Introduction

The main results of basic number theory are compelling, easy to state and have intrigued mathematicians, both professional and amateur, for *millenia.*

But in the nineteenth century, when many of the results in later chapters were stated and proved, there was a strong undercurrent of dissatisfaction among mathematicians and philosophers regarding the state of mathematical practice, the nature of its content, its applicability to observed phenomena and its foundations.

This is a story that deserves to be told, and known by the fledgling mathematician, and we give a *very* brief outline of the discord and its roots in this introduction.

In mathematics and logic, **axioms** are statements assumed to be true without proof. These are used as the building blocks in arguments whose goal is to derive previously unknown facts based on these axioms. In an ideal world axioms are supposed to be so obviously true that no one would doubt their validity.

In 1889 Giuseppe Peano (1858-1932[3]) published *The Principles of Arithmetic Presented by a New Method*[4] in which he presented axioms which were to define and codify the properties of the natural numbers (the non-negative whole numbers) clarifying the work of many others including Hermann Grassmann (1809-1877) from around 1870 and Charles Sanders Pierce (1839-1914) from 1881 and work by Richard Dedekind[5] (1831-1916) from 1888.

Whole numbers and their arithmetic were studied by many people, notably the ancient Greeks, who used geometrical methods founded on Euclid's Axioms[6] to

[3] In this text all unlabeled dates will be given in the ''Common Era'' or CE form. Earlier dates will retain the BCE designation.

[4] This very short book, originally written in Latin as *Arithmetices Principia, Nova Methodo Exposita*, is readily available on the internet. One version with parallel English translation by Vincent Verheyen may be found (currently, 2026) at `https://github.com/mdnahas/Peano_Book/blob/master/Peano.pdf`. It is also available starting at page 83 in [26].

[5] Dedekind's 1888 essay *Was Sind Und Was Sollen Die Zahlen?* (What are numbers and what should they be?) may be found starting at page 787 in [15].

[6] Euclid of Alexandria circa 300 BCE reputedly assembled and organized and improved the work of previous mathematicians in *The Elements*. Earlier mathematicians who probably contributed

construct sums, products and deduce factorization theorems. Greek mathematics based on geometry was the ‘‘gold standard’’ for logical rigor for thousands of years and this included their number theory.

So a question arises. Why were the Peano Axioms for natural numbers, an alternative to Greek arithmetic, regarded as important?

The Ancients

Euclid’s axiomatics posited an ontology[7] consisting of points, line segments, circles and other geometrical objects assumed to exist *in some sense* and an epistemology based on deductions using Aristotelian logic based on the syllogism[8] and constructions based on axioms or ‘‘postulates.’’

We determine the existence and discover properties of the objects of geometry through a construction procedure, either imagined or carried out with straightedge and compass according to rules formulated in the postulates.

Below we have a translation of a few of the definitions about numbers from Book VII of *Euclid’s Elements* [23], with a few comments from the author in parentheses.

- A **unit** is that by virtue of which each of the things that exist is called one. (So in the following discussion a unit, or **monad**, must first be selected among the things ‘‘that exist’’ to be used in subsequent definitions.)
- A **number** is a multitude composed of units. (The monad itself is technically not a number. Greek numbers start with the **dyad**, 2.)
- A number is a part of a number, the less of the greater, when it measures the greater. (We would say the part is a factor of the greater. They would deduce this by a construction to exactly cover the number by two or more copies of the part.)
- The greater number is a multiple of the less when it is measured by the less.
- A composite number is that which is measured by some number.

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included Pythagoras of Samos circa 570-495 BCE, Hippocrates of Chios 470-410 BCE and Eudoxus of Cnidus circa 408-355 BCE. It is the most successful textbook ever written, having been used continuously for over 2000 years as the primary text for mathematical instruction in Europe and the Islamic countries. The results in this book are arranged so that later theorems may be proved by deductive methods from earlier ones and, ultimately, from axioms. This is the modern style of mathematical argument, apparently invented by the ancient Greeks.

[7] The word ‘‘ontology’’ refers to the objects presumed to exist in a subject area. ‘‘Epistemology’’ on the other hand describes the means by which subject-experts acquire knowledge about these objects. Plato (circa 427-347 BCE) believed in the literal existence of abstract objects, universals, in a separate world of forms. Aristotle (384-322 BCE) believed that universals are real things but must be tied to objects in this world, with aspects apprehended by humans through sensory experience. So a universal such as ‘‘roundness’’ exists wherever it is instantiated within a round object; the same universal is present in every round object.

