A FEW OF THE EASIER PARTS OF COMPLEX ANALYSIS
(FIRST DRAFT)

August 9, 2016

LARRY SUSANKA

Contents

1. Some Geometry Related to Complex Numbers 2
2. Formal Power Series 6
3. Connectedness and Paths in C 9
4. The Complex-Derivative and the Derivative Matrix 10
5. Convergence of Power Series 13
6. Integration of Complex-Differentiable Functions Along Paths 20
7. Primitives and Goursat's Theorem 27
8. Cauchy's Integral Theorem and a Few Consequences 29
9. Morera's Theorem 33
10. The Field of Meromorphic Functions 34
11. Conformality and Complex-Differentiability 35
12. Analytic Continuation and Monodromy 36
References 38
Index 39

1. Some Geometry Related to Complex Numbers

A complex number $z = a + b\,i$ is a point (a, b) in the plane $\mathbb{R}^2$ with binary operations $+$ and $\cdot$ given by $(a+bi)+(c+di) = (a+b)+(c+d)\,i$ and $(a+b\,i)\cdot(c+d\,i) = (ac-bd)+(bc+ad)\,i$. The notation $a+b\,i$ is just an alternative symbol to denote the ordered pair (a, b). It is quite convenient.

Normally $\mathbb{R}$ is identified with the standard horizontal or x axis and the "pure imaginary" numbers with the vertical or y axis in the usual visualization of $\mathbb{R}^2$ as the xy plane.

There is an additive identity $0 = 0 + 0\,i$ and multiplicative identity $1 = 1 + 0\,i$. The operations are associative and commutative and multiplication distributes over addition, so these rules of real algebra apply to complex numbers, and so to variables with values from these numbers as well.

For $z = a + b\,i$ define $\overline{z} = a - b\,i$. Then $|\,z\,|$, defined to be $\sqrt{\overline{z}\,z} = \sqrt{a^2+b^2}$, corresponds to the usual Euclidean norm in the plane, and we take this to generate the standard concept of distance between complex numbers.

If $z \neq 0$ then the multiplicative inverse of z exists and $z^{-1} = \overline{z}/(\overline{z}\,z)$. The operations of multiplication and addition and negation are continuous, and the map $z \to z^{-1}$ is continuous away from $z = 0$.

$\mathbb{R}^2$ with this structure is denoted $\mathbb{C}$.

By an easy extension of results from real power series we find for real b that

$$e^{b\,i} = \cos(b) + i\,\sin(b).$$

And a special case of Mertens' Series Theorem (our Lemma 5.5) then gives us

$$e^{a+b\,i} = e^a e^{b\,i} = e^a[\cos(b) + i\,\sin(b)].$$

When a is real, b is an angle (measured counterclockwise) from the positive real axis and the complex number $e^{a+b\,i}$.

On the other hand, for real a and b we have the identity

$$z = a + b\,i = r\,e^{i\,\theta} = r\,[\cos(\theta) + i\,\sin(\theta)],$$

where $r = \sqrt{a^2+b^2}$ and θ is the angle of z, related to $\arctan\left(\frac{b}{a}\right)$.

So if $w = c + d\,i = s\,[\cos(\beta) + i\,\sin(\beta)]$ is another (nonzero) complex number

$$zw = rs\,[\cos(\theta+\beta) + i\,\sin(\theta+\beta)] \quad \text{and} \quad \frac{z}{w} = \frac{r}{s}\,[\cos(\theta-\beta) + i\,\sin(\theta-\beta)].$$

In words, multiplying complex numbers multiplies magnitudes and adds angles and it follows that dividing complex numbers divides magnitudes and subtracts angles.

For $z = a + b\,i = r\,e^{i\,\theta}$, if $r = 0$ the angle θ can be anything, but if $r > 0$ the angle θ is determined up to an additive constant, an integer multiple of 2π.

When complex numbers are written as $z = a+b\,i$ or complex valued functions as $f = u+i\,v$ it is presumed, unless otherwise specified, that a and b are real numbers, and u and v are real valued functions.

If $w = x+y\,i$ and $z \in \mathbb{R}$ we write (w, z) to denote $(x, y, z) \in \mathbb{R}^3$. We will identify $\mathbb{C}$, which is $\mathbb{R}^2$ with complex multiplication and addition, with the xy plane in $\mathbb{R}^3$.

Let $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$. The "North Pole" on this sphere as $N = (0, 0, 1)$.

For $w = (x, y, z) \in S - \{N\}$ define $\widetilde{w}$ by $\widetilde{w} = (w_P, 0)$ where

$$w_P = \frac{x}{1-z} + \frac{y}{1-z}\, i.$$

This map is called a **stereographic projection** of the sphere.

For each w, the point $\widetilde{w}$ is where the line containing w and N intersects the xy plane, and w_P is the complex number identified with that point.

The two maps $\widetilde{w}$ and w_P are so closely related we will only distinguish between them when strictly necessary.

Given nonzero $c = a + b\, i \in \mathbb{C}$, the point

$$c^P = \left(\frac{2a}{c\,\overline{c}+1}, \frac{2b}{c\,\overline{c}+1}, \frac{c\,\overline{c}-1}{c\,\overline{c}+1} \right) \quad \text{is in} \quad S \quad \text{and} \quad \left(c^P\right)_P = c.$$

Similarly, if $w \in S - \{N\}$ then $(w_P)^P = w$.

One can extend this correspondence of points to the whole unit sphere by adding a point "∞" to $\mathbb{C}$, binding it to $\mathbb{C}$ by making it the distinguished point in the one-point compactification of $\mathbb{C}$, called the **extended complex numbers**, and denoted $\mathbb{C}_\infty$.

Then defining $N_P = \infty$ gives a continuous extension of $\widetilde{\ }$ from all of S to the compactified complex plane. The sphere S with the implied association to $\mathbb{C}_\infty$ is called the **Riemann sphere.**

It is fairly clear that any straight line in the xy plane is the projection of a circle in the unit sphere which contains N.

To see this, suppose $w = (x, y, z)$ is a generic point on any plane in space not parallel to the xy plane and which also containing N with normal vector $B = (b_1, b_2, b_3)$. So the equation of this plane is $(w - N) \cdot B = 0$. If w is also in S, the vector $\widetilde{w} - N$ is parallel to $w - N$ so $(\widetilde{w} - N) \cdot B = b_1 x_P + b_2 y_P - b_3 = 0$ too, and the points $\widetilde{w}$ which can be obtained this way obviously lie on a line in the xy plane.

On the other hand, any point $(x, y, 0)$ in the xy plane satisfying $b_1 x + b_2 y = b_3$ is the projection of a point c^P where $c = x + y\, i$ with $(c^P - N) \cdot B = 0$:

$$\begin{aligned} &\left(\frac{2x}{c\overline{c}+1}, \frac{2y}{c\overline{c}+1}, \frac{c\overline{c}-1}{c\overline{c}+1} - 1 \right) \cdot B \\ &\quad = \frac{1}{c\overline{c}+1} \left(\ 2xb_1 + 2yb_2 + (c\overline{c}-1)b_3\ \right) - b_3 \\ &\quad\quad = \frac{1}{c\overline{c}+1} \left(\ 2b_3 + (c\overline{c}-1)b_3\ \right) - b_3 = b_3 - b_3 = 0. \end{aligned}$$

Any circle in the sphere is the intersection of the sphere with a plane of the form $w \cdot E = F$ where w is a generic point (we assume some of these points are in S) and E is normal to the plane. Divide E by $\pm|E|$ to create a **unit vector** B in place of E and a **nonegative number** k in place of F. The equivalent equation is now $w \cdot B = k$. If $k > 1$ no point w on the sphere satisfies this equation, and if $k = 1$ only the single point $w = B$ does, so we assume $0 \le k < 1$.

To reiterate: for each nontrivial circle in the sphere there is a unique $B \in S$ and k with $0 \le k < 1$ for which $w \cdot B = k$ for all points w on the circle.

Note that if $b_3 = k$ the vector equation $w \cdot B = b_3$ reduces to

$$xb_1 + yb_2 + (z-1)b_3 = 0$$

and we find that N is on the circle, and we saw above that circles in S of this kind are uniquely paired with straight lines in the complex plane.

Suppose $k \ne b_3$ and w on the circle has projection $w_P = c = X + iY$ in the complex plane. The equation $w \cdot B = k$ gives

$$\frac{2X}{c\overline{c}+1}b_1 \;+\; \frac{2Y}{c\overline{c}+1}b_2 \;+\; \frac{c\overline{c}-1}{c\overline{c}+1}b_3 \;=\; k$$

which becomes

$$2Xb_1 \;+\; 2Yb_2 \;+\; (c\overline{c}-1)b_3 \;=\; k(c\overline{c}+1)$$

from which we find

$$c\overline{c} \;+\; \frac{2b_1}{b_3-k}X \;+\; \frac{2b_2}{b_3-k}Y \;=\; \frac{k+b_3}{b_3-k}.$$

Completing the square and using $b_1^2 + b_2^2 = 1 - b_3^2$ we find

$$\left(X - \frac{b_1}{k-b_3}\right)^2 + \left(Y - \frac{b_2}{k-b_3}\right)^2 = \frac{1-k^2}{(k-b_3)^2}$$

so the projection of w is on a circle in the complex plane with radius and center given in terms of B and k.

We conclude that every circle in S not containing N projects into a circle in the complex plane.

This process can be reversed.

Suppose $(X-a)^2 + (Y-b)^2 = K^2$ is a circle in the plane. A point $c = X + iY$ on this circle is associated with a point $w = c^P$ on the sphere. Let $2F = K^2 - a^2 - b^2 - 1$. Expanding the formula for the circle we find that

$$2X(-a) + 2Y(-b) + c\overline{c} - 1 = 2F.$$

This gives

$$2X(-a) + 2Y(-b) + (c\overline{c}-1)(1+F) = (c\overline{c}-1)F + 2F = (c\overline{c}+1)F.$$

Dividing everwhere by $c\overline{c}+1$ gives

$$\frac{2X}{c\overline{c}+1}(-a) + \frac{2Y}{c\overline{c}+1}(-b) + \frac{c\overline{c}-1}{c\overline{c}+1}(1+F) = F.$$

If we let $E = (-a, -b, 1+F)$ we see that $w \cdot E = F$ where E and F depend only on the numbers a, b and K^2 and **not** on the specific point $X + iY$ on the circle in the plane.

Note that $|E| = \sqrt{K^2 + F^2}$ so $0 \le \frac{|F|}{|E|} < 1$. Setting $B = \frac{E}{\pm|E|}$ and $k = \frac{F}{\pm|E|}$ (choose "$\pm$" so that k is nonnegative) we recover the equation $w \cdot B = k$ as above.

So circles in the complex plane and circles in the sphere **not** containing N are taken onto each other via stereographic projection.

We come now to the issue of the relationship between the distance separating points $w = (x, y, z)$ and $v = (r, s, t)$ on the sphere in terms of the coordinates of their corresponding projections $w_P = c = X + Y\,i$ and $v_P = h$ in the complex plane. This provides an alternative metric for the compactified complex plane.

First, the distance between any $w \neq N$ in S and N itself is given by

$$\begin{aligned} |w - N|^2 &= x^2 + y^2 + (z-1)^2 \\ &= \left(\frac{2X}{c\overline{c}+1}\right)^2 + \left(\frac{2Y}{c\overline{c}+1}\right)^2 + \left(\frac{c\overline{c}-1}{c\overline{c}+1} - 1\right)^2 \\ &= \frac{4c\overline{c}}{(c\overline{c}+1)^2} + \frac{4}{(c\overline{c}+1)^2} = \frac{4}{c\overline{c}+1}. \end{aligned}$$

If d is the metric on the compactified complex plane induced by usual distance on the sphere we find $d(c,\infty) = \frac{2}{\sqrt{c\overline{c}+1}}$.

More generally, if w and v are points on the sphere whose stereographic projections are (non-infinite points) c and h, respectively, one calculates that

$$|w - v|^2 = \frac{4(c-h)(\overline{c}-\overline{h})}{(1+c\overline{c})(1+h\overline{h})}.$$

So the metric on the complex plane induced by ordinary distance on the sphere is given by

$$d(c,h) = \frac{2|h-c|}{\sqrt{(1+c\overline{c})(1+h\overline{h})}}.$$

It is not completely obvious, from this formula, that the triangle inequality holds for this proposed metric, but it does because the usual Euclidean metric has that property in $\mathbb{R}^3$.

Finally, we note that stereographic projection is **conformal**; that is, it preserves angles, locally.

Specifically, we presume that $w = (x,y,z)$ and $h = (a,b,c)$ are differentiable curves confined to S which meet at time $t = 0$ at some point in S other than N. These curves have stereographic projections

$$\widetilde{w} = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right) \quad \text{and} \quad \widetilde{h} = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right)$$

which are themselves differentiable curves that meet at time $t = 0$ in the plane.

