The Peano Axioms Part Two

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Why is there a Part Two?

At the end of the talk last week an audience member came up and asked me, essentially, why one would care about this alternative way of organizing our thinking about natural numbers and arithmetic.

I gave a typical kind of answer one would give to this kind of question, but this morning, about 5 hours ago, I realized that proving more theorems about PA_2 could wait till the end of today's talk, but the narrative of why mathematicians did care deeply about this at the end of the nineteenth century, and still care, is fascinating.

That will be our story today.

There can be no more interesting and colorful (literally) overview of this topic than the illustrated book LOGICOMIX An Epic Search for Truth by Apostolos Doxiadis and Christos Papadimetriou.

A fascinating podcast of an interview with Doxiadis about his book, why he and Papadimetriou created it and how they did so, can be found at

https://www.lapl.org/books-emedia/podcasts/aloud/logicomix-epic-search-truth.

His is a tale of obsession, even madness, and towering intellects.

I want to start our story even before this book to set the stage.

Euclid of Alexandria circa 300 BCE reputedly assembled and organized and improved the work of previous mathematicians in The Elements. Earlier mathematicians who probably contributed included Pythagoras circa 570-495 BCE, Hippocrates of Chios 470-410 BCE and Eudoxus of Cnidus circa 408-355 BCE. It is the most successful textbook ever written, having been used continuously in one form or another for over 2000 years as the primary text for mathematical instruction in Europe and the Islamic countries.

The Elements used strictly geometrical methods to calculate integer arithmetic, even factorization and multiplication results, prime numbers and so on. These methods used axioms, indisputable assumed truths, and logical principles to create theorems as unquestionable as the axioms used to prove them.

Greek mathematics was the "Gold Standard" of logical rigor throughout Europe and the Islamic world.

If anything at all could be known with certainty, these theorems had to be on the list.

In 825 AD Muḥammad ibn Mūsā al-Khwārizmī introduced the Book of Indian Computation outlining the Hindu-Arabic base 10 number system to the Islamic world and greatly improving the efficiency of the operations of addition and subtraction on integers.

These were adaptations of methods used by Indian mathematicians Aryabhata and Brahmagupta around 500-600 AD.

It was introduced to Europe by Fibonacci around 1202 AD.

Greek mathematics, and the fast-growing modern science based on number and accelerated by Newton and his calculus, was all about intuitions about space—geometry. And time is needed to make sense of logical arguments: strings of statements, one statement following from another by logical maneuvers.

None of this was questioned: Euclidean geometry was simply what it had to be and whole number arithmetic followed as an inevitable consequence.

One of Euclid's axioms or "postulates", his fifth, called the Parallel Postulate, is stated as follows in Book I of The Elements.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Parallel Lines are defined in Book I by:

Definition 23
Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

It is proved in Book I that Postulate 5 together with the other axioms implies that there are parallel lines.

The Parallel Postulate was widely considered to be so obvious as to be unnecessary, and attempts were made over thousands of years to show that the other axioms implied the Parallel Postulate, which could therefore be excised from the list of independent assumptions used to create Euclidean geometry.

No one doubted that this axiom was true, and the foundation of mathematics rested on this Greek edifice.

In 1829 Nikolai Lobachevsky and independently in 1831 János Bolyai published accounts of geometry that satisfied all the Postulates except the fifth. This called millennia of geometrical intuition into question. The fifth is necessary for the theorems of Greek arithmetic.

It is difficult to imagine just how shocking this was. The gold standard of logical truth, the consistency of science based on number and mathematics, was called into question.

It is in this environment that mathematicians sought to replace certainty about arithmetic based on geometry, now lost, with certainty about some new axioms not subject to doubt about the reliability of human geometrical intuition.

In 1889 Giuseppe Peano published The Principles of Arithmetic Presented by a New Method[1] in which he presented axioms which were to define and codify the properties of the natural numbers, clarifying and organizing the work of Hermann Grassmann twenty years earlier and Charles Sanders Pierce in 1881.

[1] This very short book, originally written in Latin as Arithmetices Principia, Nova Methodo Exposita, is readily available on the internet. One version with parallel English translation by Vincent Verheyen may be found (currently, 2025) at https://github.com/mdnahas/Peano_Book/blob/master/Peano.pdf.

The Axioms

Axiom One: There is a natural number which we denote 0. The set $\mathbb{N}$, whose elements are precisely these natural numbers is, therefore, nonempty.

The natural number 0 will turn out to behave as 0 should, but at this point it is merely a symbol to denote a particular element of $\mathbb{N}$, with properties to be introduced in Axioms Eight and Nine.

This gets us started. At least we are talking about something.

The next four axioms codify what we mean by natural numbers being equal and how we are authorized to use the equality binary relation between them.

Axiom Two: For every $n \in \mathbb{N}$ we have $n = n$.