[8] See **Appendix A** for a bit more on the Aristotelian and Stoic systems of logical deduction.

- A number is said to multiply a number when the latter is added as many times as there are units in the former. (Since numbers are multitudes of units we have here a multitude of multitudes, combined …*by some procedure.*)

These definitions refer to ''things that exist.'' The only things that exist in this system are those determined to exist by the Euclidean Axioms, such as line segments and circles. Numbers have no independent existence in Greek mathematics apart from a geometrical referent.

Therefore numbers correspond to particular types of geometrical constructions; existence of a number or a theorem about numbers is certified *only* when a construction can be produced corresponding to that number or that theorem.

In spite of the last item on the list above, a product of two numbers is tied to a planar figure, and a product of three numbers to a solid figure. These are not thought of as lengths, areas or volumes as we would do, using real numbers to compare them. Equality of two products of the same type is determined by cutting one figure into pieces to just cover the other, with the allowed operations of course.

Implicit in this approach is reliance on intuition about the geometry of the space around us, and an innate sense of time, in particular the concept of ''consecutive construction steps,'' used to justify existence.

One of Euclid's axioms, his fifth[9], called the Parallel Postulate, implies that parallel lines—that is, line segments that never meet, no matter how far each is extended—do exist.

Addition of segments, tied to addition of numbers, does not require this postulate. But the approach to ''equal'' for planar regions, tied to the product of numbers, relies on a construction that generally does use the Parallel Postulate.

Postulate 5 was widely considered to be unnecessary. Attempts were made over thousands of years to show that the other axioms imply the Parallel Postulate, which could therefore be excised from the list of independent assumptions used to create Euclidean geometry and its arithmetic.

Many mathematicians, particularly from the Islamic world, developed consequences of denying the hypothesis, with the intention of providing a proof of the hypothesis

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[9] The Parallel Postulate is stated as follows in Book I of *The Elements*[22].

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Parallel Lines are defined in Book I by:

Definition 23
Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

It is proved in Book I that Postulate 5 together with the other axioms implies that there are parallel lines.

by contradiction. Apparently *no one* doubted it was true[10]. The consequences, such as the sum of angles interior to a triangle which is *not* equal to two right angles, or the fact that the Pythagorean Theorem is *not* a theorem of non-Euclidean geometry, were thought to be preposterous, validating the Parallel Postulate. In fact, these mathematicians had simply been developing the theory of alternative geometries.

Philosophers

Philosophers such as John Locke[11] (1632-1704) and David Hume[12] (1711-1776) discarded the notion of the reality of abstract objects entirely. Locke proposed that humans began life as a tabula rasa (blank slate) and all subsequent thought was determined by experience and patterns of experience, a stance that brings him among the empiricist philosophers. Hume also felt that our capacity for understanding and reason is based on perceived patterns in personal experience alone.

Hume and later philosophers divided statements in various ways depending on how a statement is true, or could be known to be true. **A priori statements** are those that can be seen to be true or not by reason alone. **A posteriori statements** may be found to be true, but are contingent on experience and evidence: they cannot be derived by reason alone[13]. These distinctions are to be found in Aristotle's *Organon.* See *The Basic Works of Aristotle* [1] for this.

Hume did believe that logical deductions from assumed axioms were valid, presuming the axioms to be non-contradictory. But an axiom bound to a pattern of sensory impressions in the world could not be seen as surely true. Statements about the world derived by logical maneuvers on such axioms must remain uncertain. If it is about the world, and you didn't literally see it, there must be doubt. This brought him to reject the legitimacy of causal and inductive reasoning, putting him squarely among the most skeptical philosophers

Immanuel Kant (1724-1804), a German philosopher and pillar of the European Enlightenment, rejected the ''tabula rasa'' viewpoint of Locke. And while he credited Hume with waking him from his ''dogmatic slumber'' he built a philosophical system in which our knowledge about time, Euclidean geometry and number is innate and central.

