On Leonhard Euler

Some of his work and its subsequent developments

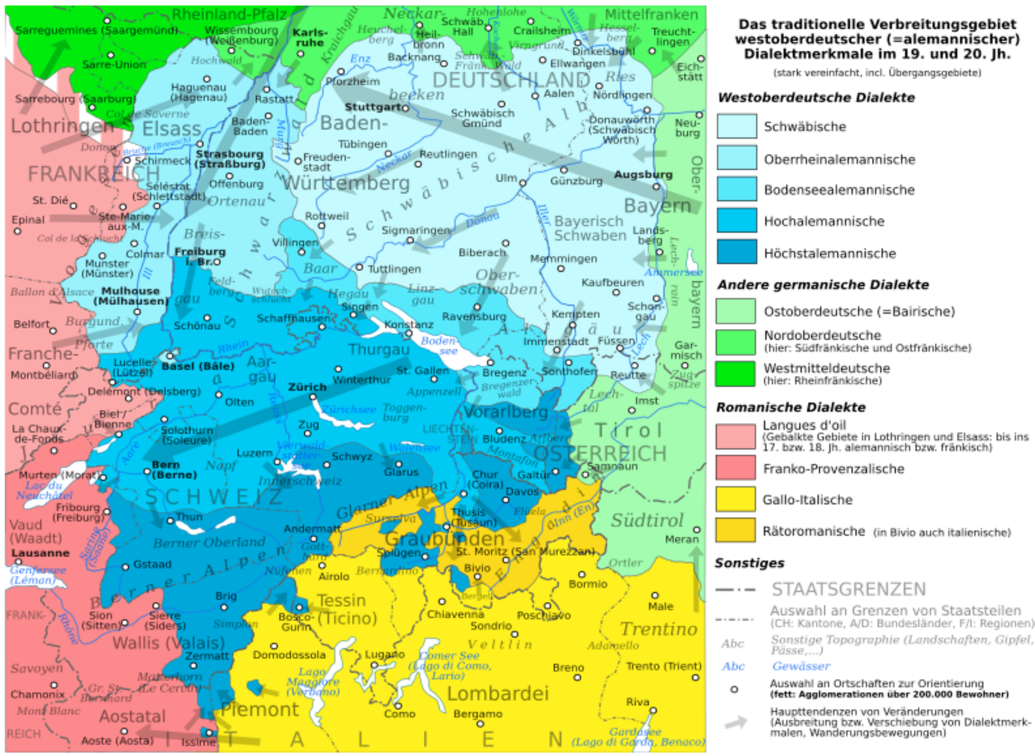
By Charles Widger

As a Person

- Family name is pronounced “oiler”
- Born 1707 in Basel, Switzerland. Died 1783 in St. Petersburg, Russia. He also worked in Berlin.
- Published 866 articles, typically in Latin. He also produced a staggering amount of correspondence.
- Known for work in mathematics, mechanics, fluid dynamics, optics, astronomy, and music theory. This presentation deals only with mathematics.
- Married twice, he had 13 children by his first wife; only 5 survived childhood. After 3 years as a widower, he married a half sister of this first wife.
- Became blind, but remained productive by using scribes. In 1775, as blind, he averaged 1 mathematical paper per week by “being less distracted by sight.”

Basel is a Linguistic Enclave

Most Swiss German dialects are different than that spoken in Basel



About Theories in General

History of Present Day Mathematical Logic

- Today, in daily parlance, a *theory* may merely be a conjecture. In physics, String Theory is not a theory in a mathematical sense. It lacks an underlying basis of axioms for all that it is applied to. Creationism is not a theory, either. One is perplexed to find any theorems of interest following from it. On the other hand, biology has such a theory. See Foundations of Biophilosophy by Martin Mahner and Mario Bunge.
- Euclid made a list of definitions, axioms, and postulates from which theorems are derived. He made errors, as became apparent in the 19th Century.
- Euler typically published work for which he had at least 3 different methods of derivation.
- Work on the foundations of mathematics began in earnest by the mid-19th century, about 75 years after Euler died. Still, Euler established the beginnings of Graphy Theory with his solution to the problem of the seven bridges of Königsberg.