Axiom Three: If k and n are natural numbers and $k = n$ then $n = k$.

Axiom Four: If $k \in \mathbb{N}$ and $k = n$ then n is a natural number too.

Axiom Five: If k is a natural number and $k = j$ and $j = n$ then $k = n$.

The next axioms assert the existence of a "successor" for each natural number and some properties of these successors.

Specifically, for each $n \in \mathbb{N}$ there is an object $S(n)$ defined for it. The association given by S has three properties.

Axiom Six: $S(n)$ is a natural number for each natural number n.

Axiom Seven: For k and n in $\mathbb{N}$, if $S(n) = S(k)$ then $n = k$.

Axiom Eight: $S(n) = 0$ is false for any natural number n.

Finally, we have the Axiom of Induction.

Axiom Nine: Suppose X is a set and $0 \in X$ and suppose also that whenever natural number n is in X then $S(n)$ is in X. Then X contains every natural number.

You will note that none of the axioms contain a hint of geometrical contamination.

Natural Numbers are Ordered

Already we can create an interesting fact about $\mathbb{N}$.

Theorem

Every natural number may be obtained by applying S to 0, iterating this application of S to the output a finite number of times.

Proof.

We will let A be the set consisting of 0 and also any natural number that can be created by applying S to 0 and then S to $S(0)$, iterating the application of S a finite number of times. 0 is assumed to be in A, and if $n \in A$ so is $S(n)$. Axiom Nine, the Axiom of Induction, implies $\mathbb{N} \subset A$ and since $A \subset \mathbb{N}$ we have $A = \mathbb{N}$. $\square$

Let's recall to mind the procedure usually used to represent natural numbers, the Hindu-Arabic numerals, involving concatenation of the ordered list of symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

The first natural number is represented by 0 and the "next one" is 1. Starting with 4329 the "next one" is 4330.

This procedure, this algorithm, does not have anything to do with addition or multiplication directly, though it might imply facts about these after some thought. If you have the base 10 symbol for any natural number it is a completely mechanical process to create the symbol for the next one. This is not arithmetic.

Based on our understanding of what natural numbers represent we would be shocked or even horrified to find an example of numerals x, y where $x + (y + 1)$ was not equal to $(x + y) + 1$, or $x(y + 1)$ was not equal to $xy + x$.

These equalities are theorems of Greek mathematics for numerals but these theorems might or might not hold in the actual world. The independence of the fifth postulate and the actual discovery of geometries where the fifth postulate was false had those who sought secure foundations reeling.

Denote $S(0)$ by the symbol 1. We will see it does behave as 1 should.

Generally if a natural number is built using iterates of S in the same way the numeral k is built starting from numeral 0 we associate that natural number with the symbol k.

Every natural number may be given a label, a numeral, that corresponds to the instructions by which it was created from 0 by application of the function S.

$$7 \longleftrightarrow S(S(S(S(S(S(S(0))))))).$$

And for natural number n associated with numeral k we find $S(n)$ is associated with the numeral $k + 1$.

As we noted in the Theorem, starting with 0 and iterating the application of S every natural number will eventually appear, and it appears only once by Axiom Seven.

If the natural number k appears before the natural number n we write $k < n$ or, synonymously, $n > k$. In words, we say that k is less than n or, synonymously, that n is greater than k.

Thus $0 < k$ for every non-zero natural number k.

It is convenient to expand on this slightly. We declare that k is less than or equal to n or, synonymously, that n is greater than or equal to k if $k < n$ or, possibly, $k = n$. This situation is denoted $k \leq n$ or, equivalently, $n \geq k$.

Theorem

Every nonempty set of natural numbers contains a least element.

Proof.

If X is a nonempty subset of $\mathbb{N}$ then it has a member n. This element must appear on the finite list of iterates of S applied to 0. Using numerals to designate natural numbers we have

$$0,\ 1,\ 2,\ 3,\ \ldots,\ n$$

Examining the elements on this list one at a time starting from 0 we will eventually encounter a member of X, possibly n itself. This first encounter is the least member of X. □

Addition for Natural Numbers

If $n, k \in \mathbb{N}$ define $n + 0 = n$ and $n + S(k) = S(n + k)$.

This is the definition of the addition symbol, and is not to be construed as something having to do with manipulation of the digits in numerals, the labels for n and k which may be used to specify these natural numbers.

In the slides to Part I we show that addition is associative, commutative with identity 0 and is cancellative: that is,

$$a + b = c + b \iff a = c.$$

Multiplication for Natural Numbers

For any a and j we define

$$a \cdot 0 = 0 \quad \text{and} \quad a \cdot S(j) = a + a \cdot j.$$

where, to reduce the number of parentheses required, we give precedence to multiplication over addition, as usual.