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[10] No one except, possibly, Sextus Empiricus, whose doubt seems universal. Sextus lived in the late second century CE. See *Sextus Empiricus Against Those in the Disciplines* [40]. Hume makes a number of references to Sextus' influential skeptical works.

[11] Among Locke's works we note his *An Essay Concerning Human Understanding* published in 1689. Locke and many other philosophers and mathematicians are mentioned in this introductory chapter. Figure 1 on page 6 may assist in keeping it all straight.

[12] Hume's *An Enquiry Concerning Human Understanding* of 1748 contains many of the ideas relevant to us.

[13] Hume actually used a different vocabulary. Hume called a priori facts ''relations of ideas'' and a posteriori knowledge ''matters of fact.'' We use the vocabulary that coincides with Kant.

Kant revised the analysis of Hume and others with an additional analytic/synthetic distinction. **Analytic statements** are those whose truth can be determined by definition of the words involved, such as the statement ‘‘All bachelors are unmarried.’’ Since a bachelor is defined to be an unmarried man this statement is true, regardless of whether or not you know what the words ‘‘bachelor’’ or ‘‘married’’ or ‘‘man’’ mean.

Synthetic statements, on the other hand, are statements that are *not* analytic. Vigorous debate continues on the possibilities for combinations of these properties. Table 1 might assist in keeping this straight.

	Analytic: truth determined by definitions alone	**Synthetic**: truth not determined by definitions alone
a priori: can be seen to be true by reason alone	all analytic true statements are a priori true	truth not determined by definitions alone but possibly by definitions plus features of the reasoning mechanism
a posteriori: truth contingent on experience and observation	Most philosophers find these conditions to be contradictory	all a posteriori true statements are synthetic true

Table 1 Combinations of some properties of statements: how they can be true, and how they may become known to be true.

Kant called innate knowledge[14] **synthetic a priori**, both independent of experience and contingent upon the nature of the world.

Human thought itself depends on synthetic a priori features of our understanding; they reflect the nature of our thought processes. Time, for example, is built into the very nature of thought. We ‘‘arrive’’ at an understanding. We ‘‘come to understand’’ an idea. We ‘‘follow’’ an argument. Without time there is no arriving.

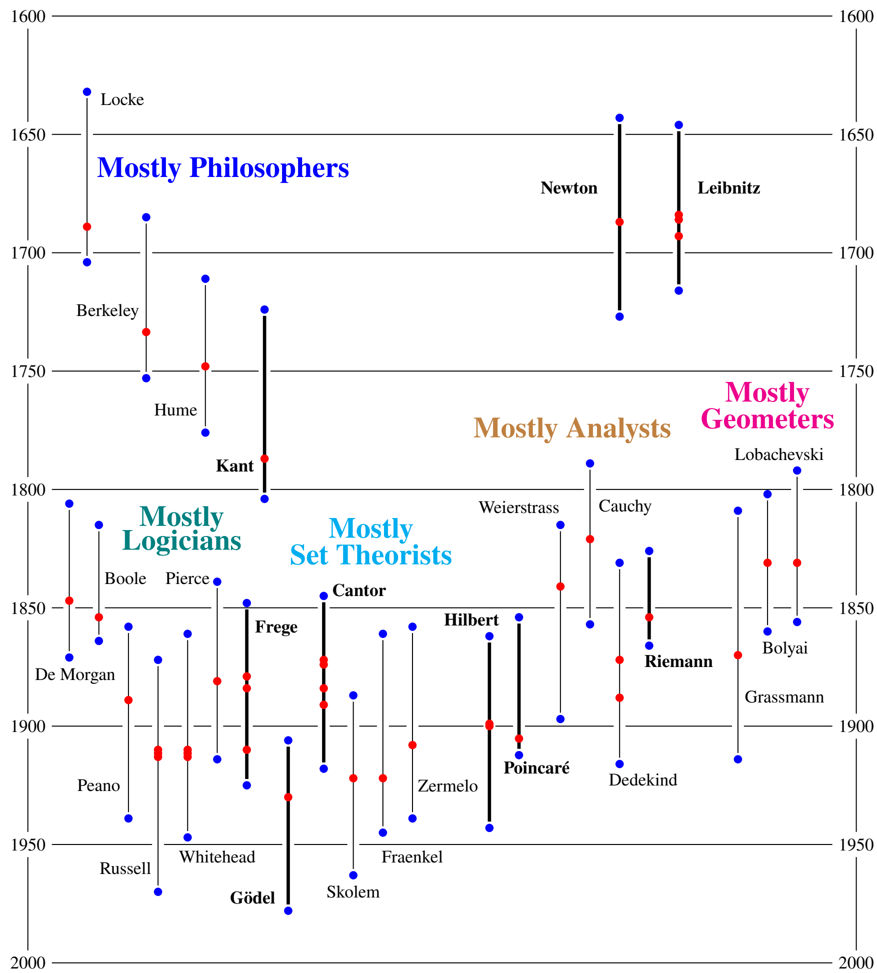
Synthetic a priori knowledge forms the boundary, even the substance, of cognition. We simply cannot conceive of *anything* without it. Though there may be objects in the world for which all synthetic a priori notions are irrelevant, they could not be important to us: humans can never think about or know of them, according to Kant.

