MORE STRUCTURE: METRICS, TOPOLOGICAL GROUPS AND UNIFORMITY

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Abstract. This is an appendix for a book I have (mostly) written on measure theory. It deals with a variety of topics with mixed structure, blending topology, algebra and other topics.

It is not really self-contained. There are references from Chapter One "Some Preliminaries" to the Axiom of Choice, the integers, the real numbers and general notational conventions. There are also references to the appendix on algebra and the appendix on topology.

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1. Metrics and Pseudometrics May 26, 2013

Topologies, which abstract certain features of the "closeness" idea among points in a set, are often derived from an explicit measure of closeness called a metric or pseudometric.

A **pseudometric** on a set X is a function $d\colon X \times X \to [0,\infty)$ with the following properties:

$d(x,x) = 0 \ \forall x \in X$ and

$d(x,y) = d(y,x) \ \forall x,y \in X$ and

$d(x,z) \leq d(x,y) + d(y,z) \ \forall x,y,z \in X$. (**The Triangle Inequality**)

The **open ball** of radius $\varepsilon \geq 0$ centered at $x \in X$ is a set $\{\, y \in X \mid d(x,y) < \varepsilon \,\}$. The notation $B(x,\varepsilon)$ is used for open balls.

The **sphere** of radius $\varepsilon \geq 0$ centered at $x \in X$ is a set $\{\, y \in X \mid d(x,y) = \varepsilon \,\}$. The notation $S(x,\varepsilon)$ can be used for this sphere. If there is more than one pseudometric around, the notation $B_d(x,\varepsilon)$ and $S_d(x,\varepsilon)$ can be used for specificity.

A pseudometric d on X is called **bounded** if $X = S(x,\varepsilon)$ for some x and sufficiently large ε. Because of the triangle inequality, the specific point x chosen is not relevant to this definition: if $X = S(x,\varepsilon)$ then $X = S(y, 2\,\varepsilon)$ for all $y \in X$.

A pseudometric d is called a **metric** if, in addition to the properties listed above we have:

$d(x,y) = 0$ implies $x = y$.

As progenitor, Euclidean distance in $\mathbb{R}^n$ constitutes an important metric.

1.1. ***Exercise.*** *Suppose $f\colon X \to Y$ is any function.*

(i) Any pseudometric d on Y induces a pseudometric D on X by $D(p,q) = d(f(p), f(q))$ for any $p,q \in X$. If d is a metric and f is one-to-one, D is a metric.

(ii) If f is one-to-one and onto, any pseudometric D on X induces a pseudometric d on Y by $d(s,t) = D(f^{-1}(s), f^{-1}(t))$ for any $s,t \in Y$. If D is a metric so is d.

The **topology generated by a pseudometric** d on X is that formed using the set of all open balls, denoted $\mathbb{B}_d$, as a subbase. We say that the pseudometric and the topology it generates are **compatible**.

Two pseudometrics are called **topologically equivalent** if they generate the same topology. This is an equivalence relation on the set of pseudometrics on X.

Any subset A of a pseudometric space X may be regarded as a pseudometric space by restricting the pseudometric on X to A. The subspace topology on A is the same as the topology induced by this restricted pseudometric.

1.2. ***Exercise.*** *(i) The open balls actually constitute a base, not just a subbase, for any topology generated by a pseudometric. So any topology which can be generated by a pseudometric is first countable.*

(ii) $\overline{B(x,\varepsilon)} \subset B(x,\varepsilon) \cup S(x,\varepsilon)$ but the reverse containment need not hold. For instance, there may **be** *no points y with $\frac{\varepsilon}{2} \leq d(x,y) < \varepsilon$. In that case, $\overline{B(x,\varepsilon)} = B(x,\varepsilon)$ but $S(x,\varepsilon)$ may well be nonempty.*

(iii) For any topology which can be generated by a pseudometric, the three conditions "second countable" and "separable" and "Lindelöf" are equivalent.

Generally, any topology on X is called a **pseudometric or metric topology** if it arises, or could have arisen, as the collection of open sets for a pseudometric or, respectively, a metric on X. A pseudometric or metric topology is described as **pseudometrizable** or, respectively, **metrizable**.

Topologically equivalent pseudometrics can have important differences, as we shall see. So the concept of pseudometric space is different from and a refinement of the concept of topological space.

1.3. ***Exercise.*** *(i) Suppose d_1 is a pseudometric. Define d_2 to be cd_1 for a fixed positive constant c. Then d_2 is a pseudometric, and topologically equivalent to d_1.*

(ii) Define d_3 by $d_3(x,y) = d_1(x,y)$ if $d_1(x,y) \leq 1$ and $d_3(x,y) = 1$ otherwise. Then d_3 is a pseudometric, and topologically equivalent to d_1. So every pseudometric is topologically equivalent to a pseudometric whose values are bounded above by 1 or any other specified positive number.

(iii) Suppose $f\colon [0,a) \to [0,b)$, where a or b could be ∞, is continuous and $f(0) = 0$. Suppose further that f is twice differentiable on $(0,a)$ and $f'(t) > 0$ and $f''(t) \leq 0$ for all $t \in (0,a)$. If the range of d_1 is contained in $[0,a)$ then $d_4 = f \circ d_1$ is a pseudometric and topologically equivalent to d_1. This gives a handy method of creating topologically equivalent metrics with particular properties. For example

$$\frac{d_1}{1+d_1} = 1 - \frac{1}{1+d_1}, \qquad Ln(1+d_1), \qquad Arctan(d_1), \quad \textit{and} \quad (d_1)^{\frac{1}{3}}$$

are all topologically equivalent to d_1. (hint: Use the mean value theorem to show that $f(t+s) - f(t) \leq f(s)$ for all positive t and s with $s+t$ in the domain of f.)

A **pseudometric space or metric space** is a pair (X, d) where d is a pseudometric or, respectively, a metric on X, along with the topology generated by d wherever that is required.

Often we will simply refer to "the metric or pseudometric space X" and a particular metric or pseudometric, assumed to be present, is not mentioned explicitly.

1.4. ***Exercise.*** *Suppose (X,d) is a pseudometric space. For each $x \in X$ let $[x]$ be the set of all points at distance 0 from x. These sets form a partition P of X. Each member of the partition is closed with respect to the topology induced by d.*

The topology on X induced by d is T_0 if and only if every member of the partition consists of one member of X, and in that case the topology is T_2. This happens exactly when d is a metric.

Define $\tilde{d}$ on P by $\tilde{d}([x],[y]) = d(x,y)$. Show that $\tilde{d}$ is well defined and a metric on P. Show that the topology induced by $\tilde{d}$ is the quotient topology. Show that each open set in X corresponds to a unique open set in P and conversely.

There are a couple of classes of functions that pop up naturally between pseudometric spaces. A function $f\colon X \to Y$ between two pseudometric spaces (X, d_1) and (Y, d_2) is called **uniformly continuous** if $\forall \varepsilon > 0\ \exists \delta > 0$ so that $a, b \in X$ and $d_1(a,b) < \delta$ implies $d_2(f(a), f(b)) < \varepsilon$.

1.5. ***Exercise.*** *(i) Uniformly continuous functions are continuous with the topologies induced by the pseudometrics.*

(ii) Continuous functions between pseudometric spaces need not be uniformly continuous. (hint: Look at $f(x) = \frac{1}{x}$ *on the open unit interval* $(0,1)$*. So* $f\colon (0,1) \to (1,\infty)$*. Give both domain and range intervals the usual metric defined by* $d(x,y) = |x-y|$*. This function is continuous but not uniformly continuous.)*

(iii) Suppose $\mathbb{O}$ *is any open cover of compact pseudometric space* (X,d)*. There is a unique largest positive number* λ *called the* ***Lebesgue number for*** $\mathbb{O}$ *for which* $B(x,\lambda)$ *is contained in a member of* $\mathbb{O}$ *for every* $x \in X$*. This result is called the* ***Lebesgue Covering Lemma****. (hint: To get started, find for each* $x \in X$ *a largest ball* $B(x,\mu_x)$ *contained in any of the members of* $\mathbb{O}$*.)*

(iii) A continuous function between pseudometric spaces with compact domain is uniformly continuous. In particular, if d_1, d_2 *are two topologically equivalent pseudometrics on compact* X *then the identity function* $i\colon (X,d_1) \to (X,d_2)$ *is uniformly continuous.*

(iv) Is it true that if if d_1, d_2 *are two topologically equivalent pseudometrics on compact* X *that there are positive numbers* α *and* β *with* $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$ *for all* $x,y \in X$*?*

(v) It is not true that a uniformly continuous function with an inverse function has a continuous inverse. (hint: Let $f\colon [0,2\pi) \to S$*, where* S *is the unit circle in the plane, and* $f(x) = (Cos(x), Sin(x))$*.)*

(vi) A function uniformly continuous with respect to one pair of metrics on domain and range ***need not be uniformly continuous if either metric is switched to a different but topologically equivalent metric.*** *(hint 1: Suppose* $f\colon (0,\pi/2) \to (0,\infty)$ *is defined by* $f(x) = tan(x)$*. Give* $(0,\pi/2)$ *the metric* $d(x,y) = |x-y|$ *and* $(0,\infty)$ *the metric* $D(x,y) = |x-y|$*. Let* $\widetilde{D}$ *be the metric* $Arctan \circ D$*.* $\widetilde{D}$ *is equivalent to* D*. Show that* f *is uniformly continuous with respect to the metrics* d *and* $\widetilde{D}$ *but* ***not*** *with respect to the metrics* d *and* D*. hint 2: Suppose* $f\colon (0,\infty) \to (0,\infty)$ *is defined by* $g(x) = x^2$*. Give* $(0,\infty)$ *the metric* $D(x,y) = |x-y|$ *in both domain and range.* g *is not uniformly continuous. Let* $\widetilde{D}$ *be the metric* $D(x,y) = |x^2 - y^2|$*.* $\widetilde{D}$ *is topologically equivalent to* D*. Show that* g *is uniformly continuous with respect to the metric* $\widetilde{D}$ *in its domain and* D *in its range.)*

Suppose $f\colon (X,d) \to (Y,e)$ is any function from one pseudometric space to another. f is called **Lipschitz** if there is a constant L, called a **Lipschitz constant for f**, so that $e(f(x), f(y) \leq Ld(x,y)$ for all $x,y \in X$. The function f is called **locally Lipschitz** if for each $x \in X$ there is a constant L_x and a neighborhood V_x of x so that $e(f(z), f(y) \leq L_x d(z,y)$ for all $y,z \in V_x$.

1.6. ***Exercise.*** *(i) The Lipschitz property implies the locally Lipschitz property which, itself, implies that* f *is continuous.*

(ii) The Lipschitz property implies uniform continuity, but the converse implication does not hold.

(iii) "Locally Lipschitz" implies "Lipschitz" when the domain is compact.

A function between two metric spaces (X, d_1) and (Y, d_2) which preserves the distance concept in the domain is called an **isometry**. Specifically, $f\colon X \to Y$ is an isometry if $d_1(a,b) = d_2(f(a), f(b))$ for every pair of points $a, b \in X$. An isometry is, obviously, uniformly continuous and one-to-one. If it is also onto, the inverse function is an isometry and X and Y with their induced topologies are homeomorphic.

The two metric spaces (X, d_1) and (Y, d_2) are called **isometric** provided there is an isometry $f\colon X \to Y$ onto Y.

1.7. ***Exercise.*** *The function* $\widetilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ *is a metric on* $X = (0,\infty)$ *and is topologically equivalent to the usual metric* $d(x,y) = |x-y|$ *on* X*. Show that* (X,d) *and* $(X,\widetilde{d})$ *are isometric.*

Some sources do not require the spaces involved in an isometry to be metric spaces—only pseudometric spaces. With this relaxed definition the quotient function of X onto P of Exercise 1.4 is an isometry onto P without, in general, an inverse. We avoid that choice.

The **diameter** of a nonempty subset A of X with respect to the pseudometric d is $\sup\{\, d(x,y) \mid x,y \in A \,\}$ if the set is bounded, and the diameter is said to be infinite otherwise. We use the notation **diam(A)** for this supremum. A set whose diameter is not infinite is said to be **bounded**. Note that if $0 < diam(A) < \infty$ and $x \in A$ then $A \subset B(x, 2\, diam(A))$.

A set A is said to be **totally bounded** if for each $\varepsilon > 0$ there is a finite list $x_1, \ldots, x_n$ of members of X so that $A \subset \cup_{i=1}^n B(x_i, \varepsilon)$. The concept of totally bounded depends on the specific pseudometric used. Topologically equivalent pseudometrics need not preserve this property.

1.8. ***Exercise.*** *(i) Compact subsets of a pseudometric space are totally bounded.*

(ii) Any totally bounded pseudometric space is separable.

The **distance between the point x and the nonempty set A** in the pseudometric space X is defined to be $\mathbf{d(x,A)} = \inf\{\, d(x,y) \mid y \in A \,\}$. The **distance between two nonempty sets A and B** in the pseudometric space X is defined to be $\mathbf{d(A,B)} = \inf\{\, d(x,A) \mid x \in B \,\}$. Finally, if A is a set and $\varepsilon > 0$ define $\mathbf{B_d(A,\varepsilon)} = \{\, x \in X \mid d(x,A) < \varepsilon \,\}$.

1.9. ***Exercise.*** *(i) If* A *and* B *are nonempty,* $d(A,B) = d(B,A)$*.*

(ii) For each nonempty A *the function* $f(x) = d(x,A)$ *is uniformly continuous.*

(iii) With f *as above,* $f^{-1}(0) = \overline{A} = \bigcap_{n=1}^{\infty} B_d(A, 1/n)$*.*

(iv) Any pseudometric space is $\mathcal{CR}$*.*

(v) Every closed set is a G_δ *in any pseudometric space.*

(vi) Every pseudometric space is paracompact. (hint: Suppose first that Y *is a metric space. For* $n \geq 1$ *let* $\mathbb{B}_n$ *denote the cover consisting of all balls of radius* $\frac{1}{n}$*. Suppose* $\mathbb{A}$ *is any open cover of* Y *and* $A \in \mathbb{A}$*. Pick* $p \in Y$ *and* $A \in \mathbb{A}$ *with* $p \in A$*. Since* A *is open there is a ball* $B\left(p, \frac{1}{n}\right) \subset A$*. Now* $Star_{\mathbb{B}_{3n}}\left(B\left(p, \frac{1}{3n}\right)\right) \subset B\left(p, \frac{1}{n}\right) \subset A$ *so the sequence* $\mathbb{B}_n$ *for integers* $n \geq 1$ *is locally starring for* $\mathbb{A}$*—in fact, this sequence is locally starring for any open cover whatsoever. So* Y *is* T_* *and since*

Y is regular this implies Y is paracompact. Now if X is a pseudometric space let Y be the quotient metric space described in Exercise 1.4 obtained by combining points in X at zero distance from each other into a single point in Y. X is paracompact and the result for Y follows.)

(vii) Any pseudometric space is perfectly T_4 and therefore completely T_4.

(viii) If A is compact and B is closed and $A \cap B = \varnothing$ then there is $\varepsilon > 0$ so that $B_d(A, \varepsilon) \cap B = \varnothing$.

(ix) In a pseudometric space, sequential compactness, countable compactness and compactness are equivalent conditions.

(x) Suppose A and B are nonempty subsets of the pseudometric space X. If A is compact there is a point $x \in A$ with $d(x, B) = d(A, B)$. If B is also compact there are points $x \in A$ and $y \in B$ with $d(x, y) = d(A, B)$.

We saw in Proposition **??** that in the presence of the T_3 property paracompactness is equivalent to the existence of a σ-locally finite open refinement $\mathbb{G} = \bigcup_{n\in\mathbb{N}} \mathbb{G}_n$ for each open cover, and pseudometric spaces are T_3 and paracompact. In fact, more is true. In a pseudometric space this countable decomposition of an open refinement can be chosen so that $\mathbb{G}$ is locally finite and each point has a neighborhood intersecting **no more than one** member of each $\mathbb{G}_n$. Engelking [**?**] refers to this result as one of the most important theorems of general topology.

1.10**.** ***Proposition. The Stone Theorem*** *In a pseudometric space each open cover has a locally finite and σ-discrete open refinement.*

Proof. Suppose given an open an cover $\mathbb{O}$ of pseudometric space (X, d). For each set $A \in \mathbb{O}$ form the countable chain of subsets A_n where for each n,

$$A_n = \left\{ x \in A \mid d(x, X - A) \geq \frac{1}{2^n} \right\}.$$

Well order $\mathbb{O}$. Define for each n and each $A \in \mathbb{O}$ the set

$$\widetilde{A_n} = A_n - \bigcup_{\substack{B\in\mathbb{O} \\ B<A}} B_{n+1}.$$

The triangle inequality (draw a sketch) implies that $d(A_n, X - A_{n+1}) \geq \frac{1}{2^{n+1}}$. So if $B < A$ then $\widetilde{A}_n \subset X - B_{n+1}$ which implies $d(\widetilde{B}_n, \widetilde{A}_n) \geq \frac{1}{2^{n+1}}$.

For each n and A define $\widehat{A}_n$ to be the set of all points in X less than $\frac{1}{2^{n+3}}$ away from $\widetilde{A}_n$. So each $\widehat{A}_n$ is open. From the last calculation we see that for distinct A and B, $d(\widehat{B}_n, \widehat{A}_n) \geq \frac{1}{2^{n+2}}$. (Draw a sketch!)

