SYSTEMS OF TWO LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Contents

1. Vectors in the Plane

Vectors in the plane will be denoted by bold capital letters with x and y coordinates indicated in columns, such as

$$\mathbf{V} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit (length) vectors in the direction of the coordinate axes

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A **linear combination** of two vectors is a vector sum $a\mathbf{V} + b\mathbf{W}$ where a and b are numbers and $\mathbf{V}$ and $\mathbf{W}$ are vectors.

So any vector, such as $\mathbf{V}$ above, can be written as a linear combination $x\mathbf{e}_1 + y\mathbf{e}_2$ of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

Two vectors $\mathbf{V}$ and $\mathbf{W}$ are called **independent** if neither is a numerical multiple of the other. The two vectors are called **dependent** if they are *not* independent.

It is not too hard to show that any vector in the plane can be written as a linear combination of any chosen independent pair of vectors.

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2. Matrix Preliminaries

Suppose we are given matrix with real entries

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

We define $\mathrm{tr}(\mathbf{A}) = a_{1,1} + a_{2,2}$ and $\det(\mathbf{A}) = a_{1,1}\, a_{2,2} - a_{1,2}\, a_{2,1}$.

These numbers are called the **trace** and **determinant** of the matrix $\mathbf{A}$, respectively.

The **characteristic polynomial** of the matrix $\mathbf{A}$ is

$$P(x) = \det\left(x\mathbf{I} - \mathbf{A}\right) \quad \text{where } \mathbf{I} \text{ is the identity matrix } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this two dimensional situation $P(x) = x^2 - \mathrm{tr}(\mathbf{A})x + \det(\mathbf{A})$.

The roots of the characteristic polynomial will be denoted λ_1 and λ_2, called the **eigenvalues** of the matrix $\mathbf{A}$. So

$$P(x) = (x - \lambda_1)(x - \lambda_2).$$

An **eigenvector** of the matrix A is a *nonzero* vector $\mathbf{V}$ for which $\mathbf{AV} = \lambda_i \mathbf{V}$ for $i = 1$ or 2. So the effect of $\mathbf{A}$ on an eigenvector is particularly simple: it acts as a "stretcher" or "shrinker" by factor λ_i. If $\mathbf{AV} = \lambda_i \mathbf{V}$, nonzero $\mathbf{V}$ is said to be an eigenvector "for" λ_i.

Note that $\mathbf{AV} = \lambda_i \mathbf{V}$ exactly when

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right)\mathbf{V} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We make three important observations, that are pretty easy to show.

First, if $\mathbf{V}$ is an eigenvector for λ_i then so is any nonzero numerical multiple of $\mathbf{V}$.

Second, there is an eigenvector for each different eigenvalue.

Third, if $\mathbf{V}$ and $\mathbf{W}$ are eigenvectors for *different* eigenvalues then $\mathbf{V}$ and $\mathbf{W}$ are independent.

We like eigenvectors in these notes because, as we shall see in a later section, they will allow us to write down simple "straight line" solutions to a linear system of differential equations. We like an independent pair of eigenvectors because we can realize any initial vector as a linear combination involving this pair, and the solution to the corresponding initial value problem is the same linear combination of these "straight line" solutions.

But that is for a bit later.

3. Cayley's Theorem

Cayley's Theorem If $P(x)$ is the characteristic polynomial of square matrix $\mathbf{A}$ then $P(\mathbf{A}) = 0$.

This important and useful theorem is proved in Linear Algebra classes for $n \times n$ matrices, but it is easy to show for 2×2 matrices by calculating

$$\mathbf{A}^2 - \operatorname{tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I}$$

and observing that it is the zero matrix.

