Felix Klein's Erlangen Program

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Évariste Galois

On the 30th of May, 1832, a twenty-year-old man lay dying in a farmer's field outside of Paris. He had been shot in a duel, abandoned by his opponent and his own seconds. The next day he was dead.

This young man, Évariste Galois, was a Republican firebrand, supporter of the anti-monarchist movement that led to the June Rebellion of 1832.[1] It is possible his death was at the hands of a political opponent; possibly it involved a certain Mademoiselle Stéphanie-Félicie Poterin du Motel.

[1] ... of Les Misérables fame.

The reason for the duel, now, will likely never be known with certainty. But what *is* understood, now, is that on that field, dying, was the most brilliant mathematician of his generation.

Of the *older* generation, Augustin-Louis Cauchy had seen two papers written by Galois in 1829 but did nothing with them. Joseph Fourier received a paper by Galois in 1830 but shortly afterward *he* died.

Siméon Denis Poisson earlier in 1830 had reviewed some of his work and declared it "incomprehensible."

Foreseeing his impending doom, he had spent his last weeks assembling and improving his mathematical notes and sent them in letters to his friend Auguste Chevalier. In these notes he told his friend to "ask Jacobi or Gauss to give their opinion, not as to truth, but to the importance of these theorems."

This did not happen.

Ten years later Joseph Liouville somehow came across the correspondence to Chevalier, realized their value, and had the rewritten notes published, finally, in his Journal de Mathématiques Pures et Appliquées in October-November 1846.

Among other things, Galois had invented and named the notion of "group" in its modern form. He knew about normal subgroups and quotient groups of a group by a normal subgroup. He invented the idea of a solvable group, and had methods to show if a group was solvable, or not.

Group G is solvable if it can be decomposed into a finite chain of subgroups, each normal in the next

$$\{e\} \ \triangleleft \ G_k \ \triangleleft \ G_{k-1} \ \triangleleft \cdots \ \triangleleft \ G_1 = G$$

where each quotient group G_{i-1}/G_i is Abelian.

He considered an equation

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$$

where p has integer (or equivalently, rational) coefficients and with $a_n \neq 0$ and irreducible—that is, it can't be factored into smaller factors with rational coefficients.

One of the outstanding problems of that era was to improve on the formulae for explicit solutions of that equation, and the goal here was to produce, for any such polynomial, a formula similar to

$$ax^2 + bx + c = 0 \quad \Longleftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Partial solutions in special cases of the quadratic formula were recorded on Babylonian cuneiform tablets and Egyptian papyri dating to 2000 BCE using geometrical methods.

Chinese books recorded solutions around 200 BCE, while the Greek Diophantus recorded solutions around 250 BCE. The Indian mathematician Brahmagupta solved the problem by producing ONE of the two roots in the real case around 628 CE and recognizably algebraic methods were used by Persian mathematician Al-Khwarizmi to solve many cases around 750 CE.

The first proof covering all cases, including complex roots, was by Simon Stevin in 1594 CE.

Work simultaneously was carried out on third and fourth degree equations

$$ax^3 + bx^2 + cx + d = 0 \quad \text{and} \quad ax^4 + bx^3 + cx^2 + dx + e = 0.$$

Solutions were published in 1545 CE by Gerolamo Cardano, attributing the solutions to Lodovico Ferrari (fourth degree) and Del Ferro and Tartaglia (third degree). Both of these required solving quadratics, and the fourth degree solution, though it came first by several years, required solutions to third degree equations, so in that sense the solution was not complete until 1594 when Simon Stevin dealt completely with quadratics.

These solutions were gigantic formulas, the fourth degree solution covering a large page in small type, filled with solutions to lower degree equations, but always using just nested roots of 4th degree or lower involving the coefficients and the four arithmetic operations.

Similar solutions of higher degree polynomials became a big target of mathematical effort . . . for the next 150 years. . . with results only in special cases.

When Galois turned to this problem he also invented the idea of a general field: a collection of numbers closed under multiplication and addition, a group with addition, commutative and associative with both operations, and whose nonzero members have multiplicative inverses.

For instance the rational numbers, $\mathbb{Q}$, form a field. And for rational k the set $\mathbb{Q}(\sqrt{k}) = \{\, a + b\sqrt{k} \mid a, b \text{ are rational} \,\}$ is a field.

The real numbers, $\mathbb{R}$, form a field. And $\mathbb{C} = \mathbb{R}(i) = \{\, a + bi \mid a, b \text{ are real} \,\}$ is a field too.

If $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is any irreducible integer-coefficient nth degree polynomial Galois saw how to create a "splitting field" for p, the smallest field $\mathbb{F}_p$ within which p could be factored into linear factors.

For instance $p(x) = x^2 - 2$ splits in $\mathbb{Q}(\sqrt{2})$.

$p(x) = x^2 + 1$ splits in $\mathbb{Q}(i)$.

Any field $\mathbb{F}$ has a collection of automorphisms which we will denote $A(\mathbb{F})$. These are invertible functions $\phi\colon \mathbb{F} \to \mathbb{F}$ that preserve the field operations.

$$\phi(ab) = \phi(a)\phi(b) \quad \text{and} \quad \phi(a+b) = \phi(a) + \phi(b).$$

They reflect the internal structure of the field and characterize the field and form a group with composition.

For instance $\mathbb{Q}(\sqrt{2}) = \{\, a + b\sqrt{2} \mid a, b \text{ are rational} \,\}$ has an automorphism group containing just two elements, the identity automorphism and the automorphism that sends 1 to 1 and $\sqrt{2}$ to -$\sqrt{2}$.

