It was 198?. Scientific calculators were new on the scene. Affordable graphing calculators were still years away. A student walked into my office — a conscientious student I knew pretty well.

"Hi, Joe. What's up?" I greeted him.

"There's something wrong with my new calculator," he complained. "Look at this."

He input $\pi \div 3$, then proceeded to punch the cosine key, repeatedly.

$$[Let\ x_0 = cos(\frac{\pi}{3}),\ x_1 = cos(x_0) = cos(cos(\frac{\pi}{3}))\ ....\ x_{n+1} = cos(x_n)]$$

$x_0 = 0.5000000000$	$x_{11} = 0.7418224609$	$x_{30} = 0.7390794413$
$x_1 = 0.8775825619$	$x_{12} = 0.7372357332$	....
$x_2 = 0.6390124942$	$x_{13} = 0.7403273476$	$x_{50} = 0.739085131$
$x_3 = 0.8026851007$	$x_{14} = 0.7382413159$	....
$x_4 = 0.6948196907$	$x_{15} = 0.7396475653$	$x_{55} = 0.739085133$
$x_5 = 0.7681958313$	$x_{16} = 0.7386993245$	$x_{56} = 0.739085133$
$x_6 = 0.7191654459$	$x_{17} = 0.7393381780$	$x_{57} = 0.739085133$
$x_7 = 0.7523522263$	$x_{18} = 0.7389077054$	$x_{58} = 0.739085133$
$x_8 = 0.7300810631$	$x_{19} = 0.7391977112$	$x_{59} = 0.739085133$
$x_9 = 0.7451123334$	$x_{20} = 0.7390022713$	$x_{60} = 0.739085133$
$x_{10} = 0.7350063090$	....	....

"See. It gets stuck!" Joe said. "And the same thing happens when I start with any other value."

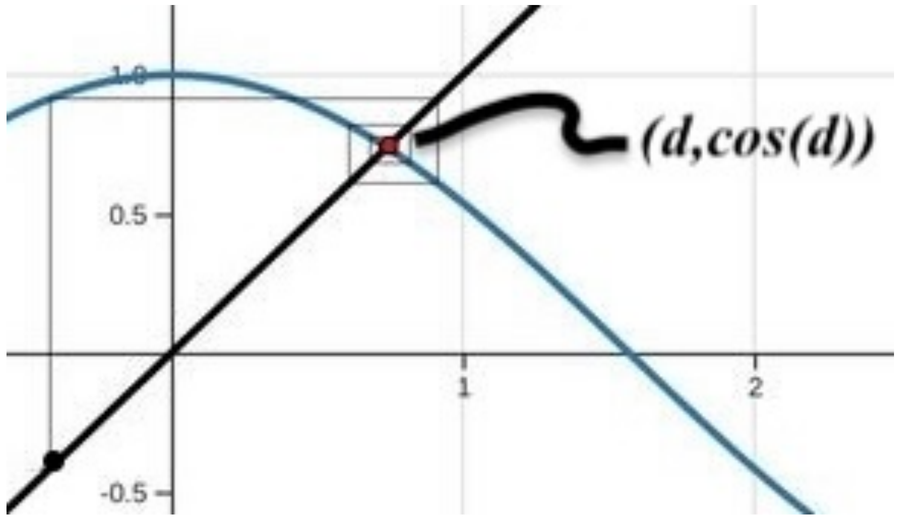
He demonstrated, and sure enough he was right

"What's going on?" he wanted to know.

*I assured Joe that his calculator was not broken. "You've just stumbled on a **fixed point** of the cosine function — in fact, the **only** fixed point of cosine: — a value, d, where cos(d) = d, ," I said.*

$$d = 0.73908513321516064165531208767387..... \approx 0.739085133$$

"Visually, $(d, \cos(d))$ is the point at which the graphs of $y = x$ and $y = \cos(x)$ intersect."



We'd discussed rational numbers a little bit in class — repeated decimal expansions, etc. Even proved that $\sqrt{2}$ is irrational.

So I wasn't surprised when Joe asked, "Is d a rational number?"

I knew the answer, but hesitated.

Joe jumped in, "It doesn't look like it, but maybe it starts repeating beyond the calculator's display."

I wouldn't bet on it," I said. "But that's a longer story."

(Check out the relationship between *d* & fractals at https://www.youtube.com/watch?v=iqvMLg93Pso.)