We are saying that the tangent vectors to w and h at $t = 0$ make the same angle as do $\widetilde{w}$ and $\widetilde{h}$ at $t = 0$. First, observe that

$$\begin{aligned} \widetilde{w}' &= \left(\frac{x'}{1-z}, \frac{y'}{1-z}, 0\right) + \left(\frac{xz'}{(1-z)^2}, \frac{yz'}{(1-z)^2}, 0\right) \\ \text{and} \quad \widetilde{h}' &= \left(\frac{a'}{1-c}, \frac{b'}{1-c}, 0\right) + \left(\frac{ac'}{(1-c)^2}, \frac{bc'}{(1-c)^2}, 0\right). \end{aligned}$$

Note too that at $t = 0$ we have $x = a$ and $y = b$ and $z = c$ and $x^2 + y^2 + z^2 = 1$. This last implies (actually, for all t) that $yy' + xx' = -zz'$ and $aa' + bb' = -cc'$.

$$\begin{aligned}
\widetilde{w}' \cdot \widetilde{h}' &= \frac{x'a' + y'b'}{(1-z)^2} + \frac{x'ac' + y'bc'}{(1-z)^3} + \frac{xz'a' + yz'b'}{(1-z)^3} + \frac{xaz'c' + ybz'c'}{(1-z)^4} \\
&= \frac{x'a' + y'b'}{(1-z)^2} + \frac{c'(x'a + y'b)}{(1-z)^3} + \frac{z'(xa' + yb')}{(1-z)^3} + \frac{z'c'(x^2 + y^2)}{(1-z)^4} \\
&= \frac{(x'a' + y'b')(1-z)}{(1-z)^3} + \frac{c'(-z'z)}{(1-z)^3} + \frac{z'(-c'c)}{(1-z)^3} + \frac{z'c'(1-z^2)}{(1-z)^4} \\
&= \frac{x'a' + y'b'}{(1-z)^3} - \frac{(x'a' + y'b')z}{(1-z)^3} - \frac{2c'z'z}{(1-z)^3} + \frac{z'c'}{(1-z)^3} + \frac{z'c'z}{(1-z)^3} \\
&= \frac{x'a' + y'b' + z'c'}{(1-z)^3} - \frac{(x'a' + y'b' + z'c')z}{(1-z)^3} = \frac{w' \cdot h'}{(1-z)^2}.
\end{aligned}$$

This tells us several interesting facts. For instance, in the special case of $w = h$ we find that the $\widetilde{w}'$ vector is longer than w' by factor $\frac{1}{1-z}$, and this stretch factor depends only on the "lattitude" of $w(0)$.

The equality of the angles between the two pairs of tangent vectors now follows immediately from the more general formula.

2. Formal Power Series

A **formal power series** (**FPS**) is nothing more than a member of $\mathbb{C}^{\mathbb{N}}$ and an interpretation. Formal power series are added or multiplied by scalars using the usual real or complex vector space structure on $\mathbb{C}^{\mathbb{N}}$.

When using this vocabulary for a member $a \in \mathbb{C}^{\mathbb{N}}$ the ultimate intent is to study functions that have a representation, for x taken from a fixed complex Banach algebra $\mathbb{B}$, as the limit of the sequence of partial sums $\mathbf{S^a_n(x)} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{a_k\, x^k}$, where the a_0 term must be 0 if the Banach algebra has no identity, and otherwise we interpret $a_0\, x^0$ to be $a_0\, e$ where e is the identity element of $\mathbb{B}$.

We want to be able to discuss properties of the sequence of partial sums whether or not the sum converges, and for now we will use either $\mathbf{a(x)}$ or $\sum_{\mathbf{k=0}}^{\boldsymbol{\infty}} \mathbf{a_k\, x^k}$, standing alone, to denote this sequence of partial sums:

$$a_0\, e, \qquad a_0\, e + a_1\, x, \qquad a_0\, e + a_1\, x + a_2\, x^2, \quad \ldots, \quad S^a_n(x), \quad \ldots .$$

When x is a variable symbol we call $\sum_{k=0}^{\infty} a_k\, x^k$ a **formal power series in x**.

We reiterate: there is no assumption of convergence of $a(x)$ for an FPS a for any particular value of x other than $x = 0$. There is certainly no assumption of convergence of $a(e)$, which is the sequence of partial sums with terms given by $\sum_{k=0}^{n} a_k\, e = (\sum_{k=0}^{n} a_k)\, e$, even if the Banach algebra *has* identity e, which it may not.

A member x of a Banach algebra is called **nilpotent** if $x^n = 0$ for some positive integer n. If two sequences of partial sums $a(x)$ and $b(x)$ are equal for even a single x which is *not* nilpotent then the two formal power series a and b are equal. In particular, if the Banach algebra has identity e and there is any nonzero complex number λ for which $a(\lambda e) = b(\lambda e)$ then a and b are equal.

a is said to **terminate** if there is a non-negative integer n for which $a_j = 0$ for $j > n$. In that case $S_n^a(x) = S_{n+k}^a(x)$ for all $k \geq 0$. The series corresponds, essentially, to an ordinary polynomial.

The **Cauchy product** of two formal power series a and b is the FPS c defined by $c_k = \sum_{j=0}^{k} a_j b_{k-j}$.

This product, denoted $\mathbf{a} \star \mathbf{b}$, is obviously commutative and the sequence $\mathbf{1_0}$, defined by $(1_0)_j = 0$ if $j \neq 0$ and $(1_0)_0 = 1$, is the identity with Cauchy product.

If a and b both terminate, at m and n respectively, then their Cauchy product terminates too, and corresponds to ordinary polynomial multiplication of $S_m^a(x)\, S_n^b(x)$ followed by gathering of like terms. $(a \star b)(x)$ is the sequence with first term $a_0\, b_0$, second term $a_0\, b_0 + (a_0\, b_1 + a_1\, b_0)\, x$ and, ultimately, at the index $m + n$ term and beyond is $S_m^a(x)\, S_n^b(x)$.

This product is associative: If $(a \star b) \star c \neq a \star (b \star c)$ then the two series would differ at some smallest subscript n. The calculations to produce these unequal nth coefficients involve no coefficient of a, b or c beyond the nth, so we can redefine a, b and c to be 0 for all coefficients larger than n and we would *still* have the troublesome inequality. But $S_n^a(x)$, $S_n^b(x)$ and $S_n^c(x)$ are polynomials of degree no higher than n and $(S_n^a(x)\, S_n^b(x))\, S_n^c(x) \neq S_n^a(x)\, (S_n^b(x)\, S_n^c(x))$ because they differ at their nth degree term, impossible given the associativity of ordinary polynomial multiplication.

So $\mathbb{C}^{\mathbb{N}}$ is a commutative algebra with Cauchy product, and easily seen to be an integral domain: $a \star b = 0$ only if $a = 0$ or $b = 0$.

If, for FPS b, we have $b_0 \neq 0$ define $a_0 = 1/b_0$ and generally $a_k = \frac{-1}{b_0} \sum_{j=0}^{k-1} a_j b_{k-j}$. Then a calculation verifies that $a \star b = 1_0$. These two sequences are called **formal reciprocals** of each other.

If, also, $a \star c = 1_0$ then $b = b \star (a \star c) = (b \star a) \star c = (a \star b) \star c = 1_0 c = c$. In other words, formal reciprocals are unique.

And if $b_0 = 0$ then there is no a with $a \star b = 1_0$. So units with Cauchy product are exactly those sequences b with nonzero b_0.

We are now going to consider composition of formal power series, but to avoid certain combinatorial issues we will do so only with series b for which $b_0 = 0$. Let $\mathbb{C}_0^{\mathbb{N}}$ denote the sequences that start with 0.

For $a, b \in \mathbb{C}_0^{\mathbb{N}}$ define the **formal composition** $\mathbf{a} \circ \mathbf{b}$ to have nth term obtained by polynomial multiplication and combining like coefficients. Specifically, $(a \circ b)_k$ is obtained as the coefficient on the kth degree term of $S_k^a(\, S_k^b(x)\,)$. Expand

$$\begin{aligned}
a_1\,(b_1\,x + b_2\,x^2 + \cdots + b_k\,x^k) &+ a_2\,(b_1\,x + b_2\,x^2 + \cdots + b_k\,x^k)^2 \\
&+ a_3\,(b_1\,x + b_2\,x^2 + \cdots + b_k\,x^k)^3 + a_4\,(b_1\,x + b_2\,x^2 + \cdots + b_k\,x^k)^4 \\
&+ \cdots + a_k\,(b_1\,x + b_2\,x^2 + \cdots + b_k\,x^k)^k \\
= \; a_1 b_1\,x &+ (a_1 b_2 + a_2 b_1^2)\,x^2 + (a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3)\,x^3 \\
&+ (a_1 b_4 + 2 a_2 b_2^2 + 3 a_3 b_1^2 b_2 + a_4 b_1^4)\,x^4 + \cdots + (a_1 b_k + M_k + a_k b_1^k)\,x^k + \cdots
\end{aligned}$$

where M_k in the last line involves *only* coefficients a_j and b_n for j and n strictly between 1 and k.

The kth degree coefficient of $S^a_m(\,S^b_n(x)\,)$ will be the same as above if $m \geq k$ and $n \geq k$, but this is only true because of our assumption that $b_0 = 0$, and is the reason for that assumption.

Note that if the first non-zero coefficient of a is at index $k \geq 1$ and the first non-zero coefficient of b is at index $n \geq 1$ then the first nonzero coefficient of $a \circ b$ is $a_k\, b_n^k$ at index $n + k$. It follows that $a \circ b = 0$ only when $a = 0$ or $b = 0$.

The sequence $\mathbf{1_1}$ defined by $(1_1)_j = 0$ if $j \neq 1$ and $(1_1)_1 = 1$ is a left and right identity with formal composition, which corresponds to the polynomial function $f(x) = x$. Specifically, $1_1(x) = 0,\, x,\, x,\, x, \ldots.$

Just as before with Cauchy product, if $(a \circ b) \circ c \neq a \circ (b \circ c)$ then the two series would differ at some smallest coefficient, say the nth. This nth coefficient is independent of any coefficient of a, b or c beyond the nth, so we could redefine a, b and c to be 0 for all coefficients larger than n and we would still have $(a \circ b) \circ c \neq a \circ (b \circ c)$. But these formal compositions have coefficients generated by actual compositions of ordinary polynomial functions, and composition of ordinary functions *is* associative.

The contradiction implied here gives us associativity of formal composition on $\mathbb{C}^{\mathbb{N}}_0$.

The coefficients produced by this process seem to be getting complex, but some patterns emerge that allow us to deduce an Inverse Function Theorem for formal power series in $\mathbb{C}^{\mathbb{N}}_0$. We are looking for a condition under which $c = a \circ b = 1_1$ or, as a power series in x, $c(x) = (a \circ b)(x) = 0,\, x,\, x,\, x, \ldots.$

First, if this is to be true we must have both a_1 and b_1 nonzero, and then these numbers are reciprocals. Second the coefficients a_n and b_n *first* occur in coefficient c_n for c, and

$$c_n = a_1\, b_n + M_n + a_n\, b_1^n$$

where M_n involves only coefficients of a and b with subscripts *smaller* than n.

So if a is given, b is determined and can be calculated explicitly by a recursive formula and, conversely, if b is given a is determined and can be found by a recursive formula.

So every member g of $\mathbb{C}^{\mathbb{N}}$ for which $g_1 \neq 0$ and $g_0 = 0$ (necessary so all the coefficient formulas are finite sums) has both a left inverse $a \in \mathbb{C}^{\mathbb{N}}_0$ and a right inverse $b \in \mathbb{C}^{\mathbb{N}}_0$ with respect to formal composition, and these inverses are uniquely determined. But then

$$b = 1_1 \circ b = (a \circ g) \circ b = a \circ (g \circ b) = a \circ 1_1 = a.$$

So these inverses are two-sided inverses.

Define the **formal derivative $\mathbf{b'}$** of FPS b by $b'_n = (n+1)b_{n+1}$ for $n \geq 0$.

2.1. ***Proposition.*** *(i) Cauchy product is an associative commutative operation on $\mathbb{C}^{\mathbb{N}}$ with identity 1_0. A member g of $\mathbb{C}^{\mathbb{N}}$ has a formal reciprocal (an inverse with respect to Cauchy product) exactly when $g(0) \neq 0$.*

(ii) Formal composition is an associative operation on $\mathbb{C}^{\mathbb{N}}_0$.

(iii) ***Inverse Function Theorem for Formal Power Series***
A member $g \in \mathbb{C}^{\mathbb{N}}_0$ has left inverse with respect to formal composition

exactly when $g'(0) \neq 0$. In that case this left inverse is unique, and is also a right inverse.

Proof. The proof is organized in the preceding remarks. $\square$

3. Connectedness and Paths in $\mathbb{C}$

When you encounter any expression of the form "$a + b\,i$" it is intended, unless indicated explicitly or by unmistakable context, that a and b represent real numbers or real-valued functions. As is customary, a complex number $a+b\,i \in \mathbb{C}$ is associated (i.e. it *is*) an ordered pair (a, b) in $\mathbb{R}^2$.

If $A \subset B \subset \mathbb{C}$ and $g\colon B \to X$ we may say that g has a certain property on A. By this we intend that the restriction of g to A has that property.

A subset A of $\mathbb{C}$ is called **connected** if there *do not* exist two disjoint open sets E and F that each have nonempty intersection with A and for which $A = (A \cap E) \cup (A \cap F)$.

A is called **path-connected** if, for every pair of distinct points $r, w \in A$, there is a continuous map $s\colon [a, b] \to A$ defined on interval $[a, b]$ with $s(a) = r$ and $s(b) = w$. s is said to **connect the starting point r to the ending point w in A.**

Continuous s is called **piecewise linear** if the domain can be broken into a finite number of subintervals $[t_k, t_{k+1}]$ so that for each piece there are complex numbers f_k and g_k for which s is of the form $s(t) = f_k + (t - t_k)\, g_k$ when $t \in [t_k, t_{k+1}]$.

Any two points in a path-connected *open* subset U of $\mathbb{C}$ are start and end points of a continuous function $s\colon [a, b] \to U$. Since $s([a, b])$ is compact, the distance between $s([a, b])$ and $U^c = \mathbb{C} - U$ is positive. So there is a "tube" of positive radius around $s([a, b])$ entirely contained in U.

With this "wiggle room" to work with, whenever we require a function s connecting r to w and confined to an open set it will be useful and possible to use in place of a generic s a function with enhanced smoothness criteria. For instance we might insist that s follow a finite number of line segments, all inside this tube.

A **polygonal path** is a continuous function $s\colon [a, b] \to \mathbb{C}$ whose image consists of a finite number of straight line segments parameterized in such a way that s is one-to-one on both $[a, b)$ and $(a, b]$. So polygonal paths, by our definition, do not cross or otherwise intersect except, possibly, to have endpoints meet. We do not require our polygonal path to be piecewise linear, though usually any explicit parameterization will be. Any polygonal path can be re-parameterized by a continuous increasing change of parameter so that the new polygonal path is piecewise linear with the same domain.