So if $\mathbb{G}_n = \{ \widehat{A}_n \mid A \in \mathbb{O} \}$, each point of X is in at most one member of $\mathbb{G}_n$.

It remains to show that $\mathbb{G} = \bigcup_{n\in\mathbb{N}} \mathbb{G}_n$ is a cover of X.

Select $x \in X$ and let A be the first member of the open cover $\mathbb{O}$ containing x. So x is in A_n for large enough n and because A is the first member of $\mathbb{O}$ containing x, we have $x \in \widetilde{A}_n$ too. So $x \in \widehat{A}_n \in \mathbb{G}_n \subset \mathbb{G}$.

Finally, we suppose $\mathbb{H}$ is a locally finite open refinement of $\mathbb{G}$. By the Refinement Lemma **??** and Exercise **??**, there is a subcover $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ and a locally finite open refinement $\widetilde{\mathbb{H}}$ of $\widetilde{\mathbb{G}}$ and a one-to-one and onto function $f\colon \widetilde{\mathbb{G}} \to \widetilde{\mathbb{H}}$ for which $f(S) \subset S$ for every $S \in \widetilde{\mathbb{G}}$.

Since $\widetilde{\mathbb{G}}$ is a subcover of a σ-discrete cover it too is σ-discrete. The properties of f ensure that $\widetilde{\mathbb{H}}$ is σ-discrete as well. □

1.11. ***Exercise.*** *Every pseudometric space has a σ-discrete base.*

1.12. ***Exercise.*** *If X is a metric space with metric d define S to be the set of all bounded nonempty closed subsets of X. Define H_d on $S \times S$ by*

$$H_d(A, B) = \sup \{\, d(x, B) \vee d(y, A) \mid x \in A,\ y \in B \,\}.$$

*(i) Show that H is a metric on S. It is called the **Hausdorff metric.***

(ii) $H_d(A, B) < \varepsilon$ exactly when $B \subset B_d(A, \varepsilon)$ and $A \subset B_d(B, \varepsilon)$.

(iii) The function $\Phi\colon X \to S$ defined by $\Phi(x) = \{\, x \,\}$ is an isometry.

*(iv) Kelley [**?**] points out that topologically equivalent metrics on X can produce **different** topologies on S. He gives the following example: Consider the nonnegative real numbers with metrics d given by $d(x, y) = \left| \frac{x}{1+x} - \frac{y}{1+y} \right|$ and e given by $e(x, y) = 1 \wedge |x - y|$. These metrics both generate the usual topology on $[0, \infty)$. Let $I_n = \{0, 1, \ldots, n-1\,\}$. Then the sequence I_n converges to $\mathbb{N}$ in the topology generated by H_d on S but not in the topology generated by H_e on S.*

1.13. ***Exercise.*** *Suppose T is a finite subset of the bounded interval $[a, b]$ containing a and b. Denote by $1 + n_T$ the number of members of T. Label the members of T as t_i for $i = 0, \ldots, n_T$ so that $a = t_0,\ t_{n_T} = b$ and $t_{i-1} < t_i$ for $i = 1, \ldots, n_T$. A selection of points from $[a, b]$ of this kind is called a **partition** of $[a, b]$.*

*Suppose $f\colon [a, b] \to X$ is a path in the metric space (X, d). f is called **rectifiable with respect to d** if the **length** of f, defined by*

$$\mathit{length}(f) = \sup \left\{ \sum_{i=1}^{n_T} d(t_{i-1}, t_i) \mid T \text{ is a partition of } [a, b] \right\},$$

is finite.

*We will suppose that X is path connected and in fact there is a **rectifiable** path in X connecting any two points in X. Define for each x and y in X the number*

$$D(x, y) = \inf \{\ \mathit{length}(f) \mid f \text{ is a rectifiable path connecting } x \text{ to } y\ \}.$$

*(i) Show that D is a metric, called the **geodesic distance** metric.*

(ii) Give the unit circle the metric it inherits as a subset of the plane with the usual Euclidean distance. What is the geodesic distance metric on the circle?

(iii) Find an example in the plane to show that D need not be topologically equivalent to d.

A set $\mathcal{F}$ of pseudometrics on X is called **separating** if for each pair of distinct points in X there is a member $d \in \mathcal{F}$ with $d(x, y) \neq 0$.

Any set of pseudometrics generates a topology formed by using as subbase $\cup_{d\in\mathcal{F}} \mathbb{B}_d$ where each $\mathbb{B}_d$ is the set of all open balls defined using the pseudometric $d \in \mathcal{F}$. The set of all finite intersections of open balls with common center and radii but differing pseudometrics is a base for this topology, and a topological space which can be produced using a *separating* family of pseudometrics is called a **gauge space.**

1.14**. *Exercise.*** *Consider the indexed collection of pseudometric spaces* (X_a, d_a) *for* $a \in A$*. The topology generated by a given* d_a *is indiscrete exactly when* d_a *is identically* 0*. Modify the pseudometrics if necessary to create topologically equivalent pseudometrics for which the diameter of each* X_a *does not exceed* 1*.*

(i) If an uncountable number of the X_a *are* ***not*** *indiscrete the product topology on* $\prod_{a\in A} X_a$ *is* ***not*** *first countable. This implies the product space is* ***not*** *pseudometrizable.*

(ii) Suppose ***only*** *a countable number of the* X_a *are* ***not*** *indiscrete. Let* S *denote the set of all indices for which* X_s *are* ***not*** *indiscrete when* $s \in S$*. For each* $s \in S$ *define the pseudometric* D_s *on* $\prod_{a\in A} X_a$ *by* $D_s(x, y) = d_s(x(s), y(s))$*. Let* n *denote a one-to-one function from* S *onto an initial segment of* $\mathbb{N}$ *if* S *is finite, and onto* $\mathbb{N}$ *itself if* S *is infinite. Finally, define* D *on* $\prod_{a\in A} X_a$ *by*

$$D(x, y) = \sum_{s\in S} \frac{D_s(x, y)}{2^{n(s)}}.$$

Show that D *is a pseudometric on* $\prod_{a\in A} X_a$ *and* D *generates the product topology there.*

(iii) A product of pseudometrizable spaces is pseudometrizable if and only if all but a countable number of factor spaces are indiscrete. It follows easily that the product is metrizable if and only if all but a countable number of factor spaces consist of a single point and every pseudometric is a metric.

1.15**. *Exercise.*** *A topology generated by a set of pseudometrics is* T_2 *if and only if the set of pseudometrics is separating.*

(ii) Suppose $\mathcal{F}$ *is any set of pseudometrics. Form a new family* $\mathcal{G}$ *by selecting one pseudometric (not necessarily one in* $\mathcal{F}$*) from each distinct equivalence class of pseudometrics related to any member of* $\mathcal{F}$*. Then* $\mathcal{G}$ *generates the same topology as does* $\mathcal{F}$*.*

(iii) Suppose that $d_0, d_1, \ldots$ *is a sequence of pseudometrics and the diameter of* X *does not exceed* 1 *for any of them. Then*

$$d(x, y) = \sum_{i=0}^{\infty} \frac{d_i(x, y)}{2^i}$$

is a single pseudometric which generates the same topology as does the set formed from the entire sequence of pseudometrics.

1.16**. *Exercise.*** *(i) Consider the product space* $[0, 1]^{\mathbb{N}}$ *where each factor space has the usual topology. The topology on this product space is a compact metric topology. Suppose that* a_n *is a positive number for each* $n \in \mathbb{N}$ *and* $\sum_{n\in\mathbb{N}} a_n$ *converges. Show that*

$$d(x, y) = \sum_{n\in\mathbb{N}} a_n \; |x_n - y_n|$$

is a compatible metric for the product space. Show that the product space is separable.

(ii) Suppose Y is any nonempty set. We define $\boldsymbol{l}^2(Y)$ to be those members f of $\mathbb{R}^Y$ which are 0 except at countably many members of Y and for which the (countable) sum $\sum_{y\in Y} f(y)^2$ converges. Define for $f,g \in \boldsymbol{l}^2(Y)$

$$d(f,g) = \sqrt{\sum_{y\in Y}(f(y)-g(y))^2}.$$

One shows that d actually converges to a number, and d satisfies the triangle inequality on $\boldsymbol{l}^2(Y)$ by examining finite sums of real numbers, showing that

$$\sqrt{\sum_{i=0}^{k}(f_i-g_i)^2} \;\leq\; \sqrt{\sum_{i=0}^{k} f_i^2} \;+\; \sqrt{\sum_{i=0}^{k} g_i^2}.$$

Once that is done, we conclude that $\boldsymbol{l}^2(Y)$ is a metric space with d as metric.

*(iii) The **Hilbert cube** is the subset of $\boldsymbol{l}^2(\mathbb{N})$ consisting of those members f for which $|f_n| \leq \frac{1}{n+1}$ for all n. Show that the Hilbert cube with subspace topology from $\boldsymbol{l}^2(\mathbb{N})$ is homeomorphic to $[0,1]^{\mathbb{N}}$ with product topology. The Hilbert cube is separable, and every subset of the Hilbert cube with subspace topology is separable too.*

2. Completeness

If d is a pseudometric on X a sequence x in X is called **Cauchy** (or d-Cauchy when there is more than one pseudometric around) if for each $\varepsilon > 0$ there is an integer N so that $d(x_n, x_m) < \varepsilon$ whenever n and m both exceed N.

When d is a pseudometric and if every Cauchy sequence in X converges to a point of X in the topology generated by d we say that X is **complete with respect to d**.

The concepts of Cauchy sequence and completeness are not topological concepts: **they depend on the specific pseudometric used in their definitions.**

A topological space X is called **topologically complete** if it is complete with respect to **some** compatible pseudometric.

2.1**. *Exercise.*** *Suppose X is pseudometrizable. Any convergent sequence is Cauchy with respect to any compatible pseudometric.*

2.2**. *Exercise.*** *Suppose X is compact and pseudometrizable and d is any compatible pseudometric. Then any d-Cauchy sequence in X converges. (hint: In a pseudometric space compactness is equivalent to both countable compactness and sequential compactness.)*

2.3**. *Proposition.*** *A pseudometrizable space X is compact if and only if it has a pseudometric with respect to which X is complete and totally bounded. In case it has such a pseudometric, X is complete and totally bounded with respect to **any** compatible pseudometric.*

Proof. In light of Exercises 1.8 and 2.2 we need only show that completeness and totally boundedness with respect to a pseudometric d implies sequential compactness.

Suppose y^0 is a sequence in X. Let $X = \cup_{j=1}^{n_1} B(x_j^1, 1)$. The sequence y^0 is in at least one of the balls in this union, which we will denote B^1, infinitely often. Let y^1 be the subsequence of y^0 formed by selecting the values of y^0 which lie in B^1.

Suppose we have sequences $y^0, \ldots, y^i$ with y^{k+1} a subsequence of y^k for $k = 0, \ldots, i-1$. Suppose further that $y^k \subset B^k$ where each B^k is a ball of radius 2^{-k+1} for $k = 1, \ldots, i$.

Find points x_j^{i+1} for $j = 1, \ldots, n_{i+1}$ for which $X = \cup_{j=1}^{n_{i+1}} B(x_j^{i+1}, 2^{-i})$. The sequence y^i is in at least one of these balls, which we denote B^{i+1}, infinitely often. Let y^{i+1} be the subsequence of y^i formed from the values of y^i which lie in B^{i+1}.

We create in this way a sequence of subsequences of y^0. The sequence x defined by $x_n = y_0^n$ is obviously Cauchy and so converges, and is a subsequence of y^0. □

2.4**. *Exercise.*** *Suppose X is a compact metric space and $\phi\colon X \to X$ is an isometry. Then X is onto. (hint: If not there is a ball $B(p,\varepsilon)$ for some $\varepsilon > 0$ in open $X - \phi(X)$. So $d(p, \phi(p)) \geq \varepsilon$. If we denote by ϕ^n the composition of ϕ with itself n times, define the sequence $p_n = \phi^n(p)$. Since X is compact there is a Cauchy subsequence p_{n_i}. For this subsequence, if $i < j$ and i is large enough $d(p_{n_i}, p_{n_i}) < \varepsilon$. But since ϕ is an isometry this means $d(p, p_{n_j - n_i}) < \varepsilon$.)*

Suppose X is a pseudometric space. A function $\phi\colon X \to X$ is called a **contraction on X** if there is a number $\varepsilon \in (0,1)$ with $d(x,y) \leq \varepsilon\, d(\phi(x), \phi(y))$ for all pairs $x, y \in X$. Contractions are, of course, continuous.

A point $p \in X$ is called a **fixed point** for any function $f\colon X \to X$ provided $f(p) = p$.

2.5**. *Exercise. Banach Fixed Point Theorem*** *If X is a complete metric space and ϕ is a contraction on X then ϕ has exactly one fixed point. (hint: First show that a contraction on a metric space can have no more than one fixed point. Then select a point $p \in X$ and form the sequence $p_n = \phi^n(p)$. Show by repeated application of the triangle inequality that this sequence is Cauchy and so must converge to some x in complete X. Continuity of ϕ implies that the sequence $\phi(p_n)$ converges to $\phi(x)$. Conclude that $x = \phi(x)$.*

To aid in the statement of next theorem, we make the the following definitions. Suppose we have a sequence $S_n,\ n \in \mathbb{N}$ of closed nonempty subsets of a pseudometric space. The sequence will be called a **Cantor sequence** if the diameter of these sets converges to 0 and $S_n \supset S_{n+1}$ for each n. The intersection of this Cantor sequence is $\bigcap_{n=0}^{\infty} S_n$.

2.6**. *Proposition. The Cantor Theorem*** *A pseudometric space X is complete exactly when every Cantor sequence in X has nonempty intersection.*

Proof. First, if X is complete and S_n for $n \in \mathbb{N}$ forms a Cantor sequence then selecting a point from each S_n produces a Cauchy sequence which converges to a point in every one of the closed sets S_n.

Now suppose every Cantor sequence in X has nonempty intersection and p is a Cauchy sequence in X. Let $S_n = \overline{\{\, p_k \mid k \geq n \,\}}$. Then the S_n form a Cantor sequence, and any point in the intersection of the sequence is a limit of p. □

2.7. ***Exercise.*** *(i) A closed subset of a complete pseudometric space is complete.*

(ii) Think of a complete subset of a complete pseudometric space that is not closed. What if the pseudometric is a metric?

(iii) Any closed subspace of $\mathbb{R}$ with the usual absolute value metric is complete.

(iv) Suppose (X, d) is complete and A, B are bounded nonempty subsets of X. It is not true that $d(A, B) = 0$ implies $\overline{A} \cap \overline{B} \neq \varnothing$. (hint: Let D be the metric in $\mathbb{R}^2$ given by $D(p, q) = 1 \wedge d(p, q)$ where d is the ordinary Euclidean distance. Let A and B denote the two components of the graph of $y = 1/x^2$.)

2.8. ***Proposition.*** *Suppose Y is a complete pseudometric space, and $A_i,\ i \in \mathbb{N}$ is a countable set of open dense subsets. Then $\bigcap_{i=0}^{\infty} A_i$ is dense.*

Proof. We will have the result if we can conclude that $W \cap (\bigcap_{i=0}^{\infty} A_i)$ is nonempty for generic nonempty open W.

We know that $W \cap A_0$ is open and nonempty so there is a ball B_1 of radius not exceeding 1 for which $\overline{B_1} \subset W \cap A_0$. B_1 is open so there is a ball B_2 of radius not exceeding $\frac{1}{2}$ with $\overline{B_2} \subset B_1 \cap A_1$. We iterate this process finding at each step a closed ball $\overline{B_i} \subset B_{i-1} \cap A_{i-1}$ of radius not exceeding $\frac{1}{i}$. We have created a Cantor sequence of closed balls $\overline{B_i}$ and since X is complete this sequence has nonempty intersection. A point in every $\overline{B_i}$ must be in every A_i and W too. The result follows. □

2.9. ***Exercise. Baire Category Theorem (Part Two)*** *A complete pseudometric space Y is of second category. More generally, subsets of Y of first category have empty interior.*

Suppose (Y, d) is any metric space and X is any topological space. Let $\mathbf{C(X, Y)}$ denote the set of continuous functions from X to Y. Suppose f is a generic function from X to Y. By analogy with the discussion before Lemma **??** we say that a net $n\colon D \to C(X, Y)$ **converges uniformly to** $\mathbf{f} \in \mathbf{Y^X}$ if

$$\forall \varepsilon > 0\ \exists \text{ terminal segment } T_\varepsilon \text{ so that } \sup\{\, d(f(x), n_t(x)) \mid t \in T_\varepsilon \,\} < \varepsilon\ \forall x \in X.$$

The proof that f **must** be continuous if it is the limit of such a net in $C(X, Y)$ is identical to the earlier argument from Lemma **??**, where the special case of $Y = \mathbb{R}$ is considered.

A subset S of C is called **uniformly closed** if S contains the limits of all uniformly convergent nets in S.