Concepts in Mathematical Logic

A quick overview of some considerations

- A theory includes:
  - The identification of a language to express the statements of the theory. A Formal Theory defines the symbols to be used and what constitutes an acceptable string of these, which is called a well-formed formula (wff, pronounced "woof"). If the theory uses a natural language, such as English, it is call a Naive Theory.
  - A Predicate Calculus which defines how to make acceptable statements from more primitive ones, such as symbols for "not", "and", "or", "implies", and for quantification, such as "for all" (∀) and "there exists" (∃). Rules of Inference, typically Modus ponens "If P then Q; P. Ergo, Q" or Modus tollens, "If P then Q; not Q. Ergo, Not P," and De Morgan's laws.
  - Axioms, that is, statements assumed to be true. Theorems; that is, statements which are either axioms or follow from previously established theorems or axioms by rules of inference.
- Important other concepts include:
  - Consistency; that is, does not contain both a statement and its negative as theorems. Euclid's geometry lacked consistency. It is also Incomplete, in that Euclid presumed that a line had more than 2 points, but this could not be established by his other axioms.
  - Models of the theory; that is, examples for which the theory applies. An example is the integers Z is a model of a group, as is the integers modulo some number. For the latter, think of the 12 numbers that appear on a clock dial, and write 0 for 12. Add as one would add hours on a clock.

Learn more of Mathematical Logic

Minimal choices

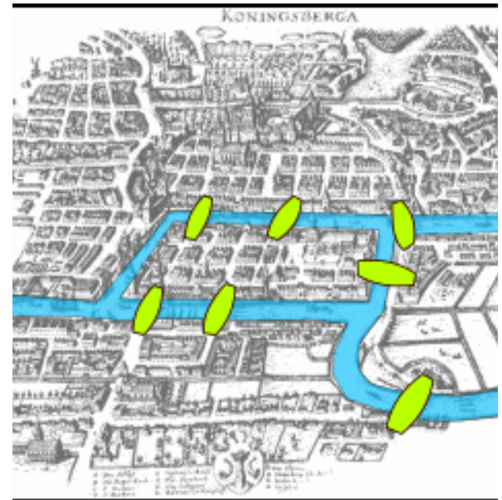
- Take a course
  - The University of Washington mathematics department no longer teaches this topic. Washington State University mathematics department also does not offer this topic.
  - The UW College of Computer Science and Engineering teaches some courses with overlap, such as a course on Digital Design, that introduces Boolean Algebras. Other overlap can be computer network security and fast algorithms, such as the Fast Fourier Transform (FFT), which reduces compution by taking advantage of symmetry between terms.
  - Princeton University and Oberlin College both list such courses.
  - Caltech lists such courses, but none are offered 2024 - 2025.
    - It offered a course for 2024 - 2025 on Computability Theory, which has some overlap.
  - Philosophy departments typically teach logic, but frequently do not cover modern concepts.
- Read a book
  - **Mathematical Logic**, Stephen Kleene, Dover
  - **An Invitation to Mathematical Logic**, David Marker, Springer
  - **Formal Languages and Their Relation to Atuomata**, John E. Hopcroft and Jeffery D. UIllman, Addison-Welsey, 1960

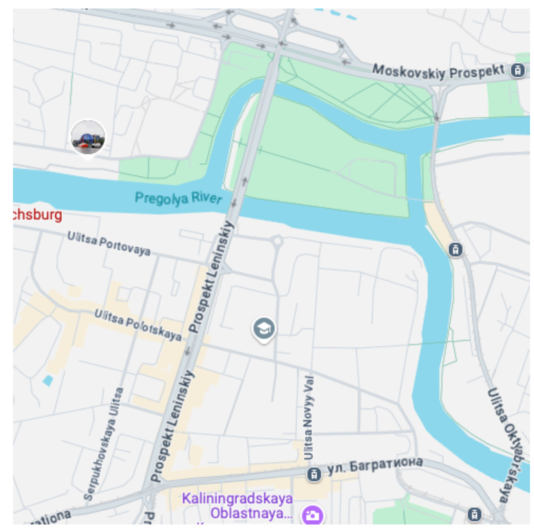
7 Bridges of Königsberg

Solved by Euler in 1736

- Find a path to walk completely across each bridge once and only once. This excludes walking across part way, turning around, walking back, and then later walking back to the turn around point from the other direction. That crosses the bridge at least one and a half times.