The reason *we* want to know this theorem is to speed up the process of finding eigenvectors. Since

$$P(\mathbf{A}) = (\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (\mathbf{A} - \lambda_2\mathbf{I})(\mathbf{A} - \lambda_1\mathbf{I})$$

this matrix must yield the zero vector when applied to **any** vector. In particular for any vector $\mathbf{V}$

$$(\mathbf{A} - \lambda_1\mathbf{I})\Big((\mathbf{A} - \lambda_2\mathbf{I})\mathbf{V}\Big) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathbf{A} - \lambda_2\mathbf{I})\Big((\mathbf{A} - \lambda_1\mathbf{I})\mathbf{V}\Big).$$

Looking at the left expression we see that (whenever it is nonzero) the vector $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{V}$ is an eigenvector for λ_1.

$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{e}_1$ is the first column of $\mathbf{A} - \lambda_2\mathbf{I}$ and $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{e}_2$ is the second column of $\mathbf{A} - \lambda_2\mathbf{I}$.

So nonzero columns of $\mathbf{A} - \lambda_2\mathbf{I}$ are eigenvectors for λ_1 and, by an identical argument, nonzero columns of $\mathbf{A} - \lambda_1\mathbf{I}$ are eigenvectors for λ_2.

4. Eigenvectors: The Real Case

First suppose $\lambda_1 \neq \lambda_2$.

$(\mathbf{A} - \lambda_1\mathbf{I})$ has a nonzero column which is an eigenvector for eigenvalue λ_2.

$(\mathbf{A} - \lambda_2\mathbf{I})$ has a nonzero column which is an eigenvector for eigenvalue λ_1.

This pair of columns forms an independent pair of vectors.

We now consider two variations on the case $\lambda_1 = \lambda_2 = \lambda$.

It is possible that $\mathbf{A} - \lambda\mathbf{I}$ is the zero matrix, so $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors. All is good, they are an independent pair of eigenvectors.

The problem case is when $\mathbf{A} - \lambda\mathbf{I}$ is nonzero. So there is a nonzero column of $\mathbf{A} - \lambda\mathbf{I}$. Any nonzero column must be an eigenvector for λ, but there cannot be two independent columns: that would imply that *any* nonzero vector is an eigenvector for λ which would imply $\mathbf{A} - \lambda\mathbf{I} = \mathbf{0}$. So there cannot be an independent pair of eigenvectors here.

5. Eigenvectors: The Complex Case

We now consider the case of complex eigenvalues. Since these are obtained by applying the quadratic formula to the characteristic polynomial, it is easy to see that they come in conjugate pairs. This is also true in higher dimensions.

Let $\lambda_1 = r + si$ and $\lambda_2 = \overline{\lambda_1} = r - si$, where the "overline" is intended to represent complex conjugation operation and r and s are real.

Any vector or matrix $\mathbf{M}$ with complex entries can be broken up into real and imaginary parts by examining each entry. For instance

$$\begin{pmatrix} 3-2i & 5+i \\ -i & 2+7i \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix} + i \begin{pmatrix} -2 & 1 \\ -1 & 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2+i \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

You should check for complex numbers $x = a+bi$ and $y = c+di$ that $\overline{xy} = (\overline{x})\,(\overline{y})$ and $\overline{x+y} = \overline{x} + \overline{y}$. Because of this, complex conjugation of products or sums of matrices can be applied to factors or summands individually too.

The first column of $\mathbf{A} - \lambda_2 \mathbf{I}$ is nonzero (since $\mathbf{A}$ is real but λ_2 is complex) and so is an eigenvector $\mathbf{V}$ for λ_1.

Note that $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{V}$ is the zero vector so

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \overline{(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{V}} = (\overline{\mathbf{A}} - \overline{\lambda_1}\ \overline{\mathbf{I}})\overline{\mathbf{V}} = (\mathbf{A} - \lambda_2\ \mathbf{I})\overline{\mathbf{V}}.$$

This means $\overline{\mathbf{V}}$ is an eigenvector for eigenvalue λ_2.

In other words, complex eigenvalues come in conjugate pairs and the associated eigenvectors do too. In particular, these eigenvectors *cannot* be real vectors.

Let $\mathbf{V} = \mathbf{V}_r + i\mathbf{V}_i$ where $\mathbf{V}_r$ and $\mathbf{V}_i$ are both real: the real and imaginary parts of the eigenvector $\mathbf{V}$.

