A Fixed Point

. . . and then some

Part 2

Bellevue College Math/Physics Colloquium
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February 2026

In general, a <u>fixed point</u> does not move or change when it undergoes some action;
i.e. when a function is applied to it.

<u>Definition</u>. Let M be some sort of space, f a function that maps M into M, and p an element of M. If $f(p) = p$, then p is called a **fixed point of f**.

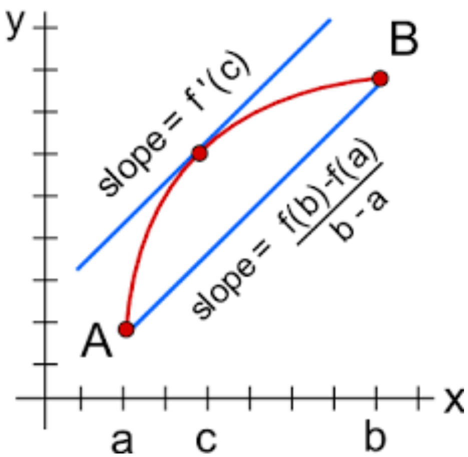
What are fixed points good for?

- Fixed point theorems guarantee solutions to problems where explicit formulas are impossible, such as proving a general equilibrium in economics or existence of solutions for differential equations.
- Many computational methods (iterative algorithms) work by finding fixed points.
- They describe equilibrium points in dynamical systems that change over time, showing where systems stabilize: like carrying capacity in population dynamics or economic models.
- In computer science they're used in proving algorithm termination, understanding stable states in processes, and even in low-cost hardware for fixed-point arithmetic.
- In engineering they're essential for calibrating instruments and analyzing complex physical phenomena.

In essence, fixed points help us find answers, understand stability, and model reality by providing powerful tools to analyze problems that seem unsolvable otherwise. This talk isn't about any of these applications . . . save one example. It's about the mathematics behind them.

Last session we discussed a fixed point of cosine in some detail: $d \approx 0.739085$, called the Dottie Number, and we saw from the graphs of $y = cos(x)$ and $y = x$ that d is cosine's only f.p. I'd like to dig a little deeper. What could we learn without a calculator? Was it just an accident that pushing the *cos* key over and over again led to d?

For example, we could have predicted the existence of a fixed point for cosine by just knowing the Mean Value Theorem for derivatives! From now on let's think of cosine defined on the closed interval $I = [\alpha, \beta]$, where $\alpha = 1^\circ = \dfrac{\pi}{180}$ radians and $\beta = 89^\circ = \dfrac{89\pi}{180}$ radians.



Let me remind you about the <u>Mean Value Theorem</u>:

If f is differentiable, then

$$\frac{f(a) - f(b}{a - b} = f'(c) \quad \text{or} \quad f(a) - f(b) = f'(c) \cdot (a - b)$$

for some c between a and b.

Specifically, if f is the *cosine* function, then <u>for every x & y in I</u>, there must be a c in I, at which $|cos(x) - cos(y)| = |-sin(c)| \cdot |x - y|$. So $|cos(x) - cos(y)| < |x - y|$, since $|-sin(c)| < 1$.

Even better, setting $r = sin(\beta) < 1$, we have $|cos(x) - cos(y)| \leq r \cdot |x - y|$,

since $|-sin(c)| = |sin(c)| < |sin(\beta)| < 1$.

(Later it will be important that this r is <u>strictly smaller</u> than 1.)

Using the first of these inequalities, we can find out a little bit about the existence of fixed points, in general.

- <u>Edelstein's Theorem</u>

If F is a strictly contractive function defined on a closed interval I $(|F(x) - F(y)| < |x - y|)$, then F has a unique fixed point.

<u>Pf.</u> Define $G(x) = |x - F(x)|$. G is continuous, so it attains a minimum; i.e. there is a value, p in I, for which $|p - F(p)| \leq |x - F(x)|$ for all x in I.
Claim: <u>p is a fixed point of F</u>, which we'll prove by contradiction.
Suppose p is not a fixed point of F. Let $a = F(p) \neq p$, and so $|p - F(p)| > 0$.
The contraction inequality implies that $|a - F(a)| = |F(p) - F(F(p))| < |p - F(p)|$ or $G(a) < G(p)$. But this contradicts $G(p)$ = minimum value of $G(x)$.
So p must be a fixed point of F.
Furthermore, <u>p is the only fixed point of F</u>. For if there were another one, say $q = F(q)$, then $|F(p) - F(q)| = |p - q|$, contradicting the contractive inequality.