- In the current city, now called Kaliningrad, 3 of the old bridges no longer exist. Such a walk is clearly possible, though maybe tiring.

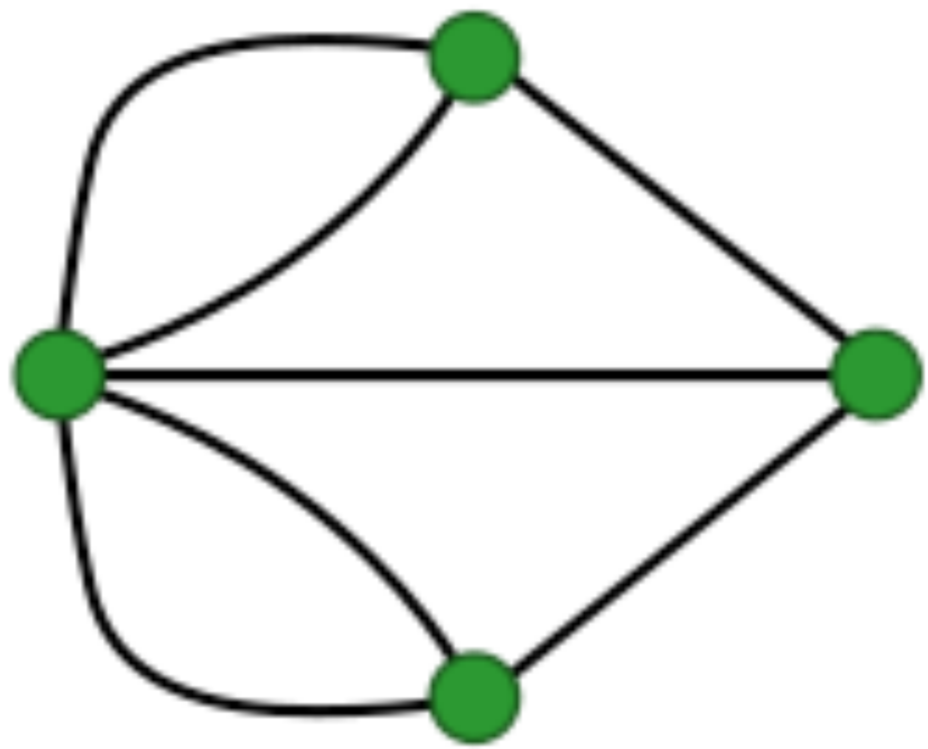




Euler's Solution

Beginnings of Graph Theory

- Observation: the route inside each land mass doesn't matter. Make each land mass a node on a graph.
- Observation: the route becomes an edge of a connection between land masses. To cross only once, the edges must be even, one for coming and one for going, except at the route end points.
- 3 of the 4 nodes have 3 edges. The last one has 5. Each has odd numbers of edges, so no circuit is possible



Sum of Reciprocals of Squares

Also known as the Basel Theorem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) = \frac{\pi^2}{6}.$$

- Euler got the right answer in 1735, but his method was a bit flawed.
  - He published a correct proof in 1741.
  - The Weierstrass factorization theorem, proved 100 years later, showed Euler's first derivation was also correct. This theorem extends the fundamental theorem of algebra: every complex polynomial p(x) is the product of linear factors, one for each root.

Euler Characteristic

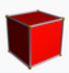
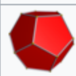
Applies to Polyhedra in R^3

$$\chi = V - E + F$$

- χ defines the *Euler characteristic,* where V is the number of vertices, E the number of edges, F the number of Faces.
- Some polyhedra are convex.
    - A set K in R^d is convex if and only if for each pair of points a, b in K the closed segment with endpoints a and b is contained in K.
- Euler showed for convex polyhedra in R^3, the characteristic is always 2.
    - A proof appears in the book **Convex Polytopes** by Branko Grünbaum, 1967, in section 8.2, pp. 134 - 137 as a result to the formula for even higher dimensions.
- For dimensions 1 and 2, the results are trivial.
    - For dimension 1, V = 2 always. The only possible connected figure is a line segment, which has two endpoints.
    - For dimension 2, The polygon does not need to be convex. Count the vertices by taking a path along its border. Each vertex has only two edges meeting at it. For each vertex, associate the edge leading to the next vertex to be counted. In this way, V= E, so V - E = 0.