Note that $\mathbf{V}_r \neq k\mathbf{V}_i$ for any constant k. That is because if there was such a k, we would have

$$\mathbf{V} = k\mathbf{V}_i + i\mathbf{V}_i \quad \text{and} \quad \overline{\mathbf{V}} = k\mathbf{V}_i - i\mathbf{V}_i$$

and so

$$\mathbf{V}_i = \frac{1}{k+i}\mathbf{V} = \frac{1}{k-i}\overline{\mathbf{V}}$$

so $\mathbf{V}_i$ would be a nonzero multiple of both eigenvectors and therefor an eigenvector for two different eigenvalues, an impossibility.

So $\mathbf{V}_i$ and $\mathbf{V}_r$ are both nonzero and an independent pair of real vectors.

Finally, one last calculation involving vectors and conjugates.

Suppose $\mathbf{Z}$ is a **real** vector and $\mathbf{Z}$ is a complex linear combination of $\mathbf{V}$ and $\overline{\mathbf{V}}$:

$$\mathbf{Z} = (x+yi)\mathbf{V} + (a+bi)\overline{\mathbf{V}} = (x+yi)(\mathbf{V}_r + i\mathbf{V}_i) + (a+bi)(\mathbf{V}_r - i\mathbf{V}_i).$$

Multiplying this out and using the fact that $\mathbf{Z}$ is real and also that $\mathbf{V}_i$ and $\mathbf{V}_r$ are independent, it is easy to see that $x = a$ and $y = -b$ and so

$$\mathbf{Z} = 2a\mathbf{V}_r + 2b\mathbf{V}_i = (a-bi)\mathbf{V} + (a+bi)\overline{\mathbf{V}}.$$

6. General Solutions to the Systems

We are interested in solving the initial value problem

$$\mathbf{Y}' \;=\; \mathbf{A}\,\mathbf{Y} \quad \text{and} \quad \mathbf{Y}(\mathbf{0}) = \mathbf{Y_0}.$$

for 2×2 real constant matrix $\mathbf{A}$ and real initial vector $\mathbf{Y}_0$.

When we have real eigenvalues λ_1 and λ_2 with two independent eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$, the general solution is simply

$$\mathbf{Y}(t) \;=\; a\,\mathrm{e}^{\lambda_1 t}\,\mathbf{V}_1 \;+\; b\,\mathrm{e}^{\lambda_2 t}\,\mathbf{V}_2.$$

To solve the initial value problem with $\mathbf{Y}(0) = \mathbf{Y}_0$ you must choose a and b so that

$$a\,\mathbf{V}_1 \;+\; b\,\mathbf{V}_2 \;=\; \mathbf{Y}_0.$$

In case of only one real eigenvalue λ and only one eigenvector the solution is

$$\mathbf{Y}(t) \;= \mathrm{e}^{\lambda t}\,\mathbf{Y}_0 \;+\; t\,\mathrm{e}^{\lambda t}\,\big(\mathbf{A} - \lambda\mathbf{I}\big)\mathbf{Y}_0.$$

You can check this is a solution using the fact that for *any* $\mathbf{Y}_0$

$$\mathbf{A}\,\big(\mathbf{A} - \lambda\mathbf{I}\big)\mathbf{Y}_0 \;=\; \lambda\,\big(\mathbf{A} - \lambda\mathbf{I}\big)\mathbf{Y}_0.$$

Finally, we come to the complex case.

We suppose $\lambda_1 = r + si$ and $\lambda_2 = r - si$.

Let $\mathbf{V} = \mathbf{V}_r + i\mathbf{V}_i$ be the first column of $\mathbf{A} - \lambda_2\mathbf{I}$.

