation, and recalling that acceleration is the 2^{nd} derivative of y, we find that $y'' = \frac{2g}{R}y$. Because the derivative of $y = e^{kt}$ is ky, this equation has a solution of $y = e^{\sqrt{\frac{2g}{R}}t}y(0)$. Plugging in values for g and R, we have $y(t) = e^{t/570}y(0)$. Thus, the value of y, the vertical error, will exponentially diverge with a time constant of 570 seconds. Because of this instability a pure inertial system must be supplemented with external information, typically barometric altitude, to prevent runaway vertical errors.

6.2 The Schuler Cycle

Another curious property of inertial systems arises because the Earth is not flat. Suppose, as before, that our inertial box sits unmoved on a table, connected to a computer. This time, suppose that its initial position is erroneous in the horizontal direction, by an amount of $x > 0$. At this position, which differs from the true position, the calculated direction of vertical will be off by an angle of x/R. In the figure below, the direction of gravity will be to the right of what the computer thinks the direction should be, and in a way that is interpreted as acceleration to the left. Checking the signs, the computer will attribute the sensor output as having a horizontal value of $mgx/R = -mx''$. Cancelling m, the resulting differential equation is

$$\frac{g}{R}x + x'' = 0 \tag{12}$$

This is identical in form to the equation for an ideal spring, whose solution, Hooke's Law, is a sinusoid motion with period of $2\pi\sqrt{R/g} = 5067$ seconds, roughly 84.4 minutes.

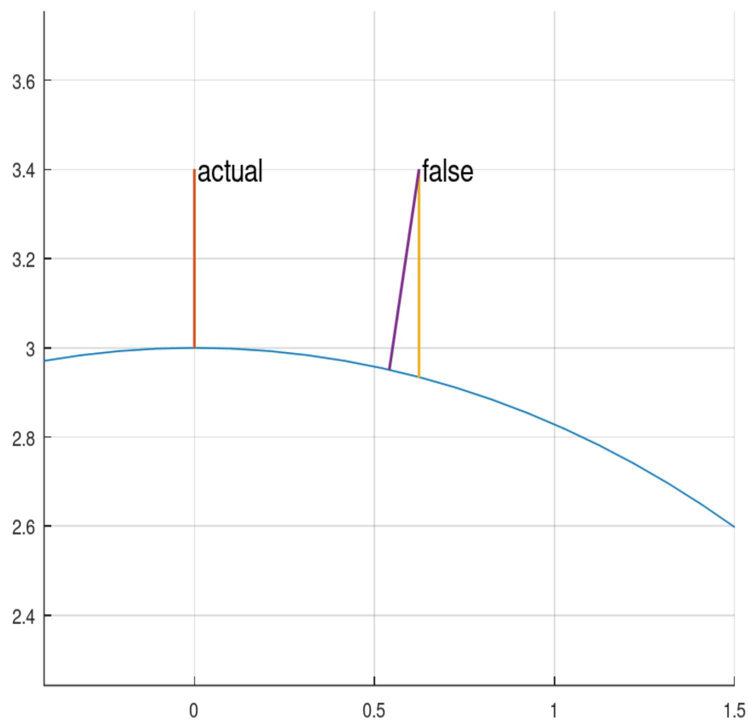

Figure 2: Level Error

In fact, if such a inertial box/computer is run as indicated, the computer will report that the box moves back and forth across the room, like a pendulum, with a period of 84.4 minutes. This behavior was first explained by Maximillian Schuler in the early 1920s, who developed the first gyroscopic compass for ships.