Some Weirdness In REAL Analysis*

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*Complex analysis is often described as more elegant and perfect than real analysis, which is often viewed as messy and chaotic.

Examples

- Rational Numbers
- Cantor Set
- Koch Snowflake
- Peano Space-filling Curve
- No Valid Taylor Series
- Lots of Corners
- Banach-Tarski Paradox

1. How big is the set of rational numbers?

The rational numbers are packed real tightly onto the real line.

That is, you can get arbitrarily close to every real number with rational numbers.

For example,

$$\sqrt{2} \approx 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421;\ 1.414213,\ \ldots\ \rightarrow\ 1.41421356237309504880168872 42\ \ldots.$$

This is called being "**dense**"; i.e. the rationals are dense in the reals.

<u>The collection of rational numbers is pretty big.</u>

2. How big is the set of rational numbers?

The set of rational numbers is **countable**, or countably infinite.
The rationals can be listed: $\mathbb{Q} = \{a_1, a_2, a_3, a_4, \ldots\}$

There are too many real numbers to list.

The reals said to be **uncountable**.

So there are far fewer rational numbers than real numbers.

<u>Maybe the collection of all rational numbers is not so big.</u>

3. How big is the set of rational numbers?

- The length/measure of the real line is infinite: $m(\mathbb{R}) = \infty$.

- Let ϵ be an arbitrarily small positive number.

  Define the open interval $A_n = \left(a_n - \frac{1}{2}\cdot\frac{\epsilon}{2^n}\ ,\ a_n + \frac{1}{2}\cdot\frac{\epsilon}{2^n}\right)$.

  The measure of A_n is $m(A_n) = \frac{\epsilon}{2^n}$. Define $B_\epsilon = \bigcup_{n=1}^{\infty} A_n$.

  $m(B_\epsilon) \ \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \ = \epsilon$. (≤ because the $A_n s$ may overlap)

  $\mathbb{Q} \ \subseteq \ B_\epsilon \ \ \rightarrow \ \ m(\mathbb{Q}) \leq \epsilon$ Since ϵ was arbitrarily small, $\underline{m(\mathbb{Q}) = 0}$.

<u>So the set of rationals is not very big at all.</u>

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How big is the set of rational numbers?

0. Infinite
1. Pretty big
2. Not so big
3. Not very big at all

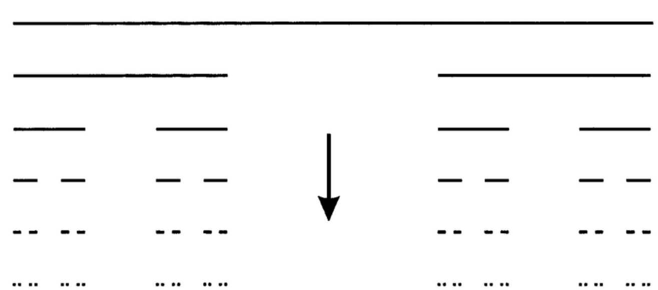
So which is it?

Infinity's a strange place!

The Cantor Set

• Start with the closed unit interval I = [0 ,1] and remove the middle third of it; i.e. throw away (1/3 ,2 /3) and keep $C_1 = [0,1/3] \cup [2/3,1]$.

• Next remove the open middle thirds of the two parts of C_1, leaving $C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$.



• Continue this process indefinitely: at each step C_n will consist of a union of 2^n closed subintervals each of length $(1/3)^{n-1}$.

The Cantor Set is C = { what's left after an infinite number of steps }.

• So what is left? At least the end points of the open intervals we removed -- a countable infinity of them. Anything else?

- The length of the stuff removed is the sum of a geometric series with common ratio 2/3:

$$(1/3) + 2\cdot(1/3)^2 + 2^2\cdot(1/3)^3 + 2^3\cdot(1/3)^4 + \cdots$$

$$= (1/3)\left[1 + (2/3) + (2/3)^2 + (2/3)^3 + (2/3)^4 + \cdots\right] = (1/3)\cdot\left[1/(1 - 2/3)\right] = \mathbf{1}$$

Therefore, the "length" (size/measure) of the Cantor Set is **0**. That is, it can't contain any intervals, it's just a bunch of "dust."

From this perspective, *C* is small.

- Look at the members of the Cantor Set in base-3 :

There are no 1s in ternary representations.