Find α and β (and a and b) so

$$\mathbf{Y}_0 = \alpha\mathbf{V}_r + \beta\mathbf{V}_i = 2a\mathbf{V}_r + 2b\mathbf{V}_i = (a - bi)\mathbf{V} \;+\; (a + bi)\overline{\mathbf{V}}.$$

Just as in any two-eigenvector case, the solution to the initial value problem is

$$\mathbf{Y}(t) \;=\; (a - bi)\,\mathrm{e}^{\lambda_1 t}\,\mathbf{V} \;+\; (a + bi)\,\mathrm{e}^{\lambda_2 t}\,\overline{\mathbf{V}}.$$

After some messy algebra using

$$\mathrm{e}^{\lambda_1 t} = \mathrm{e}^{(r+si)\,t} = \mathrm{e}^{rt}(\,\cos(st) + i\sin(st)\,)$$

we simplify this to

$$\mathbf{Y}(t) \;=\; \mathrm{e}^{rt}\Big(\;(\alpha\cos(st) + \beta\sin(st))\mathbf{V}_r \;-\; (\alpha\sin(st) - \beta\cos(st))\mathbf{V}_i\;\Big).$$

7. Another Look at the Complex Case

We refer to the complex eigenvalue solution found in the last section.

$\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \right)$ is on the unit circle, so it is of the form $(\cos(\theta), \sin(\theta))$ for an angle θ related to $\arctan\left(\frac{\beta}{\alpha}\right)$. Letting $B = \sqrt{\alpha^2+\beta^2}$ we have

$$\mathbf{Y}(t) \;=\; B\,e^{rt}\Big((\cos(\theta)\cos(st)+\sin(\theta)\sin(st))\mathbf{V}_r - (\cos(\theta)\sin(st) - \cos(\theta)\cos(st))\mathbf{V}_i \Big).$$

Using the angle difference formulas we have, finally

$$\mathbf{Y}(t) \;=\; B\,e^{rt}\Big(\cos(st-\theta)\mathbf{V}_r \;-\; \sin(st-\theta)\mathbf{V}_i \Big).$$

The term on the right is oscillatory with period $2\pi/s$. If $r=0$ the motion will be periodic. If $r>0$ it will spiral "out," away from the origin. If $r<0$ it will spiral "in."

There are two ways of rotating from initial ray $\mathbf{V}_i$ to terminal ray $\mathbf{V}_r$: clockwise or counterclockwise. One direction will represent a smaller rotation than the other. If s is positive, the oscillatory motion as seen from the origin will correspond to whichever of these directions is the smaller angle. If s is negative, this will be reversed.

8. Specific Solutions to the Systems

The case of an independent pair of real eigenvectors:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}, \qquad \mathbf{Y}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The characteristic polynomial is $P(x) = x^2 - \mathrm{tr}(\mathbf{A})x + \det(\mathbf{A}) = x^2 - x - 8$.

So the eigenvalues are $\lambda_i = \frac{1}{2}(1 \pm \sqrt{33})$.

One of these numbers is positive, the other negative. At this point we know that the origin is a saddle, solutions coming in along one eigenvector, leaving along another.

$$\begin{aligned}\mathbf{A} - \lambda_2\mathbf{I} &= \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(1-\sqrt{33}) & 0 \\ 0 & \frac{1}{2}(1-\sqrt{33}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} + \frac{1}{2}\sqrt{33} & 3 \\ 2 & \frac{-3}{2} + \frac{1}{2}\sqrt{33} \end{pmatrix}.\end{aligned}$$

Although it doesn't really look like it, the two columns of this matrix must be multiples of each other, since they are both eigenvectors for eigenvalue $\lambda_1 = \frac{1}{2}(1+\sqrt{33})$. Pick twice the first as $\mathbf{V}_1$.

