TOPICS FROM DIFFERENTIAL EQUATIONS

LARRY SUSANKA

Contents

1. Approximation

Suppose we are given an initial value problem

$$\mathbf{Y}' \;=\; \mathbf{F}(\mathbf{Y}, t) \quad \text{and} \quad \mathbf{Y}(t_0) = \mathbf{Y_0}.$$

If we want to find $\mathbf{Y}(t)$ for some $t > t_0$ we can try to solve the IVP exactly, producing some recognizable known function, but that will often fail even if $\mathbf{F}$ is differentiable. So the next best thing is an approximation. We can let, for some big integer k and integer $L \leq k$,

$$h = \frac{t - t_0}{k} \quad \text{and} \quad t_L = t_0 + Lh \quad \text{and} \quad \mathbf{Y_1} = \mathbf{Y_0} + \mathbf{F}(\mathbf{Y_0}, t_0)\, h.$$

Having found $\mathbf{Y_j}$ for $j = 1, \ldots, L < k$ let

$$\mathbf{Y_{L+1}} = \mathbf{Y_L} + \mathbf{F}(\mathbf{Y_L}, t_L)\, h.$$

Carrying on for k steps gives the approximation $\mathbf{Y_k}$ for the solution $\mathbf{Y}(t_k) = \mathbf{Y}(t)$.

A computer is really good at this kind of iterative calculation. You could write a little routine to do this on your calculator in five minutes.

Nothing we have done above restricts the dimension of $\mathbf{Y}$. $\mathbf{Y}$ could be an n-dimensional column vector (the ones we saw all quarter were 2-dimensional) and everything we wrote above would make sense.

Suppose, for example, we have the IVP

$$\frac{d^3x}{dt^3} + 7t\frac{dx}{dt} = t^2 \cos\left(x\frac{dx}{dt}\right) \text{ and } x(0) = 1 \text{ and } \frac{dx}{dt}(0) = 3 \text{ and } \frac{d^2x}{dt^2}(0) = -3.$$

Most likely there is no "named" solution to this IVP. Still, we would like to know $x(t)$, at least approximately.

To use our estimation technique in vector form, let

$$u = \frac{dx}{dt} \quad \text{and} \quad w = \frac{du}{dt} = \frac{d^2x}{dt^2}.$$

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Now the IVP we want to solve is

$$\mathbf{Y}' = \mathbf{F}(\mathbf{Y}, t) \quad \text{and} \quad \mathbf{Y}(0) = \mathbf{Y_0}$$

where we define the vector valued functions $\mathbf{Y}$ and $\mathbf{F}$ by

$$\mathbf{Y} = \begin{pmatrix} x \\ u \\ w \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{Y}, t) = \begin{pmatrix} u \\ w \\ t^2 \cos(xu) - 7tu \end{pmatrix} \quad \text{and} \quad \mathbf{Y_0} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}.$$

As in the one-dimensional setting, there are problems. For instance if $\mathbf{F}(\mathbf{Y_0}, t)$ is zero the approximation is likely to be unsatisfying. If $\mathbf{F}$ is changing rapidly along a solution a good approximation will require commensurably tiny h. But if there is no equilibrium solution near $Y(0)$ (and h is tiny enough) the approximation is likely to be a good one for a considerable amount of time.

It is important not to actually *use* an approximation technique without specific information about the size of the error, which will be discussed in later classes, but nothing is stopping you from looking at it and speculating. This method is very useful.

2. Linearization

We found, above, a way to approximate the solution of a DE or IVP away from an equilibrium point. In this section we find solutions (exact or otherwise) to an *approximation to the DE itself.* In other words, we change the DE to a *new* DE, one we can solve. The solutions to this new DE will be close to the ones we actually want, at least for some time interval.

We use methods of multivariable calculus to find a linear approximation to a function. It uses the higher dimensional analog of the idea that the tangent line is close to a curve, at least for a while.