[14] There are other ‘‘truths’’ Kant classified as synthetic a priori, such as the principle of causality: that every experience has a cause. He claimed that a Humean observer, even an intelligent one, could never acquire by sensory impressions alone the concept of causality. An awareness of regular association, yes, but the idea of causality comes from our nature as beings with agency. We act, things happen. When we see a billiard ball strike another and the second moves we ascribe that motion to the action of the first. Nowhere in the pure sensations of the event is the idea of causality hidden. That comes, according to Kant, from our cognitive apparatus: synthetic but a priori.

Time, geometry and number, including arithmetic, are on his list of synthetic a priori truths. Logical deductions based on these *could* be secure.

Kant was enormously influential. In the century after the second edition of his *Critique of Pure Reason* [30] was published in 1787 philosophers, both proponents and opponents, had to contend with his ideas. To this day he is regarded as among the most important philosophers.

Fig. 1 These are the (post ancient Greek) logicians, mathematicians and philosophers we mention in developing the story of this Introduction, with lifetime indicated by a line segment and the date of certain publications marked in red. Subject groupings refer to the works of the author as referenced. Hilbert and Poincaré, of course, belong everywhere.



Geometers

Kant's philosophical system, and the favored role for mathematical reasoning based on Euclidean geometry and number therein, was congenial to mathematicians seeking philosophical validation for their work.

But in 1829 Nikolai Lobachevsky (1792-1856) and independently in 1831 János Bolyai (1802-1860) published accounts of geometry that satisfied all of Euclid's axioms except the fifth postulate.

Lobachevsky showed that the geometry now known as hyperbolic geometry does just that, and Bolyai created a structure known as ''absolute geometry'' subsuming both Euclidean and hyperbolic geometries as cases.

This work, at first, went largely unnoticed by the mathematical community, being regarded as an unimportant curiosity. But over the next twenty years, as the significance of these results became clear, it upended millennia of confidence in the inevitability of human geometrical intuition among mathematicians.

Bernhard Riemann's (1826-1866) habilitationsschrift presentation, *On the Hypotheses which Lie at the Bases of Geometry*, delivered in 1854 but made known in a posthumous publication[15] in 1868, raised speculation that non-Euclidean geometries, such as spherical[16] or hyperbolic geometries, might have physical application and the geometry of the universe should be determined by measurement and observation, not assumption.[17]

If something as foundational to our intuition as geometry is a matter of empirical determination, not a priori fact, what other synthetic a priori ''facts'' must we question?

Calculus and its Foundation

Let's step back to the seventeenth century for another thread in this story.

Isaac Newton's (1643-1727) empirically successful—to say the least—calculus was founded on a nebulous notion of fluxions and both mathematicians and philosophers were unhappy that Newton's calculus, successful as it might be, had seemingly

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[15] See Jürgen Jost's *Bernhard Riemann On the Hypotheses Which Lie at the Bases of Geometry* [29]. Jost is much more than an editor here. He puts Riemann's habilitationsschrift in historical, mathematical and philosophical context.

[16] Spherical geometry was known to the ancient Greeks, and used by mariners. Though this geometry does not satisfy the parallel postulate, it also fails to satisfy other axioms of the Euclid system.