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(1+\sqrt{33}) & 0 \\ 0 & \frac{1}{2}(1+\sqrt{33}) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3}{2} - \frac{1}{2}\sqrt{33} & 3 \\ 2 & \frac{-3}{2} - \frac{1}{2}\sqrt{33} \end{pmatrix}.$$

The two columns are both eigenvectors for eigenvalue $\lambda_2 = \frac{1}{2}(1-\sqrt{33})$. Pick twice the first as $\mathbf{V}_2$.

$$\mathbf{Y}_0 = \alpha\mathbf{V}_1 + \beta\mathbf{V}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \left(\frac{9}{24} - \frac{\sqrt{33}}{24}\right)\begin{pmatrix} 1+\sqrt{33} \\ 4 \end{pmatrix} + \left(\frac{9}{24} + \frac{\sqrt{33}}{24}\right)\begin{pmatrix} 1-\sqrt{33} \\ 4 \end{pmatrix}.$$

Our explicit solution is

$$\mathbf{Y}(t) \;=\; \alpha\,\mathrm{e}^{\lambda_1 t}\mathbf{V}_1 + \beta\,\mathrm{e}^{\lambda_2 t}\mathbf{V}_2.$$

In trying to understand the general behavior of this motion, it would not be helpful to replace the α, $\mathbf{V}_1$, β and $\mathbf{V}_2$ by their values, calculated explicitly above. However it *is* helpful to know that λ_1 is positive and λ_2 is negative so in remarkably short order the solution will be indistinguishable from $\alpha\,\mathrm{e}^{\lambda_1 t}\mathbf{V}_1$.

The case of only one real eigenvector:

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 0 & 5 \end{pmatrix}, \qquad \mathbf{Y}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The characteristic polynomial is

$$P(x) = x^2 - \mathrm{tr}(\mathbf{A})x + \det(\mathbf{A}) = x^2 - 10x + 25 = (x-5)^2.$$

So the eigenvalue is $\lambda = 5$.

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

Any nonzero column of this matrix is an eigenvector for eigenvalue 5. This is useful for drawing the vector field of this system and otherwise understanding the nature of the solutions in aggregate. Any solution touching the line along this vector through the origin *stays* on that line. Since the eigenvalue is positive, it will move away from the origin.

Our explicit solution is

$$\mathbf{Y}(t) \;=\; \mathrm{e}^{5t}\mathbf{Y}_0 + t\,\mathrm{e}^{5t}(\mathbf{A}-5\mathbf{I})\mathbf{Y}_0 \;=\; \mathrm{e}^{5t}\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right)$$
$$= \mathrm{e}^{5t}\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 9 \\ 0 \end{pmatrix}\right) = \mathrm{e}^{5t}\begin{pmatrix} -2+9t \\ 3 \end{pmatrix}.$$

The case of two complex eigenvectors:

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}, \qquad \mathbf{Y}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The characteristic polynomial is $P(x) = x^2 - \mathrm{tr}(\mathbf{A})x + \det(\mathbf{A}) = x^2 - 6x + 11$.

So the eigenvalues are $\lambda_i = 3 \pm i\sqrt{2}$.

At this point we know that the motion will spiral out, leaving the origin roughly as e^{3t}. The period at which it rotates around the origin as viewed from the origin will be $\sqrt{2}\,\pi$ in the units favored by t.

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 - i\sqrt{2} & 0 \\ 0 & 3 - i\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 + i\sqrt{2} & -3 \\ 2 & -2 + i\sqrt{2} \end{pmatrix}.$$

The first column is $\begin{pmatrix} 2 + i\sqrt{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + i \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$, an eigenvector for λ_1.

$$\mathbf{Y}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \tfrac{3}{2} \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \tfrac{-5}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}.$$

So

$\theta = \arctan\left(\frac{\beta}{\alpha}\right) = \arctan\left(\frac{-5\sqrt{2}}{3}\right) \approx -1.17$ radians.