Suppose now (Y, d) is any **bounded** metric space. For $f, g \in C(X, Y)$ define

$$d_{sup}(f, g) = \sup\{\, d(f(x), g(x)) \mid x \in X \,\}.$$

By choosing an x so that $d(f(x), g(x))$ is very close to $d_{sup}(f, g)$ and applying the triangle inequality we see that $d_{sup}(f, g) \leq d_{sup}(f, h) + d_{sup}(h, g)$ and the other properties of a metric are obviously satisfied by d_{sup} so d_{sup} is a metric on $C(X, Y)$.

It is straightforward to see that a net $n\colon D \to C(X, Y)$ converges to f uniformly if and only if it converges to f in the metric topology from d_{sup}. We conclude that questions of closure and compactness in $C(X, Y)$ can be determined by examining sequences and subsequences in $C(X, Y)$: generic nets are not required.

We now impose the additional condition that (Y, d) be complete. In that case, if f_n for $n \in \mathbb{N}$ is a Cauchy sequence in $C(X, Y)$ and $x \in X$ then $f_n(x)$ converges to some member $g(x)$ of Y for each x. The function g is the uniform limit of the sequence and is therefore continuous itself. So $\mathbf{C}$ is complete.

We now drop the requirement that (Y, d) be a bounded metric space and instead consider the set $\mathbf{C_b(X, Y)}$ of bounded continuous functions from X to Y. Just as before, the limits of uniformly convergent nets in $C_b(X, Y)$ are in $C_b(X, Y)$. The function d_{sup} defined just as above is a metric on $C_b(X, Y)$ and a net in $C_b(X, Y)$ converges to g in the topology generated by this metric exactly when the net is uniformly convergent to g. Cauchy sequences in $C_b(X, Y)$ converge in $C_b(X, Y)$ with the d_{sup} metric when (Y, d) is complete.

In case X is compact (or countably compact), every function in $C(X, Y)$ is bounded so $C_b(X, Y)$ coincides with $C(X, Y)$.

2.10**. *Exercise*.** *Verify the statements made in the last few paragraphs.*

Suppose A is a subset of the metric space X, Y is a metric space and $g\colon A \to Y$. The **oscillation of g at** $\mathbf{x} \in \overline{\mathbf{A}}$ is

$$\mathbf{osc}(\mathbf{g})(\mathbf{x}) = \lim_{n\to\infty} \sup \left\{ e(g(z), g(y)) \mid y, z \in A \cap B_d\left(x, \frac{1}{n}\right) \right\}$$

and where we assign the "value" ∞ to $osc(g)(x)$ if the supremum involves an unbounded set for all n. When $osc(g)$ is constantly 0, g is said to have **zero oscillation**.

It is obvious that if g is the restriction to A of a function that is continuous on $\overline{A}$ then $osc(g)$ is constantly 0. It is also clear that if g is continuous with subspace topology on A then $osc(g)(x) = 0$ for every $x \in A$. There is a much more interesting partial converse.

2.11**. *Lemma*.** *Suppose (X, d) and (Y, e) are* ***complete*** *metric spaces and A has subspace metric from X and is dense in X. A function $f\colon A \to Y$ can be extended to a continuous function defined on X precisely when f has zero oscillation.*

Proof. In view of the remarks above we need only show that the zero oscillation condition allows us to extend f to a continuous function defined on all of X. Assume the condition.

There is a Cauchy sequence S^x in A converging to each $x \in X - A$. Since $osc(f)(x) = 0$ and for any n the sequence S^x is eventually in $B_d\left(x, \frac{1}{n}\right)$ we conclude that $f \circ S^x$ is Cauchy in complete Y and therefore converges to some member of Y which we will denote $F(x)$. If T^x is another Cauchy sequence in A converging to x it is easy to see that $\lim_{n\to\infty} f \circ T^x_n = \lim_{n\to\infty} f \circ S^x_n$ so F is well defined on $X - A$. We now define $F(x) = f(x)$ for $x \in A$.

If we can show that F is continuous, it will be the extension we want. Suppose $p \in X$ and Q is any sequence in X converging to p. For each $n > 0$ there is a member W_n of A with $d(W_n, Q_n) < \frac{1}{n}$ and also with $e(F(W_n), F(Q_n)) < \frac{1}{n}$. W converges to p along with Q. This implies $F \circ W = f \circ W$ converges to $F(x)$. The defining condition $e(F(W_n), F(Q_n)) < \frac{1}{n}$ forces $F \circ Q$ to converge too, and to this same limit. □

2.12***. Proposition.*** *Each metric space is isometric to a dense subset of a complete metric space.*

Proof. Let $\mathbf{B}$ denote the bounded continuous functions from the metric space (X,d) to $(\mathbb{R},e)$ where e is the usual absolute value metric. We concluded earlier that $(\mathbf{B},e_{sup})$ is a complete metric space, and convergence with respect to e_{sup} is equivalent to uniform convergence.

Select any particular point $p\in X$ and for $x\in X$ define $\Psi(x)\colon X\to\mathbb{R}$ by $\Psi(x)(y)=d(y,x)-d(y,p)$. The triangle inequality implies

$$-d(x,p)\leq d(y,x)-d(y,p)\leq d(x,p)$$

so $\Psi(x)$ is bounded. $\Psi(x)$ is obviously continuous on X so $\Psi\colon X\to\mathbf{B}$. Also, Ψ is one-to-one: if $d(y,x)-d(y,p)=d(y,z)-d(y,p)$ for all $y\in X$ then letting $y=z$ yields $d(z,x)=0$ so $x=z$.

Now suppose $x,z\in X$. So $\Psi(x)(y)-\Psi(z)(y)=d(y,x)-d(y,z)$. Once again, the triangle inequality yields

$$-d(x,z)\leq d(y,x)-d(y,z)\leq d(x,z)$$

which puts an upper bound of $d(x,z)$ on the magnitude of $\Psi(x)(y)-\Psi(z)(y)$. That upper bound is actually realized by setting $y=z$ so $\Psi\colon(X,d)\to(\mathbf{B},e_{sup})$ is an isometry.

$\Psi(X)$ is a dense subset of $\overline{\Psi(X)}$, a closed subset of the complete metric space $(\mathbf{B},e_{sup})$. $\overline{\Psi(X)}$ is, itself, complete with subspace metric. □

If (X,d) is isometric to a dense subspace of complete metric space (Y,D) we call (Y,D) a **completion of** $(\mathbf{X},\mathbf{d})$. We have just discovered that completions always exist. We now turn to the question of uniqueness of the completion.

2.13***. Proposition.*** *Suppose the metric space (X,d) is isometric to the dense subset A, with subspace metric, of the complete metric space (Y,D) and also isometric to the dense subset B, with subspace metric, of the complete metric space (Z,E). Then (Y,D) and (Z,E) are isometric.*

Proof. The isometries specified for (X,d) can be used to create an isometry $f\colon A\to Z$ with $f(A)=B$. It is easily seen (once again, the triangle inequality) that f has zero oscillation. By Lemma 2.11 the function f can be extended to a continuous function $F\colon Y\to X$.

Because continuous F is an isometry when restricted to A, and A is dense in Y, we find that F is actually an isometry on its whole domain Y.

By an identical argument, we can produce an isometry $G\colon Z\to Y$ with $G\circ f(p)=p$ for every $p\in A$. Since $G\circ F$ is the identity on the dense set A it must be the identity on all of Y, so G is onto Y and we find that Y and Z are isometric. □

2.14***. Exercise.*** *Here is an alternate construction of the completion. If (X,d) is a metric space let $\mathbf{S}$ denote the set of all Cauchy sequences in X. For $x,y\in\mathbf{S}$, write $f\equiv y$ when $\lim_{n\to\infty}d(x_n,y_n)=0$. This is an equivalence relation on $\mathbf{S}$. We will denote by $[x]$ the equivalence class containing the Cauchy sequence x. Let $\mathbb{K}$ denote the set of all these equivalence classes. Define $\Phi\colon X\to\mathbb{K}$ by $\Phi(p)=[x^p]$ where x^p is the constant (Cauchy) sequence $x^p_n=p$ for all n. Define the function D on $\mathbb{K}\times\mathbb{K}$ by $D([x],[y])=\lim_{n\to\infty}d(x_n,y_n)$. Show that D is well defined and a metric* *on $\mathbb{K}$. Show that $\mathbb{K}$ is complete with this metric. Show that Φ is an isometry. Show that $\Phi(X)$ is dense in $\mathbb{K}$.*

2.15. ***Exercise.*** *Suppose (X, d) is a pseudometric space. Motivated by the same desire, as in the metric case, to provide limits for all Cauchy sequences in X by embedding X in a "completion" and to extend nice enough functions defined on X to this "completion," what added problems do you encounter and how should you proceed?*

2.16. ***Exercise.*** *Suppose $\mathcal{F}$ is a separating family of pseudometrics on a set X, with the associated T_2 topology. Create notions of Cauchy sequence, completeness and completion in this situation, by analogy with the construction of Exercise 2.14. Is there a Baire Category Theorem here?*

Be aware that different but equivalent (in the sense that they generate the same topology) families of pseudometrics can create different notions of "Cauchy." Completeness depends on the specific family of pseudometrics in hand.

Two pseudometrics d_1 and d_2 on set X are called **pseudometrically equivalent** if there are positive constants C_1 and C_2 for which $C_1 d(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y)$ for all $x, y \in X$.

2.17. ***Exercise.*** *(i) It is obvious that pseudometrically equivalent pseudometrics are topologically equivalent, and that pseudometric equivalence is an equivalence relation on the set of pseudometrics on X.*

(ii) It is also true that the same sets are bounded with respect to either of a pair of pseudometrically equivalent pseudometrics, and the same sequences are Cauchy, and X is complete with respect to one of these pseudometrics exactly when it is complete with respect to the other.

(iii) A function $f : X \to Y$ between two pseudometric spaces (X, d_1) and (Y, e_1) retains the properties "uniformly continuous" or "Lipshitz" if either pseudometric is replaced by a pseudometrically equivalent pseudometric.

3. Metrizability

The manifold consequences of pseudometrizability and the techniques available in pseudometric spaces raise the question of when, exactly, is a topology compatible with some pseudometric. The overview essay on metrization theory in Steen and Seebach [**?**] gives some sense of the concerted effort from the years 1920-1960 devoted to understanding the correct setting and appropriate generality of these questions. Though a reasonably satisfactory theory exists, questions remain. Approaches other than those described below exist, as well as a number of popular variations obtained by rephrasing the covering conditions listed here.

3.1. ***Theorem.*** *Suppose X is a topological space.*
The following conditions are equivalent:

(i) X is metrizable.

(ii) ***The Nagata-Smirnov Metrization Theorem***
X is regular and has a σ-locally finite base.

(iii) ***The Arhangel'skiĭ Metrization Theorem***
X is T_1 and there is a single sequence of open covers that is locally starring for **any** *open cover.*

Proof. If X is metrizable with compatible metric d, the set

$$\mathbb{M}_n = \left\{ B_d\left(x, \frac{1}{n}\right) \mid x \in X \right\}$$

is an open cover of X for each integer $n > 0$.

$$Star_{\mathbb{M}_{3n}}\left(B_d\left(x, \frac{1}{3n}\right)\right) \subset B_d\left(x, \frac{1}{n}\right) \text{ for each } x \text{ and } n > 0$$

and it follows easily that the sequence $\mathbb{M}_n$ is locally starring for any open cover. We have shown that (i) implies (iii).

Assuming condition (iii), we know X is both regular and paracompact. If $\mathbb{J}_n$, $n \in \mathbb{N}$, is a fixed sequence of open covers locally starring for **any** open cover, use paracompactness to find for each n a locally finite open refinement $\mathbb{K}_n$ of $\mathbb{J}_n$.

Now suppose x is a point and $A \neq X$ is any open set containing x. Select $y \in X - A$. Since X is regular, there are nonintersecting open sets C and D with $x \in C$ and $y \in D$. So $\{ A, D \}$ is an open cover of X. By assumption, there is an open set V with $x \in V$ and integer n with $Star_{\mathbb{J}_n}(V) \subset A$. So $Star_{\mathbb{K}_n}(V) \subset A$. This implies that the union of all the $\mathbb{K}_n$ forms a base for the topology, and we have condition (ii).

We have arrived at the essential point: we need to show that if condition (ii) holds for topological space X then X is metrizable.

We presume X to be regular and that $\mathbb{A}_n$, $n \in \mathbb{N}$, is a decomposition of a base $\mathbb{A}$ for the topology as a countable collection of neighborhood finite collections of base members. It is easy to see that this implies paracompactness, and we have already shown that T_2 plus paracompactness implies normality.

In fact, X is perfectly T_4. To see this, let B be any nonempty open set. For each n let S_n be the union of all the $\overline{C}$ with $C \in \mathbb{A}_n$ and with $\overline{C} \subset B$. Since $\mathbb{A}_n$ is locally finite, each S_n is closed. Also, since X is regular and the union of the $\mathbb{A}_n$ constitute a base, each point in open B is in at least one of the S_n. We now see that $B = \cup_{n \in \mathbb{N}} S_n$ is an F_σ set, and conclude that X is perfectly T_4.

Without loss, we presume that $\mathbb{A}_n \cap \mathbb{A}_m = \varnothing$ when and only when $n \neq m$. For $A \in \mathbb{A}_n$ create Urysohn function $f_A \colon X \to [0,1]$ with $f_A^{-1}(0) = X - A$. This can be done by normality and because each closed set is a G_δ.

For each $A \in \mathbb{A}_n$ define for each $y \in X$ the number

$$g_A(y) \ = \frac{1}{n+1}\left(\frac{f_A(y)}{1 \ + \ \sum_{B \in \mathbb{A}_n} f_B(y)}\right).$$

The sum in the denominator is finite and all functions involved are continuous and nonnegative, so g_A is continuous. We now define $G \colon Y \to [0,1]^{\mathbb{A}}$ by $G(y)(A) = g_A(y)$.

We will show that actually $G(Y) \subset \boldsymbol{l}^2(Y)$ and that G is a homeomorphism of Y onto $G(Y)$ with subspace topology from the metric space $\boldsymbol{l}^2(Y)$. The conclusion, then, is that Y is metrizable.

Suppose $y \in Y$. For each n, y is in only finitely many distinct members of $\mathbb{A}_n$ and so is in only countably many members of $\mathbb{A}$. Parenthetically, we note that Y is C_I. So

$$\sum_{A\in\mathbb{A}} (G(y)(A))^2 = \sum_{n\in\mathbb{N}} \left(\sum_{A\in\mathbb{A}_n} (G(y)(A))^2 \right) \leq \sum_{n\in\mathbb{N}} \left(\sum_{A\in\mathbb{A}_n} G(y)(A) \right)^2 < \sum_{n\in\mathbb{N}} \frac{1}{(n+1)^2}.$$

Since the sum on the far right converges, $G(y) \in \boldsymbol{l}^2(Y)$.

Now suppose $x \neq y$. Then there is a member A of $\mathbb{A}$ containing x but not y. Then $G(y)(A) = 0$ but $G(x)(A) \neq 0$. So G is one-to-one.

Actually we can get a bit more from this. If W is a closed subset of Y and $x \notin W$ then by regularity there is a set $A \in \mathbb{A}_n$ for some n which contains x but does not intersect W. So $\varepsilon = G(x)(A) > 0$ and $G(y)(A) = 0$ for all $y \in W$. This implies that $d(G(x), G(W)) \geq \frac{\varepsilon}{n+1}$ where d is the metric on $\boldsymbol{l}^2(Y)$. We conclude that $G(Y) - G(W)$ is open in $G(Y)$ so $G(K)$ is closed in $G(Y)$: we have shown that $G\colon Y \to G(Y)$ is a closed function onto its image space, as a subspace of $\boldsymbol{l}^2(Y)$.

It remains to verify that G is continuous.

If **any** neighborhood of $p \in Y$ is a finite set then $\{p\}$ itself is open, so if $G(p)$ is in any relatively open set in $G(Y)$, the inverse image of that open set is a neighborhood of p.

We now assume **every** neighborhood of p is infinite.

For $n = 0$ let T_0 be any neighborhood of p in one of the $\mathbb{A}_{k(0)}$.

Having picked neighborhood T_n of p in $\mathbb{A}_{k(n)}$ let $\mathbb{V}_n$ be the collection of all members of $\mathbb{A}_j$ containing p for any $j \leq k(n)$. Let $N(n)$ be the cardinality of this set of neighborhoods. Select a neighborhood H_n of p inside the intersection of all the members of $\mathbb{V}_n$ and so that

$$|G(y)(A) - G(p)(A)| < \frac{1}{N(n)(n+1)}$$

whenever $y \in H_n$ and for all $A \in \mathbb{V}_n$. Continuity of each g_A allows this choice of H_n.

Now select T_{n+1} to be a member of some $\mathbb{A}_{k(n+1)}$ contained in H_n and where we insist $k(n+1) > k(n)$.

We create by this process a nested collection of neighborhoods $T_j,\ j \geq 0$ of p drawn from $\mathbb{A}$ and with certain useful properties.