$C = \{ x \in [0,1] : x = 0.c_1c_2c_3c_4 \ldots c_n \ldots, \text{ where } c_n = 0 \text{ or } 2 \}$

Define a function $f : C \longrightarrow I$ by

$$f(0.c_1c_2c_3 \ldots c_n \ldots_{(3)}) = 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2} \ldots \frac{c_n}{2} \ldots_{(2)}$$

Since f is onto all of I, C must be uncountable.

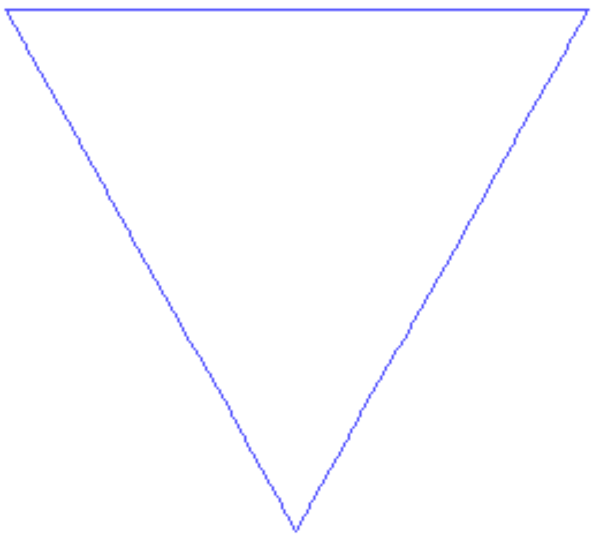
<u>In this sense, C is big.</u>

The Norwegian University of Science and Technology has produced a really nice web page that explains these, and even more, properties of the Canton Set very clearly.

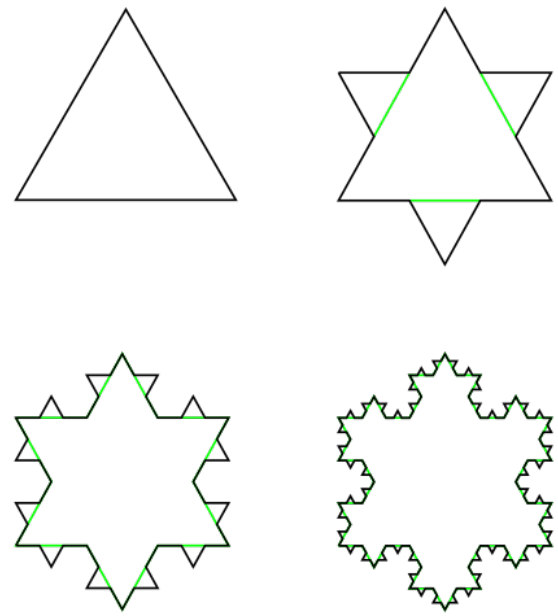
You can view it at

https://wiki.math.ntnu.no/_media/tma4225/2015h/cantor_set_function.pdf

Koch Snowflake Curve



click on the blue triangle



fractal

Looks the same at every scale.

- Personal backstory

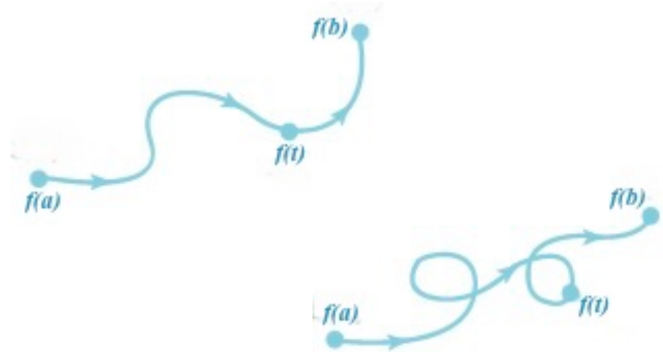
Mathematics and the Imagination
by Edward Kasner and James Newman

- Exercise: Use geometric series to show that this fractal (self-similar) curve has **infinite length** but **finite area**.