Examples of Euler Characteristic

Convex on left, Non-convex on right

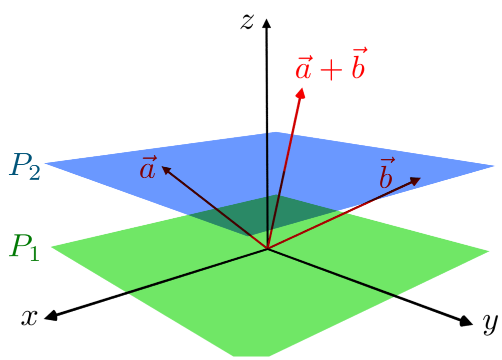
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahemihexahedron		6	12	7	**1**
Octahemioctahedron		12	24	12	**0**
Cubohemioctahedron		12	24	10	**−2**
Small stellated dodecahedron		12	30	12	**−6**
Great stellated dodecahedron		20	30	12	**2**

Concepts for Geometry

Generalizing the notion of faces of a 3 dimensional object

- The *Convex Hull* of a subset A of R^d is the intersection of all convex sets containing A.
- An *Affine Space* generalizes the notion of Euclidean space. Given the points x_j $1 \leq j \leq k$ R^k in general position, $1 \leq k \leq d$, take the points $x = \sum_{j=1}^{k} \lambda_j x_j$, those for which $\sum_{j=1}^{k} \lambda_j = 1$ to form the affine space of dimension k, also known as a *flat*.
  - The *Affine Hull* of a subset A of R^d is the intersection of all affine sets containing A.
- Given a convex set K in R^d, a point in K is an *extreme point* if it does not belong to the relative interior of any segment contained in K. The set of extreme points of K is denoted by ext K. Such a set K is a *polytope* if ext K is finite.



Generalization of Euler's Theorem

Applies to R^d, d ≥ 1; as stated in the book by Grünbaum

1. *The affine hull of the f-vectors of all members of the family $\mathscr{P}^d$ of all d-polytopes is given by*

$$\operatorname{aff} f(\mathscr{P}^d) = \{f = (f_0, \cdots, f_{d-1}) \mid \sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d\}.$$

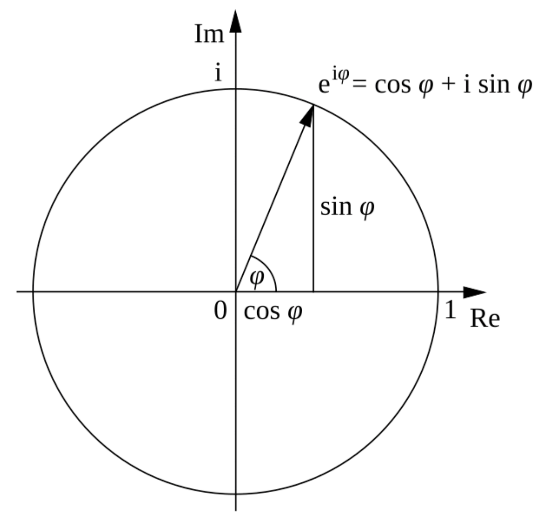
The relation $\sum_{i=0}^{d-1} (-1)^i f_i(P) = 1 - (-1)^d$, which by the theorem holds for every d-polytope P, is known as Euler's equation. Using the extended

- For d = 3, we have $f_0 - f_1 + f_2 = 1 - (-1)^3 = 2$, as Euler proved.
- For d = 4, we have $f_0 - f_1 + f_2 - f_3 = 1 - (-1)^4 = 0$.
  - Example, a Tesseract (4-D cube) has 16 vertices, 32 edges, 24 faces, and 8 3-D cells, and so, indeed, 16 - 32 + 24 - 8 = 0.