3.1. ***Exercise.*** *Prove, look up or accept the following facts. Connected subsets of $\mathbb{R}$ are intervals. If $f\colon [a, b] \to \mathbb{C}$ is connected so is the image set $f(\,[a, b]\,)$. If A and B are connected and in $\mathbb{C}$ and $A \cap B \neq \varnothing$ then $A \cup B$ is connected. If $x \in A \subset \mathbb{C}$ the union of all connected subsets of A containing x is itself connected, called a* ***connected component of A.*** *If A is open, so are all its connected components, of which there are at most countably many. For open subsets of $\mathbb{C}$, connected and path-connected are equivalent. Prove also that if A is an open connected subset of*

$\mathbb{C}$ there is a piecewise linear polygonal path connecting any two distinct points of A.

Suppose given two continuous functions s and r with common domain $[a, b]$ and with values in set $U \subset \mathbb{C}$ and for which $s(a) = r(a)$ and $s(b) = r(b)$.

s and r are called **homotopic in U** if there is a continuous map

$$H\colon [a, b] \times [0, 1] \to U$$

called a **homotopy between these two maps**, for which for every $t \in [0, 1]$ and $x \in [a, b]$

$$H(x, 0) = s(x) \text{ and } H(x, 1) = r(x) \text{ and } H(a, t) = s(a) \text{ and } H(b, t) = s(b).$$

The phrase "s can be continuously deformed into r leaving endpoints fixed and without leaving U" is used to describe this situation.

A connected subset U of $\mathbb{C}$ is called **simply connected** if *every* pair of continuous maps r and s as above are homotopic in U.

3.2. ***Exercise.*** *A connected subset U of $\mathbb{C}$ is simply connected if, whenever $s\colon [a, b] \to U$ is continuous with $s(a) = s(b)$ then s is homotopic to a constant function in U. Colloquially, loops are all homotopic to points.*

4. The Complex-Derivative and the Derivative Matrix

When using tools from linear algebra or multi-variable calculus an ordered pair $a + b\,i = (a, b)$ is represented as a column $\begin{pmatrix} a \\ b \end{pmatrix}$, a two-row one-column matrix. In that context it *is not* a one-row two-column matrix $(\,a\ b\,)$: these represent linear functionals instead.

Suppose function f is defined on an open set and p is in this set. f is called **complex-differentiable at p** if

$$\lim_{\lambda \to 0} \frac{f(p + \lambda) - f(p)}{\lambda}$$

exists, and if it does that limit is denoted $\mathbf{f'(p)}$ and called the **complex-derivative** of f at p.

And f is called **complex-differentiable on a set S** if it has a derivative at every point on some open set (possibly S itself) containing S.

A function that has complex-derivative in the whole plane is called **entire.**

If f' exists and is continuous on an open set containing S we say f is **holomorphic on S**. We will see later that *every* function that is complex-differentiable on an open set is holomorphic on that set, but that surprising result will require considerable preliminary work to prove.

For each positive k and $p \in \mathbb{C}$ let $\mathbf{D_k(p)}$ denote the closed disk in $\mathbb{C}$ centered at p of radius k. Let $\mathbf{B_k(p)}$ denote the interior of that disk.

When f has a complex-derivative on $B_k(p)$ and $B_k(p)$ is contained in the domain of function g, f is called a **local primitive for g on $\mathbf{B_k(p)}$** if $g(x) = f'(x)$ for all

$x \in B_k(p)$. More generally, any complex-differentiable f is called a **primitive for g** if $g(x) = f'(x)$ for every x in the domain of g.

Many of the important results of this subject depend explicitly on the domain under consideration. For instance, a result might fail unless the domain is connected. Or it might fail unless the *complement* of the domain is connected. A function defined and differentiable on a set might have an extension to a differentiable function on larger set. It is possible that if $A \subset B$, a function g defined on B might have a primitive (i.e. an antiderivative) when restricted to A but *fail to have* a primitive on its original domain B. Or the function g might have a nonzero derivative, and important properties that follow from this, on A but not on B.

For that reason we will make explicit mention of domain in definitions and statements of results and this acknowledgment is essential, not perfunctory.

Just as in the real case, when g is complex-differentiable at p,

$$\lim_{\lambda \to 0} \frac{f(p+\lambda) - f(p) - f'(p)\lambda}{\lambda} = 0$$

and so the difference between $f(p+\lambda)$ and its **local linear approximant** $f(p) + f'(p)\lambda$ is small *even in comparison to* λ when λ is sufficiently small.

As mentioned before, a function that is complex-differentiable on a set actually has continuous derivative there, and so the derivative has a maximum magnitude on any line segment. So in the following lemma, there actually will be, always, a maximum magnitude of the derivative. But since we have not proved that result yet, we assume the maximum magnitude given as part of the lemma.

4.1. ***Lemma.*** *Suppose g is complex-differentiable at each point on a line segment with endpoints p and q. Suppose that M is the maximum magnitude of $|g'|$ on this segment. Then $|\, g(p) - g(q)\,| \le M\,|\, p - q\,|$.*

Proof. Suppose $\varepsilon > 0$. For each particular x in the segment there is a $\delta > 0$ for which $|\, g(x+\lambda) - g(x) - g'(x)\lambda\,| < \varepsilon\,|\lambda|$ whenever $|\lambda| < \delta$. But since the line segment is compact, a single δ can be chosen that "works" for every x in the segment.

Let $x_0, \ldots, x_n$ be a list of points on the segment moving along the segment from $x_0 = q$ to $x_n = p$ where the segments between x_{i-1} and x_i for different i values don't overlap and with $|\, x_i - x_{i-1}\,| < \delta$ for $i = 1, \ldots, n$.

$$\begin{aligned} |g(p) - g(q)| &= \left| \sum_{i=1}^{n} g(x_i) - g(x_{i-1}) \right| \le \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ &\le \sum_{i=1}^{n} \left(\,|\, g'(x_i)\,(x_i - x_{i-1})\,| + \varepsilon\,|\, x_i - x_{i-1}\,|\, \right) \le (M + \varepsilon)\,|\, p - q\,|. \end{aligned}$$

Since ε can be chosen to be arbitrarily small, the result follows. $\square$

The existence of a complex-derivative of a function $g(z) = u(z) + i\,v(z)$ (u and v are real valued) is far more restrictive than the existence of the derivative matrix of the map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Introducing notation

$$z = x + i\,y = \begin{pmatrix} x \\ y \end{pmatrix}, \quad g(z) = u(z) + i\,v(z), \quad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u\begin{pmatrix} x \\ y \end{pmatrix} \\ \\ v\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix},$$

we have a representation for g when complex notation is suppressed. If g has a complex-derivative then it definitely has a derivative matrix

$$g'(p) = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix},$$

but with the additional property that multiplication by this matrix corresponds to complex multiplication, which itself corresponds to multiplication by a non-negative constant followed by a rotation. This implies that the derivative matrix has certain symmetry properties.

Specifically, a function g that has a *derivative matrix* at p has, also, a complex-derivative there exactly when the real and complex parts of g satisfy the **Cauchy-Riemann equations** at p:

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

That these equations are necessarily satisfied by a complex-differentiable function can be seen by using the definition of complex-derivative and observing we must have equality of the limits as you approach p from the two axis directions, yielding

$$g' = \frac{\partial g}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{1}{i}\,\frac{\partial g}{\partial y} = \frac{1}{i}\,\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}.$$

And any pair of functions u and v defined around p that satisfy the Cauchy-Riemann equations at p and that, individually, have matrix derivatives there can be used as real and complex parts of a function that is complex-differentiable at p. And then the matrix form of the derivative is

$$g'(p) = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}$$

which generates two forms of the complex derivative

$$g' = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}.$$

This is all closely related to properties of **harmonic functions**, which are the solutions to **Laplace's equation**:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

If a solution F to Laplace's equation has second partials on an open domain and if the mixed second partials are equal (if second partials are continuous this will be true) then the pair of functions $u = \frac{\partial F}{\partial y}$ and $v = \frac{\partial F}{\partial x}$ form a solution to the Cauchy-Riemann Equations on the domain.

On the other hand, if a pair of functions u and v satisfy the the Cauchy-Riemann Equations on an open domain and if they have second partial derivatives and the mixed partials are equal (if second partials are continuous this will be true) then both u and v are solutions to Laplace's Equation.

4.2. ***Proposition.*** *The chain, product, quotient and linearity rules all apply for complex-derivatives with the usual provisos from ordinary calculus. In particular, if $f(z) = z^n$ then $f'(z) = n\, z^{n-1}$ for any integer n.*

Proof. This is left as an exercise. □

5. Convergence of Power Series

We note for later a useful, purely algebraic, fact.

Suppose Δ is the **difference operator** which when applied to any sequence s_j produces the difference sequence $\Delta s_j = s_j - s_{j-1}$.

Consider the **summation by parts** formula, the algebraic identity

$$\sum_{k=m}^{n} a_k\, \Delta B_k = \sum_{k=m}^{n} a_k(B_k - B_{k-1}) = a_{n+1}B_n - a_m B_{m-1} - \sum_{k=m}^{n} B_k(a_{k+1} - a_k)$$
$$= a_{n+1}B_n - a_m B_{m-1} - \sum_{k=m}^{n} B_k\, \Delta a_{k+1}.$$

We now suppose that sequence B_k has the particular form $B_k = \sum_{j=0}^{k} b_k$. If we are interested in the series $\sum_{j=0}^{\infty} a_j b_j = \sum_{j=0}^{\infty} a_j\, \Delta B_j$ we can transform the partial sum $\sum_{j=1}^{n} a_j b_j$, by a procedure referred to as the **Abel transform**, into the form

$$a_{n+1}B_n - a_1 b_0 - \sum_{k=1}^{n} B_k\, \Delta a_{k+1} = a_{n+1}B_n - a_1 b_0 - \sum_{k=1}^{n} B_k\,(a_{k+1} - a_k).$$

Just as in applications of integration by parts, we transform one sequence of partial sums, $\sum_{j=1}^{n} a_j b_j$, which may be hard to work with directly, into two sequences B_n and $a_{n+1}B_n - a_1 b_0 - \sum_{k=1}^{n} B_k\,(a_{k+1} - a_k)$ with which we hope to make progress. Although this last sum may seem messier, we will see examples (notably the proof of Abel's Second Theorem) where the Abel transform is tractable and the initial sum is not.

5.1. ***Theorem. Abel's First Theorem***
If $\sum_{j=0}^{\infty} a_j\,(z_0 - p)^j$ converges then $\sum_{j=0}^{\infty} a_j\,(z - p)^j$ converges absolutely and uniformly on $D_k(p)$ for each $k < t = |\, z_0 - p\,|$.

Proof. To see this, let $M = \sup\{\, |a_n|\,|\, z_0 - p\,|^n \mid n \geq 0\,\}$. Assume temporarily that none of the $a_n = 0$ and select $z \in D_k(p)$. Then

$$\sum_{n=0}^{\infty} |a_n|\,|\, z - p\,|^n \leq \sum_{n=0}^{\infty} \frac{M}{|a_n|\,|\, z_0 - p\,|^n}\; |a_n|\,|\, z - p\,|^n$$
$$= M \sum_{n=0}^{\infty} \frac{|\, z - p\,|^n}{|\, z_0 - p\,|^n} \leq\; M \sum_{n=0}^{\infty} \left(\frac{k}{t}\right)^n < \infty.$$

Modify slightly to account for possible terms with $a_n = 0$ and note that the magnitude of any tail sequence $\sum_{n=j}^{\infty} a_j\,(\,z-p\,)^n$ cannot exceed $M\sum_{n=j}^{\infty}\left(\frac{k}{t}\right)^n$ and the proof is complete. □

5.2. ***Corollary.*** *Suppose the domain of function f is the set of those z for which the power series $\sum_{j=0}^{\infty} a_j\,(z-p)^j$ converges, and define*

$$f(z) = \sum_{j=0}^{\infty} a_j\,(z-p)^j$$

on this domain. Suppose z_0 is a particular member of the domain of f, and $z_0 \neq p$. Suppose also that $0 < k < |\,z_0 - p\,|$.

Then $D_k(p)$ is entirely contained in the domain of f, and the sequence of partial sums $f_n(z) = \sum_{j=0}^{n} a_j\,(z-p)^j$ converges absolutely and uniformly to f in $D_k(p)$.

Proof. This is a rephrasing of Abel's First Theorem. □

Abel's Second Theorem is also of interest. It says that "nontangential " limits exist. Specifically, suppose $z_0 = p + r\,e^{i\,\mu}$ for positive r and $\sum_{j=0}^{n} a_j\,(z_0-p)^j$ converges. Then we know that $f(z) = \sum_{j=0}^{\infty} a_j\,(z-p)^j$ exists, and the convergence is absolute, whenever $|z-p| < r$ by Abel's First Theorem.

Abel's Second Theorem states that for any ζ with $0 < \zeta < \pi/2$ and any sequence θ_n with $-\pi/2+\zeta < \theta_n < \pi/2-\zeta$ and sequence t_n with $0 < t_n < r$ and $\lim_{n\to\infty} t_n = 0$

$$\lim_{n\to\infty} f\left(z_0 - t_n\,e^{i\,(\theta_n+\mu)}\right) \quad \text{exists and equals } f(z_0).$$

Note that if $\theta = \pm\pi/2$ then $e^{i\,(\theta+\mu)}$ is actually perpendicular to the angle of z_0 as seen from p, so if we allowed such values for θ_n we would be approaching z_0 from *outside* the closed disk $D_r(p)$. With the specified restriction on θ_n we approach z_0 (when t_n is small enough) from *inside* $D_r(p)$ *and between two rays symmetric to the line from p to z_0*. The sequence $z_0 - t_n\,e^{i\,(\theta_n+\mu)}$ is said to approach z_0 **within a Stoltz angle;** dependency on the parameter ζ is an issue when considering the *rate* of convergence.

For f and z_0 and ζ and r as above, define $g(z) = f\left(p + r\,z\,e^{i\,\mu}\right) - f(z_0)$. So $g(0) = f(p) - f(z_0)$ and $g(1) = 0$ and points in $B_1(0)$ corresponds, for g, to the points in $B_r(p)$, which are all in the domain of f.