[17] This was subsequently found to be true in dramatic fashion: such geometries are the mathematical heart of the General Theory of Relativity. Its creator, Albert Einstein (1879-1955), was heavily influenced by, and greatly admired, the work of both Hume and Kant. But his successful theory of 1915 confirmed doubts about Kant's list of synthetic a priori truths.

abandoned the standards of rigor of Euclidean geometry by including these undefined, possibly undefinable, entities.

The approach used by Gottfried Wilhelm Leibniz (1646-1716) with his infinitesimals had the same problem.

The philosopher George Berkeley (1685-1753) in his 1734 critique *The Analyst* [2] referred to infinitesimals as ‘‘the ghosts of departed quantities’’ and claimed their use invalidated any argument. He also criticized the use of ‘‘the actual infinite’’ by which he meant a completed infinite object, such as an infinite set, as a legitimate independent entity. The complete title of this work is *The Analyst: A Discourse Addressed to an Infidel Mathematician: Wherein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, Than Religious Mysteries and Points of Faith.* Berkeley was a profoundly religious man, and evidently argued, partly, to counter the skeptical critics of his religious ruminations. His claim was that *even mathematicians* appeal to unproved foundations in their work.

Newton’s defenders had trouble rebutting Berkeley’s claims, and were reduced to logically weak responses. For instance they argued by asking ‘‘If calculus is incorrect why are its methods so successful?’’ Thinking about this question might convince a critic that *something* was correct about calculus … but what?

Berkeley’s criticisms proved to be prescient: loosely allowing such entities *does* lead to contradictions. Mathematicians *do* care about the nature of mathematical truth, the rigor of their arguments, the applicability of mathematical truths and the limits of mathematical knowledge. Berkeley, from his earliest publications to the last, had much to say about all this.

All this could not have been satisfying for those who saw themselves, as mathematicians (usually) do, as dealing with universal truth, beyond mere superstition.

One Foot Back in Paradise

This unfortunate, possibly lamentable, situation persisted for over 130 years until Augustin-Lois Cauchy (1789-1857) in his 1821 *Cours D’Analyse* [5] and Karl Weierstrass (1815-1897), finishing the job twenty years later, showed how to use real number arithmetic to re-establish mathematical rigor in analysis in what came to be the study of limits in its modern form of ‘‘ϵ-δ proofs.’’

Then Richard Dedekind and Georg Cantor (1845-1918) using different approaches showed, in 1872, how to use (infinite) sets of natural numbers and their arithmetic to define real numbers and their arithmetic.

It remained only to solidify the basis for natural number arithmetic.

Logicians and Set Theorists

In 1879 Gottlob Frege (1848-1925) published *Begriffsschrift: Eine der Arithmetischen Nachgebildete Formelsprache des Reinen Denkens*.

This may be translated as *Concept-Writing, a Formula Language, Modeled Upon That of Arithmetic, for Pure Thought* and is found in J. van Heijenoort's *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* [26].

In this book Frege essentially invented[18] the modern theory of axiomatic predicate logic, including the first explicit use of the universal and existential quantifiers.

Frege also criticized Kant's classification of whole number arithmetic as innate. In his 1884 book *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number* [17] he sharply differed with Kant that an infinite number of facts about numbers are so obvious they should all be taken as axioms. Is it self-evident, for instance, that $135,664 + 37,863 = 173,527$? Shouldn't that and similar facts about multiplication be *theorems* rather than axioms?

His works were not widely known at the time, but Peano *did* know of them[19].

Revelations by Cantor in the period from 1874 to 1884 on infinite sets, such as the set of natural numbers, and the various ''sizes of infinity'' available to such sets, were also controversial. Some mathematicians accepted and others rejected his approach, but *everyone* realized that infinite sets could not be handled in a cavalier fashion. Many thought that merely accepting the existence of infinite collectives as sets, available for all set operations, would provide contradictions.

The Other Foot

So this was the situation in the late nineteenth century, a *particularly* exciting and confusing time to be a mathematician.

Dissatisfaction with the roots of number theory and concerns, generally, about the relationship between facts derived from axioms and the true nature of the world, and severe uncertainty about how to handle the idea of infinite sets, led many mathematicians to contemplate alternative axiomatics for natural numbers. Preferably these would not rely, to the extent possible, on human intuition. Such a theory should retain, so far as possible, the logical rigor of Greek geometry, and

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[18] Works by George Boole (1854-1864) such as his 1854 *An Investigation of the Laws of Thought* [3] and that of his friend Augustus De Morgan (1806-1871) such as *Formal Logic: or, The Calculus of Inference, Necessary and Probable* [7] of 1847 set the stage for this.