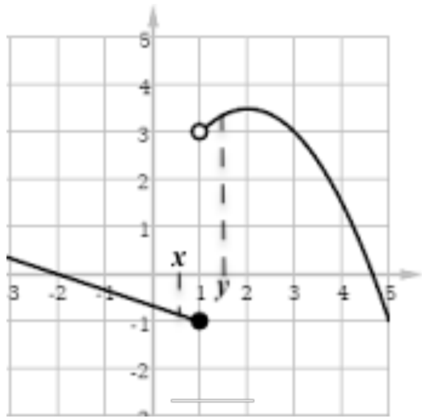
Since $|cos(x) - cos(y)| < |x - y|$, Edelstein's Theorem guarantees the existence of d and its uniqueness. But it tells us nothing about how we might go about determining the value of d.

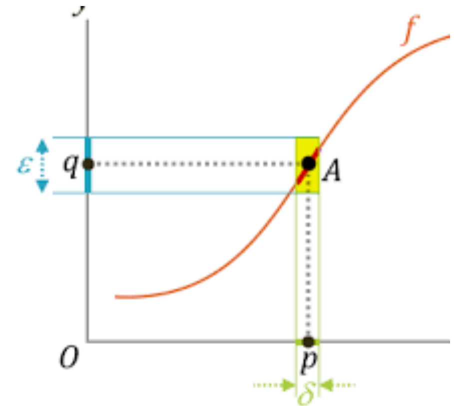
For that we'll need some more machinery — some review, some new.

(1) Continuity

- **Continuous** means "you can guarantee that $f(x)$ and $f(y)$ are as close together as you want by choosing x and y close enough together."

For all $\epsilon > 0$ there is a $\delta > 0$ for which $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.





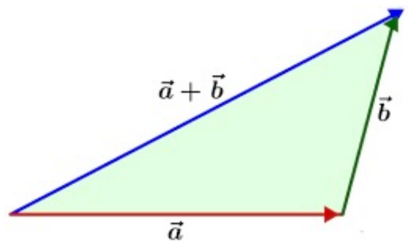
Discontinuity: no natter how close you squeeze x & y together, the distance between $f(x)$ & $f(y)$ never gets ay smaller than 4.

Continuous: given ϵ around $f(p)$. Choose δ around p.

- If $x_1, x_2, x_3, \ldots . x_n \ldots \to L$ and f is continuous, then $f(x_1), f(x_2), f(x_3), \ldots . f(x_n) \ldots \to f(L)$. We say that "continuity preserves" the convergence of sequences."

Pf. Let ϵ be an arbitrarily small number. There is a value δ for which $|f(x) - f(y)| < \epsilon$, whenever x and y are within δ of each other. You can make the tail end of $\{x_n\}$ as close to L as you want by taking n large enough, say $x_N, x_{N+1}, x_{N+2}, x_{N+3+} \ldots$ are all within δ of L.
Thus, for all $n \geq N$ $|f(x_n) - f(L)| < \epsilon$.

(2) <u>Triangle Inequality</u>: $|a+b| \le |a|+|b|$



<u>Pf.</u> $(|a|+|b|)^2 = |a|^2 + 2|a||b| + |b|^2$

$$\ge a^2 + 2ab + b^2\ , \text{ since } |a| \ge a \text{ and } |b| \ge b$$
$$= (a+b)^2$$

Since the $\sqrt{\ }$ function is increasing, taking the square root of both sides preserves the inequality; i.e. $|a|+|b| \ge |a+b|$.

(3) <u>Cauchy Sequences</u>

- A sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ is called **Cauchy**, if eventually it's terms get arbitrarily close together.

No matter how small ϵ is, there is a number N_ϵ for which $|a_i - a_j| < \epsilon$, whenever $i, j > N_\epsilon$.

Another way of saying this is: $|a_i - a_j| \to 0$ as $i \,\&\, j \to \infty$.

- If a sequence of real numbers $\{a_0, a_1, a_2, a_3, \ldots\}$ converges, then the sequence must be Cauchy.

<u>Pf.</u> Let $\epsilon > 0$ and $\lim_{n\to\infty} a_n = L$. Choose N large enough so that $|a_k - L| < \dfrac{\epsilon}{2}$ for all $k > N$.

Then if m and n ≥ N we have (by the Triangle Inequality)

$|a_n - a_m| = |(a_n - L) + (L - a_m)| \le |a_n - L| + |L - a_m| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2}$, so that $|a_n - a_m| < \epsilon$ for all $m, n \ge N$, which proves that the sequence is Cauchy.

The converse is also true.

- **If a sequence of real numbers $\{a_0, a_1, a_2, a_3, \ldots\}$ is Cauchy, then the sequence is convergent.**

<u>Pf.</u> *(Too long and technical to present here.)*

This is a really important result! The property "Cauchy implies convergent" is called completeness. The set of real numbers is said to be complete.