Letting $b_0 = a_0 - f(z_0)$ and, for $k > 0$, $b_k = a_k\,r^k\,e^{k\,i\,\mu}$ we have $\sum_{k=0}^{\infty} b_k = g(1) = 0$ and, generally, if z is in $B_1(0)$ then

$$\begin{aligned}\sum_{k=0}^{\infty} b_k\,z^k &= -f(z_0) + \sum_{k=0}^{\infty} a_k\,\left(r\,z\,e^{i\,\mu}\right)^k = -f(z_0) + \sum_{k=0}^{\infty} a_k\,\left(\left(p + r\,z\,e^{i\,\mu}\right) - p\right)^k \\ &= f\left(p + r\,z\,e^{i\,\mu}\right) - f(z_0)\end{aligned}$$

where $p + r\,z\,e^{i\,\mu}$ is a member of $B_r(p)$.

The sequence $1 - t_n\,e^{i\,(\theta_n)}$ is in a Stoltz angle for g, $D_1(0)$, ζ and domain point 1 for large enough n, and each such point corresponds to a point in a Stoltz angle for f, $D_r(p)$, ζ and domain point z_0.

The transformation from $f - f(z_0)$ to g involves composition with a continuous function (linear, actually) so convergence of $g\left(1 - t_n\, e^{i\,\theta_n}\right)$ to 0 will imply the convergence of $f\left(z_0 - t_n\, e^{i\,(\theta_n+\mu)}\right)$ to $f(z_0)$.

5.3. ***Theorem. Abel's Second Theorem***

Suppose the domain of function f is the set of those z for which $\sum_{j=0}^{\infty} a_j\,(z-p)^j$ converges, and define

$$f(z) = \sum_{j=0}^{\infty} a_j\,(z-p)^j$$

on this domain. Suppose z_0 is in the domain of f and $|\,z_0 - p\,| = r$.

Then for any ζ with $0 < \zeta < \pi/2$ and any sequence θ_n with

$$-\pi/2 + \zeta < \theta_n < \pi/2 - \zeta$$

and any sequence t_n with $0 < t_n < r$ and $\lim_{n\to\infty} t_n = 0$ we have

$$\lim_{n\to\infty} f\left(z_0 - t_n\, e^{i\,(\theta_n+\mu)}\right) \quad \textit{exists and equals } f(z_0).$$

Proof. We will prove the theorem for the function g related to f as described above, and draw the conclusion for f by the obvious extension.

Suppose $x = 1 - t\,e^{i\,\theta}$ and $0 < \theta < \zeta < \frac{\pi}{2}$ and t is positive and small enough so that x is in $B_1(0)$. Note that if t is made smaller (but still positive) x will remain in $B_1(0)$.

We examine the fraction

$$\frac{|\,1-x\,|}{1-|x|} = \frac{t}{1-\sqrt{(1-t\,e^{i\,\theta})\,(1-t\,e^{-\,i\,\theta})}} = \frac{t}{1-\sqrt{1-2t\,\cos(\theta)+t^2}}$$

$$= \frac{t\,(\,1+\sqrt{1-2t\,\cos(\theta)+t^2}\,)}{2t\,\cos(\theta)-t^2} = \frac{1+\sqrt{1-2t\,\cos(\theta)+t^2}}{2\,\cos(\theta)-t} < \frac{1}{\cos(\theta)-t/2}.$$

Suppose γ is any angle strictly between ζ and $\pi/2$. Then t, which is the distance between x and 1, can be chosen sufficiently small (depending on γ) so that

$$\frac{|\,1-x\,|}{1-|x|} < \frac{1}{\cos(\theta)-t/2} < \sec(\gamma)$$

regardless of the value of θ, so long as x is in the Stoltz angle defined by ζ.

So suppose $x = 1 - t\,e^{i\,\theta}$ to be in that Stoltz angle for g, $D_1(0)$, ζ and 1 and require t to be small enough so $|1-x| = t < \sec(\gamma)\,(1-|x|)$ for a fixed choice of γ (it doesn't matter which one) strictly between ζ and $\pi/2$.

$\sum_{k=0}^{n} b_k\, z^k$ converges to $g(1) = 0$ at $z = 1$, and also converges absolutely for z in the in $B_1(0)$. Our point x is in this set.

Let $G_n = \sum_{k=0}^{n} b_k\, x^k$. Our goal is to show that $g(x) = \lim_{n\to\infty} G_n = \sum_{k=0}^{\infty} b_k\, x^k$, which we know converges absolutely, can be made arbitrarily small by choosing t small enough.

Let $B_n = \sum_{k=0}^{n} b_k$ so that $\Delta B_n = B_{n+1} - B_n = b_{n+1}$. By our assumptions, B_n converges to $g(1) = 0$, though that convergence need not be absolute. In any case, there is a constant W so that $|B_n| < W$ for all n.

Using the summation by parts formula we have

$$\begin{aligned}
G_n &= \sum_{k=0}^{n} b_k\, x^k = b_0 + \sum_{k=1}^{n} b_k\, x^k = b_0 + \sum_{k=1}^{n} x^k\, \Delta B_{k-1} \\
&= b_0 + x^{n+1} B_n - x\, B_0 - \sum_{k=1}^{n} B_k\, \Delta x^k \\
&= b_0(1-x) + x^{n+1} B_n - \sum_{k=1}^{n} B_k\, (x^{k+1} - x^k) \\
&= x^{n+1} B_n + (1-x) \sum_{k=0}^{n} B_k\, x^k.
\end{aligned}$$

We know $G_n \to g(x)$ and $x^{n+1} B_n \to 0$ so we have a new series representation

$$\begin{aligned}
g(x) &= (1-x) \sum_{k=0}^{\infty} B_k\, x^k \\
&= (1-x) \sum_{k=0}^{n} B_k\, x^k + (1-x)\, x^{n+1} \sum_{k=0}^{\infty} B_{k+n+1}\, x^k.
\end{aligned}$$

For any $\varepsilon > 0$ choose n so large that $|B_{n+1+k}| < \varepsilon$ for all $k \geq 0$. Then we have

$$\begin{aligned}
|\, g(x)\, | &\leq |1-x| \sum_{k=0}^{n} W\, |x|^k \;+\; |1-x|\, |x|^{n+1} \sum_{k=0}^{\infty} \varepsilon\, |x|^k \\
&= |1-x| \frac{W\,(1-|x|^{n+1})}{1-|x|} \;+\; |1-x|\, |x|^{n+1} \varepsilon\, \frac{1}{1-|x|} \\
&< \sec\gamma\, \left(W\,(1-|x|^{n+1}) + \varepsilon \right).
\end{aligned}$$

The number $\sec(\gamma)$ depends on a choice involving the maximum angle from which x can approach 1, not x itself. The number W and choice of n depends on the series for $g(1)$, not x. The term $1 - |x|^{n+1}$ cannot exceed $(n+1)\, t$. So by insisting that $t < \frac{\varepsilon}{(n+1)W}$ we find that $|g(x)| < 2\, \varepsilon \sec\gamma$.

The necessary conclusion follows. □

5.4. ***Theorem. The Cauchy-Hadamard Theorem***

The radius of convergence R of a power series $\sum_{j=0}^{\infty} a_j\, (z-p)^j$ is the reciprocal of $\limsup |a_n|^{1/n}$, or ∞ if the limit is 0, or 0 if the limit is ∞.

Convergence is absolute on the interior of the disk of convergence and uniform on any $D_k(p)$ where $k < R$.

Proof. Suppose $L = \limsup |a_n|^{1/n}$ is neither 0 nor ∞ and let $R = 1/L$.

If $0 < r < R$ then $0 < \limsup r\, |a_n|^{1/n} < 1$. So there is a number $C < 1$ and integer N for which $0 < r < \frac{C}{|a_n|^{1/n}}$ whenever $n > N$. But then if $z \in D_r(p)$ we have

$$\left| \sum_{j=N+1}^{\infty} a_j\, (z-p)^j \right| \leq \sum_{j=N+1}^{\infty} |a_j|\, r^j < \sum_{j=N+1}^{\infty} C^j$$

and the last sum converges to 0 as $N \to \infty$. So $\sum_{j=0}^{\infty} a_j\,(z-p)^j$ converges absolutely and uniformly on $D_r(p)$.

The last calculation works without change for *any* positive r if $\limsup |a_n|^{1/n} = 0$ so the power series converges on the whole complex plane in that case.

Similarly, if $|\,z-p\,| = r > R$ then $|\,a_n\,(z-p)^n\,|$ exceeds 1 in magnitude infinitely often, so the series $\sum_{j=0}^{\infty} a_j\,(z-p)^j$ cannot converge.

The last calculation works without change for *any* positive r if $\limsup |a_n|^{1/n} = \infty$, so the power series cannot converge unless $z = p$. $\square$

5.5. ***Lemma. Mertens' Series Theorem (and Corollaries)***

Suppose $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ converge to limits A and B, respectively, and at least one of the two converges absolutely.
Define $c_j = \sum_{n=0}^{j} a_n\,b_{j-n}$ for each integer $j \geq 0$.
(So c is the Cauchy product $a \star b$ of the formal power series a and b.)
Then $\sum_{j=0}^{\infty} c_j$ converges, and its limit C satisfies $C = A\,B$.

If both constituent series converge absolutely, so too does $\sum_{j=0}^{\infty} c_j$.

This applies immediately to power series.

Suppose $\sum_{i=0}^{\infty} a_i\,(z-p)^i$ and $\sum_{i=0}^{\infty} b_i\,(z-p)^i$ converge and at least one of the two converges absolutely. Then $\sum_{i=0}^{\infty} c_i\,(z-p)^i$ converges, and

$$\sum_{i=0}^{\infty} c_i\,(z-p)^i = \left(\sum_{i=0}^{\infty} a_i\,(z-p)^i\right)\left(\sum_{i=0}^{\infty} b_i\,(z-p)^i\right).$$

When both of the original power series converge absolutely, so too does the Cauchy product power series.

If it is known only that the three series for A, B and C converge (no absolute convergence assumed) then $C = AB$.

Proof. Assume the first conditions of the lemma, and suppose the series for A converges absolutely. Let $A_n = \sum_{i=0}^{n} a_i$ and $B_n = \sum_{i=0}^{n} b_i$ and $C_n = \sum_{i=0}^{n} c_i$.

Let $H = \sum_{i=0}^{\infty} |a_i|$.

Let K denote the supremum of the numbers $|\,t_n\,|$ where t_n denotes the tail sequence $t_n = B - B_n = \sum_{i=n+1}^{\infty} b_i$ which converges to 0 by assumption.

Consider the sequence of numbers $x_n = \sum_{i=0}^{n} a_i t_{n-i}$ and suppose $\varepsilon > 0$.

Select N so large that whenever $n \geq N$ we have $\sum_{i=n+1}^{\infty} |a_i| < \varepsilon/K$ and also $|t_n| < \varepsilon/H$. For any such n we find that

$$\begin{aligned} |x_{2n}| &\leq \sum_{i=0}^{n} |\,a_i\,|\,|t_{2n-i}\,| + \sum_{i=n+1}^{2n} |\,a_i\,|\,|t_{2n-i}\,| \\ &\leq \frac{\varepsilon}{H}\sum_{i=0}^{n} |\,a_i\,| + \sum_{i=n+1}^{2n} |\,a_i\,|\,K \leq \frac{\varepsilon}{H}\,H + \frac{\varepsilon}{K}\,K = 2\varepsilon. \end{aligned}$$

Our conclusion then is that $x_{2n} \to 0$, and it follows immediately that $x_n \to 0$.

Now we have

$$C_j = \sum_{n=0}^{j}\sum_{k=0}^{n} a_k\, b_{n-k} = \sum_{k=0}^{j}\sum_{m=0}^{j-k} a_k b_m = \sum_{k=0}^{j} a_k \sum_{m=0}^{j-k} b_m$$
$$= \sum_{k=0}^{j} a_k B_{j-k} = \sum_{k=0}^{j} a_k\,(B - t_{j-k}) = A_j B - x_j.$$

Taking the limit on j we find $C = AB$.

The conclusion of absolute convergence when both constituent series converge absolutely follows easily by comparison of $|C_j|$ to the partial sums of the bigger series (which also converges) formed as the limit of $\sum_{n=0}^{j}\sum_{k=0}^{n} |a_k|\,|b_{n-k}|$.

Finally, if it is known that the series for A, B and C converge define $f(z)$, $g(z)$ and $h(z)$ as above. So the series for these three functions converge absolutely for $|z| < 1$. By Abel's Second Theorem and *real* x

$$\lim_{x\to 1^-} f(x) = f(1) = A \qquad \lim_{x\to 1^-} g(x) = g(1) = B \qquad \lim_{x\to 1^-} h(x) = h(1) = C$$

and for each x with $0 \le x < 1$ we have $h(x) = f(x)g(x)$. So $C = AB$. □

5.6. ***Proposition.*** *Suppose f is a function with open domain whose value $f(z)$ can be given by the power series $f(z) = \sum_{n=0}^{\infty} a_n\,(z-p)^n$ for those z in the domain of f for which the series converges.*

Suppose that the power series has nonzero radius of convergence R and that $|z-p| < R$.

Then the function f is infinitely differentiable at z and if $f^{(n)}$ denotes the nth derivative of f then

$$f^{(n)}(z) = \sum_{j=n}^{\infty} \frac{j\,!}{(j-n)\,!}\, a_j\,(z-p)^{j-n} = \sum_{k=0}^{\infty} \frac{(k+n)\,!}{k\,!}\, a_{k+n}\,(z-p)^k.$$

In other words, these derivatives can be calculated by differentiating the series "term-by-term," and this corresponds to the derivative we defined previously for formal power series.

The radius of convergence of such series is not changed by this procedure.

And a_n is therefore seen to be $\frac{f^{(n)}(p)}{n\,!}$ for each n.

This implies that for each p in the domain of a function there can be at most one representation of the function as a power series in $z-p$ valid on a disk centered at p with positive radius.

Proof. We need only prove the result for $n = 1$. The general formula and the interpretation of a_k follow by an induction.

