Martingales

With application to Brownian Motion

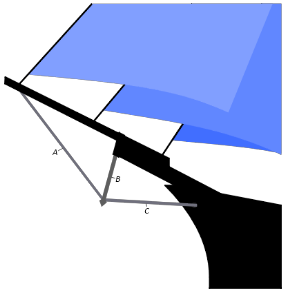
by Charles Widger, 5/19/2026 ver 1.0

Martingale

Source of word: French from Occitan

- Probability distributions (our use, here)
- Betting scheme in 18th century France
- Horse tack (used to control the height of the head when ridden)
- Clothing term
- Collar for dogs and other animals (used when the neck is larger than the head)
- Sailing
- Sword strap (used in fencing to retrieve sword when dropped)
- Theater lighting (obsolete, now called a twofer, a parallel wiring junction)

When unbuttoned, garment folds out to be a blanket



a) martingale stay, b) martingale, c) bobstay

References

Sources for Learning More

- **Random Walks and Electrical Networks**, Peter G. Doyle and J. Laurie Snell (1925-2011), Mathematical Association of America (MAA), 1984, 159 pages, ISBN 978-0883850244. Suitable for undergraduates (#22 of the MAA Carus Monograph series, despite having two authors).
- **An Introduction to Probability Theory and its Applications**, William Feller (1906-1970), Volume I, Second Edition, 1957, John Wiley & Sons, ISBN 978-0471257080. Volume II, 1966, John Wiley & Sons, ISBN 978-0471257097. Volume I covers the discrete case, Volume II the continuous case. Several applications given.
- **Brownian Motion, Martingales, and Stochastic Calculus**, Jean-François Le Gall, Springer-Verlag, 2013, ISBN 978-3-319-31088-6. Graduate Texts in Mathematics 274, translated from French. Not for beginners.
- **Introduction to Concepts and Theories in Physical Science**, Gerald Holton (1922 -), Addison-Wesley, 1952, 650 pages. Second edition, 1973, 608 pages, ISBN 978-0201029710, available used at Amazon.com. Undergraduate level text. Speaks to Brownian motion. Author specialized in the history of physics.
- **Classical Potential Theory and Its Probabilistic Counterpart**, Joseph L. Doob (1910-2004), Springer-Verlag, 1984, 846 pages, ISBN 0-978-3-540-41206-9. Grundlehren der mathematischen Wissenschaften 262. Not for beginners. Commentary by Ronald Getoor (1929-2017).

Stochastic Processes at Bellevue College

Occurrences of the topic

- Math Club, ca. 2024
  - Question: Consider the half plane of pairs of integers with non-negative x values and unrestricted y values. Pick any point as a starting point. Make a random walk, where a decision is to go either left or right, or up or down, with equal probability. Will the walk ever reach a point where the x-value is 0, so the walk must terminate?
- My Presentation on Information Theory, 2025
  - A source for the talk, a Great Course, claimed Information Theory guarantees a gambler of infinite wealth will always recover all losses and get even more to boot. This not included in my talk then because the hypothesis is always false, as I will discuss here.

Definitions

Probability Space and Conditional Probability and Expectation

- A probability space consists of three elements $(\Omega, \mathcal{F}, P)$, where
  - Ω is the *sample space* of all possible outcomes a *random process*, also known as a *stochastic process*.
  - $\mathcal{F}$ is the *event space* of the subsets of interest of Ω.
    - Example 1: The die is thrown once and displays a side.
    - Example 2: The die displays an even number of dots.
  - P is the probability that the event will happen,
    - Example: A die is a cube, with six sides with dots on each side indicating the side of the die. When the die is thrown, it lands with one side displayed. The number of dots displayed is the result of the throw.
      - For a fair die, the probability that any particular side is displayed is 1/6.
      - For a fair die, the probability that an even number is displayed is $3/6 = 1/2$.
  - Many random processes can be indexed by subset of the natural numbers or by intervals of the real numbers. Those indexed by natural numbers are called *discrete* processes. Those indexed by intervals of real numbers are called *continuous*.
- The conditional probability of an event A will occur, given the event B has occurred is defined by $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$. There is a corresponding conditional expectation.
  - The game show **Let's Make a Deal** featured illustrations of this concept when the deal behind one of the 3 curtains is revealed. The probability of winning the best deal is increased if one changes one's bet, because one of the curtains has been eliminated.