Suppose y is any point of T_{n+1}. We would like to discover that $G(y) \in B_d\left(G(p), \varepsilon(n)\right)$ for some expression $\varepsilon(n)$ that converges to 0 as n becomes large. That would imply that the inverse image under G of any open set containing $G(p)$ is a neighborhood of p. Since the only restriction we placed on p was that all of its neighborhoods are infinite, and we took care of the finite neighborhood case above, we could deduce that G is continuous.

$$(d(G(p),G(y)))^2 = \sum_{j=0}^{\infty} \sum_{A\in\mathbb{A}_j} (G(p)(A) - G(y)(A))^2$$

$$= \sum_{j=0}^{k(n)} \sum_{A\in\mathbb{A}_j} (G(p)(A) - G(y)(A))^2 + \sum_{j=k(n)+1}^{\infty} \sum_{A\in\mathbb{A}_j} (G(p)(A) - G(y)(A))^2$$

$$\leq \sum_{j=0}^{k(n)} \sum_{A\in\mathbb{A}_j} (G(p)(A) - G(y)(A))^2 + \sum_{j=k(n)+1}^{\infty} \frac{1}{(j+1)^2}$$

$$\leq N(n) \left(\frac{1}{N(n)(n+1)} \right)^2 + \sum_{j=k(n)+1}^{\infty} \frac{1}{(j+1)^2}$$

Both terms in this sum can be made as small as necessary by choosing n large enough and the theorem is proved. □

This theorem provides some insight into the distinction between paracompact spaces and metrizable spaces.

- We saw earlier that a space is regular and paracompact exactly when it is T_1 and T_*: that is, T_1 and each open cover has a locally starring sequence.
- The Arhangel'skiĭ condition says that metrizability is equivalent to T_1 plus the existence of a **single** sequence that is locally starring for **any** open cover.
- In a similar vein, the Nagata-Smirnov condition states that metrizability is equivalent to regularity and the existence of a σ-locally finite **base**. A selection of members of this base could be used to produce a σ-locally finite open refinement for any open cover you might encounter.
- Previously, we showed that in the presence of the T_3 property, paracompactness was equivalent merely to the existence of a σ-locally finite open refinement for each open cover.

3.2. ***Exercise.*** *The Nagata-Smirnov condition can be modified to produce an even more compelling result. A regular topological space X is metrizable if and only if it has a σ-discrete base: that is, there is a base $\mathbb{A}$ which can be decomposed as a countable union of families $\mathbb{A}_n$ where each point in X has a neighborhood* ***that touches no more than one set in*** $\mathbb{A}_n$ *for each n. This is called the* ***Bing Metrization Theorem.*** *Review The Stone Theorem, Proposition 1.10, and prove the Bing Metrization Theorem.*

3.3. ***Corollary. The Urysohn Metrization Theorem***
Among C_{II} topological spaces, regularity
and metrizability are equivalent conditions.

Proof. This theorem follows by invocation of the Bing (or the Nagata-Smirnov) Metrization Theorem. □

3.4. ***Exercise.*** *A C_{II} topological space is metrizable exactly when it is homeomorphic to a subspace of the Hilbert cube* $[0,1]^{\mathbb{N}}$.

3.5. ***Exercise.*** *If a topological space X is to be pseudometrizable, it is necessary that $\overline{\{x\}}$ is either equal to or disjoint from $\overline{\{y\}}$ for each $x, y \in X$, because all pseudometric spaces have this property. If this condition holds, the set of all these closed sets $\overline{\{x\}}$ forms a partition P of X, which we give the quotient topology. Conclude that X is pseudometrizable if and only if P is metrizable.*

3.6. ***Exercise.*** *Theorem 10.4 yields an important metrization result.*

Suppose $\mathbb{A}_n$, $n \in \mathbb{N}$, is a countable family of open covers of X whose union is a base for a T_2 topology. Suppose for each n there is an m so that $\mathbb{A}_m$ is a barycentric refinement of $\mathbb{A}_n$. Suppose that for any pair of these covers there is a common refinement on the list of covers. Then X with this topology is metrizable.

3.7. ***Exercise.*** *Dugundji* [**?**] *characterizes Tychonoff spaces as precisely those whose topology is generated by a separating set of pseudometrics. Kelley* [**?**] *rephrases this by describing Tychonoff spaces as precisely those homeomorphic to a subspace of a cube $[0,1]^Y$ for **some** index set Y, and later as exactly all subspaces of compact T_2 spaces. The reader is invited to review the proof of Theorem 10.4 and consider the equivalence of these conditions.*

4. Two Topologies on Sets of Continuous Functions

Suppose X and Y are topological spaces. $C(X,Y)$, the set of continuous functions from X to Y, is typically a rather sparse subset of Y^X, the set of **all** functions from X to Y. There are many topologies used for one reason or another on $C(X,Y)$ and we will consider two of them.

The **compact-open topology** on $C(X,Y)$ is that with subbase given by all sets of the form $\mathbf{\mathcal{N}_{A,B}} = \{\, f \in C(X,Y) \mid f(A) \subset B \,\}$ where A is any compact set in X and B is any open set in Y.

If $A_1 \supset A_2$ then $\mathcal{N}_{A_1,B} \subset \mathcal{N}_{A_2,B}$ and if $B_1 \supset B_2$ then $\mathcal{N}_{A,B_1} \supset \mathcal{N}_{A,B_2}$.

4.1. ***Exercise.*** *Suppose $\mathbb{B}$ is a subbase of Y and $\mathbb{A}$ is a collection of compact sets in X with the property that whenever $\varnothing \neq A \subset V \subset X$ and A is compact and V is open there is an integer n and sets A_i, $i = 1, .., n$ in $\mathbb{A}$ with $A \subset \bigcup_{i=1}^n A_i \subset V$.*

Then the set of all $\mathcal{N}_{A,B}$ with $A \in \mathbb{A}$ and $B \in \mathbb{B}$ is a subbase for the compact-open topology.

The **pointwise topology** on $C(X,Y)$ is that with subbase given by all sets of the form $\mathcal{N}_{A,B}$ where A is any one point set in X and B is any open set in Y. Since one point sets are compact, the compact-open topology is finer than the pointwise topology. The pointwise topology is the subspace topology of $C(X,Y)$ as a subspace of Y^X with product topology. Though the topology on Y^X does not depend on that of X, the set $C(X,Y)$ obviously does.

We will denote by $\mathbf{C(X,Y)_{co}}$ the set $C(X,Y)$ with compact-open topology, and by $\mathbf{C(X,Y)_{pw}}$ this same set with pointwise topology.

4.2. ***Exercise.*** *(i) If $\Phi\colon C(X,Y)_{pw} \to Z$ is continuous, then $\Phi\colon C(X,Y)_{co} \to Z$ is too.*

(ii) If $f\colon E \to C(X,Y)_{co}$ is a net converging to $\phi \in C(X,Y)$ then $f\colon E \to C(X,Y)_{pw}$ also converges to ϕ.

(iii) It is not true that if a net $f\colon E \to C(X,Y)_{pw}$ converges to a continuous function ϕ that $f\colon E \to C(X,Y)_{co}$ also converges to ϕ. Even if both X and Y are compact metric spaces, pointwise convergence does not imply compact-open convergence. Let $X = Y = [0,1]$ with the usual absolute value metric. The sequence f given by

$$f_n(x) = \begin{cases} (n+2)x, & \text{if } 0 \le x \le \frac{1}{n+2};\\ -(n+2)x+2, & \text{if } \frac{1}{n+2} < x \le \frac{2}{n+2};\\ 0, & \text{otherwise.} \end{cases}$$

provides a counterexample. f converges to the zero function in $C(X,Y)_{pw}$, but none of the functions f_n is in the compact-open subbasic neighborhood $\mathcal{N}_{[0,1],[0,1/2)}$ of the zero function.

*(iv) The linear ordering of $\mathbb{R}$ provides a **very** useful avenue to evade the issue of part (iii) above. A net $f\colon E \to C(X,\mathbb{R})$ is called **monotone** if either $a \ge b$ implies $f_a(x) \ge f_b(x)$ for all $x \in X$ or $a \ge b$ implies $f_a(x) \le f_b(x)$ for all $x \in X$.*

*Show that if X **is compact and the monotone net** $f\colon E \to C(X,\mathbb{R})$ **converges pointwise to a member of** $C(X,\mathbb{R})$ then the net converges uniformly to this limit function.*

Define the **evaluation map** $e\colon X \times C(X,Y) \to Y$ by $e(x,f) = f(x)$, and for each x define the **evaluation map at x** to be $e_x\colon C(X,Y) \to Y$ via $e_x(f) = e(x,f)$.

Define the **composition function** $\circ : C(Y,Z) \times C(X,Y) \to C(X,Z)$ to be the obvious choice: send (f,g) to $f \circ g$.

4.3. ***Lemma.*** *(i) $\circ : C(Y,Z)_{co} \times C(X,Y)_{co} \to C(X,Z)_{co}$ is continuous when X, Y and Z are all T_2 and Y is locally compact.*

(ii) Each $e_x\colon C(X,Y)_{pw} \to Y$ is continuous. So each $e_x\colon C(X,Y)_{co} \to Y$ is continuous.

(iii) $e\colon X \times C(X,Y)_{co} \to Y$ is continuous when X is locally compact.

Proof. Suppose $f \in C(X,Y)$ and $g \in C(Y,Z)$. Let $\mathcal{N}_{A,B}$ be a subbasic compact-open neighborhood of $g \circ f$: that is, A is compact in X and B is open in Z and $g \circ f(A) \subset B$. Note that $g^{-1}(B)$ is open in Y and $f(A)$ is compact and contained in $g^{-1}(B)$. Since Y has a compact neighborhood base, $f(A)$ can be covered by a finite collection of open sets whose closures lie in $g^{-1}(B)$. Let W be the union of this finite cover of open sets. It follows that $\mathcal{N}_{\overline{W},B}) \times \mathcal{N}_{A,W}$ is a neighborhood of (g,f) taken by $\circ$ into $\mathcal{N}_{A,B}$. Continuity of $\circ$ follows. Items (ii) and (iii) are left as exercises. □

With any function $F\colon X \times Y \to Z$ there is associated a unique function $\widetilde{F}\colon X \to Z^Y$, related by $\widetilde{F}(x)(y) = F(x,y)$. This association sets up a one-to-one and onto map $\widetilde{\ }\colon Z^{X\times Y} \to (Z^Y)^X$. Identified functions are called **associates** and the identification is called the **association map**.

4.4. ***Lemma.*** *If X, Y and Z are T_2 and Y is locally compact then $F \in C(X \times Y, Z)$ if and only if $\widetilde{F} \in C(X, C(Y,Z)_{co})$.*

Proof. The proof is left as a (rather difficult) exercise. See Dugundji [**?**]. □

4.5. ***Proposition.*** *If X, Y and Z are T_2 and Y is locally compact, the association map $\widetilde{\ }\colon C(X \times Y, Z)_{co} \to C(X, C(Y,Z)_{co})_{co}$ is a homeomorphism.*

Proof. The proof is also left as another (hard) exercise. Again, for reference we recommend Dugundji [**?**]. □

4.6**.** ***Exercise.*** *The following exercises refer to pointwise addition and multiplication of functions, restricted to the subset* $C(Y,\mathbb{R})$ *of* $\mathbb{R}^Y$*.*

(i) Are $+ : C(Y,\mathbb{R})_{co} \times C(Y,\mathbb{R})_{co} \to C(Y,\mathbb{R})_{co}$ *and* $+ : C(Y,\mathbb{R})_{pw} \times C(Y,\mathbb{R})_{pw} \to C(Y,\mathbb{R})_{pw}$ *continuous?*

(ii)Are $\cdot : C(Y,\mathbb{R})_{co} \times C(Y,\mathbb{R})_{co} \to C(Y,\mathbb{R})_{co}$ *and* $\cdot : C(Y,\mathbb{R})_{pw} \times C(Y,\mathbb{R})_{pw} \to C(Y,\mathbb{R})_{pw}$ *continuous?*

4.7**.** ***Exercise.*** *Under certain circumstances,* $C(X,Y)_{co}$ *is a metric topology.*

(i) If X *is compact and* Y *has a topology generated by metric* d *then, as we saw in the remarks preceding Exercise 2.10, there is a metric topology on the set* $C(X,Y)$ *denoted* d_{sup} *and defined by* $d_{sup}(f,g) = \sup\{\, d(f(x),g(x)) \mid x \in X \,\}$*. Show that this metric is compatible with the compact-open topology on* $C(X,Y)$*. Convergence in this metric space is equivalent to uniform convergence.*

(ii) Even if X *itself is* ***not*** *compact, when* Y *is a metric topology and if* K *is compact in* X *you do have a metric* d_K *on* $C(K,Y)$ *given by* $d_K(f,g) = \sup\{\, d(f(x),g(x)) \mid x \in K \,\}$*. With this definition and if* X *is locally compact, convergence of a net* ν *in the compact-open topology is equivalent to uniform convergence of the net of restricted functions* $\nu|_K$ *with respect to* d_K *for* ***every*** *compact subset* K *of* X*. It is for this reason that the compact-open topology on* $C(X,Y)$ *is sometimes called "****the topology of uniform convergence on compacta.****" Local compactness of* X *is required to guarantee that a function which is a uniform limit on each compact subset* K *(and therefore continuous when restricted to each compact* K *with subspace topology) is actually continuous on all of* X*.*

(iii) Suppose X *is* σ*-compact and locally compact and* Y *has a topology generated by metric* d*. Then there is a chain of compact sets* K_i *for* $i \in \mathbb{N}$ *and with* K_i *contained in the interior of* K_{i+1} *for each* i *and with* $X = \bigcup_{i\in\mathbb{N}} K_i$*. For each* i *define* d_i *to be the pseudometric* $d_i(f,g) = \sup\{\, d(f(x),g(x)) \mid x \in K_i \,\}$*. Show that the set of all these pseudometrics is separating. Consider Exercise 1.16 and conclude this is a metric topology. Show that this metric is compatible with the compact-open topology.*

5. The Gelfand-Kolmogoroff Theorem and Other Stories

The following propositions, of particular interest to analysts, relate topological properties on a topological space to algebraic and order properties on the ring of real valued continuous functions on the space.

We call upon a number of standard results from algebra (including facts about the prime ideal space of a ring with the Zariski topology from Section **??**) and topology during the discussion. Proofs of these may be found in Appendices **??** and **??**.

We will find that when a space X is compact and T_2, purely algebraic properties of the ring of continuous real valued functions on X carry enough information to determine the topology. A review of the related Stone-Čech compactification results and Exercise **??** would be useful here.

Suppose $C(X,\mathbb{R})$ is the ring of continuous real valued functions on any topological space X. This ring has multiplicative identity function which we denote 1_X. We will collect a few elementary facts about this ring as a "warm-up" to the main discussion.

5.1. ***Lemma.*** *Suppose X and Y are any pair of topological spaces. If $H\colon C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ is a ring homomorphism then*

$$H(cf) = cH(f) \text{ for any } f \in C(Y,\mathbb{R}) \text{ and } c \in \mathbb{R}.$$

In other words, ring homomorphisms are actually algebra homomorphisms for rings of this type.

Proof. To see this note first for nonzero integers p and q that

$$qH\left(\frac{p}{q}1_Y\right) = H(p1_Y) = pH(1_Y)$$

so the result holds for rational c and the function 1_Y. Now suppose a, b are any real numbers and $a < b$. There is a c with $b - a = c^2$. Then

$$H(b1_Y) - H(a1_Y) = H(c1_Y c1_Y) = (H(c1_Y))^2$$

which is a nonnegative function so $H(a1_Y) \leq H(b1_Y)$. Since $rH(1_Y) = H(r1_Y)$ for rational r this inequality implies $aH(1_Y) = H(a1_Y)$ for all a. Finally, if $f \in C(Y,\mathbb{R})$ and $c \in \mathbb{R}$ we find that

$$cH(f) = cH(1_Y)H(f) = H(c1_Y)H(f) = H(c1_Y f) = H(cf).$$

□

5.2. ***Exercise.*** *Suppose X and Y are any pair of topological spaces and $H\colon C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ is a ring homomorphism.*

(i) H is order preserving: that is, $f \leq g$ implies $H(f) \leq H(g)$. (hint: If $0 \leq g \in C(Y,\mathbb{R})$ then $\sqrt{g}$ is continuous.)

(ii) If f is a bounded function then so is $H(f)$. (hint: The function $H(1_Y)$ can only have two possible values: 0 or 1. If $0 \leq f \leq n1_Y$ for positive integer n then $n1_Y - f \geq 0$.)

5.3. ***Exercise.*** *Suppose X is a topological spaces and $\mathcal{S}_X$ is the set of all nonzero ring homomorphisms $G\colon C(X,\mathbb{R}) \to \mathbb{R}$ with the pointwise topology. $\mathcal{S}_X$ is a topological subspace of the product space $\mathbb{R}^{C(X,\mathbb{R})}$.*

(i) $Ker(G)$ is a maximal ideal in $C(X,\mathbb{R})$ for every $G \in \mathcal{S}_X$, and every maximal ideal in $C(X,\mathbb{R})$ is the kernel of some member of $\mathcal{S}_X$.

(ii) $G(1_X) = 1$ for each $G \in \mathcal{S}_X$.

For each $p \in X$ consider the evaluation homomorphisms $e_p\colon C(X,\mathbb{R}) \to \mathbb{R}$ defined by $e_p(f) = f(p)$. Since $c1_X$ is continuous for each real constant c, each e_p is onto the field $\mathbb{R}$ so $Ker(e_p)$, which we will denote M_p, is a maximal ideal in $C(X,\mathbb{R})$. We find that the Jacobson radical consist of the zero function alone in this ring.