For convenience we consider the autonomous case, though by increasing the dimension of the vectors by one we can consider the nonautonomous case in the same way. (Can you show this?)

Suppose $\mathbf{Y}$ is an n-dimensional vector valued function of time and $\mathbf{G}$ is a differentiable vector valued function of a vector variable, all in dimension n.

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \quad \text{and} \quad \mathbf{G}(\mathbf{Y}) = \begin{pmatrix} G_1(\mathbf{Y}) \\ G_2(\mathbf{Y}) \\ \vdots \\ G_n(\mathbf{Y}) \end{pmatrix}.$$

To say that $\mathbf{G}$ is differentiable means (definition of differentiable) that for each vector $\mathbf{Y_0}$ there is an n by n matrix $\mathbf{M}$ (which varies from $\mathbf{Y_0}$ to $\mathbf{Y_0}$ just like the usual derivative does) that can be used to produce a good approximation to $\mathbf{G}$ near $\mathbf{Y_0}$. Specifically, if $\mathbf{h}$ is a small vector then

$$\mathbf{G}(\mathbf{Y_0} + \mathbf{h}) \;\approx\; \mathbf{G}(\mathbf{Y_0}) \;+\; \mathbf{M}\,\mathbf{h}.$$

The entries of M can be calculated as partial derivatives given that the entries of $\mathbf{G}$ are sufficiently smooth. It is called the **Jacobian matrix** for $\mathbf{G}$ at $\mathbf{Y_0}$. In multivariable calculus classes you find that

$$\mathbf{M} = \begin{pmatrix} \frac{\partial G_1}{\partial Y_1}(\mathbf{Y_0}) & \frac{\partial G_1}{\partial Y_2}(\mathbf{Y_0}) & \cdots & \frac{\partial G_1}{\partial Y_n}(\mathbf{Y_0}) \\ \frac{\partial G_2}{\partial Y_1}(\mathbf{Y_0}) & \frac{\partial G_2}{\partial Y_2}(\mathbf{Y_0}) & \cdots & \frac{\partial G_2}{\partial Y_n}(\mathbf{Y_0}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial G_n}{\partial Y_1}(\mathbf{Y_0}) & \frac{\partial G_n}{\partial Y_2}(\mathbf{Y_0}) & \cdots & \frac{\partial G_n}{\partial Y_n}(\mathbf{Y_0}) \end{pmatrix}.$$

Now suppose given the IVP

$$\mathbf{Y}' = \mathbf{G}(\mathbf{Y}) \text{ and } \mathbf{Y}(\mathbf{0}) \text{ is a chosen vector sufficiently near to } \mathbf{Y_0}.$$

The idea in this section is to approximate the solution to this IVP by $\mathbf{X} + \mathbf{Y_0}$ where $\mathbf{X}$ is a solution to the *linear* IVP

$$\mathbf{X}' = \mathbf{G}(\mathbf{Y_0}) + \mathbf{M}\mathbf{X} \text{ and } \mathbf{X}(\mathbf{0}) = \mathbf{Y}(\mathbf{0}) - \mathbf{Y_0} \text{ is near } \mathbf{0}.$$

The point is, we have practice with solving equations of this type, and as long as we are near $\mathbf{Y_0}$ solutions to this linearized equation must change in a way *that is very close to the way the solutions we actually want* change. We reiterate that far away from $\mathbf{Y_0}$ the solutions will **not be similar at all,** necessarily. The most useful points $\mathbf{Y_0}$ to consider are the equilibrium points of the original DE because the approximation techniques of the last section don't work well there.