A Fixed Point*

. . . and then some

Bellevue College Math/Physics Colloquium
Larry Curnutt
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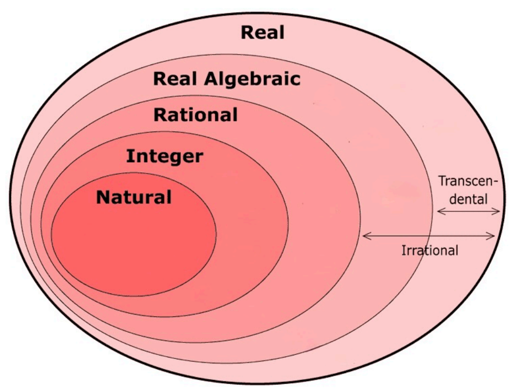
** In general, a fixed point does not move or change when it undergoes some action.*

<u>*Definition*</u>. *Let M be some sort of space, f a function that maps M into M, and p an element of M. If f(p) = p, then p is called a* ***fixed point of f****.*

Algebraic Numbers

A complex number, a, is **algebraic**, if $P(a) = 0$ for some polynomial, $P(x)$, with rational coefficients. If there is no such polynomial, then a is **transcendental**.

- All rational numbers are algebraic.
- $\dfrac{5 \pm \sqrt{17}}{2}$ and $\sqrt[n]{prime}$ are all irrational and algebraic. $i = \sqrt{-1}$ is algebraic.
- $p(x) = x^5 - x - 2$ has only one real root, $\alpha \approx 1.296$ by numerical methods.
  α is algebraic by definition, but does not involve radicals (Galois).
- Lindemann-Weierstrass Theorem (very deep*)
  If $T(a) = 0$ for some polynomial $T(x)$ with rational coefficients, then e^a is transcendental.
  (This is true whether a is real or imaginary.)
- π, e, $\ln(2)$, and $L = \displaystyle\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001 \ldots$ are transcendental.



(Proof is long and difficult; uses machinery outside scope of statement; connects disparate branches.)*

Structure of Algebraic Numbers

The set of algebraic numbers, along with addition and multiplication, form a **field**. Obviously, they share familiar algebraic properties with the reals, such as commutativity, associativity, and identities (0 and 1). It's not so clear that negatives, reciprocals, sums, and products of algebraic numbers are also algebraic.

- If a is algebraic, then $P(a) = 0$, for some polynomial, $P(x)$, with rational coefficients. It's fairly straight forward to check that if $F(x) = P(-x)$ and $G(x) = x^n \cdot P(x)$, where n is the degree of $P(x)$, then $F(-a) = 0$ and $G(1/a) = 0$. So $-a$ and $1/a$ are algebraic.

Simple example: $P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$ and $\mathrm{P(a)} = 0$.

$$F(x) = P(-x) = p_0 - p_1 x + p_2 x_2 - p_3 x^3 \text{ and } G(x) = p_0 x^3 + p_1 x^2 + p_2 x + p_3$$

$$F(-a) = p_0 - p_1(-a) + p_2(-a)^2 - p_3(-a)^3 = p_0 + p_1 a + p_2 a^2 + p_3 a^3 = P(a) = 0$$

$$G(\frac{1}{a}) = p_3 + p_2(\frac{1}{a}) + p_1(\frac{1}{a^2}) + p_0(\frac{1}{a^3}) = \frac{p_3 a^3 + p_2 a^2 + p_1 a + p_0}{a^3} = \frac{1}{a^3} \cdot P(a) = 0$$

- If a and b are algebraic with $\mathrm{R}(a) = 0$ and $\mathrm{S}(b) = 0$. It's possible to construct polynomials R* and S* from R and S for which $\mathrm{R}^*(a + b) = 0$ and $\mathrm{S}^*(a \cdot b) = 0$, ensuring that $a + b$ and $a \cdot b$ are algebraic. But it's too lengthy and trusty to do here, so you'll have to trust me.

Here's a little bonus that has nothing to do with properties of fields.

- If a is algebraic, then $\sqrt{a}$ is algebraic?

Define $H(x) = P(x^2)$

$H(a) = P([\sqrt{a}]^2) = p_0 + p_1(\sqrt{a})^2 + p_2(\sqrt{a})^4 + p_3(\sqrt{a})^6 = p_0 + p_1(a) + p_2(a)^2 + p_3(a)^3 = P(a) = 0.$

- If $P(r) = 0$, there is a polynomial, $Q(x)$ with rational coefficients, for which $Q(i \cdot r) = 0$.

Define $G(x) = P(x) \cdot P(-x)$. Since $G(x) = G(-x)$, $G(x)$ has only even powers of x.
So $G(x) = H(x^2)$. Define $Q(x) = H(-x^2)$.
Then $Q(ir) = H(-[ir]^2) = H(r^2) = G(r) = P(r) \cdot P(-r) = 0 \cdot P(-r) = 0.$

The coup de grace

- **The Dottie number (the fixed point of cosine) is transcendental.**

Pf. We assume that $d = cos(d)$ is algebraic and try to deduce a contradiction. d algebraic means that $i \cdot d$ is algebraic, too. By the Lindemann-Weierstrass Theorem, e^{id} has to be transcendental. Euler's Theorem implies that $e^{id} = cos(d) + i \cdot sin(d) = d + i\sqrt{1 - d^2}$, since $cos(d) = d$, and $sin^2(d) = 1 - cos^2(d)$. Now, d and $i\sqrt{1 - d^2}$ are both algebraic, so their sum, and hence e^{id} is algebraic. But this contradicts the transcendentality/transcendentalness(?) of e^{id}.
Therefore, d must be transcendental.