In the slides to Part I we show that multiplication is associative, commutative with identity 1 and is cancellative: that is,

$$a \cdot b = c \cdot b \iff a = c.$$

We also have the distributive property:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Two Key Theorems for Number Theory

With the remaining theorems from Part I connecting order and the operations we can prove two key theorems and go on to develop all the basic theorems of Number Theory, including prime factorization, modular arithmetic and so on.

Theorem

Archimedean Order Property:

For each pair a, b in $\mathbb{N}$ with $a > 0$ there is a least n such that

$$a \cdot n > b.$$

Theorem

The Division Algorithm:

For each pair a, b of nonzero natural numbers there is a unique pair of natural numbers q and r for which

$$a = b \cdot q + r \quad \text{and} \quad 0 \leq r < b.$$

The Connection between Numerals and Natural Numbers

So as not to beg the question, we will use $\odot$ and $\oplus$ to denote the ordinary operations on numerals and use $S(0)$ to denote the natural number which we have previously designated as 1, retaining the use of 1 for numerals. If, for numerals j and k, we always have

$$(j \oplus k) \oplus 1 = j \oplus (k \oplus 1)$$

then numerals can be used to calculate sums of natural numbers. Specifically, suppose j and k are numerals associated with natural numbers m and n. Then

$j \oplus k$ is the numeral associated with natural number $m + n$.

Proof.

Fix j and associated natural number m. Let A be the set of those natural numbers n, with associated numeral k, for which $j \oplus k$ is the numeral associated with $m + n$. 0 is obviously in A. And if $n \in A$ then

Proof.

Fix j and associated natural number m. Let A be the set of those natural numbers n, with associated numeral k, for which $j \oplus k$ is the numeral associated with $m + n$. 0 is obviously in A. And if $n \in A$ then

$m+S(n) = S(m+n)$ which is associated with $(j \oplus k) \oplus 1 = j \oplus (k \oplus 1)$.

So $S(n) \in A$ and we conclude that $A = \mathbb{N}$. $\square$

If, for numerals j and k, we always have

$$j \otimes (k \oplus 1) = (j \otimes k) \oplus j$$

then numerals can be used to calculate products of natural numbers. Specifically, suppose j and k are numerals associated with natural numbers m and n. Then

$j \otimes k$ is the numeral associated with natural number $m \cdot n$.

Proof.

Fix j and associated natural number m. Let A be the set of those natural numbers n, with associated numeral k, for which $j \otimes k$ is the numeral associated with $m \cdot n$. 0 is obviously in A. And if $n \in A$ then

$$m \cdot S(n) = m \cdot (n + S(0)) = m \cdot n + m.$$

$m \cdot n$ is associated with $j \odot k$ and m is associated with j so by Proposition 25

$$m \cdot S(n) \text{ is associated with } (j \odot k) \oplus j = j \otimes (k \oplus 1).$$

Since $k + 1$ is associated with $S(n)$ we have $S(n) \in A$ and we conclude that $A = \mathbb{N}$. $\square$

We can now drop the annoying distinction between operations on numerals and operations on natural numbers.

With Propositions 25 and 27 we have shown that if you can prove, or believe, that for numerals we have

$$(\mathrm{j}+\mathrm{k})+1=\mathrm{j}+(\mathrm{k}+1) \text{ and } \mathrm{j}\cdot(\mathrm{k}+1)=(\mathrm{j}\cdot\mathrm{k})+\mathrm{j}$$

then arithmetic among natural numbers can be performed by doing the corresponding arithmetic on their representative numerals.

With that conclusion in hand, we could then go on to deduce that this natural number structure is categorical. That is a term used by philosophers to indicate that there is, essentially, only one structure defined by a logic and a collection of axioms.

PA_2 is categorical.

Trouble in Paradise

So it seems we have proved that the arithmetic developed by the ancients, founded in geometry, could be founded instead on "algebraic" axioms, the Peano Axioms, involving only logic and apparently rather harmless set theory.

This happy state of affairs was not to last.

Gottlob Frege spent many years formalizing mathematical foundations of logic and set theory. His 1903 Volume 2 of Foundations of Arithmetic had, as pointed out by Bertrand Russell shortly before its publication, a logical error. Frege wrote a note that appeared as an afterword in that book noting it and expressing hope that a "fix" could be found.

The fix is not easy, nor is it without its own controversy.

Trouble in Paradise

And as mentioned in Part I Kurt Gödel proved in 1930 that there are sensible statements in any system rich enough to be useful, and which describes natural number arithmetic with addition and multiplication, cannot be proved to be true or false from within the model. And one of those statements is "This collection of axioms is consistent."

For more on all this, I highly recommend Foundations of Set Theory by A.A. Fraenkel, Y. Bar-Hillel, and A. Levy.

And anyone with an interest in computability, logic and mathematics should absolutely check out Logicomix and the podcast interview of its author at

https://www.lapl.org/books-emedia/podcasts/aloud/logicomix-epic-search-truth.