In fact, if X is compact any maximal ideal is of the form M_p for some p. For if J is an ideal in $C(X,\mathbb{R})$ and there is no point in X at which all members of J are 0, then for each $p \in X$ there is a continuous function f_p and a neighborhood

N_p of p with $f_p(N_p) \subset [1,\infty)$. The set of all these N_p cover X, and since X is compact there is a finite subcover $N_{p_1}, \dots, N_{p_n}$ of X. So the positive function $f = \sum_{i=1}^n f_{p_i}^2$ is a member of J. Also, the reciprocal g of f is a member of $C(X,\mathbb{R})$. So $gf = 1_X \in J$ and we conclude that $J = C(X,\mathbb{R})$. So there is a point p in X at which every member of any maximal ideal M (or any other proper ideal, for that matter) vanishes. Since M_p is a maximal ideal and contains M we find $M = M_p$.

With these facts in hand, we proceed to the main result of the section. The following proposition tells us, foremost, that the algebraic structure on $Max(C(X,\mathbb{R}))$ determines the topology on X if X is compact. Second, it provides a way of thinking about the Zariski topology on $Max(C(X,\mathbb{R}))$ in terms of the more transparent pointwise topology on $\mathcal{S}_X$, with which $Max(C(X,\mathbb{R}))$ is shown to be homeomorphic in the case of compact X.

5.4. ***Proposition.*** *(The* ***Gelfand-Kolmogoroff Theorem****) Suppose X is a compact T_2 topological space.*

The function $\Phi_X \colon X \to Max(C(X,\mathbb{R}))$ via $\Phi_X(p) = M_p$ is a homeomorphism.

The function $\Psi_X \colon X \to \mathcal{S}_X$ via $\Psi_X(p) = e_p$ is a homeomorphism.

Proof. We saw in the remarks above that Φ_X and Ψ_X are one-to-one and onto.

Sets of maximal ideals of the form $\mathbb{A}_f = Out(f) \cap Max(C(X,\mathbb{R}))$ for $f \in C(X,\mathbb{R})$ constitute, by definition, a base for the topology on $Max(C(X,\mathbb{R}))$.

Because X is normal, sets of the form $\mathbb{B}_f = X - f^{-1}(0)$ for $f \in C(X,\mathbb{R})$ constitute a base for the topology on X.

Note that $\Phi_X(\mathbb{B}_f) = \mathbb{A}_f$. We conclude that Φ_X is a homeomorphism.

Note that $\mathcal{S}_X$ is T_2 because X is compact and T_2: all members of $\mathcal{S}_X$ are of the form e_p, and a net in $\mathcal{S}_X$ converges exactly when the corresponding net of "evaluation points" converge in X. Continuity of Ψ_X is easy to show, and because Ψ_X is one-to-one and onto with T_2 range and compact domain it is a homeomorphism. □

Let's consider now a continuous function $W \colon X \to Y$ for generic X and Y. W induces a ring homomorphism

$$\widetilde{W} \colon C(Y,\mathbb{R}) \to C(X,\mathbb{R}) \text{ defined by } \widetilde{W}(f) = f \circ W.$$

The homomorphism $\widetilde{W}$ itself induces a continuous function

$$W^* \colon Spec(C(X,\mathbb{R})) \to Spec(C(Y,\mathbb{R}))$$
$$\text{defined by } W^*(P) = \widetilde{W}^{-1}(P) = \{\, f \mid f \circ W \in P \,\}.$$

It is immediate that if W is a homeomorphism then $\widetilde{W}$ is an isomorphism.

Below we reprise some of the material in Exercise **??** in our specific context.

5.5. ***Exercise.*** *We will presume X and Y are both compact and T_2 and $W \colon X \to Y$ is continuous.*

(i) If W is one-to-one then W is a homeomorphism onto its image $W(X)$ (see Exercise **??***) which is closed in normal Y. The Tietse Extension Theorem then implies that $\widetilde{W}$ is onto $C(X,\mathbb{R})$: that is, every continuous $f \colon X \to \mathbb{R}$ can be written as $g \circ W$ for some continuous $g \colon Y \to \mathbb{R}$.*

(ii) When $\widetilde{W}$ is onto $C(X,\mathbb{R})$, $W^*(M)$ is a maximal ideal in $C(Y,\mathbb{R})$ whenever M is a maximal ideal in $C(X,\mathbb{R})$. We find that $W^*\colon Spec(C(X,\mathbb{R})) \to Spec(C(Y,\mathbb{R}))$ is one-to-one and continuous, and is also one-to-one and continuous as a function from the subspace $Max(C(X,\mathbb{R}))$ into the subspace $Max(C(Y,\mathbb{R}))$.

*(iii) If W is **not** onto Y then there is a continuous function defined on Y which is zero on the closed set $W(X)$ but nonzero somewhere on $Y - W(X)$. This means that $Ker\left(\widetilde{W}\right)$ contains more than the zero function $0\,1_Y$ and we find that $\widetilde{W}$ is not an isomorphism onto its image.*

*(iv) If $Ker\left(\widetilde{W}\right)$ contains a function f other than the zero function, then there is a maximal ideal which does not contain f and in particular, $Ker\left(\widetilde{W}\right)$ is not a subset of $\mathfrak{Jac}(C(Y,\mathbb{R})) = \{0(1_Y)\}$. This means the image of W^* fails to contain this maximal ideal or, in fact, any prime ideal containing f. (Note: any $f \in Ker\left(\widetilde{W}\right)$ must be zero **somewhere** unless $\widetilde{W}$ is the zero homomorphism.)*

*(v) On the other hand, if W **is** onto Y then $\widetilde{W}$ has trivial kernel and we find that $W^*(Max(C(X,\mathbb{R})) = Max(C(Y,\mathbb{R}))$.*

(vi) If W is both one-to-one and onto then it is a homeomorphism and W^* is a homeomorphism.

Finally, we have an interesting result adapted from the treatment of Dugundji regarding the sole source of ring homomorphisms between function spaces of the kind we are examining.

5.6. ***Exercise.*** *Suppose $\theta\colon C(X,\mathbb{R}) \to C(Y,\mathbb{R})$ is any function. Give*

$$\mathbb{R}^{C(Y,\mathbb{R})} \quad \text{and} \quad \mathbb{R}^{C(X,\mathbb{R})}$$

the product topology: that is, the topology of pointwise convergence.

(i) The map

$$\Theta\colon \mathbb{R}^{C(Y,\mathbb{R})} \to \mathbb{R}^{C(X,\mathbb{R})} \quad \text{defined by} \quad \Theta(\omega) = \omega \circ \theta$$

*is continuous. (hint: See Exercise **??**.)*

(ii) If $\Theta(\mathcal{S}_Y) \subset \mathcal{S}_X$ then the restriction of Θ to $\mathcal{S}_Y$ is continuous.

5.7. ***Corollary.*** *Suppose X and Y are compact T_2 topological spaces and $\theta\colon C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ is a ring homomorphism with $\theta(1_Y) = 1_X$. Then there is a unique $W \in C(X,Y)$ with $\theta = \widetilde{W}$. Moreover, if θ is an isomorphism then W is a homeomorphism.*

Proof. For each $e_x \in \mathcal{S}_X$ the function $\Theta(e_x)$ on $C(Y,\mathbb{R})$ defined by sending f to $e_x(\theta(f))$ is a real valued ring homomorphism on $C(Y,\mathbb{R})$. $e_x(\theta(1_Y)) = 1$ so this homomorphism is not trivial and corresponds to a member $e_y \in \mathcal{S}_Y$. Define $W(x)$ to be this y: that is,

$$W = \Psi_Y^{-1} \circ \Theta \circ \Psi_X$$

which is the composition of continuous functions and therefore is itself continuous.

The proof that W is a homeomorphism if θ is an isomorphism we leave as an exercise. □

5.8. ***Exercise.*** *(i) To what extent is the result from above true when the condition $H(1_Y) = 1_X$ is dropped?*

*(ii) Show that X is connected exactly when $C(X, \mathbb{R})$ is **not** the internal direct sum of two nontrivial proper ideals. Rephrasing, this means there exist two proper ideals J, K of $C(X, \mathbb{R})$ so that for each $f \in C(X, \mathbb{R})$ there exist unique members $j \in J$ and $k \in K$ with $f = j + k$ exactly when X is not connected.*

Complete success in adapting the program from above to deal with non-compact spaces cannot be expected. An example from Dugundji, [**?**], illustrates this. Let Ω denote the first uncountable ordinal and endow intervals of ordinals with order topology. In Exercise **??** we found that $C([0, \Omega), \mathbb{R})$ is essentially identical to $C([0, \Omega], \mathbb{R})$, because any member of $C([0, \Omega), \mathbb{R})$ is eventually constant. Yet $[0, \Omega]$ is compact while $[0, \Omega)$ is not, so these underlying spaces are not homeomorphic. In this case $[0, \Omega)$ itself has nice properties. It is, for instance, normal and so Tychonoff.

However, any Tychonoff space will have a Stone-Čech compactification and the results above apply to this compactification. Maximal ideals there correspond to sets of functions with a common 0 at individual points of the compactification. Using this and other tools, the beautiful results of this section have been pushed much further and the reader is invited to examine two different approaches to these issues (different from each other and from that of this section) in Gillman and Jerison, *Rings of Continuous Functions* [**?**] and Beckenstein, Narici and Suffel, *Topological Algebras* [**?**].

We find in those sources, for instance, that the ideal structure of $C_b(X, \mathbb{R})$, the bounded real valued continuous functions on non-compact X, can be markedly more complex than that of $C(X, \mathbb{R})$. If X is Tychonoff, we know that each member of $C_b(X, \mathbb{R})$ can be extended to a unique member of $C(Y, \mathbb{R})$, where Y is the Stone-Čech compactification of X. These two sets of functions are ring isomorphic. We have seen in this section that $Max(C(Y, \mathbb{R}))$ is homeomorphic to the compactification Y.

6. Ascoli's Theorem

We will now suppose that X is any topological space and Y is a metric space and suppose given a nonempty subset $\mathfrak{F}$ of $C(X, Y)$. We endow $\mathfrak{F}$ with subspace topology from $C(X, Y)_{co}$. We will be concerned with $\overline{\mathfrak{F}}$ and by that we will mean the compact-open closure in $C(X, Y)_{co}$.

If $\varepsilon > 0$, an open set $N \subset X$ is called a **neighborhood of ε-equicontinuity for $\mathfrak{F}$** if $diam(f(N)) < \varepsilon$ for all $f \in \mathfrak{F}$.

$\mathfrak{F}$ is called **equicontinuous at p** if p has a neighborhood which is a neighborhood of ε-equicontinuity for $\mathfrak{F}$ for every $\varepsilon > 0$.

Finally, $\mathfrak{F}$ is called **equicontinuous** if it is equicontinuous at each $p \in X$.

6.1. ***Lemma.*** *Suppose $\mathfrak{F}$ is equicontinuous and $f \colon E \to \mathfrak{F}$ is a net in $\mathfrak{F}$. Suppose further that for each $x \in X$ the net $e_x \circ f$ converges to some point $\phi(x) \in Y$.*

(i) *ϕ is continuous.*
(ii) *If N_p is a neighborhood of ε-equicontinuity for $\mathfrak{F}$ then N_p is a neighborhood of 3ε-equicontinuity for $\{\phi\}$.*

(iii) *f converges to ϕ in $C(X,Y)_{co}$.*

Proof. Fix $\varepsilon > 0$. Let N be a neighborhood of ε-equicontinuity for $\mathfrak{F}$. Suppose y and z are in N. Since the nets $f(y)$ and $f(z)$ converge to $\phi(y)$ and $\phi(z)$, respectively, there is $a \in E$ so that $b \geq a$ implies both $d(f_b(y), \phi(y)) < \varepsilon$ and $d(f_b(z), \phi(z)) < \varepsilon$. We find

$$d(\phi(y), \phi(z)) \leq d(\phi(y), f_b(y)) + d(f_b(y), f_b(z)) + d(f_b(z), \phi(z)) < 3\varepsilon.$$

So $d(\phi(y), \phi(z)) < 3\varepsilon$. We have shown both (i) and (ii).

Now suppose ϕ is in the basic compact-open neighborhood $\mathcal{N}_{A,B}$. Since $\phi(A)$ is compact the distance between $\phi(A)$ and $Y - B$ is 5ε for some $\varepsilon > 0$. For each $z \in A$ let N_z be a neighborhood of z and a neighborhood of ε-equicontinuity for $\mathfrak{F}$. Extract a finite subcover N_{z_i} for $i = 1, \ldots, n$ of A. Find $a \in E$ so that $b \geq a$ implies $d(f_b(z_i), \phi(z_i)) < \varepsilon$ for all $i = 1, \ldots, n$. So if $y \in A$ it must be in one of the N_{z_i} and we have

$$d(f_b(y), \phi(y)) \leq d(f_b(y), f_b(z_i)) + d(f_b(z_i), \phi(z_i)) + d(\phi(z_i), \phi(y)) < \varepsilon + \varepsilon + 3\varepsilon.$$

So $b \geq a$ implies $f_b \in \mathcal{N}_{A,B}$. So the net is eventually in this compact open neighborhood of ϕ. So f converges to ϕ in this topology. □

6.2**. *Exercise.*** *If a net f in equicontinuous $\mathfrak{F}$ converges to ϕ in $C(X,Y)_{co}$ then f converges to ϕ in $C(X,Y)_{pw}$ so the lemma above applies. So if N is a neighborhood of ε-continuity for $\mathfrak{F}$ then N is a neighborhood of 3ε-continuity for $\overline{\mathfrak{F}}$, the compact-open closure of $\mathfrak{F}$. So $\overline{\mathfrak{F}}$ is equicontinuous exactly when $\mathfrak{F}$ is equicontinuous.*

6.3**. *Theorem. (Ascoli's Theorem)***

Suppose $\mathfrak{F} \subset C(X,Y)_{co}$ and $\overline{\{\, f(p) \mid f \in \mathfrak{F} \,\}}$ is compact in Y for each $p \in X$. Then $\overline{\mathfrak{F}}$ is compact in $C(X,Y)_{co}$.

Proof. Suppose f is a universal net in $\overline{\mathfrak{F}}$. Since evaluation at a point is continuous we have

$$\{\, f(x) \mid f \in \overline{\mathfrak{F}} \,\} \subset \overline{\{\, f(x) \mid f \in \mathfrak{F} \,\}}.$$

Thus $f(x)$ is universal in compact $\overline{\{\, f(x) \mid f \in \mathfrak{F} \,\}}$ for all $x \in X$. So $f(x)$ converges to a point $\phi(x) \in \overline{\{\, f(x) \mid f \in \mathfrak{F} \,\}}$ for each $x \in X$.

By the exercise above, $\overline{\mathfrak{F}}$ is equicontinuous, and by the lemma above ϕ is continuous and f converges to ϕ in the compact-open topology. Since $\overline{\overline{\mathfrak{F}}} = \overline{\mathfrak{F}}$ we find that $\phi \in \overline{\mathfrak{F}}$. So universal nets in $\overline{\mathfrak{F}}$ converge in $\overline{\mathfrak{F}}$, and we conclude that $\overline{\mathfrak{F}}$ is compact. □

7. The Stone-Weierstrass Theorem

We begin with a very special case of the main theorem.

For each ε with $0 \leq \varepsilon < 1$, the domain of the functions in $C([\varepsilon, 1], \mathbb{R})$ is compact and the range is complete. So the topology of uniform convergence is a complete metric topology, generated by metric d_ε defined by $d_\varepsilon(f,g) = \sup\{\, |f(x) - g(x)| \mid x \in [\varepsilon, 1] \,\}$.

We are going to be interested in

$$\mathcal{C}_\varepsilon = \left\{ f \in C([\varepsilon, 1], \mathbb{R}) \;\middle|\; \frac{1}{2}\mathcal{I}(x) \leq f(x) \leq \frac{1}{2} + \frac{1}{2}\mathcal{I}(x) \ \forall x \in [\varepsilon, 1] \right\}$$

where $\mathcal{I}$ is the identity function on $[0, 1]$. Note: we restrict $\mathcal{I}$ or any other function defined on $[0, 1]$ to smaller intervals as appropriate and without notational indication.

$\mathcal{C}_\varepsilon$ is a closed subset of $C([\varepsilon, 1], \mathbb{R})$, and so is itself a complete metric space with subspace metric.

7.1. ***Lemma.*** *With $\mathcal{C}_0$ as above, there is a sequence of members of $\mathcal{C}_0$ which are polynomials (with no constant term) in the function $\mathcal{I}$ and which converge uniformly to $\sqrt{\mathcal{I}}$ in $\mathcal{C}_0$.*

Proof. Suppose $0 < \varepsilon < 1$. For each $f \in \mathcal{C}_\varepsilon$ define $T_\varepsilon(f) = f + \frac{1}{2}(\mathcal{I} - f^2)$.

If $H(t) = t + \frac{1}{2}(x - t^2)$ then for $0 \leq t \leq 1$ we find that $\frac{dH}{dt}(t) = 1 - t \geq 0$. Since $H(0) = \frac{1}{2}x$ and $H(1) = \frac{1}{2} + \frac{1}{2}x$, it follows that for any t between 0 and 1, $\frac{1}{2}x \leq H(t) \leq \frac{1}{2} + \frac{1}{2}x$.