Let's consider a couple of examples, following the text presentation of this subject. First:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2x + 2x^2 \\ -3x + y + 3x^2 \end{pmatrix}.$$

The equilibrium points for this system, the places where $\mathbf{G}$ is the zero vector, are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The Jacobian matrix is

$$\mathbf{M} = \begin{pmatrix} -2 + 4x & 0 \\ -3 + 6x & 1 \end{pmatrix}.$$

Near the first equilibrium point, the solutions should behave similarly to the solutions of

$$\mathbf{X}' = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix} \mathbf{X}.$$

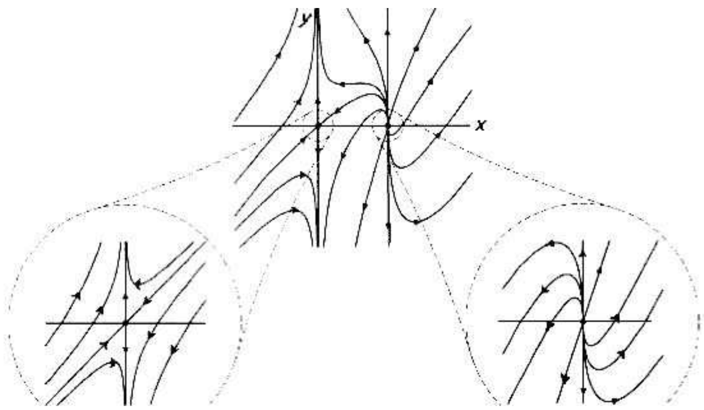
Near the second equilibrium point, the solutions should be close to

$$\mathbf{X} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\mathbf{X}$ is a solution to

$$\mathbf{X}' = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \mathbf{X}.$$

The picture in the text on page 448 shows a phase plane for the original system with a magnified look at the equilibrium points. The solutions to the linearized systems near the origin are, essentially, identical to the original system around each equilibrium point. I reproduce that picture below.



The second example is that of a pendulum with a rigid "wire." A pendulum in a resistive medium, which creates a velocity dependent friction term, corresponds to

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -\alpha y - \sin(x) \end{pmatrix}$$

where x represents the angular displacement from its equilibrium position, hanging straight down, and y represents angular velocity. The parameter $\alpha \geq 0$ is a measure of the "stickiness" of the medium.

The Jacobian matrix for the pendulum is

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -\cos(x) & -\alpha \end{pmatrix}.$$

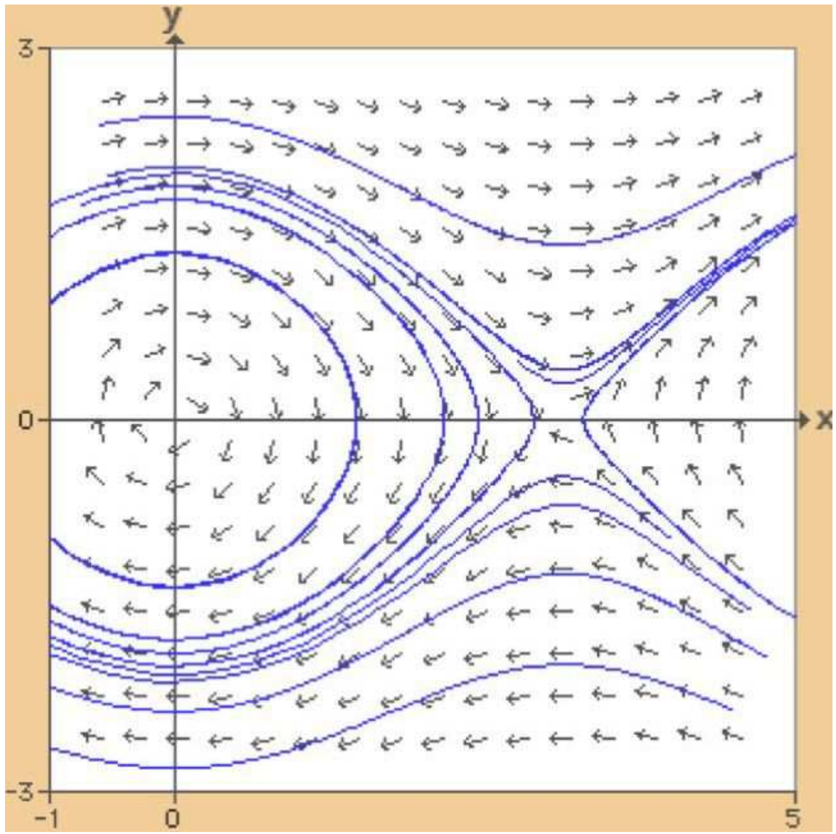
We consider here only the case of $\alpha = 0$: that is, no friction.