Me: "So yes, Joe. After all these years, $d \approx 0.739085133...$ is irrational!"

Why is $d \approx 0.739085...$ called "the Dottie Number"?

A French professor, known as **Dottie**, noticed that no matter what number she started with, hitting the cosine button on her calculator over and over again always produced the same value on her screen. Much later [years?], in 2007 **Samuel R. Kaplan** published an expository paper in the MAA's <u>Mathematics Magazine</u>, in which he named this constant "Dottie," after the French professor.

- I can't locate any biographical information about the French professor.

- On the other hand, Sam Kaplan has had a distinguished, if not famous, career as a mathematician at UNC Asheville. He's won several awards for excellence in teaching and community service. He's done research in differential equations, dynamical systems and chaos theory (3-body problem), and voting theory.

How big is the set of algebraic numbers?

<u>Review</u>

1. If the elements of a set can be listed ($a_1, a_2, a_3, a_4, \dots.$), it's called **countable**.
   If it is impossible to list the elements of a set, then it's **uncountable**.
2. The set of natural numbers, N, is countable.
3. The set of real numbers, R, is uncountable. (Cantor's diagonal)
4. The collection of all subsets of N, denoted by $P(N)$, is uncountable. (mimic Cantor's diagonal)

<u>Theorem</u>. If $S_1, S_2, S_3, \dots$ are all countable, then $S = \bigcup_{i=1}^{\infty} S_i = S_1 \cup S_2 \cup S_3 \cup \cdots$ is countable.

<u>Pf</u>. List the elements of each of the A_ns in an infinite matrix, $[a_{j,k}]$, where $a_{j,k}$ is the kth element in A_j. Define the *height of* $a_{j,k}$ to be $j + k$. All the elements of A can be listed in order of their heights:

$\underbrace{s_{1,1}}_{ht=2} \quad \underbrace{s_{2,1} \quad s_{1,2}}_{ht=3} \quad \underbrace{s_{1,3} \quad s_{2,2} \quad s_{2,2}}_{ht=4} \quad \underbrace{s_{4,1} \quad s_{3,2} \quad s_{2,3} \quad s_{1,4}}_{ht=5} \dots$ So S is countable.

$$
\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & \cdots \\
 & \swarrow & & \nearrow & & & & & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & \cdots \\
\downarrow & \nearrow & & & & & & & \\
a_{51} & & a_{52} & & a_{53} & & a_{54} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

(From now on this listing procedure will be referred to as *"Cantor's Zig-Zag"*.)

Corollary 1. The set of rational numbers, Q, is countable.

Pf. Let E_n = {all fractions with denominator n}. E_n is countable. So $Q = \bigcup_{n=1}^{\infty} E_n$ is countable.

Corollary 2. The set of all n-tuples of rational numbers, $Q^n = Q \times Q \times \cdots \times Q$, is countable.

Outline of Pf. First, build a grid where rows are elements of one copy of Q and columns are elements the a second copy of Q.

Use Cantor's Zig-Zag to list all the 2-tuples (ordered pairs) of rational numbers.

This verifies that $Q^2 = Q \times Q$ is countable.

Now, use math induction to extend this result to Q^n.

Corollary 3. The set of algebraic numbers, A, is countable.

Pf. (The coefficients of all polynomials in this argument are are assumed to be rational.)

Every polynomial of degree n can be uniquely identified with a member of Q^{n+1} (its n-tuple of coefficients). Therefore, according to Corollary 2, P_k = {all polynomials of degree k} is countable.

By the Theorem $P = \bigcup_{k=1}^{\infty} P_k$ = {all polynomials} is countable. Finally, for each polynomial $p_m(x) \in P$ define A_m = {all the real roots of $p_m(x)$}. Since all the A_ms are finite (hence, countable), $\bigcup_{m=1}^{\infty} A_m = A$ must be countable.

But A = {all algebraic numbers}, so the set of algebraic numbers, A, is countable.

Corollary 4. The set of transcendental numbers, T, is uncountable.*

Pf. Suppose T is countable. Then $A \cup T$ would be countable. But $A \cup T = R$ is uncountable. So T must be countable.

*There are a lot more trans. numbers (like π & e) than there are alg. numbers (like $\frac{4}{5}$ & $\sqrt{2}$ & i). There are far fewer algebraic numbers than reals.