Probability Density Function

Acronym: PDF

- For our purposes, a random variable X has *probability density* f_X where f_X is a integrable non-negative function, where $P[a \leq X \leq b] = \int_a^b f_X(x)dx$. The notion of an integral as taught in calculus can be generalized by Measure Theory. In this case, the event space needs to be a σ-ring. We will use the *probability mass function of a discrete random variable* as the density function over the counting measure.
  - For the counting measures, define the *Kronecker delta function* by $\delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$ for all $i, j \in \mathbb{N}$. Then, for any countable set S and any measurable function $f : S \mapsto \mathbb{R}$, define $\int_S f(\delta)d\delta = \sum_{i \in S} f(i)$.
  - The *cumulative distribution function* of X is then defined by $F_X(x) = \int_{-\infty}^{x} f_X(u)du$.
  - Part of the theory of stochastic functions involves structuring the event space. An example is the notion of a *filtration*.
- If a random variable X has a probability density function f, then the *expected value* of X is given by $\mathrm{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$.

Development of Martingales

Source of name

- Martingales originally referred to a class of betting strategies in 17th Century France.
    - Example:
        - A gambler at a casino places a bet by placing money, probably as tokens, into an area called the pot. The house matches the bet in the pot. The game is to flip a fair coin. If the result is Heads, the gambler collects all the pot contents. If the result is Tails, the casino collects the pot contents.
        - Strategy: double the stake if a loss occurs. This recoups the loss if win occurs and pays out the additional amount of the bet as well.
        - The strategy fails because the exponential growth of the debt eventually bankrupts the player.

Martingales

Mathematical Definition

- Paul Lévy (1886-1971) in 1934 introduced the concept of martingales, and Jean André Ville (1910-1989) so named them in 1939. Joseph Leo Doob (1910-2004) did much work in the area. More on the name, here.
- The gambling game can be modeled as a martingale.
  - Define a sequence of coin toss outcomes as a set $X_i, 1 \leq i \leq n, i, j \in \mathbb{N}$, where each $X_i = \begin{cases} 1, & \text{if flip is Heads} \\ 0, & \text{if flip is Tails} \end{cases}$.
  - Further, set the Expectation as $\mathbf{E}(X_{n+1} \,|\, X_1, X_2, \cdots, X_n) = X_n$. That is, the conditional expectation of the next outcome, given all the previous outcomes, is just the last outcome. A discrete martingale also needs $\mathbf{E}(\,|\,X_i\,|\,) < \infty$ for all i, as this one does.
- This set of definitions is a martingale because the coin is defined to be fair. If the coin were unfair in favor of the casino, it would be a *supermartingale*. If in favor of the gambler, it would be a *submartingale*. If one wished to speak of either of these without naming which, one says *semimartingale*.
- These concepts can be extended to continuous stochastic processes. Most presentations on this topic require knowledge of measure theory. It turns out that a continuous martingale can be approximated by discrete ones.
  - The theory allows one to consider ¨stopping times;¨ for example, when the gambler goes bust or just quits in disgust.