[19] Without a doubt, Frege deserves to be more widely known. *The Frege Reader* [18] contains much of his work as well as letters debating foundational issues with such luminaries as Bertrand Russell.

improve upon its expressive power[20]. And this theory should not rely, excessively, on infinite sets.

Thus, Peano's 1889 replacement logical structure for natural number arithmetic based on very simple algebraic relations, with its careful handling of infinite sets, answered a need.

Peano's Axioms are statements about natural numbers and, in two cases, sets of natural numbers.

A ''model'' of the natural numbers, as conceived by Peano, must contain objects which are to behave as natural numbers do. Declarative statements about natural numbers are called ''first order'' statements in this model. But the model must contain other objects, sets, which might have these natural numbers as members. It is due to these last objects, the reference to sets of natural numbers in the axioms and the means by which they may be deduced to exist, that the structure Peano created is called ''second order.''

The nine axioms listed by Peano are not statements to be proved: they are intended to represent essential features of what we imagine integers *to be*. From these axioms the ordering on the natural numbers and the arithmetical operations of addition and multiplication with their interlinked properties are *generated*, not assumed. The theorems we propose do reflect our human intuition about numbers, and our purpose in proposing this structure. But their *proofs* do not depend on intuition beyond that enshrined in the axioms, and no trace of Euclidean geometry remains there.

The particular model we develop in **Chapter 1** is nowadays denoted PA_2 and was regarded as important progress in the goal to re-establish logical rigor in mathematical practice.

The Story Continues…It's *Complicated*

This, of course, is not the end of the story.

In 1899 David Hilbert (1862-1943) re-organized the principles of Greek geometry in 20 axioms. Hilbert added axioms to include features thought by the Greeks to be too obvious to mention, and clarified the role of the Parallel Postulate. The book in which he outlined his system, translated as *The Foundations of Geometry* [28], was a tour de force[21], but this work did not address all of the philosophical issues mentioned above.

A year later at the International Congress of Mathematicians in Paris Hilbert proposed a list of 23 problems which he saw as key to mathematical progress in

[20] It is not possible, for instance, to express Euclid's Lemma ''There are an infinite number of prime numbers.'' directly in the Euclidean system.

[21] Robin Hartshorne's *Geometry: Euclid and Beyond* [21] gives a thorough and modern description of this system and how it may be used to create number fields, and much more.

the coming century. On that list as Problem 2 was the call to prove the compatibility of the arithmetical axioms. Several other problems on his list refer to number theory, some still unresolved.

In 1903 Frege finally finished volume 2 of his *Grundgesetze der Arithmetik* (Basic Laws of Arithmetic) but, in a famous incident, he included as an addendum a criticism[22] he had received at the last minute from Bertrand Russell (1872-1970) who, upon examining a pre-print version of the book, found a contradiction regarding infinite sets there. These sets were essential to Frege's construction and this fully corroborated the fears of earlier mathematicians and philosophers regarding unrestricted set formation.

Beginning in 1905, and with various modifications upon republishing, Henri Poincaré (1854-1912) wrote a collection of three articles with title *Les mathématiques et la logique* (Mathematics and Logic) in Revue de métaphysique et de morale with a particularly clear critique of foundational matters as they stood at the time, arguing that the Peano axioms should not be taken as ''obviously'' non-contradictory.

This whole series of articles is fascinating, with Poincaré's take on the varied worries of the day and the stances of the mathematicians and logicians involved. Of particular pertinence to us is section XVI which may be found in [15] beginning on page 1036.

In 1908 Ernst Zermelo (1871-1953) provided an axiomatization of set theory, seeking to answer the challenge posed by paradoxes such as the one pointed out to Frege by Russell.

Russell and Alfred North Whitehead (1861-1947) struggled to repair the logical and mathematical issues at the roots of arithmetic in their *Principia Mathematica* in three volumes (1907 *dense* pages) published in the period 1910–1913. Their theory of types[23] made considerable but incomplete progress in this direction.