So $T_\varepsilon \colon \mathcal{C}_\varepsilon \to \mathcal{C}_\varepsilon$. Also, nonnegativity of the derivative above implies that $T_\varepsilon(f)(x) \geq f(x)$ for each $x \in \mathcal{C}_\varepsilon$.

Note also that if $f, g \in \mathcal{C}_\varepsilon$ then

$$\begin{aligned} d_\varepsilon(T_\varepsilon(f), T_\varepsilon(g)) &= \sup\left\{ \left| f(x) - \frac{1}{2}f^2(x) - g(x) + \frac{1}{2}g^2(x) \right| \;\middle|\; x \in [\varepsilon, 1] \right\} \\ &= \sup\left\{ \left| (f(x) - g(x))(1 - \frac{1}{2}(f(x) + g(x))) \right| \;\middle|\; x \in [\varepsilon, 1] \right\} \\ &\leq (1 - \varepsilon)\ d(f, g). \end{aligned}$$

Since T_ε is a contraction, it has a unique fixed point, which is obviously $\sqrt{\mathcal{I}}$. The sequence $f_i = T_\varepsilon^i(\mathcal{I})$ must converge uniformly to this fixed point in $\mathcal{C}_\varepsilon$. Each f_i is created explicitly as a polynomial (with no constant term) in $\mathcal{I}$.

The sequence f_i is monotone and converges pointwise to $\sqrt{\mathcal{I}}$ on compact $[0, 1]$ so the convergence is uniform. □

7.2. ***Corollary.*** *If X is compact and S is a uniformly closed sub$\mathbb{R}$-algebra of $C(X, \mathbb{R})$ and if $f, g \in S$ then $f \vee g$ and $f \wedge g$ are in S.*

Proof. Suppose $f \in S$. Let K be a positive number for which K^2 exceeds the maximum value of f^2 on X. By the lemma above, every uniform neighborhood of the function

$$\frac{|f|}{K} = \sqrt{\frac{f^2}{K^2}}$$

contains a polynomial (with no constant term) evaluated at f, and all these polynomials are in the algebra S. Since S is closed, $K\frac{|f|}{K} = |f| \in S$. So $f \vee 0 = \frac{f+|f|}{2}$ and $f \wedge 0 = \frac{f-|f|}{2}$ are in S.

So if $f, g \in S$, $f \vee g = f + (g - f) \vee 0$ and $f \wedge g = f + (g - f) \wedge 0$ are in S. □

7.3. ***Theorem. (The Stone-Weierstrass Theorem)*** *Suppose S is a sub$\mathbb{R}$-algebra of $C(X, \mathbb{R})$ containing 1_X and suppose S distinguishes points. Then S is dense in $C(X, \mathbb{R})$ with compact-open topology.*

Proof. Note that since addition and multiplication, as a functions from $C(X,\mathbb{R})_{co} \times C(X,\mathbb{R})_{co}$ to $C(X,\mathbb{R})_{co}$, are continuous the compact-open closure $\overline{S}$ is a sub $\mathbb{R}$-algebra of $C(X,\mathbb{R})$ whenever S is.

Also, if $\overline{S}$ is dense in $C(X,\mathbb{R})$ then $C(X,\mathbb{R}) = \overline{S}$. So if we can find a member of S in every basic open neighborhood of $C(X,\mathbb{R})$ **when** $\mathbf{S} = \overline{\mathbf{S}}$ we will have proven the theorem.

As a second simplification, it suffices to prove the theorem when X is compact. For if we have that, and if X is a more general topological space and if f is in the compact-open basic neighborhood $N = \bigcap_{i=1}^{n} \mathcal{N}_{A_i,B_i}$ let ε be the least distance of compact set $f(A_i)$ to closed set $\mathbb{R} - B_i$ for $i = 1, \ldots, n$.

From our simplifying assumption, we can find a member g of S within a uniform ε-neighborhood of f, where all functions involved are thought of as restricted to the compact set $\bigcup_{i=1}^{n} A_i$. But then $g \in N$.

So we assume $S = \overline{S}$ and X is compact.

Suppose $f \in C(X,\mathbb{R})$. Since the algebra S distinguishes points and contains 1_X, for each $x, y \in X$ there is a function $H_{x,y} \in S$ with $H_{x,y}(x) = f(x)$ and $H_{x,y}(y) = f(y)$. For $\varepsilon > 0$ let $N_{x,y} = \{\, x \in X \mid |H_{x,y}(x) - f(x)| < \varepsilon \,\}$. For each fixed y and various x, these sets cover X. Extract a finite subcover $N_{x_i,y}$ for $i - 1, \ldots, n_y$.

Let $H_y = H_{x_1,y} \wedge \cdots \wedge H_{x_{n_y},y}$. Each H_y is in S because S is uniformly closed. By construction, $H_y(x)$ cannot exceed $f(x) + \varepsilon$ anywhere on X. And for x in the set $N_y = \bigcap_{i=1}^{n_y} N_{x_i,y}$ the numbers $H_y(x)$ cannot be less than $f(x) - \varepsilon$.

Extract a finite subcover $N_{y_1}, \ldots, N_{y_n}$ of X from among these N_y.

Once again, because S is closed the function $H = H_{y_1} \vee \cdots \vee H_{y_n}$ is in S and by construction is within ε of f on all of X. The result follows. □

7.4. ***Exercise.*** *What can be said if the condition $1_X \in S$ is removed from the conditions in the Stone-Weierstrass Theorem?*

7.5. ***Exercise.*** *Suppose X is a compact subset of $\mathbb{R}^n$ and the set $C(X,\mathbb{R})$ is given the compact-open topology: the topology of uniform convergence. Let X_k be the function that returns the kth coordinate of a point in X. Then the algebra generated by the set $\{\, 1_X, X_1, \ldots, X_n \,\}$ is dense in $C(X,\mathbb{R})_{co}$: that is, any continuous function can be uniformly approximated by a polynomial on any compact subset of $\mathbb{R}^n$.*

It is one thing to assert that there is a polynomial uniformly close to any continuous function $f \colon [0,1] \to \mathbb{R}$ and quite another to actually produce such a function. Note that for positive integer n

$$1 = (x + (1-x))^n = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

is a partition of unity on $[0,1]$. Weighting the polynomial summands by $f\left(\frac{i}{n}\right)$ yields the **nth Bernstein polynomial for f**,

$$\sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right)$$

which can be shown by a clever but completely elementary argument (see, for example, George F. Simmons' book *Introduction to Topology and Modern Analysis* [**?**]) to converge to f.

7.6**.** ***Exercise.*** *Suppose X is a compact T_2 space. X is metrizable if and only if $C(X,\mathbb{R})_{co}$ is separable.*

8. Elementary Results on Topological Groups

For the material of the next two sections the primary references are Pontryagin, *Topological Groups, 2nd Ed.* [**?**] and Nachbin, *The Haar Integral* [**?**] and Chevalley, *Theory of Lie Groups* [**?**].

A **topological group** is a triple $(\mathbf{G},\cdot,\mathbb{T})$ where $(G,\mathbb{T})$ is a topological space and $(G,\cdot)$ is a group and for which

$$\mathbf{Mult}\colon G\times G\to G \text{ defined by } \mathbf{Mult}(h,g)=h\cdot g \text{ is continuous and}$$

$$\mathbf{Inv}\colon G\to G \text{ defined by } \mathbf{Inv}(g)=g^{-1} \text{ is a homeomorphism}$$

where $G\times G$ has product topology.

8.1**.** ***Exercise.*** *The assumption of continuity for the group operation and the inverse operation can be replaced, equivalently, by the single assumption*

$$\mathbf{Combo}\colon G\times G\to G \textit{ defined by } \mathbf{Combo}(h,g)=h\cdot g^{-1} \textit{ is continuous.}$$

Any subgroup of a topological group is itself a topological group with subspace topology. A product group formed from an indexed set of topological groups is a topological group with product topology.

In this section and the next we presume that G is the underlying set of a topological group with identity e.

If A and B are subsets of G we define AB and gA and Ag in the obvious ways and define $A^{-1}=\{g^{-1}\mid g\in G\}$. So $(AB)^{-1}=B^{-1}A^{-1}$ and $(A^{-1})^{-1}=A$ and $(AB)C=A(BC)$.

If $\mathbb{A}\subset\mathbb{P}(G)$ and $g\in G$ define $g\mathbb{A}=\{gA\mid A\in\mathbb{A}\}$ and define $\mathbb{A}g$ similarly.

We define $\mathfrak{N}$ to be the set of **open** neighborhoods of e. If $g\in G$ and A is a neighborhood of g then Ag^{-1} is a neighborhood of e. It follows then that $\mathfrak{N}g$ is the set of *all* open neighborhoods of g, and $g\mathfrak{N}=\mathfrak{N}g$.

8.2**.** ***Exercise.*** *In this exercise A and B are subsets of G and $x,y\in G$.*

(i) A is open if and only if A^{-1} is open.

(ii) $A\in\mathfrak{N}y$ exactly when $A^{-1}\in\mathfrak{N}y^{-1}$.

(iii) A is a neighborhood of x if and only if $x^{-1}A$ is a neighborhood of e.

(iv) If A is open then AB and BA are open.

(v) If A contains x and B contains y and A or B is open then $AB\in\mathfrak{N}xy$.

(vi) $\mathfrak{N}x=\{xA\mid A\in\mathfrak{N}\}=\{Ax\mid A\in\mathfrak{N}\}$.

(vii) For each neighborhood C of xy there is $A\in\mathfrak{N}x$ and $B\in\mathfrak{N}y$ with $AB\subset C$.

(viii) If A is closed then so is A^{-1}.

(ix) If A is closed and B is compact then AB is closed.

(x) If A is compact then so are A^{-1} and xA and Ax.

(xi) If A is connected then so are A^{-1} and xA and Ax.

(xii) A net n in G converges to p if and only if np^{-1} converges to e.

We point out here that the Exercise 8.2 implies that a neighborhood base at e entirely determines the topology on G. This has powerful implications.

8.3. ***Exercise.*** *(i) If e has a compact neighborhood then G is locally compact.*

(ii) If e has a neighborhood base consisting of connected sets then G is locally connected.

(iii) If V is the connected component containing e then $V^{-1} = V$ and gV is the connected component containing g.

(iv) If V is the connected component containing e then V is a topologically closed and an algebraically normal subgroup of G. Note: We distinguish by context the group theoretic and topological meanings of the word "normal" in a topological group.

(v) Suppose e has a compact neighborhood and G is separable. Then G is σ-compact.

8.4. ***Exercise.*** *(i) If A and B are compact in G then AB is compact. (hint: Apply Tychonoff's Theorem and use the fact that the image under a continuous function of a compact set is compact.)*

(ii) Suppose A is compact and A generates G. This means that $G = \bigcup_{i=1}^{\infty} A^i$. So G is σ-compact.

(iii) If A is any neighborhood of e and G is connected then $G = \bigcup_{i=1}^{\infty} A^i$.

(iv) If A is any connected neighborhood of e then $\bigcup_{i=1}^{\infty} A^i$ is the connected component containing e.

Let $\mathfrak{M}$ denote the members $A \in \mathfrak{N}$ with $A = A^{-1}$. These are called **symmetric open neighborhoods of e**. $\mathfrak{M}$ is, actually, the set of all $A \cap A^{-1}$ for $A \in \mathfrak{N}$.

8.5. ***Exercise.*** *For this exercise A and B are subsets of G and $g, x, y \in G$*

(i) $g\mathfrak{M} = \mathfrak{M}g$, and this set is a neighborhood base at g.

(ii) If e has a countable neighborhood base then e has a countable nested neighborhood base of symmetric open sets.

(iii) If e has a countable neighborhood base then G is C_I.

(iv) For each $x, y \in G$ and each neighborhood C of xy there is a single set $A \in \mathfrak{M}$ for which all six of the sets $xyAA$, $xAyA$, $xAAy$, $AxyA$, $AxAy$ and $AAxy$ are contained in C. (hint: Use continuity of **Mult** *to find a set in $\mathfrak{M}$ of each type and take the intersection of all of them.)*

(v) In particular, since $ee = e$ this implies that for any $C \in \mathfrak{N}$ there is an $A \in \mathfrak{M}$ with $AA \subset C$.

(vi) For any $C \in \mathfrak{N}$ there is an $A \in \mathfrak{M}$ with $A^n \subset C$.

(vii) Suppose $C \in \mathfrak{N}$. Then $\overline{C} \subset CC^{-1}$. (hint: Pick $p \in \overline{C}$. Then $pC \in \mathfrak{N}p$ so $(pC) \cap C \neq \varnothing$. So there are $x, y \in C$ with $px = y$. So $p = yx^{-1} \in CC^{-1}$.)

(viii) For any $C \in \mathbf{N}$ there is an $A \in \mathbf{M}$ for which $\overline{A} \subset AA \subset C$.

(ix) It now is easy to show that topological groups are T_3. It is more difficult to prove but true (implied by Proposition 9.9 in the next section) that topological groups are $\mathcal{CR}$.

(x) If G is T_0 then G is T_2.

(xi) G is T_2 if and only if $\{e\}$ is closed.

8.6. ***Exercise.*** *(i) If $\mathbb{B}$ is any neighborhood base at e and $S \subset G$, show that*

$$\overline{S} = \bigcap_{B \in \mathbb{B}} BS.$$

(hint: If $p \in \overline{S}$ then every neighborhood of p contains a point of S. Each $B^{-1}p$ for $B \in \mathbb{B}$ is a neighborhood of p, so there is $s \in S$ and $b \in B$ with $s = b^{-1}p$. So $p \in BS$. Since this can be done for every $B \in \mathbb{B}$ we conclude that $\overline{S} \subset \bigcap_{B\in\mathbb{B}} BS$. On the other hand, suppose $p \in \bigcap_{B\in\mathbb{B}} BS$ and $J \in \mathbf{N}p$. If we can find a member of S in J we will have $\overline{S} \supset \bigcap_{B\in\mathbb{B}} BS$. Since J is open there is a member M of $\mathbf{M}$ with $Mp \subset J$. So there is a $B \in \mathbb{B}$ with $B \subset M$ and then we have both $Bp \subset Mp \subset J$ and also $B^{-1}p \subset Mp \subset J$. By assumption $p = bs$ for some $s \in S$ and $b \in B$ so $b^{-1}p = b^{-1}bs = s \in J$.)

(ii) If G is C_I then every closed set is a G_δ set.

8.7. ***Exercise.*** *Suppose H is a subgroup of G.*

(i) $\overline{H}$ is a subgroup of G.

(ii) If H is a normal subgroup of G then $\overline{H}$ is a normal subgroup of G.

(iii) The center of G is closed.

(iv) If H is open then H is closed.

(v) If H is normal and discrete and G is connected then H is a subset of the center of G. (hint: If H is normal then $gHg^{-1} = H$ for each $g \in G$. If $h \in H$ find $A \in \mathbf{M}$ with $AhA = \{h\}$. In particular, $aha^{-1} = h$ for all $a \in A$. Since $G = \bigcup_{i=1}^{\infty} A^i$ we find that h commutes with all members of G.)

Both continuous functions and homomorphisms between topological groups are important, as you might expect.

8.8. ***Exercise.*** *(i) Suppose $f \in Hom(G, H)$ where G and H are topological groups. If f is continuous and H is T_2 then $Ker(f)$ is closed.*

*(ii) For $a \in G$ define $L_a \colon G \to G$ by $L_a(g) = ag$ and $R_a \colon G \to G$ by $R_a(g) = ga$. These maps are called, respectively, the **left and right translations by a**. These functions are homeomorphisms. The set of left translations forms a group with composition. So does the set of right translations.*

(iii) Inner automorphisms are homeomorphisms on any topological group.

(iv) Suppose $f \in Hom(G, H)$ where G and H are topological groups. f is continuous if and only if $f^{-1}(A) \in \mathbf{N}$ for each $A \in \mathbf{M}$, where in this case $\mathbf{M}$ denotes the symmetric open neighborhoods of e in H and $\mathbf{N}$ denotes the open neighborhoods of e in G.

Any topological space X is called **homogeneous** if for each pair $x, y \in X$ there is a homeomorphism $h\colon X \to X$ with $h(x) = y$. The quality of being a homogeneous topological space is referred to as **homogeneity.**

Any set of homeomorphisms from X onto X generates a group of homeomorphisms. In the context of algebraic group theory, this set acts on X by function evaluation. Using that vocabulary, X is homogeneous if and only if there is a group of homeomorphisms for which this action is transitive.

Examining the effect of the function $L_{yx^{-1}}$ on G we see that any topological group is homogeneous. A transitive action is created by using the group of left or the group of right translations.

A function $f\colon G \to \mathbb{R}$ is called **uniformly continuous with respect to the group operation** if for each $\varepsilon > 0$ there is a set $V_\varepsilon \in \mathfrak{M}$ so that $xy^{-1} \in V_\varepsilon$ implies $|f(x) - f(y)| < \varepsilon$.