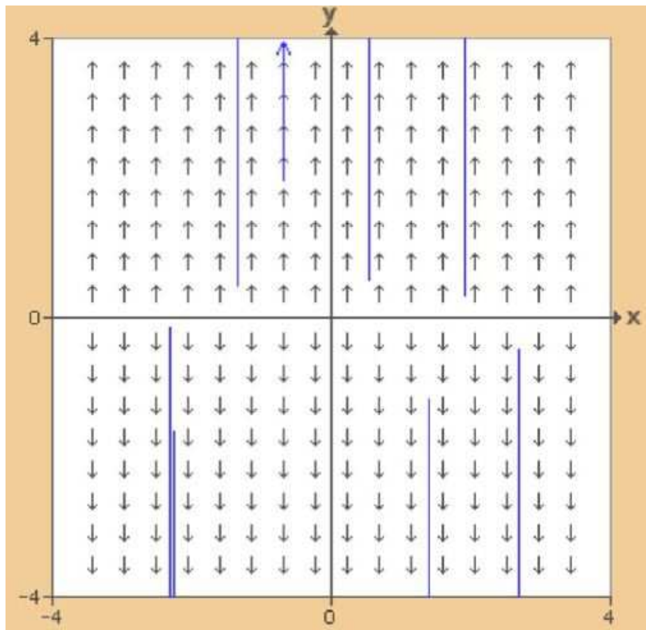
$B = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{9}{4} + \frac{25}{2}} = \frac{1}{2}\sqrt{59}$.

Finally, our explicit solution is

$$\mathbf{Y}(t) \;=\; \frac{1}{2}\sqrt{59}\, e^{3t} \left(\cos(\sqrt{2}\,t - \theta) \begin{pmatrix} 2 \\ 2 \end{pmatrix} \;-\; \sin(\sqrt{2}\,t - \theta) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \right).$$

The motion is clockwise as seen from the origin.

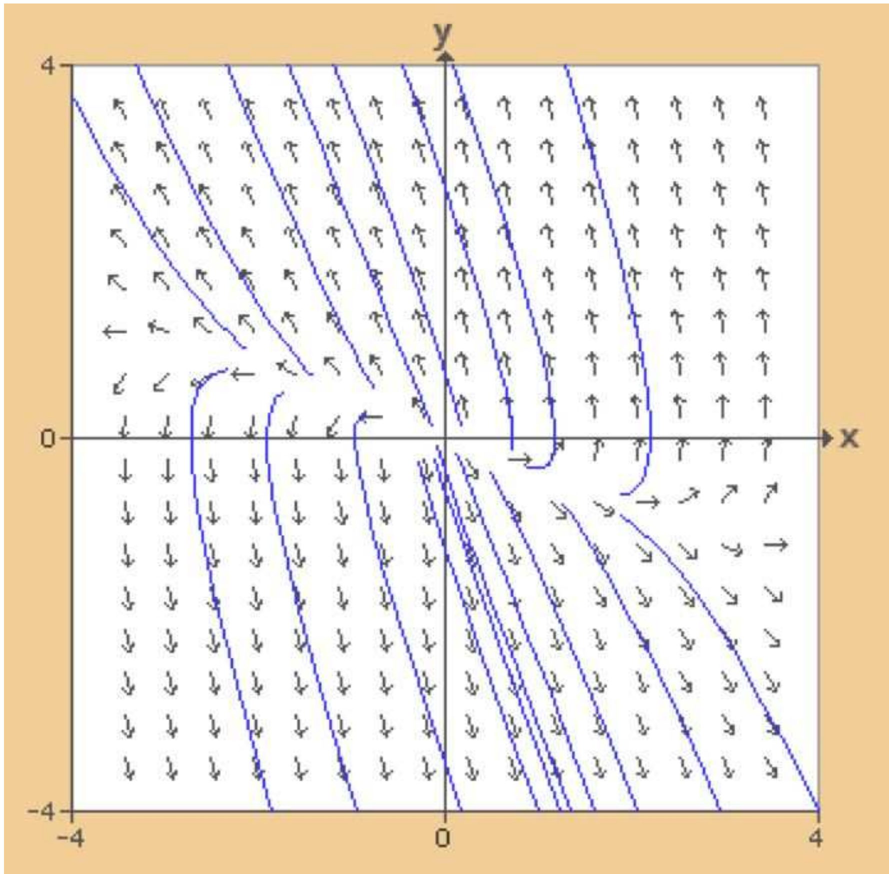
9. Examples with Graphs



$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ y(t) \end{pmatrix} \qquad P(\lambda) = \lambda^2 - \lambda.$$

Eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for eigenvalue 0 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for eigenvalue 1.