<u>Note</u>

> *The set of rational numbers is <u>not</u> complete, because there are Cauchy sequences of rational numbers that do converge to rational numbers. For example,*
> *{1, 1.4, 1.41, 1.414, . . . } is Cauchy, but converges to* $\sqrt{2}$ *, which is irrational;*
> $\{[1+(1/n)]^n\}$ *is Cauchy, but converges to e* $\approx 2.71828....$ *, which is not rational.*

Here are two more things about Cauchy sequences for you to chew on.

- The Cauchy criterion is much stronger than just saying that *"the gaps between successive terms get arbitrarily small."* $|a_n - a_{n-1}| \to 0$ as $n \to \infty$ does not mean that $\{a_n\}$ is Cauchy or that the sequence converges. For instance, let $h_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. $\{h_n\}$ is the sequence of partial sums of the harmonic series, which we know diverges; i.e., $\{h_n\}$ is divergent and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n} = \infty$.

But $|h_n - h_{n-1}| = \dfrac{1}{n} \to 0$.

- Given our penchant for counting and listing and such, human beings seem to be hardwired with the whole numbers.

  Rational numbers grow naturally out of the physical world as parts or ratios of real-life things.

  But real numbers just seem to magically appear to plug infinitesimal gaps in the rationals. Cauchy sequences can be used to "construct" the reals and make them "concrete."

Maybe somebody will talk about this construction process at this seminar one of these days.

(4) **Geometric Series**: If $r < 1$, then $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

Pf. Let S_n stand for the *nth* partial sum of the series.

Then $S_n = 1 + r + r^2 + \cdots + r^n$ and $r \cdot S_n = r + r^2 + \cdots + r^n + r^{n+1}$.

So $S_n - r \cdot S_n = (1 - r) \cdot S_n = 1 - r^{n+1}$.

Dividing by $(1 - r)$ yield $S_n = \dfrac{1 - r^{n+1}}{1 - r}$.

Finally, since $r < 1$, $r^{n+1} \to 0$, and $S_n \to \dfrac{1}{1-r}$. i.e., the sum of the series is $\dfrac{1}{1-r}$.

I think we're finally in position to say something about the recipe for locating the Dottie Number.

(5) Banach's or Picard's Fixed Point Theorem

- If I is a closed interval, $C : I \to I$, and there is an $0 < r < 1$ for which
$$|C(x) - C(y)| \le r \cdot |x - y| \quad \forall\, x, y \,\epsilon\, I\,,$$
then there one and only one point, x^*, such that $C(x^*) = x^*$.

Pf. (In this argument exponents on C stand for repeated composition, not multiplication.)
First note that C must be continuous (given $\epsilon > 0$, choose $\delta = \epsilon$).
Apply the inequality above to $C(x)$ and $C(y)$:
$|C(C(x)) - C(C(y))| \le r \cdot |C(x) - C(y)| \le r^2 \cdot |x - y|$ or $|C^2(x) - C^2(y)| \le r^2 \cdot |x - y|$.
Applying the inequality n times we get $|C^n(x) - C^n(y)| \le r^n \cdot |x - y| \;\; \forall\, x, y \,\epsilon\, I$.
Now, select any $x_0 \,\epsilon\, I$, and define $x_1 = C(x_0),\ x_2 = C(x_1), \ldots,\ x_{n+1} = C(x_n)$.
In general, $x_n = C^n(x_0)$. We're going show that $\{x_n\}$ is Cauchy. To this end, let m and n be any two whole numbers with $m > n$, say $m = n + p$. Then by the Triangle inequality we have

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n+p}| \\
&= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} - x_{n+3} + x_{n+3} + \cdots + x_{n+p-1} - x_{n+p}| \\
&\le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + |x_{n+2} - x_{n+3}| + \cdots + |x_{n+p-1} - x_{n+p}| \\
&= |C^n(x_0) - C^n(x_1)| + |C^{n+1}(x_0) - C^{n+1}(x_1)| + \cdots + |C^{n+p-1}(x_0) - C^{n+p-1}(x_1)| \\
&\le r^n \cdot |x_0 - x_1| + r^{n+1} \cdot |x_0 - x_1| + \cdots + r^{n+p-1} \cdot |x_0 - x_1|| \\
&\le r^n \cdot |x_0 - x_1| \cdot (1 + r + r^2 \cdots + r^{n+p}) \\
&< r^n \cdot |x_0 - x_1| \cdot (1 + r + r^2 + r^3 \cdots) \\
&= r^n \cdot |x_0 - x_1| \cdot \frac{1}{1 - r} \to 0\,, \text{ since } r < 1.
\end{aligned}$$

We've shown that $|x_n - x_m| \to 0$ as m & n $\to \infty$. So $\{x_n\}$ is Cauchy, and must converge. Let $\lim_{n\to\infty} x_n = x^*$.

Also, the continuity of C implies that $\lim_{n\to\infty} C(x_n) = C(x^*)$. But $\{C(x_n)\}$ is just $\{x_n\}$ with the first term lopped off. So $\{C(x_n)\}$ must coverage to the same thing as $\{x_n\}$.