8.9. ***Exercise.*** *(i) $f\colon G \to \mathbb{R}$ is continuous if and only if for each $x \in G$ and each $\varepsilon > 0$ $\exists V_{x,\varepsilon} \in \mathfrak{M}$ so that $yx^{-1} \in V_{x,\varepsilon}$ implies $|f(x) - f(y)| < \varepsilon$.*

(ii) Suppose $f\colon G \to \mathbb{R}$ is uniformly continuous with respect to the group operation. Then f is continuous. (hint: Suppose f is uniformly continuous with respect to the group operation and $f(a) = r$ and $\varepsilon > 0$. Find V_ε so $xy^{-1} \in V_\varepsilon$ implies $|f(x) - f(y)| < \varepsilon$. Then $a \in V_\varepsilon a \subset f^{-1}((r - \varepsilon, r + \varepsilon))$.)

(ii) If G is compact and $f\colon G \to \mathbb{R}$ is continuous then f is uniformly continuous with respect to the group operation. (hint: If G is compact and f continuous and $\varepsilon > 0$ find for each $g \in G$ open sets $A_g, B_g \in \mathfrak{M}$ with $A_g A_g g \subset B_g g \subset f^{-1}((f(g) - \varepsilon, f(g) + \varepsilon))$. The collection of sets of the form $A_g g$ covers G so select a finite subcover $A_{g_i} g_i$ for $i = 1, \ldots, n$. Let $W \in \mathfrak{M}$ be the intersection of all the A_{g_i}. Suppose $xy^{-1} \in W$. So $y \in A_{g_j} g_j$ for some j. Then $xg_j^{-1} = xy^{-1}yg_j^{-1} \in WA_{g_j} \subset A_{g_j} A_{g_j} \subset B_{g_j}$. So $|f(x) - f(y)| \le |f(x) - f(g_i)| + |f(y) - f(g_i)| < 2\varepsilon$.)

(iii) Consider the positive real numbers $(0, \infty)$ with the usual topology as a topological group with multiplication as group operation, and also as a metric space with the usual metric. Consider a function $f\colon (0, \infty) \to \mathbb{R}$ where $\mathbb{R}$ has the usual topology and the usual metric. We have created two concepts of uniform continuity for a function such as f: uniform continuity with respect to the group operation in the domain and uniform continuity between these two metric spaces. Are these properties equivalent for f? Is one condition stronger than the other?

9. The Homogeneous Space of Cosets in a Topological Group

In this section we will be working with cosets of a subgroup K in a topological group G, and doing so in a way that emphasizes topological rather than purely algebraic properties. New notation will simplify the details of some proofs.

If A and B are subsets of G we let $\mathbf{A} * \mathbf{B} = \{aB \mid a \in A\}$. So AB is the union of all the left cosets found in $A * B$.

If $\mathbb{A} \subset \mathbb{P}(G)$ define $\mathbb{A} * \mathbf{B}$ to be $\{A * B \mid A \in \mathbb{A}\}$. So $\mathbb{A} * B \subset \mathbb{P}(G * B)$.

If K is any subgroup of the topological group G we give $G * K$ the quotient topology: that is, $\mathbb{A}$ is open in $G * K$ exactly when the union of the members of

$\mathbb{A}$ is open in G. This is the largest topology on $G * K$ for which the quotient map $q: G \to G * K$ given by $q(g) = gK$ is continuous.

Though we use only left coset spaces in this section, the reader is invited to observe that all results proved for left coset spaces hold for right coset spaces as well.

The quotient map is open: $q(A) = A * K$ is open exactly when the union of its members, which is AK, is open in G. We note that when A is open so is AK. If $\mathbb{T}$ is the topology on G then $\mathbb{T} * K$ is the quotient topology on $G * K$.

9.1**. *Exercise.*** *Suppose K is any subgroup of G. Then $G * K$ is a homogeneous space. (hint: Show that the function L_{hK} defined by $L_{hK}(cK) = hcK$ is a homeomorphism.)*

9.2**. *Exercise.*** *If $\mathbb{A}$ is an open cover of S in G then $\mathbb{A} * K$ is an open cover of $S * K$ in $G * K$.*

9.3**. *Exercise.*** *If $A \subset G$ and K is any subgroup of G and $B = AK$ then $BK = B$ and $B * K = A * K$. Every open set in $G * K$ can be written in one and only one way as $B * K$ where B is open in G and consists of a union of cosets of K.*

9.4**. *Exercise.*** *Suppose $f: G \to Y$ is continuous, where Y is any topological space. If f is constant on every coset in $G*K$ then $\widetilde{f}: G*K \to Y$ defined by $\widetilde{f}(cK) = f(c)$ is continuous.*

9.5**. *Exercise.*** *(i) If $\mathbb{B}$ is any neighborhood base at e in G then $\mathbb{B} * K$ is a neighborhood base at K in $G * K$.*

(ii) If $\mathbb{B}$ is an open neighborhood base at e and $\mathbb{A} \subset G * K$ then

$$\overline{\mathbb{A}} = \bigcap_{B \in \mathbb{B}} (B * K)\mathbb{A}.$$

(iii) If G is C_I then every closed subset of $G * K$ is a G_δ set.

(iv) $G * K$ is T_2 if and only if K is closed. (hint: Suppose $K \neq dK$. K closed implies there is an open set A containing d but not e, and $B = A - K$ is open and contains d but no member of K. So BK is open, contains d and contains no member of K. So $q(BK) = B * K$ is open in $G * K$ and contains dK. Now appeal to the homogeneous nature of $G * K$.)

9.6**. *Exercise.*** *(i) If $n: E \to G$ is a net in G and nK converges to pK in $G * K$ then there is a net $m: E \to K$ for which nm converges to p in G.*

(ii) If K and $G * K$ are both compact then G is compact.

9.7**. *Exercise.*** *If K is any subgroup of G and both K and $G * K$ are connected then G is connected. (hint: Suppose $G = A \cup B$ where A and B are open. So $A*K$ and $B * K$ are open in connected $G * K = (A * K) \cup (B * K)$. So there is a coset $cK \in (A * K) \cap (B * K)$. So there is a member of A in cK and also a member of B in cK. So $A \cap (cK)$ and $B \cap cK$ are nonempty and relatively open in connected cK, and so these sets have nonempty intersection. So $A \cap B$ is nonempty.)*

9.8**. *Lemma.*** *Suppose K is any subgroup of G. Then the homogeneous space $G*K$ is T_3.*

Proof. Suppose $A * K$ is an open subset of $G * K$ containing the member K of $G * K$, where A is open in G and $AK = A$. So A contains a member of K and it follows that $e \in A$.

We want to find an open set $B * K$ in $G * K$ with $K \in B * K \subset \overline{B * K} \subset A * K$.

Since $e \in A$ we know from Exercise 8.5 there is a set $B \in \mathcal{M}$ with $\overline{B} \subset BB \subset A$.

Suppose $b \in \overline{BK}$. Then since Bb is open and contains b there is a point in $Bb \cap (BK)$ which evidently can be written as both $b_1 b$ and $b_2 k$ for $b_1, b_2 \in B$ and $k \in K$. So $b = b_1^{-1} b_2 k \in BBK \subset AK = A$.

$K \in q(BK) \subset q\left(\overline{BK}\right) \subset A * K$.

By continuity of q we know that $q\left(\overline{BK}\right) \subset \overline{q(BK)}$. If we can show that $q\left(\overline{BK}\right)$ is closed we will have the reverse containment and hence equality of these two sets.

For each $k \in K$ the homeomorphism R_k takes BK onto BK. This implies that R_k also takes $\overline{BK}$ onto $\overline{BK}$. Thus $\overline{BK}K = \overline{BK}$. We conclude that $\overline{BK}$ is a union of cosets of K. Since every point of G is in exactly one coset, the set $W = G - \overline{BK}$ is open and also a union of cosets. So $WK = W$ and $q(W)$ is open and is the complement of $q\left(\overline{BK}\right)$ which is therefore closed.

So K can be separated in $G * K$ from closed sets in $G * K$ which do not contain K. The conclusion now follows by invocation of homogeneity. □

The following result is stronger than the one given above. It's proof is reminiscent of Urysohn's Lemma and is patterned after the proof found in Pontryagin [**?**]. We show here that any coset space formed from any subgroup of a topological group is $\mathcal{CR}$. Since G itself is homeomorphic to the coset space $G/\{e\}$ the result below applies to G: any topological group is $\mathcal{CR}$.

9.9**. *Proposition.*** *Suppose K is any subgroup of G. The homogeneous space $G * K$ is $\mathcal{CR}$.*

Proof. We will prove that if $C * K$ is open and $K \in C * K$ where $CK = C$ is open in G then the "point" K and the closed set $G * K - C * K$ can be separated by a Urysohn function. The homogeneity of $G * K$ allows us then to conclude that $G * K$ is $\mathcal{CR}$.

We note as above that $e \in C$. We saw in Exercise 8.5 that for any member of $\mathcal{N}$, such as C, there is an $A \in \mathcal{M}$ for which $\overline{A} \subset AA \subset C$.

An appeal to induction allows us to conclude that there are sets $A_k \in \mathcal{M}$ for $k \in \mathbb{N}$ with $A_{k+1} \subset \overline{A_{k+1}} \subset A_{k+1}A_{k+1} \subset A_k$ for all k and where $A_0 A_0 \subset C$.

For any $0 \leq k < n$ we find that $A_n \cdot (A_n A_{n-1} \cdots A_{k+1}) \subset A_k$. To see this apply the fact that $A_j A_j \subset A_{j-1}$ for each j to pairs on the list, starting on the leftmost pair $j = n$ and working your way to the right. Note also that if any set in the list is missing, or if any set on the list is replaced by a set with larger subscript, the product cannot contain more elements and we still have containment.

Denote the dyadic rationals in the open unit interval by

$$D = \left\{ \frac{p}{2^q} \;\middle|\; p, q \text{ are positive integers and } 0 < p < 2^q \right\}.$$

Each dyadic in this set has a representation as

$$\frac{p}{2^q} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_q}{2^q}$$

where each a_i is 0 or 1, and the representation can be extended to a sum of terms with arbitrarily large denominators by choosing $a_j = 0$ for all $j > q$. With this convention, there is a unique sequence of numerators associated with each r.

Define for each r the set $F(r) = A_q^{a_q} A_{q-1}^{a_{q-1}} \cdots A_1^{a_1} K$ where the corresponding factor is replaced by $\{e\}$ when $a_i = 0$.

Each $F(r) \subset A_1 A_1 K \subset A_0 K \subset C$. Each $F(r)$ contains K, and actually is an open set consisting of a union of cosets of K. The union of *all* the $F(r)$ is then an open set consisting of a union of cosets of K, and contained in $A_0 K \subset C$.

We will show that if $r < s$ then $F(r) \subset F(s)$.

If $r < s$ then the dyadic representations of r and s differ at some first term corresponding to subscript j. So s and r can be represented as

$$\begin{aligned} r =& \frac{a_1}{2} + \cdots + \frac{a_{j-1}}{2^{j-1}} + \frac{0}{2^j} + \frac{a_{j+1}}{2^{j+1}} + \cdots + \frac{a_q}{2^q} \\ s =& \frac{a_1}{2} + \cdots + \frac{a_{j-1}}{2^{j-1}} + \frac{1}{2^j} + \frac{b_{j+1}}{2^{j+1}} + \cdots + \frac{b_q}{2^q}. \end{aligned}$$

That means $F(s)$ and $F(r)$ look like

$$\begin{aligned} F(r) =& A_q^{a_q} \cdots A_{j+1}^{a_{j+1}} A_j^0 A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K \\ F(s) =& A_q^{b_q} \cdots A_{j+1}^{b_{j+1}} A_j A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K. \end{aligned}$$

We noted above that $A_q^{a_q} \cdots A_{j+1}^{a_{j+1}} \subset A_j$ and the conclusion $F(r) \subset F(s)$ is immediate.

We now define $\phi\colon G \to [0,1]$ as follows. If g is in *any* of the $F(r)$ we let $\phi(g) = inf\{r \in D \mid g \in F(r)\}$. If g is in *none* of these sets we let $\phi(g) = 1$. We note first that since each $F(r)$ is comprised of a union of cosets of K that ϕ is constant on each coset. Note also that if $\phi(g) \le r < 1$ then $g \in F(r)$.

Also, if x is in any of the cosets of $G * K - C * K$ then $\phi(x) = 1$ and if $x \in K$ then $\phi(x) = 0$. It only remains to show that ϕ is continuous: in fact, we show below that ϕ is uniformly continuous with respect to the group operation. An appeal to Exercise 9.4 then shows that ϕ will induce a Urysohn function $\widetilde{\phi}$ on $G * K$ with the necessary properties.

Suppose $f(x) < 1$ and $k > 0$. Find an integer p with $0 < p < 2^k - 1$ and with $\frac{p-1}{2^k} \le f(x) \le \frac{p}{2^k} = r$. Note that $x \in F(r)$.

$r = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_k}{2_k}$ where each a_i is 0 or 1. Because $p < 2^k - 1$, not all the a_i can be 1. Let j correspond to the subscript of the last zero value of a_i and define $s = \frac{a_1}{2} + \cdots + \frac{a_{j-1}}{2^{j-1}} + \frac{1}{2^j}$. So $s - r = \frac{1}{2^j} - \sum_{L=j+1}^{k} \frac{1}{2^L} = \frac{1}{2^k}$.

We note that $x \in F(r) \subset F(s)$. Observe that

$$\begin{aligned} F(r) =& A_k \cdots A_{j+1} A_j^0 A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K \\ F(s) =& A_j A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K \end{aligned}$$

$$\text{so} \quad A_k F(r) = A_k (A_k \cdots A_{j+1}) A_j^0 A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K \subset A_j A_{j-1}^{a_{j-1}} \cdots A_1^{a_1} K = F(s).$$

With this construction and observation out of the way, we will proceed to show that ϕ is uniformly continuous with respect to the group operation.

Suppose x, y are elements of G with yx^{-1} in the neighborhood A_k of e for integer $k > 1$. By symmetry of A_k we have xy^{-1} also in A_k so without loss presume that $0 \leq \phi(x) \leq \phi(y) \leq 1$. We will show that $\phi(y) - \phi(x) < \frac{1}{2^{k-1}}$.

As above, select integer p with $\frac{p-1}{2^k} \leq f(x) \leq \frac{p}{2^k} = r$. If $p = 2^k$ or $p = 2^k - 1$ the desired inequality is obvious, so we presume that $p < 2^k - 1$. Since $0 < r < s = r + \frac{1}{2^k} < 1$ we have defined above both $F(r)$ and $F(s)$ and $x \in F(r) \subset F(s)$.

Since $yx^{-1} \in A_k$ we have $yx^{-1}F(r) \subset A_kF(r) \subset F(s)$. Since $x \in F(r)$ this gives $y \in F(s)$, which implies $\phi(y) \leq s = r + \frac{1}{2^k}$. □

Recall that when K is a normal subgroup left and right cosets coincide and $G*K$ is a group, usually denoted G/K.

9.10**. *Lemma.*** *Suppose K is a normal subgroup. Then G/K is a topological group.*

Proof. We need to show that in G/K multiplication and inversion are continuous. We will let **Mult** and **Inv** denote multiplication and inversion in G/K.

Let $S * K$ be an open neighborhood of cK, where S is open in G and $SK = S$. Suppose $aKbK = cK$. $S = q^{-1}(S * K)$ is open and contains ab so there is $B \in \mathfrak{M}$ with $aBbB \subset S$. So $\mathbf{Mult}(q(aB) \times q(bB)) \subset S * K$. This means there is an open neighborhood of each point in $\mathbf{Mult}^{-1}(S * K)$ inside of $\mathbf{Mult}^{-1}(S * K)$. We conclude that **Mult** is continuous in G/K.

Continuity of **Inv** is left to the reader. □

Suppose given a continuous homomorphism $f\colon G \to H$ **onto** H and let $K = Ker(f)$.

If B is open in H then $A = f^{-1}(B)$ is open in G. Moreover, if $a \in A$ and $k \in K$ then $f(ak) = f(a)f(k) = f(a)e = f(a)$ so $AK = A$ and we conclude that A is the union of cosets of K. Since these cosets form a partition of G, the complement of A is a union of cosets too.

We know that as groups G/K and H are isomorphic by the isomorphism $\tilde{f}\colon G/K \to H$ defined by $\tilde{f}(xK) = f(x)$. This isomorphism is continuous by the previous paragraph.

However $\tilde{f}$ need not be a homeomorphism. Suppose $f(A)$ fails to be open in H for some open set A in G. Then $q(A) = A * K$ is open in G/K because the union of its members is the open set AK. But $\tilde{f}(q(A)) = f(AK) = f(A)$ is not open. So $\tilde{f}$ cannot have continuous inverse.