Euler's Number

Basis for Natural Logarithms

- In 1683, Jacob Bernoulli (also born in Basel, and a mutual acquaintance of Leonhard Euler), showed compounding interest n times in one year yields a rate of $r_n = (1 + 1/n)^n$. He showed this sequence converges, it being monotonically increasing, (and bounded above), but he did not name the $\lim_{n\to\infty} r_n$.
- Euler named the value of Bernoulli's limit as *e*. He then showed: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- Approximately, $e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352$.
- Euler also introduced $e^{ix} = cosx + isinx$, where $i = \sqrt{-1}$.



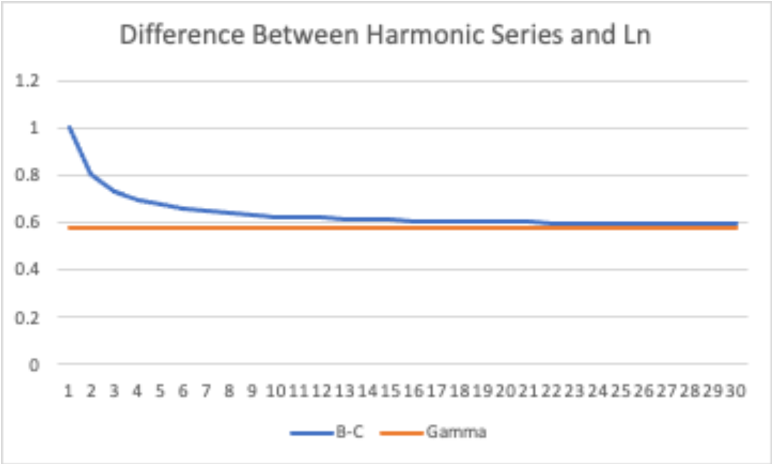
Euler's Constant

The symbol used these days is lowercase gamma

- Euler showed the harmonic series grows as fast as the natural logarithm; that is,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - log(n)\right) = C$$

- Sometime afterwards, lowercase gamma γ was used as a symbol for this constant. Euler used either C or O.
- Current approximate value for γ is 0.57721 56649 01532 86060 65120 90082



Gamma Function

Several equivalent definitions have been given

- Initially defined by Daniel Bernoulli, b, 8 February 1700 in Groningen, Dutch Republic, d. 27 March 1782, in Basel, Swiss Republic.
- The Gamma Function has the property that for every positive integer n, Γ(n) = (n-1)! There are other such functions, but this was investigated first.
- Commonly defined for all complex numbers z, with strictly positive real part, by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$
- An equivalent definition made by Karl Weierstraß is $\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}(1+\frac{z}{n})^{-1}e^{\frac{z}{n}}.$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- Harold Bohr and Johannes Mollerup proved around 1922 that the Gamma Function is the unique solution to the factorial recurrence relation that is positive, logarithmically convex for positive z, and has value 1 at 1. The Bourbaki group defines Garmma this way.

More on Numbers

On properties of *e, π, and γ*

- Both e and π are known to be transcendental. That is, neither is rational nor algebraic. An algebraic number is the root of a non-zero polynomial with integer coefficients. The square root of 2 is an irrational number, but which is algebraic, since it is the root of $x^2 - 2 = 0$. There are a countable number of algebraic numbers and an uncountable number of transcendentals.
    - It is not yet known whether either $(e + \pi)$ or (e times π) is transcendental, but it is known that at least one of these expressions is transcendental.
- It is not yet known if γ is rational, algebraic or transcendental.
    - It is known that if γ is rational, its denominator must greater than 10^{244663}.

Learn More about Geometry

Suggested books

- An elementary book:
  - **Notes on Geometry** by Elmer G. Rees, 1983, Springer-Verlag.
    - Author at the University of Edinburgh
    - Cheaper to buy the Japanese version on Amazon
- A more advanced text:
  - **The Foundations of Geometry and the Non-Euclidean Plane** by George E. Martin, 1975, Springer-Verlag
    - Author at State University of New York at Albany
    - Cheapest to buy the hardcover on Amazon