In 1922 Abraham Fraenkel (1891-1965) and, independently, Thoralf Skolem (1887-1963) modified Zermelo's system in various ways and added the Axiom of Replacement. Normally, mathematicians these days also include the Axiom of Choice to form ZFC set theory, with extremely powerful—but carefully delineated—set formation resources.

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[22] If you allow any explicit property to define a set, as did Frege, then the collective consisting of all sets that do not contain themselves is a set. This set either contains itself or not, and either case leads to a contradiction. Modern set theory puts restrictions on the way properties can be used to define sets to (try to) avoid this type of problem. Most mathematicians feel the effort has succeeded. A similar but not identical problem, involving ordinary language, is the famous story of the Spanish village. The town barber, a man who lives in the town, shaves every man in town who does not shave himself. So who shaves the barber? This contradictory situation is easy to see, but if the contradiction were to be buried under several layers of verbiage it would be very difficult to notice. But all reasoning based on that situation would be insecure.

[23] In theoretical computer science type theories are often used to validate consistency of computer programs or in proof-validation assistants, currently a very active area of research.

And then, surprisingly, in 1930 Kurt Gödel (1906-1978) proved that there are sensible statements in any system rich enough to describe natural number arithmetic with addition and multiplication which cannot be proved to be true or false from within the model. Assuming the system to be consistent[24], one of those statements is ''This collection of axioms is consistent.'' This is an unwelcome answer to Hilbert's problem 2: No proof of the consistency of arithmetic can be devised.

Very few[25] mathematicians believe that the model which blossoms from the simple Peano Axioms, which we develop, with our restrictions on set formation, could be inconsistent: that they could be used to prove the same statement to be both true and false. But Gödel proved that we can never find a proof of that.

This places a permanent, if tiny, ''asterisk'' on *any* attempt to reformulate arithmetic on absolutely unimpeachable logical grounds. Since ZFC set theory can be used to create a model of natural number arithmetic, modern set theory cannot escape this slightly disturbing issue.

The reader may find this to be an unsettling step in a long chain of such discoveries. The Earth is *not* the center of creation. Humans are *not* apart from animals, but among them. Quantum mechanics removes certainty from causal relations, *in principle*.

And now we find that even ordinary arithmetic cannot be taken as absolute. We must take from this yet another lesson in humility, certainly. But if growth can be measured, in part, by the abandonment of cherished but flawed beliefs we may also take pride in how far we have come, as a species, in a mere 3000 years.

Utility and Fascination

Number theory has historically been the province of the ''pure'' mathematician. Mathematicians such as G. H. Hardy reveled in its ''uselessness'' which prevented it from being applied in potentially harmful ways.

That relegation of number theory to the ivory tower has become inapt.

The National Security Agency (NSA), an agency of the United States government, is a major employer of number theorists, whose working journals such as the Cryptologic Quarterly have articles not available to the public. Public key cryptography to secure online banking and blockchain distributed-ledger technologies are ubiquitous. Digital communications over noisy channels are improved by error-correcting codes. All of these and many more require number theory, whose study begins with the material of this book.

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[24] The proof that no proof of consistency can be created in the model *uses* arithmetic and hence relies, itself, on the consistency of arithmetic.

[25] Very few, but not zero. For instance Edward Nelson (1932-2014) apparently believed the system (and with it all of modern mathematics) to be inconsistent. See his intriguing partially completed manuscript (with notes by Sam Buss and Terence Tao) *Inconsistency of Primitive Recursive Arithmetic* [35] at the arXiv for more.

But these current applications, and our own story of paradise lost and (mostly) regained, can't explain the obsession with the *problems* of number theory by countless mathematicians and mathematical hobbyists over *at least* 2500 years of recorded history.

Many books of the ancient Greek, Hindu, Chinese and Arabic mathematicians are on the topic. Some have been lost to time, but are mentioned in those that have survived the centuries.

Sir Thomas Heath comments that Plutarch describes Archimedes, painting a dire picture of a man so enthralled by the subject that ''he neglected to eat and drink and took no care of his person; that he was often carried by force to the baths.''

More ''recent'' work of Euler and Gauss is, perhaps equally, testimony to the compelling nature of the problems in the subject. Legions of current-day mathematicians continue the work.