By the Cauchy-Hadamard Theorem, the derivative series

$$G(z) = \sum_{k=0}^{\infty} k\, a_{k+1}\,(z-p)^k$$

does have the same radius of convergence as the original series. Our goal is to prove that $f'(z) = G(z)$.

For z with $|z-p| < r < R$ require λ to be in the interior of the disk around 0 of radius $r - |z-p|$. So $z+\lambda$ as well as z are in a disk $D_r(p)$ of absolute and uniform convergence of both series.

We assume that $p = 0$ for notational clarity and let $S_N = \sum_{k=0}^{N} a_k\, z^k$ be the Nth partial sum of the series for f and $T_N = \sum_{k=N+1}^{\infty} a_k\, z^k$ the corresponding tail sequence.

$$\begin{aligned}
\left|\frac{T_N(z+\lambda)-T_N(z)}{\lambda}\right| &= \left|\frac{1}{\lambda}\sum_{k=N+1}^{\infty} a_k\,(\,(\,z+\lambda\,)^k - z^k\,)\right| \\
&= \left|\sum_{k=N+1}^{\infty} a_k\,\left(\,(\,z+\lambda\,)^{k-1} + (\,z+\lambda\,)^{k-2}z + \cdots + (\,z+\lambda\,)z^{k-2} + z^{k-1}\right)\right| \\
&\leq \sum_{k=N+1}^{\infty} |a_k|\, r^{k-1}.
\end{aligned}$$

This is the tail of a convergent series, so for each $\varepsilon > 0$ an integer N can be chosen so that for all relevant λ this ratio is smaller than ε. We also require N to be large enough so that $|\,S_N'(z) - G(z)\,| < \varepsilon$. Now we have

$$\begin{aligned}
&\frac{f(z+\lambda)-f(z)}{\lambda} \;-\; G(z) \\
&\quad= \left(\frac{S_N(z+\lambda)-S_N(z)}{\lambda} \;-\; S_N'(z)\right) + (\,S_N'(z) - G(z)\,) + \frac{T_N(z+\lambda)-T_N(z)}{\lambda}.
\end{aligned}$$

The first group in the line above converges to 0 with λ. And N was chosen so neither the second group nor the ratio involving the tail sequence can exceed ε, which can be chosen to be as small as desired. $\square$

A power series $\sum_{k=0}^{\infty} a_k\,(z-p)^k$ is said to **represent f around p** if there is some $t > 0$ for which $f(z) = \sum_{k=0}^{\infty} a_k\,(z-p)^k$ for every $z \in B_t(p)$. In particular, this assumes that the series has positive or infinite radius of convergence.

A function that can be represented by a power series around every point in its domain is called **analytic.**

We have found that an analytic function is not just complex-differentiable, but holomorphic and infinitely differentiable on its entire domain.

5.7. ***Proposition.*** *Suppose z_n is a sequence of distinct points in connected open U converging to a point p which is also in U. Suppose also that f and g are analytic and $f(z_n) = g(z_n)$ for all n. Then $f = g$ on all of U.*

Proof. $h = f - g$ is analytic and $h(z_n) = 0$ for all n. Since $p \in U$, h has a series representation $h(z) = \sum_{n=0}^{\infty} a_n\,(z-p)^n$ that converges on some $B_\varepsilon(p)$, and we may assume that $z_n \in B_\varepsilon(p)$ and $z_n \neq p$ for all n. Since h is continuous, $h(p) = 0$, so $a_0 = 0$.

If h is not the zero function on $B_\varepsilon(p)$, then $a_k \neq 0$ for some least $k \geq 1$. Then

$$h(z) = (z-p)^k\,\left(a_k + (z-p)\sum_{n=0}^{\infty} a_{n+k+1}\,(z-p)^n\right)$$

It follows then that for every n

$$\frac{a_k}{p - z_n} = \sum_{n=0}^{\infty} a_{n+k+1} \, (z_n - p)^n.$$

But the series on the right corresponds to function values of a function analytic on $B_\varepsilon(p)$ and this function must be bounded on any compact neighborhood of p inside $B_\varepsilon(p)$. And z_n is eventually in such a neighborhood and the sequence $a_k/(p - z_n)$ has unbounded magnitude, by our assumption that $a_k \neq 0$. Since that assumption leads to a contradiction, we have $a_k = 0$ for all k and, in particular, h is 0 on all of $B_\varepsilon(p)$.

Now suppose $h(q) \neq 0$ at some point $q \in U$. Since U is connected there is a continuous path $r \colon [0,1] \to U$ from p to q. Let t_0 be the infinum of all those $t \in [0,1]$ for which $h(r(t)) \neq 0$. By continuity of h and r we know that $t_0 < 1$, and since $h = 0$ on $B_\varepsilon(p)$ we know $t_0 > 0$.

But then there is an increasing sequence a_n converging to t_0 for which $h(r(a_n)) = 0$ for all n. That means, by the earlier result, that $h = 0$ on some disk around $h(r(t_0))$, and so there will be some $\eta > 0$ for which $h(r(t)) = 0$ for every $t \in [0, t_0 + \eta]$, contrary to definition of t_0.

We conclude that no such q exists: h is 0 on all of U, so $f = g$ on all of U. $\square$

6. Integration of Complex-Differentiable Functions Along Paths

In this section we discuss integrals of complex-valued functions along paths in the plane using, essentially, the Riemann integral, and we will rely on the standard facts and vocabulary about such integrals.

We do not strive here for greatest generality.

A **path in open connected U** is a piecewise continuously differentiable function $s \colon [a,b] \to U$ for nonempty interval $[a,b] \subset \mathbb{R}$ and open $U \subset \mathbb{C}$.

Piecewise differentiability entails that there is a partition $P = \{ t_0, \dots, t_n \}$ of $[a,b]$ for which s is continuously differentiable on each (t_i, t_{i+1}) and s has right and left one-sided derivatives at each internal partition member and derivative from the right at a and from the left at b. This implies that s is continuous.

If $s(a) = s(b)$ we call this path a **loop**. The vocabulary **"closed path"** is often used to describe loops.

If $g = u + i\,v$ is a continuous complex valued functions with domain U and $s = x + i\,y$ is a path in U define

$$\begin{aligned} \int_a^b g\, ds \quad \text{to be} \quad & \int_{t=a}^{t=b} g(s(t))\, s'(t)\, dt \\ = & \left(\int_{t=a}^{t=b} u(s(t))\, x'(t) - v(s(t))\, y'(t)\, dt \right) \\ & \qquad + i \left(\int_{t=a}^{t=b} u(s(t))\, y'(t) + v(s(t))\, x'(t)\, dt \right) \end{aligned}$$

where, if s fails to be differentiable at one or more members of finite set $\{ t_0, \ldots, t_n \}$, where $a = t_0 < t_1 < \cdots < t_n = b$, we define

$$\int_a^b g\, ds = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} g\, ds.$$

Using $|ds|$ to denote the symbols $\sqrt{(x')^2 + (y')^2}\, dt$, we have

$$\left| \int_a^b g\, ds \right| \leq \int_a^b |g|\, |ds| = \int_{t=a}^{t=b} |g \circ s| \sqrt{(x')^2 + (y')^2}\, dt.$$

When g is the constant function $g \equiv 1$ the integral on the right above, a number L, is just the distance traversed by moving along the curve using this parameterization. If pieces of the curve are not traversed more than once, this number is the arclength along the curve. For more general g and s, since the image of s is compact and g continuous, $g \circ s$ has a finite maximum magnitude M and the integral on the right will not exceed ML.

As a trivial special case we allow the "point" $s : \{c\} \to U$ or constant path $s \colon [a, b] \to U$, $s(t) = p\ \forall t \in [a, b]$, and define the integral with respect to any such path to be 0.

Typical integration theorems hold. For constant k and continuous f and g

$$\int_a^b kg + f\, ds = k \int_a^b g\, ds + \int_a^b f\, ds.$$

Also if $c \in [a, b]$ then

$$\int_a^b g\, ds = \int_a^c g\, ds + \int_c^b g\, ds \quad \text{and} \quad \int_a^b g\, ds = -\int_b^a g\, ds.$$

If f has continuous complex-derivative then a fundamental theorem of calculus holds

$$f(s(b)) - f(s(a)) = \int_a^b f'\, ds.$$

We rephrase this critical result below using a slightly different vocabulary. We assume in the statement of this theorem that g is continuous. However we will soon see that any complex-derivative defined on any open domain must be continuous, so this assumption is actually superfluous.

6.1. ***Theorem. Fundamental Theorem of Calculus***

Suppose $s \colon [a, b] \to U$ is a path in U and F is a primitive for continuous g in U.

Then $$F(s(b)) - F(s(a)) = \int_a^b g\, ds.$$

In particular, if s is a loop and g has a primitive on some open set containing $s([a, b])$ then

$$\int_a^b g\, ds = 0.$$

Proof. Suppose $s = x + i\,y$ and $F = u + i\,v$ and then $g = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$.

$$\begin{aligned}
\int_a^b g\,ds &= \int_a^b \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right)\left(\frac{dx}{dt} + i\,\frac{dy}{dt}\right)dt \\
&= \int_a^b \frac{\partial u}{\partial x}\,\frac{dx}{dt} + i\,\frac{\partial v}{\partial x}\,\frac{dx}{dt} + i\,\frac{\partial u}{\partial x}\,\frac{dy}{dt} - \frac{\partial v}{\partial x}\,\frac{dy}{dt}\,dt \\
&= \int_a^b \left(\frac{\partial u}{\partial x}\,\frac{dx}{dt} + \frac{\partial u}{\partial y}\,\frac{dy}{dt}\right) + i\,\left(\frac{\partial v}{\partial x}\,\frac{dx}{dt} + \frac{\partial v}{\partial y}\,\frac{dy}{dt}\right)dt \\
&= u(s(b)) - u(s(a)) + i\,(v(s(b)) - v(s(a))) \\
&= F(s(b)) - F(s(a)).
\end{aligned}$$

□

If $r\colon [c,d] \to [a,b]$ is continuously differentiable with positive derivative and $r(c) = a$ and $r(d) = b$ then if s is a path so is $s \circ r$ and for any continuous g

$$\int_a^b g\,ds = \int_c^d g\,d(s \circ r).$$

A path can be reparameterized by a linear change of parameter to conform to any convenient parameter interval.

Generally, reparameterizing the path of integration by a change of parameter with non-negative derivative doesn't change any integral computed using that path.

For this reason we frequently denote an integral $\int_a^b g\,ds$ by

$$\int_a^b g\,ds = \int_\gamma g(z)\,dz$$

where γ is intended to denote the **image set** $\{\, s(t) \mid t \in [a,b] \,\}$ together with an **orientation**, which is, essentially, an assumption that this set actually is the image of at least one path s and an agreement to choose to calculate the integral using a path that "goes in the same direction" and "traverses all parts of the curve the same number of times" as does s.

Specifically, two paths s and w with the same image set have the property we require provided that each one can be obtained from the other by a change of parameter which is piecewise differentiable, but not necessarily continuous, with non-negative derivative wherever that derivative exists, and which can be made one-to-one and onto by removing a finite number of points from domain and range intervals. Such paths are called **compatible** with γ.

The importance of the "not necessarily continuous" and "remove finite number of points" conditions is to allow compatible parameterizations to traverse the pieces of a self-intersecting image set in different orders, so long as each piece is traversed the same number of times in the same direction. Piecewise differentiability does restrict such "switchings around" to be finite in number. And the two paths involved are both, themselves, assumed from the outset to be continuous.

6.2. ***Exercise.*** *Suppose $s : [-1,1] \to \mathbb{C}$ is defined by $s(t) = t + i\,t^3 \cos\left(\frac{\pi}{t}\right)$ for $t \in [-1,0)$ and $s(t) = -t + i\,t^4$ if $t \in [0,1]$. Then s is a piecewise differentiable loop that self-intersects an infinite number of times.*

The letter z in the integral involving γ above is present to distinguish the variable in a formula for g, which in practice might have several parameters, at which the intended integration is to take place.

Any single parameterization can be used to define the whole equivalence class of parameterizations with which it is compatible. When called upon to actually calculate an integral on γ, any compatible parameterization may be used and will give the same result.

γ as described above, with its image set together with an orientation (an equivalence class of mutually compatible paths with this image set) is called a **curve**.

If one of the compatible paths defining γ is a loop then all compatible paths are loops, and we call the curve **closed**.

6.3. ***Exercise.*** *In this exercise we will consider generic paths, continuous but without differentiability assumptions.*

*If U is open in $\mathbb{C}$, a path $s\colon [a,b] \to U$ of this kind is called **rectifiable** provided there is a number L so that for every $\varepsilon > 0$ there is a number $\delta > 0$ so that $\left|\, L - \sum_{k=1}^{n} |\, s(t_k) - s(t_{k-1})\,|\,\right| < \varepsilon$ for every partition $\{\, t_0, \ldots, t_n \,\}$ of $[a,b]$ with mesh less than δ.*

If s is piecewise differentiable (as all of the paths in the text are, unless the contrary is made explicit) then it is rectifiable.

If f is continuous on the image of rectifiable s then there is a real number M with $|\, f(s(t))\,| \leq M$ for all $t \in [a,b]$, and $f \circ s$ is uniformly continuous on $[a,b]$. It follows that there is a number I so that for every $\varepsilon > 0$ there is a number $\delta > 0$ so that $\left|\, I - \sum_{k=1}^{n} f(s(c_k))\,(\, s(t_k) - s(t_{k-1})\,)\,\right| < \varepsilon$ for every partition $\{\, t_0, \ldots, t_n \,\}$ of $[a,b]$ with mesh less than δ and any choice of c_k with $c_k \in [t_{k-1}, t_k]$ for $k = 1, \ldots, n$.

The number I is unique and denoted $\int_a^b f\, ds$. The use of this symbol agrees with the previous usage when s is piecewise differentiable.

Two rectifiable paths $s\colon [a,b] \to U$ and $r\colon [c,d] \to U$ are called equivalent, notation $s \simeq r$, provided $\int_a^b f\, ds = \int_c^d f\, dr$ for every function which is continuous on $s([a,b]) \cup r([c,d])$. It is fairly easy to show (Urysohn's Lemma) that $s([a,b]) = r([c,d])$ whenever $s \simeq r$. Reparameterizing s with a continuous and non-decreasing change of parameter will produce an equivalent rectifiable path, though if portions of the image are traversed more than once that may not be the only way to produce equivalent paths.