Derived Martingales

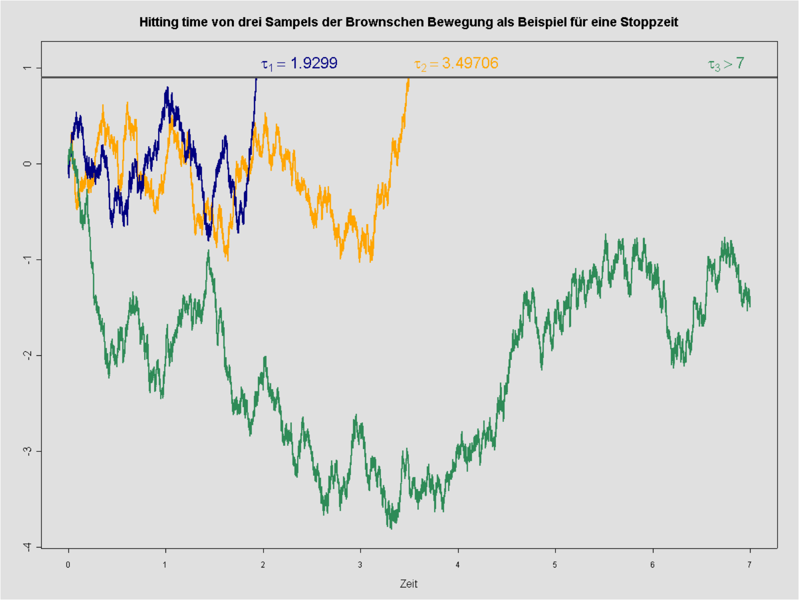
Two Applications

- Start with a process X in which the gambler wins \$1 if the fair coin is Heads, and loses \$1 if tails. Define a process Y by $Y_n = X_n^2 - n$. This captures doubling the bet on a loss and is the amount retained on a win.
  - Note the process Y is still a martingale and the debt indeed grows exponentially in respect to the number of games played.
  - Also note $Y_n = X_n^2$ is a supermartingale. In general, a convex function of a martingale is a supermartingale.
- Definition: given two processes X and Y, Y is said to be a martingale with respect to X when $\mathbf{E}[\,|Y_n|\,] < \infty$ and $\mathbf{E}[Y_{n+1}\,|\,X_1, \cdots, X_n] = Y_n$ for all n.
  - Likelihood testing: A random variable X is thought to be distributed by either a function f or g. Define the likelihood ratio by $Y_n = \prod_{i=1}^{n} \frac{g(X_i)}{f(X_i)}$ for all n. Then, if X is distributed by f rather than g, then Y is a martingale with respect to X.

Stopping Time

As Applied to the Gambler

- One kind of stopping time is to stop the martingale when one of its random processes hits a particular value. An example with Brownian motion is illustrated.
- Suppose the gambler has initial fortune m. Define martingales X and Y as before. Stop at time τ, where $X_\tau = |m|$. The Doob Optional Stopping Theorem says $\tau < \infty$. So the probability that the gambler goes broke is 1.
  - See Reference by Doob, pp. 413 ff.



Hitting times for 3 Brownian motion samples

Brownian motion

Development of Explanations

- Robert Brown (1773-1858) reported motion of pollen grains in water as observed by microscope in 1827. This kind of movement was named for him.
    - Thinking at the time attributed this motion to life in the particles. Later observations noted the same motion with non-living particles, such as dust.
- In 1900, Louis Bachelier (1870-1946) modeled Brownian motion as a stochastic process of particle position in his Ph.D. thesis at the University of Paris under Henri Poincaré (1854-1912). He also applied his methods to valuing stock on the Paris Bourse. This led to the field of mathematical finance.
- In 1905, Albert Einstein (1879-1955) published an article on the topic.
- In 1908, Jean Perrin (1870-1942), provided experimental evidence to show the Einstein predictions are correct. He won the Nobel Prize in Physics in 1926 for this work.
- In 1923, Norbert Wiener (1894-1964) provided a rigorous mathematical basis for this motion. He based it on what is now called *The Wiener Process,* a continuous time martingale.

The Wiener Process

Definition

- The Wiener Process can be defined by its probability density function: $f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$.
- Some derived results include:
  - *Expectation:* $\mathbf{E}[W_t] = 0$, *Variance*: $\mathrm{Var}(W_t) = t$.
  - The *Brownian increment decomposition* $W_t = W_s + (W_t - W_s), s \leq t$.
- The Wiener Process can also be characterized by:
  - $W_0 = 0$ almost surely; that is: with probability 1.
  - W has *independent increments*: for all t>0, $W_{t+u} - W_t, u \geq 0$ are independent of past values $W_s, s < t$.
  - W has Gaussian increments; that is, a time step u results in an increment that is normally distributed with mean 0 and variance u.
  - W_t is almost surely continuous in t.

Expanding Martingale Theory

Applications

- Potential theory deals with harmonic functions; that is, twice differentiable functions $f : U \mapsto \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open and $div(grad(f)) = \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial^2 x_i} = 0$; that is, they satisfy the Pierre-Simon Laplace (1749-1827) equation.
    - There are analogues in the two theories between harmonic functions and martingales. Basically, harmonic functions correspond to martingales, subharmonic functions to submartingales, and superharmonic functions to supermartingales. Specific examples appear in the Reference by Doob, Part 3, pp. 705 ff.
- Queueing Theory
- Stochastic Differential Equations
    - Associated Calculi
        - Itô Integral developed by Kiyosi Itô (1915-2008).
        - Stratonovich Integral developed by Rusian Stratonovich (1930-1997).
- Stochastic Geometry
    - Example: Stochastic models of wireless networks.
- Mathematical Finance

On Learning More

- The University of Washington offers undergraduate courses on Stochastic Processes: MATH 491 A, which covers the discrete case, and MATH 492 A, which covers a part of the continuous case. Fuller details are offered on the graduate level.
- Various articles in Wikipedia.
- Graduate studies at many universities throughout the world.
- The referenced books by William Feller, Jean-François Le Gall, and Joseph L. Doob.