Writers such as Martin Gardner, whose *Scientific American* column *Mathematical Games* ran from 1957-1979, have introduced many to the pleasures of number theory.

A variety of recreational mathematics magazines and countless websites are devoted to mathematical puzzles which often cross the line to problems of interest to research mathematicians.

One of the wonderful things about number theory is that it is often possible to express such problems in ordinary language. This is a feature number theory shares with graph theory, along with much content. These days it can take years of study at graduate level before one can even understand the statement and context of a mathematical research problem, and it is refreshing to encounter a subject area where this may not be obligatory.

In **Chapters 3** through **6** the reader will engage with many of the classic older results—including unsolved problems. And in **Chapter 7** we see a modern application.

Steps Beyond This Book

For those who wish to extend their understanding of how formulas and statements involving axioms can be created and simplified, and the nature of structures that may be derived from axioms, John Stillwell's excellent books *The Story of Proof Logic and the History of Mathematics* [42] and *Reverse Mathematics* [41] are great starting places. In the first book Stillwell gives an overview of the idea of proof itself in mathematics, historically and to this day. In the second he discusses a hierarchy of five models starting with one simpler than PA_2 and with increasing power to create sets. In these models an increasing fraction of analysis can be stated and then, at the next level, proved. *And the newly provable mathematical fact is equivalent, given the axioms of the previous level, to the new axiom that allowed its proof!*

Along with analysis, most of applied mathematics is available at the top level, which is still vastly less complex than the default ZFC universe whose axioms are used without comment in modern mathematical practice.[26]

Much of the rumination in this introduction may be classified as ‘‘philosophical,’’ rooted in the considerations of the ancients and, more recently, Immanuel Kant. But among the living my favorite philosopher is Penelope Maddy who thinks and writes about the Second Philosopher and the philosophy of mathematics. For those who are intrigued by where this might go I recommend her books *A Plea for Natural Philosophy and Other Essays* [34] and *Defending the Axioms On the Philosophical Foundations of Set Theory* [33] as good starting points.

The reader is referred to Shawn Hedman’s *A First Course in Logic* [25] and Herbert Enderton’s *A Mathmatical Introduction to Logic* [11] to supplement our rather naive approach to logic and model theory. It is my hope that some of our comments and worries about matters of logic (e.g., finite lists) will inspire readers to follow up in that direction. For historical context and *much* else see *Foundations of Set Theory* [16] by A.A. Fraenkel, Y. Bar-Hillel, and A. Levy.

For closer engagement with primary sources consult William Ewald’s *From Kant to Hilbert A Source Book in the Foundations of Mathematics* Volumes I [14] and II [15]. Jean van Heijenoort’s previously mentioned *From Frege to Gödel A Source Book in Mathematical Logic, 1879-1931* [26] is also fascinating reading.

It never occurred to me that I might recommend, in a math text, a book that made #1 on the New York Times softcover graphic novels list, but that day has arrived. I heartily recommend *LOGICOMIX An Epic Search for Truth* [8] by popular science author Apostolos Doxiadis and UC Berkeley computer scientist Christos Papadimitriou. This is an illustrated novelization of the life and times of a young Bertrand Russell, and includes fascinating details of many of the issues we address. A student loaned me a copy and I was skeptical but sat down to give it a look. Three hours later I realized I had read the whole thing and was left wondering why more of our stories, the exciting and compelling stories of mathematicians, logicians, philosophers and other mad-persons, are not told this way. If you find a copy of this inexpensive book you will not regret it.

Having set the stage, we return to the immediate business of **Chapter 1**, the axioms of Giuseppe Peano.

Given the centuries of discord and uncertainty involving mathematical foundations from many directions, as we have briefly described, it is apparent that an instinct for parsimony in assumptions is a good one, limiting the possibilities for internal contradiction. This led Peano, and now leads us, to the system PA_2.

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[26] Most mathematicians are concerned with proving hard theorems in their specialized subject area. They adopt the practices of this area, whose methods may be validated by reference to a foundation such as ZFC set theory. Usually the practitioner is far more interested in these higher-level results than their foundations, trusting that it has all been checked by … *someone.* And of course that may well be true. Or not. See Imre Lakatos’ *Proofs and Refutations: The Logic of Mathematical Discovery* [32].