In other words, $C(x^*) = x^*$; that is, x^* is a fixed point of C.

To see that x^* is the only f. p., suppose that it's not. Then there's a $y^* \neq x^*$ with $C(y^*) = C(x^*)$ and $|x^* - y^*| = |C(x^*) - C(y^*)| \leq r \cdot |x^* - y^*|$. Dividing everything by $|x^* - y^*|$ implies that $1 \leq r$, which contradicts $r < 1$.

Ta–da! or Voila! or Whew! Q.E.D. or ■ or ▮ or The proof is complete!*

This theorem, actually it's proof, shows that the process we used to discover d = the Dottie Number was not just an accident: iterating any function that satisfies $|C(x) - C(y)| \leq r \cdot |x - y|$ (like *cos*) will produce a fixed point.

In fact, suppose M is any space with a notion of distance that satisfies the Triangle Inequality defined on it (called a *metric space*). If M is complete, you can replace the closed interval, I, with M in the statement of the Banach/Picard Theorem, and the result will still hold. The proof is identical.

["The symbol ▮ is definitely not my invention – it appeared in popular magazines (not mathematical ones) before I adopted it, but I seem to have introduced it into mathematics. It is used to indicate an end, usually the end of a proof. It is frequently called the 'tombstone', but at least one generous author referred to it as* ***'the halmos'****."]*

This quote is from I Want To Be A Mathematician — an autopathography by Paul R. Halmos.]

(6) **<u>An application of the B/P Theorem to differential equations</u>**

Here's a typical first order differential equation problem: find a function, $y = y(x)$, that satisfies

$$\frac{dy}{dx} = p(x, y) \text{ with } y(x_0) = y_0$$

A solution to this problem would be a function $\phi(x)$ with $\phi(x_0) = y_0$ and $\phi'(x) = p[x, \phi(x)]]$,

or by integration $$\phi(x) = y_0 + \int_{x_0}^{x} p[t, \phi(t)]dt .$$

What does this have to do with fixed points?

Well, let M be the set of continuous, real-valued functions on a closed interval that contains x_0.
With distance between functions in M defined by $||f - g|| = \max |f(x) - g(x)|$, it turns out that M is a complete metric space.

Define $C : M \to M$ by $C(\phi) = y_0 + \int_{x_0}^{x} p[t, \phi(t)]dt$.

Then ϕ satisfies our d.e. if and only if $C(\phi) = \phi$. i.e., ϕ is a fixed point of C.

It can be shown that C satisfies the hypothesis of the B/P Theorem.* So C must have a fixed point, and we know that a solution exists.

* *p must satisfy a Lipschitz condition on a rectangle containing (x_0, y_0):*
$|p(x, y_1) - p(x, y_2)| \le K|y_1 - y_2|$ *for some $K > 0$, also.*

(7) Probably the most famous result about fixed points is Brouwer's Fixed Point Theorem.

- If you gently stir cup of coffee (no splashing!), then at least one point on the coffee's surface will return to it's original location.

- Take two identical maps of a country. Lay them on the ground, one on top of the other, so that their boundaries match-up exactly. Crumple the top map up, and drop it onto the one left on the ground. At last one point of the crumpled map will be directly above its original location on the ground.

Does this seem/feel surprising or counter-intuitive to you? It does to me.

- If $f : D \to D$ is continuous, where D is a closed disc, then f has at least one fixed point.

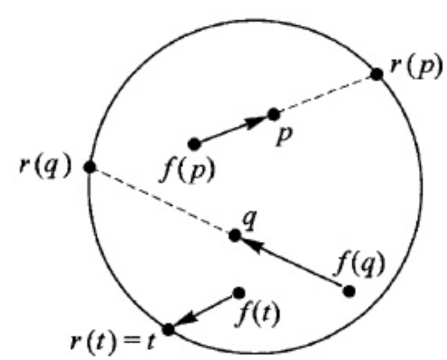
  A careful proof requires algebraic topology. All I can offer is a heuristic argument that may, in fact, be circular (no pun intended). I'm not sure.

  Step 1. Imagine a circular piece of rubber with its edge nailed to a table. It's impossible to slide, stretch, compress or otherwise deform the rubber so that every single molecule ends up on the nailed-down edge without ripping the interior. It's impossible to create a continuous function from a closed disk to its boundary (a circle, C) that leaves every point on the boundary fixed.

  Step 2. Assume f has no fixed points. Then $f(x) \neq x \;\; \forall\, x \in D$. Define $r : D \to C$ according to the diagram.

  Step 3. r is continuous and leaves the boundary fixed.

  Step 4. We just constructed something that was supposed to be impossible. So there must be some fixed points.



(*Note: Brouwer's Fixed Point Theorem is true in all dimensions.*)