Show that if s is continuous there is a number $\delta > 0$ so that the tube

$$T_{s,\delta} = \{\, z \in \mathbb{C} \mid |\, z - s(t)\,| \leq \delta \,\}$$

is a compact subset of U.

Show that for each rectifiable s and each $\varepsilon > 0$ and for each function f defined and continuous on $T_{s,\delta} \subset U$ there is a piecewise linear[1] *path $r\colon [a,b] \to U$ for which*

$$|\, r(t) - s(t)\,| \leq \delta \text{ and } \left|\, \int_a^t f\, ds - \int_a^t f\, dr \,\right| \leq \varepsilon \quad \forall t \in [a,b].$$

[1] This parameterization may cross or retrace segments, and if you recall we defined "polygonal path" to forbid this: technically, r might not be a polygonal path.

If a curve γ is given using a particular compatible path s we define $-\gamma$ to have the same set as γ but with opposite orientation: specifically, that given by the reverse path $\tilde{s}$ which has domain $[a,b]$ and is defined by $\tilde{s}(t) = s(b+a-t)$.

For any continuous g

$$\int_{-\gamma} g(z)\,dz = -\int_{\gamma} g(z)\,dz.$$

If γ and τ are two curves and γ ends where τ starts we define $\gamma+\tau$ using a parameterization $s\colon [0,2] \to U$ for which s restricted to $[0,1]$ is a compatible parameterization of γ and s restricted to $[1,2]$ is a compatible parameterization of τ. It is not hard to see that if μ is a curve that starts where τ ends we can define $(\gamma+\tau)+\mu$ and $\gamma+(\tau+\mu)$ and these curves are equal.

If γ is a closed curve then $\gamma+\gamma$ is defined, and the notation $2\,\gamma$ may be used for this. Generally, if n is any nonzero integer we leave it as an exercise to define $n\,\gamma$. We resist the urge to engage in further arithmetic with closed curves.

And for any continuous g, when $\gamma+\tau$ is defined

$$\int_{\gamma+\tau} g(z)\,dz = \int_{\gamma} g(z)\,dz + \int_{\tau} g(z)\,dz.$$

Note that $\gamma+\tau$ and $\tau+\gamma$ can only *both* be defined if the sum is a closed curve, in which case for any continuous g

$$\int_{\gamma+\tau} g(z)\,dz = \int_{\tau+\gamma} g(z)\,dz.$$

There are a few curves that come up often enough to warrant names. For any positive b and complex number p define the curve $\boldsymbol{\mu_{b,p}}$, called the **circle of radius b centered at p**, to be the curve given by the parameterization $s_{b,p}(t) = p + b\,e^{i\,t}$ for $0 \le t \le 2\,\pi$. Note that the interior of the circle is on the left as you walk along the image of $s_{b,p}$ using this parameterization, and the curve is traversed once, counterclockwise, over the parameter interval.

A **polygonal curve** is a curve having a polygonal path with that curve's orientation. Of particular interest are the **segments**, which have parameterizations onto a single piece of a straight line, and **triangles** and **rectangles** which are closed polygonal curves of the specified shape that are the sum of three and four segments, respectively, and whose parameterizations traverse the segments in the counterclockwise direction: the interior of the triangle or rectangle is on the left as the parameter increases.

It is important to note that any compatible parameterization of a segment, as we have defined it, is one-to-one. By our definition, polygonal paths traverse their image just once. If you want a curve of triangular shape that traverses the image three times you would use $3\,\gamma$ where γ is a triangle. $-2\,\gamma$ traverses the image twice, clockwise.

As a general convention, whenever we use curves to carve out and identify a piece of the complex plane, we will (unless the contrary is made explicit) choose boundary curves, and parameterization of these boundary curves, so that the interior of the intended part of the plane is on the left as you move in the direction of increasing parameter.

For complicated shapes it might be arduous to keep track of this. In the cases seen in virtually every application, the required parameterization is obvious.

We have a **substitution identity**: if f is holomorphic on the path s and if $f(s(t))$ is always in the domain of continuous g then

$$\int_a^b (g \circ f)\, f'\, ds = \int_a^b g\, d(f \circ s).$$

Also, If f and g are both holomorphic, we have **integration by parts**

$$\int_a^b f\, g'\, ds = f(s(b))g(s(b)) - f(s(a))g(s(a)) - \int_a^b f'\, g\, ds.$$

6.4. ***Lemma.*** *A complex valued function with zero derivative and defined on a connected open domain must be constant. Two such functions with equal complex-derivatives differ by a constant and a function on connected open domain whose $(n+1)$st complex-derivative is 0 must be an nth degree polynomial.*

Proof. This is left for the reader, using integration by parts. □

6.5. ***Exercise.*** *A complex valued function with constant magnitude and defined on a connected open domain must be constant.*

Several classical integrability results are useful.

6.6. ***Lemma.*** *Suppose g_n is a sequence of continuous complex valued functions defined on a common domain U converging uniformly to continuous g and $s\colon [a,b] \to U$ is a path in U.*

Then $\int_a^b g_n\, ds$ converges, and the limit is $\int_a^b g\, ds$.

Proof. The proof is left for the reader. □

6.7. ***Lemma.*** *Suppose $s_n\colon [a,b] \to U$ is a sequence of continuously differentiable paths that converge uniformly to a function $s\colon [a,b] \to U$ and s_n' converges uniformly to s'. Suppose also that g is a continuous complex valued function defined on domain U.*

Then $\int_a^b g\, ds_n$ converges, and the limit is $\int_a^b g\, ds$.

Proof. Since s_n converges to s uniformly, the sequence is eventually in some closed "tube" T around s which is entirely contained in U. T is compact so g has bounded magnitude on T. The remainder of the proof is left for the reader. □

The second result is actually somewhat more general than it appears, as can be seen by thinking about the following fact.

6.8. ***Lemma.*** *Suppose $s_n\colon [a,b] \to U$ is a sequence of continuously differentiable paths. Suppose the sequence of numbers $s_n(a)$ converges to a number z_0 and the sequence of derivatives s_n' converges uniformly to a function $w\colon [a,b] \to U$. Then w is continuous and s_n converges uniformly to a continuously differentiable function $s\colon [a,b] \to U$ and $s' = w$.*

Proof. A uniformly convergent sequence of continuous functions defined on a compact interval converges to a continuous limit, so w is continuous. Now define

$$s(p) = z_0 + \int_a^p w(h)\, dh \quad \text{and} \qquad f_n(p) = z_0 + \int_a^p s_n'(h)\, dh \text{ for } p \in [a, b].$$

Each f_n differs from s_n by a constant but that constant converges to 0. And it is easy to show that f_n converges to f uniformly. The required conclusion follows. □

The third result is a bit more involved.

6.9. ***Lemma.*** *Suppose $g\colon [a, b] \times [c, d] \to \mathbb{C}$ is continuous with continuous partial derivative in its second domain factor, and we denote this derivative with notation $\frac{\partial g}{\partial t}$. Now define*

$$H\colon [c, d] \to \mathbb{C} \quad \textit{by} \quad H(t) = \int_a^b g(s, t)\, ds.$$

Then H is continuously differentiable (one-sided derivatives at the endpoints) and

$$H'(t) = \int_a^b \frac{\partial g}{\partial t}(s, t)\, ds.$$

Proof. Suppose $\varepsilon > 0$.

Since $\frac{\partial g}{\partial t}$ is continuous on a compact domain it is uniformly continuous. So there is a $\delta > 0$ so small that $t, t' \in [c, d]$ and $|\, t - t'\,| < \delta$ then $\left| \frac{\partial g}{\partial t}(s, t') - \frac{\partial g}{\partial t}(s, t) \right| < \varepsilon$.

Now select $t \in (c, d)$ and for this t require $h \neq 0$ to be so small that $t \pm h \in (a, b)$ and also $|h| < \delta$. Finally, for this t and h select $\mu > 0$ so small that it too is smaller than δ and also any Riemann sum for

$$\int_a^b \left| \frac{g(s, t + h) - g(s, t)}{h} - \frac{\partial g}{\partial t}(s, t) \right| \, ds$$

using partition $P = \{\, s_0, \ldots\, s_n \,\}$ whose mesh is less than μ differs from the integral by an amount less than ε.

For any such partition, by the mean value theorem there are points t_i between t and $t + h$ for which

$$\begin{aligned}
\left| \frac{H(t+h) - H(t)}{h} - \int_a^b \frac{\partial g}{\partial t}(s, t)\, ds \right| &\le \int_a^b \left| \frac{g(s, t + h) - g(s, t)}{h} - \frac{\partial g}{\partial t}(s, t) \right| \, ds \\
< \varepsilon + \sum_{i=1}^n \left| \frac{g(s_i, t + h) - g(s_i, t)}{h} - \frac{\partial g}{\partial t}(s_i, t) \right| \, \Delta\, s_i \\
= \varepsilon + \sum_{i=1}^n \left| \frac{\partial g}{\partial t}(s_i, t_i) - \frac{\partial g}{\partial t}(s_i, t) \right| \, \Delta\, s_i < \varepsilon + \sum_{i=1}^n \varepsilon\, \Delta\, s_i = \varepsilon + \varepsilon(b - a).
\end{aligned}$$

We conclude that the limit of the difference quotient for H exists and is the indicated integral for these t. The one-sided limits at the endpoints are handled similarly. Continuity of H' follows from the uniform continuity of $\frac{\partial g}{\partial t}$. □

7. Primitives and Goursat's Theorem

7.1. ***Proposition.*** *Suppose complex valued continuous g with connected open domain U has the property that $\int_\gamma g(z)\,dz = 0$ for every closed polygonal curve γ in U.*

Then g has a primitive on U and so $\int_\gamma g(z)\,dz = 0$ for every

closed curve γ (polygonal or not) in U.

Proof. Select $p \in U$. For each $q \in U$ let γ_q be a polygonal curve in U that starts at p and ends at q. We propose to define the primitive F on U by $F(q) = \int_{\gamma_q} g(z)\,dz$. First, note that this definition does not depend on the specific choice of γ_q, since if τ is another polygonal curve that starts at p and ends at q then $\gamma_q + (-\tau) = \gamma_q - \tau$ is a closed polygonal curve so

$$0 = \int_{\gamma_q - \tau} g(z)\,dz = \int_{\gamma_q} g(z)\,dz - \int_\tau g(z)\,dz$$

and we have $\int_{\gamma_q} g(z)\,dz = \int_\tau g(z)\,dz$.

So any choice of γ_q would do just as well to define F.

Now select q and closed disk $D_k(q)$ so small that it is entirely contained in U.

If any starting place $\overline{p}$ other than p had been used in the definition of F the new function would differ from the original one by the constant $\int_\gamma g(z)\,dz$ where γ is a polygonal curve connecting $\overline{p}$ to p. So the complex derivative of this new function, if it exists, would be the same as the complex-derivative of F. So we are free to assume $p = q$ when calculating the derivative at q, and we make that choice now.

Suppose $|h| < k$. Define path $s\colon [0,1] \to D_k(q)$ by $s(t) = q + t\,h$. So s is a straight-line path connecting q to $q + h$ and can therefore be used to calculate $F(q + h)$.

Since g is continuous, for each $\varepsilon > 0$ there is a $\delta > 0$ so that $|\,g(q+h) - g(q)\,| < \varepsilon$ whenever $|h| < \delta$. We insist $|h| < \delta$ as well. Define $M(h) = g(q + h) - g(q)$.

$$\begin{aligned}
&\frac{1}{h}\,(\,F(q+h) - F(q)\,) - g(q) = \frac{1}{h}\,F(q+h) - g(q) \\
&= \frac{1}{h}\int_0^1 g(s(t))\,ds - g(q) = \frac{1}{h}\int_0^1 g(q + t\,h)\,h\,dt - g(q) \\
&= \int_0^1 g(q + t\,h)\,dt - g(q) = \int_0^1 g(q) + M(\,t\,h\,)\,dt - g(q) = \int_0^1 M(\,t\,h\,)\,dt
\end{aligned}$$

The magnitude of the last term cannot exceed ε, which can be made as small as desired by choosing $|h|$ small enough. So

$$\lim_{h\to 0} \frac{F(q+h) - F(q)}{h} \quad \text{exists and equals} \quad g(q).$$

□

7.2. ***Theorem. Goursat's Theorem for Triangles and Rectangles***

If S is a closed triangle or rectangle entirely contained in the domain of function f, and if f is complex-differentiable on S then $\int_\gamma f(z)\,dz = 0$ where γ is a boundary curve for S.

Proof. We suppose first that S is triangular and γ is the curve that traverses the boundary of S once, counterclockwise. By assumption, f is complex-differentiable on an open set containing S and, in particular, at every point in γ or in the interior of S. Assume $\int_\gamma f(z)\,dz = M$.

Bisect the three sides of γ to obtain $four$ smaller triangles a, b, c and d. Integrals over internal segments are counted twice but with opposite orientation, yielding $\int_\gamma f(z)\,dz = \int_a f(z)\,dz + \int_b f(z)\,dz + \int_c f(z)\,dz + \int_d f(z)\,dz$. And then

$$\left| \int_\gamma f(z)\,dz \right| \leq \left| \int_a f(z)\,dz \right| + \left| \int_b f(z)\,dz \right| + \left| \int_c f(z)\,dz \right| + \left| \int_d f(z)\,dz \right|$$

So one of those integrals has magnitude at least $M/4$.

Call that one γ_1, which has perimeter $L/2$.

Proceed in this way next with γ_1 in place of γ, ultimately creating a sequence of triangles γ_i for which γ_i has perimeter $L/2^i$ and $\left| \int_{\gamma_i} f(z)\,dz \right| \geq M/4^i$. These triangles form nested sequence of compact sets and so have nonempty intersection p in the domain of f.