The general solution is: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\, \mathrm{e}^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

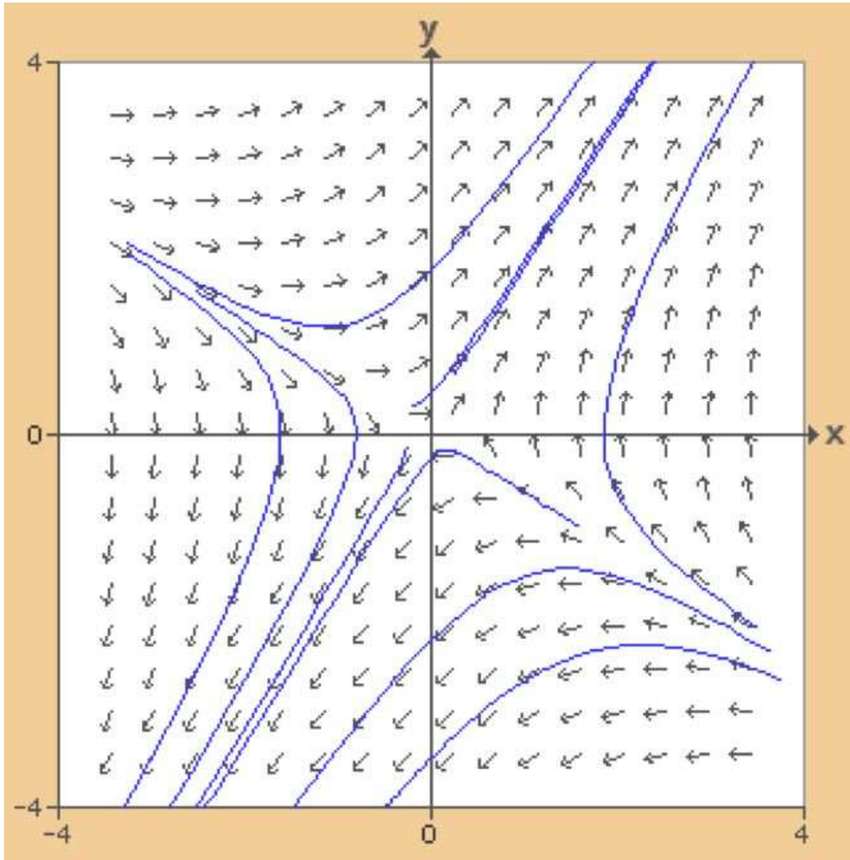


$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -y(t) \\ x(t) + 3y(t) \end{pmatrix} \qquad P(\lambda) = \lambda^2 - 3\lambda + 1.$$

There are two positive eigenvalues, $\dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$.

Eigenvectors: $\begin{pmatrix} 3-\sqrt{5} \\ -2 \end{pmatrix} \approx \begin{pmatrix} .38 \\ -2 \end{pmatrix}$ for $\dfrac{3}{2}+\dfrac{\sqrt{5}}{2}$ and $\begin{pmatrix} 3+\sqrt{5} \\ -2 \end{pmatrix} \approx \begin{pmatrix} 5.24 \\ -2 \end{pmatrix}$ for $\dfrac{3}{2}-\dfrac{\sqrt{5}}{2}$.

The general solution is: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a\, e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \begin{pmatrix} 3-\sqrt{5} \\ -2 \end{pmatrix} + b\, e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \begin{pmatrix} 3+\sqrt{5} \\ -2 \end{pmatrix}.$

