A non-Measurable set

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A primary goal of measure theory is to extend the notion of length to complicated subsets of the line. Ideally, a **measure** is a function with the following properties:

- Every set $A \subset \mathbb{R}$ has a non-negative (possibly infinite) measure: m(A)
- The measure of an interval is its length, i.e. $m([a, b]) = b - a$
- The measure of a set shifted by a fixed amount doesn't change: m(A+x) = m(A). (translation invariance)
- The measure of a set composed of countably many disjoint pieces is the sum of the measures of each piece (countable additivity)

As it happens, no such measure can exist. Here is how to construct a non-measurable set.

To start, recall that an **equivalence relation** generalizes of the idea of equality. Such a relation must satisfy these 3 properties:

- Everything is equivalent to itself. ($x \equiv x$)
- The relation is symmetric (if $x \equiv y$ then $y \equiv x$)
- The relation is transitive (if $x \equiv y$ and $y \equiv z$, then $x \equiv z$)

One well known equivalence relation (the clock equivalence) is $x \equiv y \iff 12|(x - y)$. This is also written $x \equiv y \pmod{12}$. Because of self equivance, every element x will be in some set of things equivalent to x, and any two sets of equivalent things will be either disjoint or identical. They are called the **equivalence classes**. For any element x, the equivalence class that it sits in is also call the **orbit of x**.

Consider now the interval [0,1) and the equivalence relation $x \equiv y \iff (x - y) \in \mathbb{Q}$. This breaks the interval into disjoint equivalence classes. Using the Axiom of Choice, form a set P by choosing one member from each class. Also, define addition $\oplus$ on the interval as $x \oplus y = x + y \pmod 1$. In other words, if the sum of two numbers lies outside the interval, reduce their sum by 1. This effectively makes the interval into a circle of circumference 1.

Notice now that every shifted set of the form $P \oplus q$, where $q \in \mathbb{Q} \bigcap [0, 1)$, will have the same measure. Also, these sets are disjoint and their union is [0,1). Disjoint because if $p_1 + q_1 = p_2 + q_2$ then $p_1 - p_2 \in \mathbb{Q}$, and there is only one member of P from each class. Also, each x in [0,1) is in some class, so differs from the chosen number from P by a rational.

Now we have broken [0,1) into a countable disjoint union of subsets ($\mathbb{Q} \bigcap [0, 1)$ is countable), all having the same measure. By countable additivity of measure, the length of [0,1) would be either 0 (if m(P)=0) or ∞ (if m(P) > 0). Both situations are impossible. Therefore P is not measurable.

This means that there is a subset of [0,1] with no meaningful size. This bothers some people more than others. Perhaps you are unhappy about the Axiom of Choice?

References

[1] Halsey Royden, "Real Analysis"