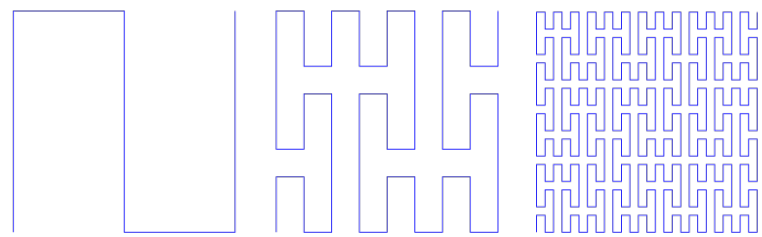
- Read about fractals on the web or at the library.

Space-Filling Curves

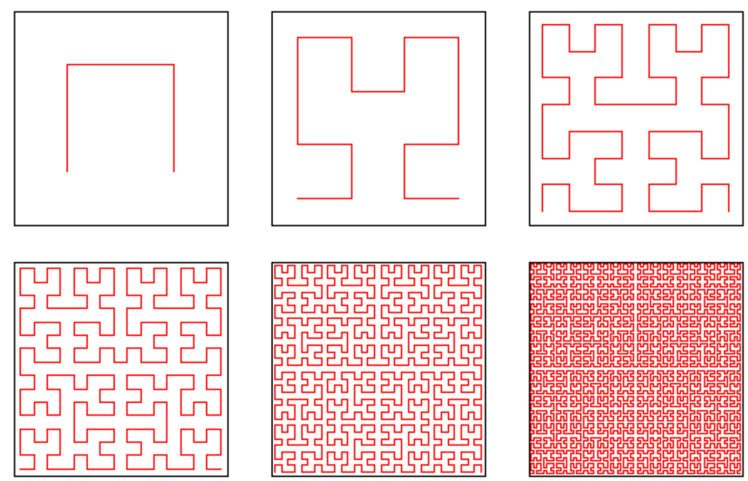
• We usually think of a curve as the path of a continuously moving point, a squiggle or a piece of string tossed carelessly on a table. More formally, if I is the closed interval [a,b] and f is continuous real-valued function on I, then the graph of f is a curve. In any case a curve is a one dimensional object with no area.



• In 1890 Giuseppe Peano produced a continuous function on the unit interval whose graph filled up a whole square. **HUH ? !**



• A year or so later David Hilbert published another example of a space-filling curve.

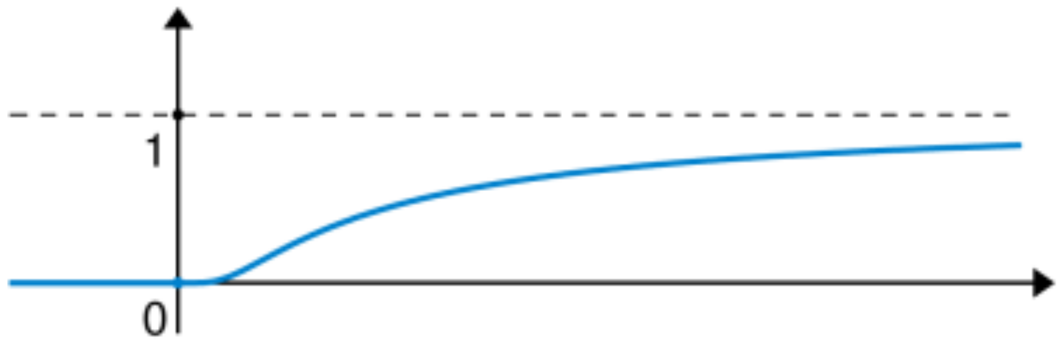


Lack of Power Series

In complex analysis every analytic function is infinitely differentiable and has a valid power series representation.

In real analysis a function may not have a valid power series representation even if it is infinitely differentiable. Here's an example.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$



has continuous derivatives of all orders.

$$f^{(k)}(x) = \begin{cases} \frac{p_k(x)}{x^{2k}} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$p_k(x) =$ a polynomial of degree $k - 1$

$$\textit{Maclaurin series for } f(x) \;=\; \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k \;\equiv\; 0 \quad \forall\, x,$$

But $f(x) \neq 0$, *when* $x > 0$, *so the Maclaurin series is NOT a valid representation of* $f(x)$.