Suppose $\varepsilon > 0$. n can be chosen so large that for every z on or inside γ_n

$$|\, f(z) - f(p) - f'(p)(z-p)\,| < \varepsilon\,|z-p|.$$

Since polynomials have antiderivatives we know that $\int_{\gamma_n} f(p)+f'(p)(z-p)\,dz = 0$ which means

$$\frac{M}{4^n} \leq \left| \int_{\gamma_n} f(z)\,dz \right| \leq \varepsilon \int_{\gamma_i} |\,z-p|\,dz \leq \varepsilon\,\frac{L}{2^n}\,\frac{L}{2^n} = \varepsilon\,\frac{L^2}{4^n}.$$

The conclusion $\frac{M}{L^2} < \varepsilon$ for any $\varepsilon > 0$ requires M to be 0 and we have proved this theorem for triangular S.

Now suppose S is rectangular and that $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is the curve on the boundary of S and where each γ_i is the appropriate curve on the four edges. We can break the rectangle into two triangles by adding a curve σ along the diagonal. Then $\tau_U = \gamma_1 + \gamma_2 + \sigma$ is the bounding curve for the upper triangle, and $\tau_L = \gamma_3 + \gamma_4 - \sigma$ is the bounding curve for the lower triangle. So

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \sum_{k=1}^{4} \int_{\gamma_k} f(z)\,dz + \int_\sigma f(z)\,dz - \int_\sigma f(z)\,dz \\
&= \int_{\tau_U} f(z)\,dz + \int_{\tau_L} f(z)\,dz = 0 + 0.
\end{aligned}$$

□

The proof of this theorem does not use the continuity of* f$'$*. Instead, coupled with results below, continuity of* f$'$ *will follow from this theorem.

8. Cauchy's Integral Theorem and a Few Consequences

An open set U is called **star-shaped** if it possesses a point p for which the entire line segment from p to x is in U for every $x \in U$. We refer to p as a central point of U. Any star-shaped open set is, of course, connected.

Examples of star-shaped regions are rectangles, half-planes, triangles, the region between two parallel lines, the region between two line segments that meet at a vertex, a disk and a sector of a disk. Any convex open set is star-shaped, since any of its points can serve as a central point.

8.1. ***Theorem. Cauchy's Integral Theorem for Star-Shaped Domains***
If g is complex-differentiable on star-shaped open domain U then g has a primitive on U so $\int_\tau g(z)\,dz = 0$ for every closed curve γ in U.

Proof. Suppose U is star-shaped with central point p. We define, for each $q \in U$ the number $F(q) = \int_{\gamma_q} f(z)\,dz$ where γ_q is a polygonal curve connecting p to q by a single straight-line segment. For each q the distance between the compact set γ_q and the complement of U is positive, so there is a number $k > 0$ so that $D_k(x) \subset U$ for every point x on γ_q. In particular, if $q + h \in D_k(q)$ every point inside and on the triangular curve with corners p, q and $q + h$ is in U. Goursat's Theorem applied to this triangular curve μ tells us that $\int_\mu f(z)\,dz = 0$ and it follows that $F(q + h) - F(q)$ is just $\int_0^1 g(s(t))\,ds$ where $s\colon [0, 1] \to D_k(q)$ by $s(t) = q + t\,h$.

The calculation to show that F is a primitive for g now proceeds just as in Proposition 7.1. □

This theorem is also true for domains other than star-shaped ones, but these domains become arduous to describe as their shapes become more complex. In fact, the theorem is true for open domains whose complements are connected. We will not need this generality now. However, there is one such shape, the keyhole, which we will need for one of our more important results.

We suppose disk $D_\varepsilon(z_0)$ is contained in $B_k(p)$. Connect z_0 to point q on the boundary of $D_k(p)$ with line segment $L(z_0, q)$. Suppose $\delta < \varepsilon$ and define $\text{Corridor}(z_0, q, \delta)$ to be the points in $\mathbb{C}$ whose distance from $L(z_0, q)$ does not exceed δ.

$$\mathbf{Keyhole}(\mathbf{p}, \mathbf{z_0}, \mathbf{q}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\delta}) \\ = B_k(p) - (\ \text{Corridor}(z_0, q, \delta) \cup D_\varepsilon(z_0)\ ).$$

The boundary of the keyhole is a closed curve consisting of two arcs-of-circles and two line segments.

An **annulus** is the set $\mathbf{Annulus}(\boldsymbol{\varepsilon}, \mathbf{R})$, where $\varepsilon \geq 0$ and $R > \varepsilon$, given by

$$Annulus(\varepsilon, R) = \{\, z \in \mathbb{C} \mid \varepsilon < |z| < R \,\}.$$

An **indented semicircle** is the set $\mathbf{Indent}(\boldsymbol{\varepsilon}, \mathbf{R})$, where $\varepsilon \geq 0$ and $R > \varepsilon$, given by

$$Indent(\varepsilon, R) = \{\, z = x + i\,y \in \mathbb{C} \mid \varepsilon < |z| < R \text{ and } y > 0 \,\}.$$

8.2. ***Exercise.*** *Adapt the construction above to prove Cauchy's Integral Theorem for keyholes and indented semicircles. Show that the theorem does not hold for annuli.*

One of the principal pieces of complex-variables-technology used to prove facts about holomorphic functions is (the easiest version of) Cauchy's Integral Formula. Recall that the curve $\mu_{k,p}$ is the circle of radius k centered at p.

8.3. ***Lemma.*** *Suppose $D_k(p)$ is entirely contained in the domain of complex-differentiable f and z is in the domain but $z \notin D_k(p)$. Then*

$$0 = \int_{\mu_{k,p}} \frac{f(y)}{y-z}\, dy$$

Proof. For fixed z the function $\frac{f(y)}{y-z}$ is complex-differentiable on an open disc slightly larger than $D_k(p)$, and the result follows from Cauchy's Integral Theorem. □

8.4. ***Theorem. Cauchy's Integral Formula***

$$f(z) = \frac{1}{2\,\pi\, i} \int_{\mu_{k,p}} \frac{f(y)}{y-z}\, dy$$

where we assume that $D_k(p)$ is entirely contained in the domain of complex-differentiable f and z is in $B_k(p)$.

Proof. Select z as above and define for those w in the domain of f (but $w \neq z$)

$$G(w) = \frac{f(w)}{w-z} = \frac{f(w)-f(z)}{w-z} + \frac{f(z)}{w-z}.$$

We assume f to be complex-differentiable on $D_k(p)$ so the function $M(w) = \frac{f(w)-f(z)}{w-z}$ can be extended (define it to be $f'(z)$ at $w = z$) to a continuous function on $D_k(p)$. So the magnitude of $M(w)$ cannot exceed some number C for w in compact $D_k(p)$.

Let γ be a parameterization of the boundary of $\mathbf{Keyhole}(\mathbf{p}, \mathbf{z}, \mathbf{q}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ where the point q is the closest point to z on the boundary, or chosen at random if $p = z$.

Since f is complex-differentiable on a disk slightly larger than $D_k(p)$, G is complex-differentiable on a keyhole slightly larger than the one given above for any choice of δ and ε subject only to $0 < \delta < \varepsilon$. So Cauchy's Integral Theorem applies for any G and γ: $\int_\gamma G(y)\, dy = 0$.

γ can be represented as

$$\gamma = \gamma_k + \gamma_{down} + \gamma_\varepsilon + \gamma_{up}$$

where γ_k is a counterclockwise curve on the arc of the disk $D_k(p)$, γ_{down} is the part of γ that runs down the corridor, γ_ε runs clockwise on the arc of the disk $D_\varepsilon(z)$ and γ_{up} runs back up the corridor to close the curve at the boundary of $D_k(p)$.

By continuity of G, choosing the corridor width δ small enough forces

$$\int_{\gamma_{down}} G(y)\, dy \approx - \int_{\gamma_{up}} G(y)\, dy.$$

This implies then that for small enough δ

$$\int_{\gamma_k} G(y)\,dy \approx -\int_{\gamma_\varepsilon} G(y)\,dy.$$

As δ approaches 0, integrals over γ_k approach integrals along the path $s_{k,p}$ parameterizing all of $\mu_{k,p}$, and integrals over $-\gamma_\varepsilon$ approach integrals along the path $s_{\varepsilon,z}$.

Therefore, for any small ε

$$\int_{\mu_{k,p}} G(y)\,dy = \int_{\mu_{\varepsilon,z}} G(y)\,dy.$$

We can write

$$\begin{aligned}\int_{\mu_{\varepsilon,z}} G(y)\,dy &= \int_{\mu_{\varepsilon,z}} M(y)\,dy + \int_{\mu_{\varepsilon,z}} \frac{f(z)}{y-z}\,dy \\ &= \int_{\mu_{\varepsilon,z}} M(y)\,dy + f(z)\int_{\mu_{\varepsilon,z}} \frac{dy}{y-z}.\end{aligned}$$

The first of the two integrals on the far right cannot exceed $C\,2\pi\varepsilon$. The second integral is

$$f(z)\int_{\mu_{\varepsilon,z}} \frac{dy}{y-z} = f(z)\int_0^{2\pi} \frac{\varepsilon\, i\, e^{i\,t}}{z+\varepsilon\, e^{i\,t}-z} = f(z)\,2\pi\, i.$$

Since ε can be chosen as small as required, we have

$$\int_{\mu_{k,p}} \frac{f(y)}{y-z}\,dy = \int_{\mu_{\varepsilon,z}} \frac{f(z)}{y-z}\,dy = f(z)\,2\pi\, i$$

and the result we claimed in the statement of the theorem follows. □

For function f we let $f^{(n)}$ refer to the nth derivative of f. (If $n = 0$ this is intended to denote the function itself.)

8.5. ***Theorem.*** *Suppose $D_k(p)$ is entirely contained in the domain of complex-differentiable f and z is in $B_k(p)$.*
Let γ denote the boundary curve of $D_k(p)$ which is parameterized by a one-to-one counterclockwise parameterization.

Then f has complex-derivatives of all orders, which can be calculated on $B_k(p)$ by

$$f^{(n)}(z) = \frac{n!}{2\,\pi\, i}\int_\gamma \frac{f(y)}{(y-z)^{n+1}}\,dy.$$

Proof. We have the result for $n = 0$, which is Cauchy's Integral Formula. Suppose we have proved this result in the case $n-1$ for positive integer n. Then

$$\begin{aligned}\frac{f^{(n-1)}(z+h)-f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi\, i\, h}\int_\gamma \frac{f(y)}{(y-z-h)^n} - \frac{f(y)}{(y-z)^n}\,dy \\ &= \frac{(n-1)!}{2\pi\, i\, h}\int_\gamma f(y)\,\frac{(y-z)^n-(y-z-h)^n}{(y-z-h)^n(y-z)^n}\,dy \\ &= \frac{(n-1)!}{2\pi\, i}\int_\gamma f(y)\,\frac{\sum_{j=0}^{n-1}(y-z)^{n-1-j}(y-z-h)^j}{(y-z-h)^n(y-z)^n}\,dy\end{aligned}$$

The numerator in the integral converges uniformly on γ as $h \to 0$ to $n\,(y-z)^{n-1}$ while the denominator converges uniformly to $(y-z)^{2n}$, and this denominator limit is bounded away from 0 on γ. So

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi\, i} \int_\gamma f(y) \frac{n\,(y-z)^{n-1}}{(y-z)^{2n}}\, dy = \frac{n!}{2\pi\, i} \int_\gamma \frac{f(y)}{(y-z)^{n+1}}\, dy.$$

□

8.6. ***Corollary. Cauchy's Estimate***
Suppose $D_k(p)$ is entirely contained in the domain of complex-differentiable f and $|f(y)| \leq M$ for y in the boundary circle of $D_k(p)$. Then

$$|\, f^{(n)}(p) \,| \leq \frac{n!\, M}{k^n}.$$

Proof. This follows immediately from Theorem 8.5. □

8.7. ***Corollary. Liouville's theorem***
A bounded entire function is constant.

Proof. Suppose f is entire and bounded, and $|f(z)| \leq M$ for all z. By Corollary 8.6, for any $p \in \mathbb{C}$ and *any* positive k we have $|\, f'(p) \,| \leq \frac{n!\, M}{k^n}$.

That means $f'(p) = 0$ for all p. So f is constant. □

8.8. ***Theorem.*** *A function f defined on an open set U is complex-differentiable on U if and only if it is analytic on U.*

Any power series centered at p that represents complex-differentiable f converges on any $B_k(p)$ contained in the domain of f.

In particular, f is entire (i.e. f is complex-differentiable and $U = \mathbb{C}$) if and only if every power series that represents f on any disk has infinite radius of convergence.

Proof. Suppose f is complex-differentiable and $D_k(p)$ is any disk entirely contained in the domain of f. By Cauchy's Integral Formula

$$f(z) = \frac{1}{2\,\pi\, i} \int_{\mu_{k,p}} \frac{f(y)}{y-z}\, dy = \frac{1}{2\,\pi\, i} \int_{\mu_{k,p}} \frac{f(y)}{(y-p)}\, \frac{dy}{1 - \frac{z-p}{y-p}}.$$

There is some r with $0 < r < 1$ so that $\left| \frac{z-p}{y-p} \right| < r$ for every y on $\mu_{k,p}$.

That means that $\sum_{j=0}^{\infty} \left(\frac{z-p}{y-p} \right)^j$ converges uniformly to $\frac{1}{1-\frac{z-p}{y-p}}$ on $\mu_{k,p}$.

This justifies exchanging the order of limit-taking and integration, and convergence of the final series, in

$$\begin{aligned} f(z) &= \frac{1}{2\,\pi\,i}\int_{\mu_{k,p}} \frac{f(y)}{(y-p)} \sum_{j=0}^{\infty} \left(\frac{z-p}{y-p}\right)^j dy \\ &= \frac{1}{2\,\pi\,i}\sum_{j=0}^{\infty}\int_{\mu_{k,p}} \frac{f(y)}{(y-p)} \left(\frac{z-p}{y-p}\right)^j dy \\ &= \frac{1}{2\,\pi\,i}\sum_{j=0}^{\infty}\left(\int_{\mu_{k,p}} \frac{f(y)}{(y-p)^{j+1}}\,dy\right)(z-p)^j. \end{aligned}$$

The remaining statements in the theorem follow from uniqueness of power series representations. □

8.9. ***Proposition.*** *Suppose g_k is a sequence of complex-differentiable functions whose domains all contain a disk $D_s(a)$ for some $s > r > 0$, and suppose the domain of complex-differentiable f also contains $D_s(a)$. Suppose that the sequence g_k converges uniformly to f on $\mu_{s,a}$, the boundary circle of $D_s(a)$.*

(i) For each $n \geq 0$, $g_k^{(n)}$ converges uniformly to $f^{(n)}$ on all of $D_r(a)$.