9.11**. *Lemma.*** *Suppose $f \in Hom(G, H)$ and f is onto H. The induced function $\tilde{f}\colon G/K \to H$ is a homeomorphism exactly when f is both open and continuous.*

Proof. See the remarks above. □

9.12**. *Exercise.*** *Suppose V is the connected component of e in G. Show that G/V is totally disconnected. If G is locally connected then G/V is discrete.*

9.13. ***Exercise.*** *(i) The set* $\overline{\{e\}}$ *is a normal subgroup of* G*, and* $G/\overline{\{e\}}$ *is* T_2*. It is now obvious (Exercise 9.5) that if* H *is normal and* G/H *is* T_2 *then* $\overline{\{e\}} \subset H$*.*

(ii) Recall the commutator subgroup $G^{(1)}$ *of* G *from Exercise* **??**. *It is the smallest subgroup containing all finite products of elements of the form* $aba^{-1}b^{-1}$ *for* $a, b \in G$*.* $G^{(1)}$ *is normal and if* H *is any normal subgroup of* G *then* G/H *is abelian if and only if* $G^{(1)} \subset H$*. The topological group* $G/\overline{G^{(1)}}$ *is abelian and* T_2*. If* G/H *is abelian and* T_2 *for any normal* H *then* $\overline{G^{(1)}} \subset H$*.*

9.14. ***Exercise.*** *Page through Appendix* **??** *and find twenty five topological groups which are not discrete. Find twenty five subgroups, not all discrete. Think about the homogeneous spaces produced as coset spaces of your groups. In particular, consider the topological groups from Exercise* **??**.

10. Uniformities

The concept of a uniform structure follows naturally from the discussion of topological groups above, abstracting a concept of "uniform closeness" over the whole space that is encapsulated in that earlier context in the ability of the symmetric neighborhoods of the identity to capture global topological properties.

We begin with some notation involving elements of $\mathbb{P}(G \times G)$ for a nonempty set G. In this section we let $\mathbf{\Delta} = \{ (x, x) \mid x \in G \}$, the **diagonal** of the product space. If $A \in \mathbb{P}(G \times G)$ we define A^{-1} to be the set of those (y, x) for which $(x, y) \in A$ and $A[x] = \{ y \mid (x, y) \in A \}$. Finally, for $A, B \in \mathbb{P}(G \times G)$ define $A \circ B$ to be those (x, z) for which there is a $y \in G$ with $(x, y) \in A$ and $(y, z) \in B$.

A **uniform structure** on G is a collection of sets $\mathcal{U} \subset \mathbb{P}(G \times G)$ with the following properties:

(i) $\Delta \subset A \ \forall A \in \mathcal{U}$.
(ii) $A \in \mathcal{U} \Rightarrow A^{-1} \in \mathcal{U}$.
(iii) $A \in \mathcal{U} \Rightarrow \exists C \in \mathcal{U}$ with $C \circ C \subset A$.
(iv) $A, B \in \mathcal{U} \Rightarrow \exists C \in \mathcal{U}$ with $C \subset A \cap B$.
(v) $A \in \mathcal{U}$ and $A \subset B \subset G \ \Rightarrow \ B \in \mathcal{U}$.

Items (i) and (ii) can be, roughly, translated as "points are arbitrarily close to themselves" and "if x is close to y then y is close to x. Item (iii) could be construed as the analog of the statement "every sphere of given radius contains a sphere with half that radius."

If $\mathcal{U}$ only satisfies (i) through (iv) it is called a **uniformity**. Each uniformity is contained in a unique smallest uniform structure, called the **uniform structure generated by the uniformity.** Two uniformities are called **equivalent** if they generate the same uniform structure.

The analog of the T_2 property is the following.

(vi) $\Delta = \bigcap_{A \in \mathcal{U}} A$.

If a uniformity $\mathcal{U}$ satisfies (vi) it is called **separating**.

If $\mathcal{U}$ is a uniformity and $x \in G$ let $\mathbb{T}_x = \{ A[x] \mid A \in \mathcal{U} \}$. Declare $S \subset G$ to be open if for each $s \in S$ there is a member of $\mathbb{T}_s$ contained in S. Then the collection of all these open sets constitute a topology on G, and each $\mathbb{T}_x$ is a neighborhood base at x for this topology, called **the topology generated by** $\mathcal{U}$.

It is not true that every $A[x]$ is an open set with this topology. However the collection of interiors $A[x]^o$ of sets of this form constitute a base for the topology.

Also, if $G \times G$ has the product topology with respect to this topology then the collection of interiors $\{ A^o \mid A \in \mathcal{U} \}$ is also a uniformity, equivalent to $\mathcal{U}$.

Any two uniformities generating the same uniform structure also generate the same topology, but the converse is quite false.

Dugundji has a particularly clear approach to uniform structures, characterizing their properties through covers.

Let $\boldsymbol{\mathcal{A}}$ be a set of covers of a set G. $\boldsymbol{\mathcal{A}}$ will be called a **uniformizing family of covers of G** if, whenever $\mathbb{A}, \mathbb{B} \in \boldsymbol{\mathcal{A}}$ there is a member $\mathbb{E}$ of $\boldsymbol{\mathcal{A}}$ which is a barycentric refinement of both $\mathbb{A}$ and $\mathbb{B}$. The family is called **separating** if whenever $x, y \in G$ and $x \neq y$ then there is a member $\mathbb{B} \in \boldsymbol{\mathcal{A}}$ with $x \notin Star_{\mathbb{B}}(\{y\})$.

The topology generated by a uniformizing family $\boldsymbol{\mathcal{A}}$ is that generated by

$$\{ Star_{\mathbb{A}}(\{y\}) \mid \mathbb{A} \in \boldsymbol{\mathcal{A}},\ y \in G \}.$$

This collection of sets is actually a base for a topology. In fact, if S is any set open with respect to this topology then for each $s \in S$ there is a member $\mathbb{A}_s \in \boldsymbol{\mathcal{A}}$ with $Star_{\mathbb{A}_s}(\{s\}) \subset S$. Then

$$S = \bigcup_{s \in S} Star_{\mathbb{A}_s}(\{s\}).$$

The connection between uniformizing families and uniformities is as follows. First, if $\boldsymbol{\mathcal{A}}$ is a uniformizing family and $\mathbb{B} \in \boldsymbol{\mathcal{A}}$ let $V_{\mathbb{B}} \subset G \times G$ denote the union of all $B \times B$ for $B \in \mathbb{B}$. Define $\widetilde{\boldsymbol{\mathcal{A}}}$ to be the collection of all these sets for $\mathbb{B} \in \boldsymbol{\mathcal{A}}$.

Condition (i) for a uniformity is satisfied because each $\mathbb{B}$ covers G and (ii) follows by construction. Now suppose $V_{\mathbb{B}} \in \widetilde{\boldsymbol{\mathcal{A}}}$. Let $\mathbb{E}$ be a barycentric refinement of $\mathbb{B}$ in $\boldsymbol{\mathcal{A}}$. So if there is a point $y \in G$ with $(x, y),\ (y, z) \in V_{\mathbb{E}}$ then both x and z are in $Star_{\mathbb{E}}(\{y\})$, which must be in a member of $\mathbb{B}$. It follows that $V_{\mathbb{E}} \circ V_{\mathbb{E}} \subset V_{\mathbb{B}}$. Condition (iv) follows similarly, using the common refinement property. So $\widetilde{\boldsymbol{\mathcal{A}}}$ is a uniformity.

On the other hand, suppose $\mathcal{U}$ is a uniformity. Let $\widehat{\mathcal{U}}$ denote the family of covers $\mathcal{U}_A$ for $A \in \mathcal{U}$ where $\mathcal{U}_A = \{ A[y] \mid y \in G \}$.

Condition (iv) for uniformity implies that pairs of members of $\widehat{\mathcal{U}}$ have common refinements in $\widehat{\mathcal{U}}$, while condition (iii) guarantees the existence of barycentric refinements in $\widehat{\mathcal{U}}$. So $\widehat{\mathcal{U}}$ is a uniformizing family.

Two uniformizing families are called **equivalent** exactly when their associated uniformities are equivalent.

10.1. ***Exercise.*** *(i) Show that* $\widetilde{\widehat{\mathcal{U}}}$ *is equivalent to* $\mathcal{U}$ *and* $\widehat{\widetilde{\boldsymbol{\mathcal{A}}}}$ *is equivalent to* $\boldsymbol{\mathcal{A}}$.

(ii) Show that the topologies generated by a uniformity and an associated uniformizing family are the same.

(iii) Show that this topology is T_2 *exactly when the uniformity is separating, which happens exactly when the associated uniformizing family is separating.*

10.2**. *Exercise.*** *Suppose G is a topological group with identity e and $\mathcal{B}$ is any neighborhood base at e consisting of symmetric open sets. For each $A \in \mathcal{B}$ let $\mathbb{B}_A$ denote the cover $\{\, xA \mid x \in G \,\}$. This family of open covers $\boldsymbol{\mathcal{L}}$ constitutes a uniformizing family, and the topological group topology on G is that generated by this uniformizing family.*

The family of covers $\boldsymbol{\mathfrak{R}}$ consisting of $\{\, Ax \mid x \in G \,\}$ for $A \in \mathcal{B}$ is also a uniformizing family and generates the same topology, but it need not be equivalent to $\boldsymbol{\mathcal{L}}$.

10.3**. *Exercise.*** *(i) Let d denote a pseudometric on G. For each integer $n > 0$ let $\mathbb{A}_n$ denote the set of all open balls with common radius 2^{-n}. This collection of covers constitutes a uniformizing family of covers, and this family is separating exactly when d is a metric. The topology generated by the pseudometric is, obviously, that generated by the uniformizing family of covers.*

(ii) It is possible for topologically equivalent metrics d_1 and d_2 on G to generate inequivalent uniformizing families.

(iii) For each $n > 0$ let $A_n \subset G \times G$ denote $\{\, (x, y) \mid d(x, y) < 2^{-n} \,\}$. Then the collection of all these sets is a uniformity on G and the topology generated by this uniformity is the pseudometric topology.

(iv) Let $\mathcal{D}$ denote a family of pseudometrics on G. For each integer $n > 0$ let $\mathbb{A}_n$ denote the set of all finite intersections for various pseudometrics from $\mathcal{D}$ of open balls with common radius 2^{-n} and common center.

$$\mathbb{A}_n = \left\{ \bigcap_{i=1}^{k} B_{d_i}(x, 2^{-n}) \;\middle|\; k > 0,\ d_i \in \mathcal{D},\ x \in G \right\}.$$

*This collection of covers constitutes a uniformizing family of covers, the **uniformizing family generated by $\boldsymbol{\mathcal{D}}$**. This family is separating exactly when $\mathcal{D}$ is separating. The topology generated by the family of pseudometrics is, obviously, that generated by the uniformizing family of covers.*

10.4**. *Theorem.*** *Suppose $\boldsymbol{\mathcal{A}}$ is a uniformizing family of open covers on a set G which generates the topology on G. Each open cover in the family can be used to create a pseudometric so that the family $\mathcal{D}$ of these pseudometrics generates the topology on G. $\boldsymbol{\mathcal{A}}$ is equivalent to the uniformizing family generated by $\mathcal{D}$.*

Proof. For each cover $\mathbb{B} \in \boldsymbol{\mathcal{A}}$ we will create a sequence of covers (via inductive construction) taken from $\boldsymbol{\mathcal{A}}$ as follows.

Let $\mathbb{B}_0 = \{G\}$ and $\mathbb{B}_1 = \mathbb{B}$. Having found $\mathbb{B}_k$ for $0 \le k \le n-1$ let $\mathbb{B}_n$ be a member of $\boldsymbol{\mathcal{A}}$ that star refines $\mathbb{B}_{n-1}$. In words, for each open set S from the cover $\mathbb{B}_n$, the union of all members of $\mathbb{B}_n$ that touch S is contained in a member of $\mathbb{B}_{n-1}$.

For $x, y \in G$ define

$$F(x, y) = \inf\{\, 2^{-n} \mid x \in Star_{\mathbb{B}_n}(\{y\}) \,\}.$$

Note that $x \in Star_{\mathbb{B}_n}(\{y\})$ exactly when $y \in Star_{\mathbb{B}_n}(\{x\})$, so $F(x, y) = F(y, x)$. $F(x, y)$ could be as large as 1, but only if $x \notin Star_{\mathbb{B}}(\{y\})$. Also, $F(x, x) = 0$.

For $x, y \in G$ let $M_{x,y}$ denote the set of all sequences $P \colon \mathbb{N} \to G$ with $P(0) = x$ and so that there is an n with $P_k = y$ whenever $k \ge n$. These sequences, eventually, have constant value y. Note that there is a one-to-one correspondence between members of $M_{x,y}$ and members of $M_{y,x}$, obtained by reversing the order of the "nonconstant part" of each sequence. Now define $d_{\mathbb{B}}$ by

$$d_{\mathbb{B}}(x,y) = \inf \left\{ \sum_{i=0}^{\infty} F(P_i, P_{i+1}) \;\middle|\; P \in M_{x,y} \right\}.$$

It is obvious that $d_{\mathbb{B}}(x,x) = 0$ and $d_{\mathbb{B}}(x,y) = d_{\mathbb{B}}(y,x)$ for each $x, y \in G$.

Also, the members of $M_{x,y}$ that "pass through" $z \in G$ constitute a subset of $M_{x,y}$ which can be formed by patching a member of $M_{z,y}$ onto the "end" of a member of $M_{x,z}$. It follows that $d_{\mathbb{B}}(x,y) \leq d_{\mathbb{B}}(x,z) + d_{\mathbb{B}}(z,y)$ so $d_{\mathbb{B}}$ is a pseudometric on G.

Let $\mathcal{D} = \{\, d_{\mathbb{B}} \mid \mathbb{B} \in \boldsymbol{\mathcal{A}} \,\}$.

If $x \in Star_{\mathbb{B}_{n+1}}(\{y\})$ for $n \geq 1$ then $F(x,y)$, and hence $d_{\mathbb{B}}(x,y)$, cannot exceed 2^{-n-1}. So the $d_{\mathbb{B}}$-ball of radius 2^{-n} centered at y contains the set $Star_{\mathbb{B}_{n+1}}(\{y\})$. It follows that the topology on G is finer than that generated by $\mathcal{D}$. We now need the converse conclusion.

To help with this we show that if $P \in M_{x,y}$ then $F(x,y) \leq 2\sum_{i=0}^{\infty} F(P_i, P_{i+1})$.

The result is obviously true when there is but a single nonzero term in the sum. We hypothesize that we have the result for all x and y when the nonzero terms correspond to summation index less than k for some positive k.

Let $c = \sum_{i=0}^{k+1} F(P_i, P_{i+1})$ where for this sequence $P \in M_{x,y}$ the nonzero terms correspond to summation index less than $k+1$.

Let L denote the largest index with $\sum_{i=0}^{L-1} F(P_i, P_{i+1}) \leq \frac{c}{2}$. Since

$$\sum_{i=0}^{k} F(P_i, P_{i+1}) = \left(\sum_{i=0}^{L-1} F(P_i, P_{i+1}) \right) + F(P_L, P_{L+1}) + \left(\sum_{i=L+1}^{k} F(P_i, P_{i+1}) \right)$$

(one summation on the right could be empty and hence 0) we have also that

$$\sum_{i=L+1}^{k} F(P_i, P_{i+1}) \leq \frac{c}{2}.$$

Positing that neither sum on the right is empty, our inductive assumption yields

$$F(x, P_L) \leq 2\frac{c}{2} \quad \text{and} \quad F(P_{L+1}, P_{k+1}) \leq 2\frac{c}{2}.$$

Pick the smallest value of j for which $2^{-j} \leq c$. We conclude that

$$F(x, P_L) \leq 2^{-j} \quad \text{and} \quad F(P_L, P_{L+1}) \leq 2^{-j} \quad \text{and} \quad F(P_{L+1}, P_{k+1}) \leq 2^{-j}.$$

So there is a member A of the open cover $\mathbb{B}_j$ containing both P_L and P_{L+1}. $Star_{\mathbb{B}_j}(A)$, therefore, contains these two points and both x and P_{k+1} as well. So $F(x, P_{k+1})$ cannot exceed $\frac{1}{2^{j-1}} = 2\frac{1}{2^j} \leq 2c$.

The case where one of the sums is empty is left to the reader.

Suppose $x \in G$ and $j > 1$. If $d_{\mathbb{B}}(x,y) < 2^{-j}$ then from above we have $F(x,y) < 2^{-j+1}$ so $y \in Star_{\mathbb{B}_{j-1}}(\{x\})$. So the open ball of radius 2^{-j} formed using $d_{\mathbb{B}}$ centered at x is contained in $Star_{\mathbb{B}_{j-1}}(\{x\})$, and sets of this form constitute a base for the topology formed using the uniformizing family of open covers. Coupled with the earlier observation, we conclude that the topology generated by this family of pseudometrics is identical to that topology.

We leave the proof that $\mathcal{D}$ generates a uniformizing family equivalent to $\boldsymbol{\mathcal{A}}$ as an exercise. □

10.5. ***Exercise.*** *(i) If there is a uniformizing family of covers which generates a given topology then that topology is completely regular. If the family is separating, G is Tychonoff.*

(ii) In the theorem above, the cardinality of $\mathcal{D}$ will not exceed that of $\boldsymbol{\mathcal{A}}$. We conclude that if there is a countable separating uniformizing family of covers which generates a given topology, then that topology is metric. In particular, a C_I and T_2 topological group is metrizable.

10.6. ***Exercise.*** *A metric d on a topological group G is called* ***left invariant*** *if $d(xy, xz) = d(y, z)$ for all $x, y, z \in G$. The metric is called* ***right invariant*** *if $d(yx, zx) = d(y, z)$ for all $x, y, z \in G$.*

Use a countable symmetric open neighborhood base at the identity e in conjunction with Theorem 10.4 to show that if G is T_2 and C_I there is a left invariant metric which generates the topology on G. Show that there is a right invariant metric too.

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