Smoothness

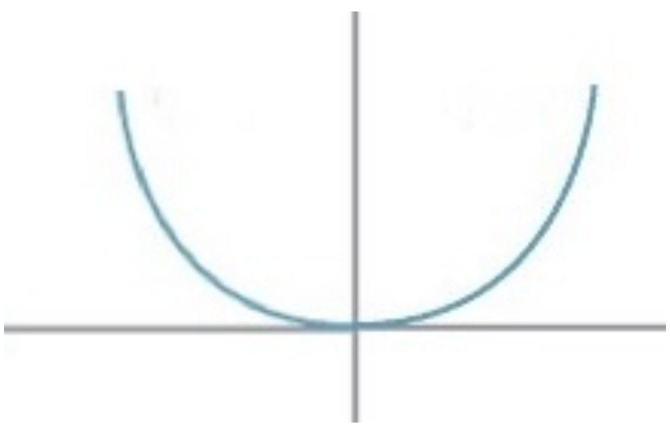
Continuous ≈ “no breaks or gaps”   Differentiable ≈ “no sharp corners”

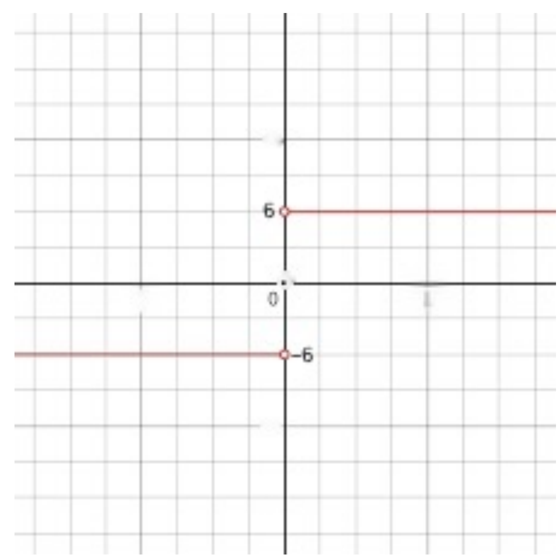
Differentiable functions are automatically continuous.

The level of “smoothness” of a function is measured by the number of continuous derivatives it has. The more derivatives, the smoother.

$$f(x) = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$$

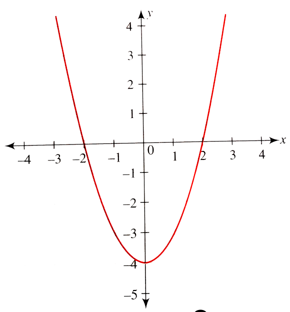
$$f^{(3)}(x) = \begin{cases} 6 & x > 0 \\ -6 & x < 0 \end{cases}$$



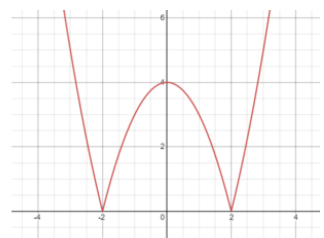


Here are a few examples for you to ponder.

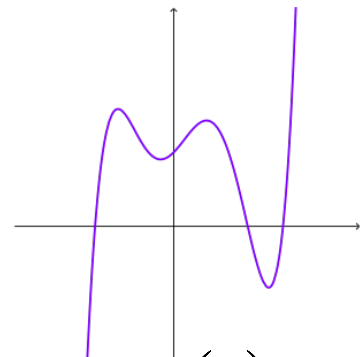
- How many points of discontinuity do they have?
- At how many points are they not differentiable?
- How many derivatives do they have?