Fields that have isomorphic automorphism groups are isomorphic as fields. The automorphism group $A(\mathbb{F})$ reflects the internal symmetries in the field $\mathbb{F}$, and *characterize* the field.

And the symmetries of $\mathbb{F}_p$ reflect the nature of the roots of irreducible p.

Let $\mathbb{F}_p$ denote the splitting field of irreducible rational polynomial $p(x)$.

Galois showed that the automorphism group $A(\mathbb{F}_p)$ is solvable if and only if $p(x)$ has roots of the required types: they can be written in terms of various arithmetic operations and nth root operation applied to the coefficients of the polynomial.

It was now easy to find polynomials of fifth degree that failed this criteria and, incidentally, solved (in a negative sense) the ancient problem of trisecting a general angle, which can be reduced to an "unsolvable" polynomial equation of this type.

NONE of these ideas and definitions had been considered in this way before. He invented it all, and set the course of much of abstract algebra and analysis to this day.

Bernhard Riemann

In 1853 George Friedrich Bernard Riemann was a student of Gauss at Göttingen. Gauss offered him a choice of topics for his Habilitationschrift and he chose "The Hypotheses Which Lie at the Bases of Geometry."

In this incredible talk delivered in June of 1854 he outlined what would become Riemannian geometry and presaged the work of Einstein and others on the geometrization of physics, and the role of a metric and its variations.

The book *Bernhard Riemann on the Hypotheses Which Lie at the Bases of Geometry* edited by Jürgen Jost is much more than a translation, and puts this work in context of the time and later, revealing the depth and subtlety of this early work of Riemann.

Felix Klein

In 1872 Felix Klein outlined his Erlangen program to classify geometries, and later any mathematical structure, by their symmetries. Group theory, specifically the group of symmetries of the structure, was to be the tool to accomplish this.

For instance if $\langle \cdot, \cdot \rangle$ is an inner product (such as dot product of Euclidean space or the Minkowski inner product in the relativistic description of the world) the group he had in mind is the group of isometries, namely those functions ϕ for which

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y.$$

It is a fact that any map of this kind is linear, corresponding to rotations in Euclidean space and something harder to describe called "Lorentz transformations" in Minkowski space-time.

Sophus Lie

Klein's close friend Sophus Lie was a full participant in this program. In three volumes published from 1888-1893, he gave substance and technique to a subject which had begun as a philosophical "to-do" list, and his analysis of these groups, many of which are now called Lie groups, was critical to its success.

In particular, he saw that a Lie group could often be replaced in an analysis by a much simpler object called a Lie algebra, the "infinitesimal generators" of the group.

In 1890 Wilhelm Killing essentially finished the project of classifying the finite-dimensional simple Lie groups, a powerful tool in understanding the symmetries they represent.

Emmy Noether

Amalie Emmy Noether was an amazingly gifted algebraist, who, with her many students who adored her, built the foundation of modern abstract algebra. But in her earlier life in Göttingen she worked with Hilbert and others on problems of mathematical physics.

She proved, in 1918, two theorems (Noether's first and second) in which she revolutionized how people understood the mechanics of physical systems. These theorems state that whenever you have a symmetry of physical law, such as time symmetry or rotational symmetry or translational symmetry, you will have a constant of the time-evolution of the physical system.

In Hamiltonian mechanics it is understood that these invariants may be used as coordinates, and decrease the dimension of the system.

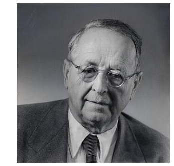
At a critical point Einstein was at Göttingen working with Hilbert and trying to reconcile his new General Relativity with the fact that it didn't *seem* to conserve energy, a show-stopper.

She showed Hilbert how to correct this misimpression.

The symmetries here come from the Lie group associated with the relativistic Hamiltonian. The techniques work for *any* Hamiltonian.

The Gruppenpest

Hermann Weyl was a philosopher, a physicist and a mathematician and an important figure at Göttingen. In 1918 he made an attempt to model the electromagnetic field and the gravitational field as geometrical properties of spacetime.



Though this didn't work (people are still trying) these group-theoretic methods led to a flurry of papers by Weyl and Eugene Wigner in theoretical physics. All of them required understanding of group theory and few physicists understood the subject due, partially, to the fact that there were no books with which one could learn it. Even linear algebra was not part of the PhD curriculum in physics before 1920.

In a famous quote, MIT physicist John Slater said "I had what I can only describe as a feeling of outrage at the turn which the subject had taken…"

Quoting from the Gruppenpest page of the ncatlab.org website:

"Eventually this resistance vanished and turned into its opposite in theoretical fundamental physics, in the classification of fundamental particles by unitary representations of the Poincaré group introduced by Hermann Weyl, in the description of gauge theory in terms of associated bundles given by representations of gauge groups.

Today almost the first thing that one wants to know about any proposed physical theory is its global symmetry-group and gauge group and their relevant representations."

The Eightfold Way

Beginning in 1961 and continuing into 1964, Murray Gell-Mann (and, independently, Yuval Ne'eman) created the quark model of particle physics which Gell-Mann called the "eightfold way" in reference to both the "eightfold path"

of Buddhism and and the 8-dimensional Lie group SU(3) which encodes the symmetries and interactions of these multi-quark systems of particles.

And then there is string theory …

Last Thoughts ...

This is the incredible tree that can grow from a mathematical idea. Other ideas are still out there, waiting for you, waiting for your inspiration and, yes, obsession to bring them light. Truth is worth obsessing about.

And Galois? His last *words* were to his younger brother, recorded as “Don’t weep, Alfred! I need all my courage to die at twenty!”

But I bet his last *thoughts* were about mathematics.