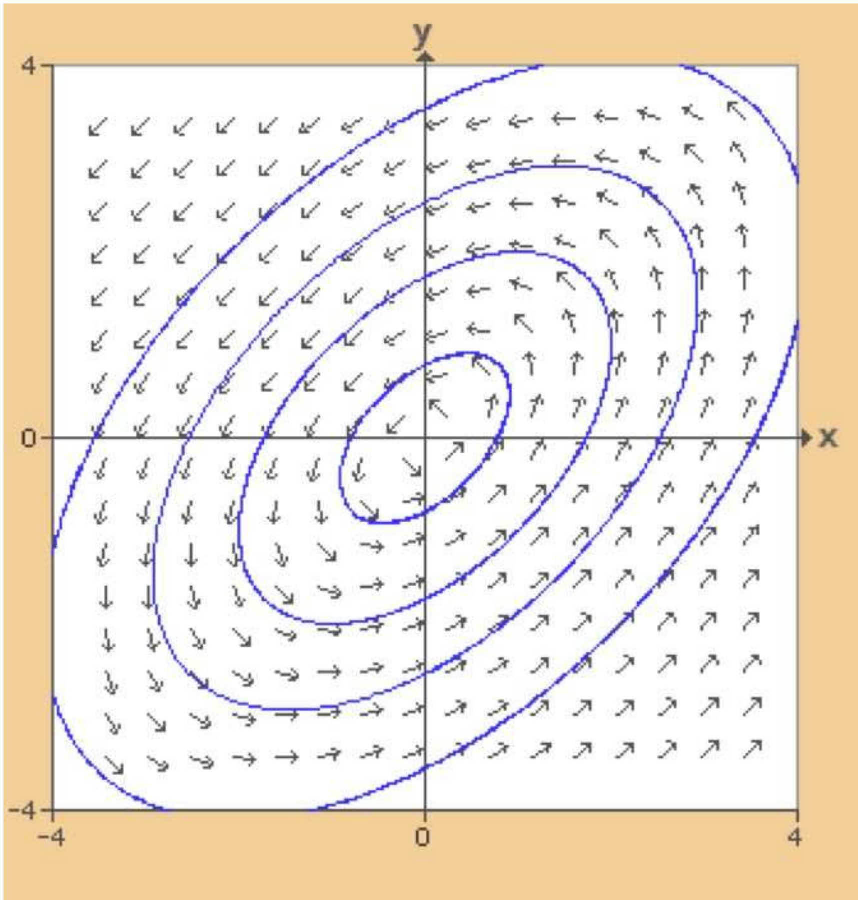


$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ x(t) + y(t) \end{pmatrix} \qquad P(\lambda) = \lambda^2 - \lambda - 1.$$

There are two eigenvalues, one positive and one negative, $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$.

Eigenvectors: $\begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} \approx \begin{pmatrix} 1.2 \\ 2 \end{pmatrix}$ for $\frac{1}{2}+\frac{\sqrt{5}}{2}$ and $\begin{pmatrix} \sqrt{5}+1 \\ -2 \end{pmatrix} \approx \begin{pmatrix} 3.2 \\ -2 \end{pmatrix}$ for $\frac{1}{2}-\frac{\sqrt{5}}{2}$.

The general solution is: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a\, \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} + b\, \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} \begin{pmatrix} \sqrt{5}+1 \\ -2 \end{pmatrix}.$



$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) - 2y(t) \\ 2x(t) - y(t) \end{pmatrix} \qquad P(\lambda) = \lambda^2 + 3.$$

There are two complex eigenvalues with zero real part, $\pm i\sqrt{3}$. So the motion will be periodic with period $2\pi/\sqrt{3}$.

$$\mathbf{V}_r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{V}_i = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}.$$

The general solution is:

$$\begin{aligned} \mathbf{Y}(t) &= B\left(\cos(\sqrt{3}\,t - \theta)\mathbf{V}_r \; - \; \sin(\sqrt{3}t - \theta)\mathbf{V}_i \right) \\ &= B\left(\cos(\sqrt{3}\,t - \theta)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \; - \; \sin(\sqrt{3}t - \theta)\begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \right) \end{aligned}$$

For initial condition $\mathbf{Y}_0$ choose B and θ so that

$$\mathbf{Y}_0 = B\left(\cos(\theta)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \; + \; \sin(\theta)\begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \right).$$