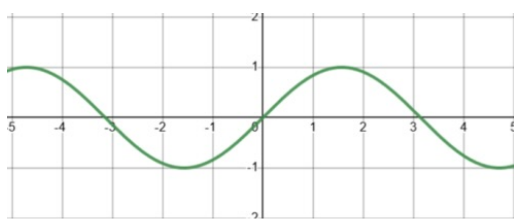
$p(x) = x^2 - 4$



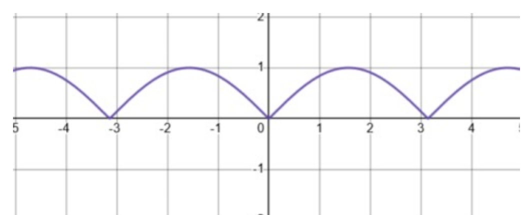
$v(x) = |x^2 - 4|$



$y = g(x)$



$s(x) = sin(x)$



$a(x) = |sin(x)|$

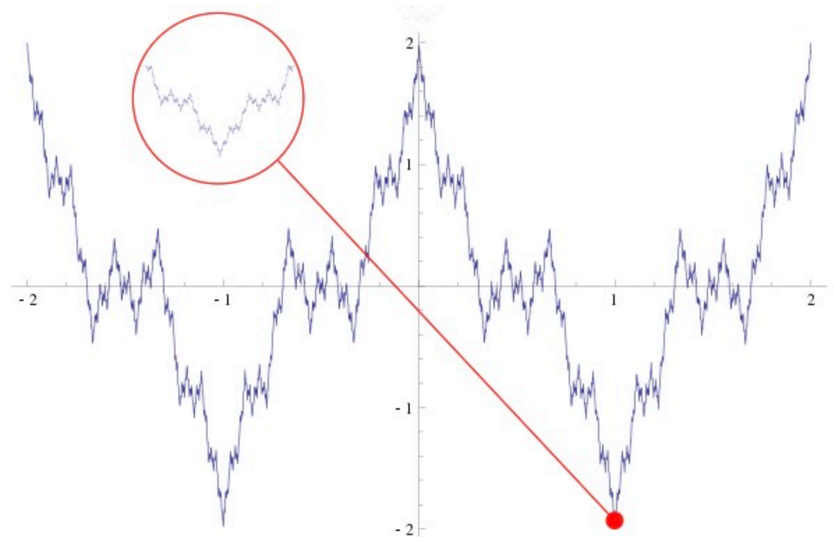
More Sharp Corners Than You Can Imagine

How unsmooth can you get?

In 1872 Weierstrass published an example of a function that is continuous at every point on the real line, but fails to have a derivative at every point of the real line, the so called "**everywhere continuous but nowhere differentiable function**."

$$W(x) = \sum_{n=o}^{\infty} \frac{\cos(3^n x)}{2^n}$$

(Notice that $\cos(3^n x)$ wiggles really rapidly for large values of n.)



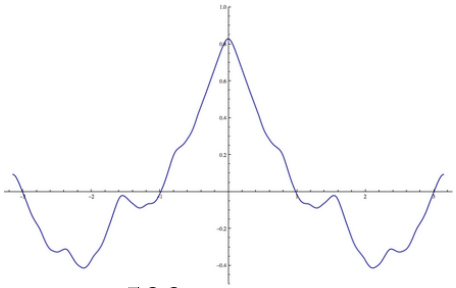
<u>Every point on the "graph" of Weierstrss's function is a sharp corner!!!</u>

According to Paley and Wiener, a Fourier series with "random coefficients" describes Brownian motion on $[0,2\pi]$.

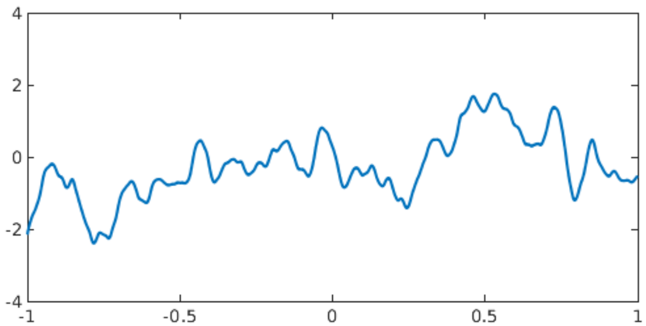
In the early 2000s, while fooling around in the coffee room, a group at Oxford used these "random Fourier" series to invent a function $S(x)$ that possesses Taylor series at every point of a closed interval, but <u>not one</u> of these Taylor series coverages to $S(x)$.

Since then, a whole class of such function have been produced. With apologies to the fruit and beverage industry, they call them a *smoothies*. Here are a couple of samples.

$$S(x) = \sum_{k=o}^{\infty} e^{-\sqrt{2^k}} \cos(2^k x)$$



2^{500} *terms*



Banach-Tarski Paradox*

A bowling ball can be "cut up" into a small finite number of pieces in such a way that the pieces can be carefully reassembled into two bowling balls identical to the original one!

Another version of the paradox asserts that "a marble can be chopped into chunks that can be put back together to form the Moon."

*In these procedures, the **Axiom of Choice** is used to create the fragments of bowling ball and marble, and it takes an uncountable number of steps. The fragments themselves are not ordinary solids, but scattered, **"nonmeasurable"** sets of points.