(ii) Every coefficient in the series representations for the g_k at a converges to the corresponding coefficient for the limit function f.

Proof. Suppose $\varepsilon > 0$ and choose N so large that $|\,g_k - f\,|$ cannot exceed ε on the boundary circle of $D_s(a)$ for $k \geq N$. For such k and $z \in B_r(p)$ and $n \geq 0$

$$\begin{aligned} |\,g_k^{(n)}(z) - f^{(n)}(z)\,| &= \frac{n!}{2\,\pi}\left|\int_{\mu_{s,a}} \frac{g_k(y)-f(y)}{(y-z)^{n+1}}\,dy\right| \\ &\leq \frac{n!}{2\,\pi}\int_0^{2\,\pi} \frac{\varepsilon}{(s-r)^{n+1}}\,s\,dt = \frac{n!\,s}{(s-r)^{n+1}}\,\varepsilon. \end{aligned}$$

Item (ii) is now just an observation. □

9. Morera's Theorem

9.1. ***Theorem. Morera's Theorem***
Suppose complex valued continuous g with domain U has the property that for every $x \in U$ there is a disk $D_k(p)$ entirely contained in U and containing x in its interior for which $\int_\gamma g(z)\,dz = 0$ for every closed triangle γ in $D_k(p)$.

Then g has a local primitive on each $B_k(p)$ and it follows that g is analytic at every point in U.

Proof. By Proposition 7.1 g has a local primitive F_p defined on each $B_k(p)$. By Theorem 8.8 that local primitive, and hence g itself, is analytic in $B_k(p)$, and hence at each $x \in U$. □

9.2. ***Corollary.*** *Suppose f_n is a sequence of holomorphic functions converging uniformly to continuous f on $B_k(p)$.*

Then f is holomorphic on $B_k(p)$.

Proof. By Cauchy's Integral Theorem $\int_\gamma f_n(z)\,dz = 0$ for each n and every closed curve γ in $B_k(p)$. By uniform convergence on the image set for γ, we have

$$0 = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

Now Morera's Theorem has f holomorphic. □

9.3. ***Corollary.*** *Suppose $g\colon B_k(p)\times[c,d]\to\mathbb{C}$ is continuous and $g(\cdot,t)\colon B_k(p)\to\mathbb{C}$ is holomorphic for each $t\in[c,d]$.*

Then the function $H\colon B_k(p)\to\mathbb{C}$ given by $H(s)=\int_c^d g(s,t)\,ds$ is holomorphic.

Proof. Suppose γ is a closed curve in $B_k(p)$ with compatible parameterization $w\colon[a,b]\to B_k(p)$. The interval $[a,b]$ can be written as the finite union of n intervals $[a_j,b_j]$ upon which w is continuously differentiable, and

$$\int_a^b H(w(r))\,w'(r)\,dr = \sum_{j=1}^n \int_{a_j}^{b_j} H(w(r))\,w'(r)\,dr.$$

So $h_j\colon[a_j,b_j]\times[c,d]\to\mathbb{C}$ defined by $h_j(s,t)=g(w(s),t)\,w'(s)$ is continuous on the compact product space $[a_j,b_j]\times[c,d]$ and so Fubini's theorem is justified for this function. Then we have

$$\begin{aligned}
\int_\gamma H(z)\,dz &= \sum_{j=1}^n \int_{a_j}^{b_j} H(w(r))\,w'(r)\,dr = \sum_{j=1}^n \int_{a_j}^{b_j}\int_c^d g(w(r),t)\,w'(r)\,dt\,dr\\
&= \sum_{j=1}^n \int_c^d\int_{a_j}^{b_j} g(w(r),t)\,w'(r)\,dr\,dt = \int_c^d \left(\sum_{j=1}^n \int_{a_j}^{b_j} g(w(r),t)\,w'(r)\,dr\right)dt\\
&= \int_c^d \left(\int_\gamma g(z,t)\,dz\right)dt = 0
\end{aligned}$$

where the final parenthesized integral is 0 because each $g(\cdot,t)$ is holomorphic.

We invoke Morera's Theorem to conclude that H too is holomorphic. □

10. The Field of Meromorphic Functions

Suppose $f(z)=\sum_{j=0}^\infty a_j\,(z-p)^j$ has radius of convergence exceeding k. Let M denote the maximum magnitude of $|f(z)-f(p)|$ for $z\in D_k(p)$. Define $g(z)=f(z+p)-f(p)$ and suppose $f'(p)=g'(0)\neq 0$. So g is defined on all of $D_k(0)$ and the maximum magnitude M of continuous g on this disk is actually attained at some $z_0\in D_k(0)$.

$$g(z)=a_1\,z+a_2\,z^2+\cdots=\sum_{j=1}^\infty a_j\,z^k \qquad a_1=g'(0)\neq 0.$$

According to Proposition 2.1 the formal power series for g has inverse with respect to formal composition of formal power series as discussed before that proposition, given by a recursive formula

$$b_1 = \frac{1}{a_1}, \quad b_2 = \frac{a_2 b_1^2}{a_1}, \quad b_3 = \frac{2a_2 b_1 b_2 + a_3 b_1^3}{a_1}, \quad \cdots$$

That construction says nothing about convergence of the sequence of partial sums for these power series in z. However, we do know that

$$\left| \sum_{j=1}^{\infty} b_j \, (\, g(z_0)\,)^j \right| = |z_0| \leq k.$$

10.1. ***Proposition.***

Proof. □

11. Conformality and Complex-Differentiability

A function f with open domain U is called **conformal** if, whenever s and w are continuously differentiable paths in U with domain $(-1, 1)$ for which $s(0) = w(0)$, and if neither $s'(0)$ nor $w'(0)$ are zero vectors, then the paths $f \circ s$ and $f \circ w$ have nonzero derivative at 0, these derivatives are continuous, and the counterclockwise angle from $(f \circ s)'(0)$ to $(f \circ w)'(0)$ is the same as the counterclockwise angle from $s'(0)$ to $w'(0)$.

The existence and continuity of these derivatives for any path implies that f has a continuous derivative matrix, and the angle-preservation property requires that this matrix is a nonzero multiple of a rotation matrix. So the real and complex parts of f satisfy the Cauchy-Riemann equations and f is holomorphic on U, with nonzero complex-derivative.

Conversely, if f is holomorphic with derivative never 0 the Cauchy-Riemann equations imply that f', represented as a 2 by 2 matrix, is a non-zero multiple of a rotation matrix so f is conformal.

Intuitively, when "moved" by f, tiny polygonal figures don't change their shape because their interior angles all remain the same, though the figure might be rotated or made larger or smaller. Continuity of the complex-derivative is needed to complete this visualization: we don't want one corner of a tiny polygon to be rotated by a different angle than another.

Conformality is a geometric condition equivalent to complex-differentiability *on neighborhoods where the function has nonzero derivative.*

12. Analytic Continuation and Monodromy

The magnitude $|f(z)|$ attains both maximum and minimum on any closed disk $D_k(a)$ in its domain, and this maximum cannot exceed $\sum_{i=0}^{\infty} |a_i|\, k^i$ when the series for f centered at a has coefficients a_i.

Any local maximum of $|f(z)|$ on $D_k(a)$ must occur at a boundary point of $D_k(a)$, a fact called the **Maximum Modulus Principle**. (Also, any one-to-one conformal map from the disk to itself is linear. Also, Schwartz reflection principle.)

Local maxima *cannot* occur in the interior unless the function is constant. Local minimum values of $|f(z)|$ on $D_k(a)$ cannot occur in the interior either unless the function is constant or that local minimum value is 0, a result obtained by examining $1/f$.

Two complex-differentiable functions which agree at a sequence of points converging to a limit point which is in both domains agree everywhere on the connected component of the intersection of their domains within which the limit point resides.

So if $f'(z_n) = 0$ on a sequence of domain members z_n that converges to a point of the domain of f then f is constant.

This also implies that a complex-differentiable function defined on a disk has isolated zeros unless it is constantly zero on that disk. (In fact, the set of points $f^{-1}(c)$ cannot have a limit point in the disk for *any* c unless f is constant.)

More, if $g(a) = 0$ and g is complex-differentiable and is defined and not constant on some disk around a then there is a positive exponent n and complex-differentiable function f with $f(z) \neq 0$ on some disk around a and so that $g(z) = (z-a)^n f(z)$ for every z in the domain of g. In other words, **zeros of a holomorphic function are not only isolated but have "finite order" as well.**

Finally, we note that non-constant holomorphic functions (remember, our domains are connected) are open maps: $f(U)$ is an open set whenever U is open in the domain. That implies that a one-to-one holomorphic map has holomorphic inverse.

For positive number k let $\boldsymbol{\mathcal{F}_k}(p)$ denote the set of functions which possess Taylor series centered at p whose radius of convergence **exceeds** k. Compactness of $D_k(p)$ implies that any member of $\mathcal{F}_k(p)$ is complex-differentiable on an open set containing $D_t(p)$ for some $t > k$, and this allows for convenient framing of various convergence results.

We quote (again without proof) a handy theorem due to Osgood.[2]

12.1. ***Lemma.*** *Suppose $g_n \in \mathcal{F}_t(p)$ for each n and the sequence converges pointwise to a function f, and $t > k > 0$. Then $f \in \mathcal{F}_k(p)$ and the sequence converges uniformly to f on $D_k(p)$.*

---

[2]See W. F. Osgood, *Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, with corresponding results for definite integrals*, Ann. Math. 3 (1901), 25-34 and R. Remmert *Classical Topics in Complex Function Theory*, Springer-Verlag 1998 page 151.

Suppose $f(z) = \sum_{i=0}^{\infty} a_i\,(z-p)^i \in \mathcal{F}_k(p)$ and $f_n(z) = \sum_{i=0}^{n} a_i\,(z-p)^i$.

By Lemma 12.1 and Abel's First Theorem we know the series converges absolutely, and convergence is uniform, on $D_k(p)$, and in particular $\sum_{i=0}^{\infty} |a_i|\,k^i < \infty$.

For $n > m \geq 0$ and any z in this closed disk,

$$|f_n(z) - f_m(z)| \leq \sum_{i=m+1}^{n} |a_i|\,k^i \quad \text{and also} \quad \|f(z)\| \leq \sum_{i=0}^{\infty} |a_i|\,k^i < \infty.$$

12.2. ***Lemma.*** *Suppose $f \in \mathcal{F}_t(p)$ and $g \in \mathcal{F}_k(q)$ are complex-valued functions of one complex variable and $g(z) = \sum_{i=0}^{\infty} b_i\,(z-q)^i$. Suppose further that $g(q) = p$ and $\sum_{n=0}^{\infty} |b_n|\,k^n < t$.*

Then $f \circ g \in \mathcal{F}_k(q)$.

Proof. The maximum value of $|g(z) - p|$ on an open disk slightly larger than $D_k(q)$ is less than t, so the derivative of $f \circ g$ exists everywhere on this slightly larger open set and can actually be calculated by the chain rule. So $f \circ g$ has Taylor series centered at q with radius of convergence exceeding k. □

Kelley *General Topology* [3], Dugundji, *Topology* [1], Steen and Seebach, *Counterexamples in Topology* [4] and Engelking, *General Topology* [2].

References

[1] Dugundji, J., *Topology.* Allynn and Bacon, Inc., Boston, MA, 1966
[2] Engelking, R., *General Topology.* Heldermann Verlag, Berlin, 1989
[3] Kelley, J. L., *General Topology.* D. Van Nostrand Company, Inc., New York, 1955.
[4] Steen, L. A.. and Seebach, J. A., *Counterexamples in Topology.* Springer-Verlag, Inc., New York, 1978

Index

1_0, 7
1_1, 8
$B_k(p)$, 10
$D_k(p)$, 10
$\mu_{b,p}$, 24
$a \star b$, 7

Abel transform, 13
Abel's
  First Theorem, 13
  Second Theorem, 15
analytic, 19
annulus, 29

Cauchy Product, 7
Cauchy's
  Estimate, 32
  Integral Formula, 30
  Integral Theorem, 29
Cauchy-Hadamard Theorem, 16
Cauchy-Riemann equations, 12
circle, 24
closed
  curve, 20, 23
compatible
  parameterization, 22
complex
  derivative, 10
  differentiable, 10
conformal, 35
connect, 9
connected, 9
  component, 9
  path, 9
  simply, 10
curve, 23
  circle, 24
  polygonal, 24
  rectangle, 24
  segment, 24
  triangle, 24

derivative
  complex, 10
difference operator, 13
Dugundji, J., 38

Engelking, R., 38
entire, 10

formal
  composition, 7
  reciprocals, 7
formal power series, 6
  in x, 6
FPS, 6
Fundamental Theorem of Calculus, 21

Goursat's theorem, 28

harmonic, 12
holomorphic, 10
homotopic, 10
homotopy, 10

image set, 22
indented semicircle, 29
Inverse Function Theorem, 8

Kelley, J. L., 38

Laplace's equation, 12
Liouville's theorem, 32
local linear approximant, 11
local primitive, 10
loop, 20

maximum modulus principle, 36
Mertens' Series Theorem, 17
Morera's Theorem, 33

nilpotent, 6

orientation, 22

path, 20
  polygonal, 9
path-connected, 9
piecewise
  differentiable, 20
  linear, 9
power series
  formal, 6
primitive, 10

rectangle, 24
rectifiable, 23
represent, 19

Seebach, J. A., 38
segment, 24
simply connected, 10
star-shaped, 29
Steen, L. A., 38
summation by parts, 13

triangle, 24