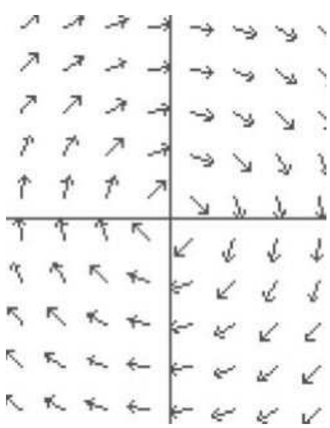
There are two types of equilibria for this system. First, a stable one such as the "initial-position-and-velocity-zero" solution. The second type is unstable such as the solution with velocity zero but the pendulum at the top of its arc, at angle π. The solutions near these two types of equilibria look like the solutions of one or the other of the linearized equations

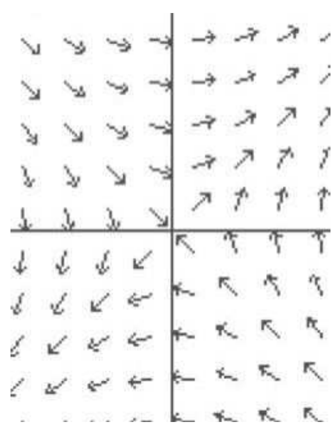
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

near the origin.

Find below a phase portrait of the nonlinearized system and then slope fields for the two linearized systems centered at the origin.







Much can be done with this kind of analysis, but that is for later classes.

3. Several Nonlinear DEs

In this section we consider examples of differential equations which occur in applications and which are not linear.

Bernouli's Equation is of the form, for $n \neq 1$,

$$\frac{dy}{dx} \ + \ P(x)\, y \ = \ Q(x)\, y^n.$$

This equation can be solved by using the substitution $u = y^{1-n}$ which will yield the linear equation

$$\frac{du}{dx} \ + \ (1-n)P(x)\, u \ = \ (1-n)Q(x).$$

The second DE is **Ricatti's Equation**, which is of the form

$$\frac{dy}{dx} \ = \ P(x) \ + \ Q(x)\, y \ + \ R(x)\, y^2.$$

Solution of this equation hinges on our ability to find or guess some particular solution y_1. Letting $y = u + y_1$ this reduces to

$$\frac{du}{dx} - (\, Q(x) + 2y_1 R(x)\,)\, u \ = \ R(x)\, u^2$$

which is a Bernouli Equation with $n = 2$.

Solutions of **Clairaut's Equation**

$$y \ = x\, \frac{dy}{dx} \ + \ g\!\left(\frac{dy}{dx}\right)$$

consist of the family of lines $y(x) = kx + g(k)$ together with (if g is differentiable) a solution given parametrically by

$$x(t) = -g'(t) \qquad y(t) = g(t) - t\, g'(t).$$

Lagrange's Equation is of the form,

$$y \ = x\, f\!\left(\frac{dy}{dx}\right) \ + \ g\!\left(\frac{dy}{dx}\right).$$

So Clairaut's Equation is a special case of the Lagrange's Equation.

Differentiate both sides of this equation with respect to x and use the substitution $u = \frac{dy}{dx}$. Solve for $\frac{dx}{du}$, yielding the linear equation

$$\frac{dx}{du} \ + \ \frac{f'(u)}{f(u) - u}\, x \ = \ \frac{g'(u)}{u - f(u)}.$$

Once $x(u)$ has been found and inverted to yield u as a function of x, the equation $y \ = x\, f\,(u(x)) \ + \ g\,(u(x))$ gives the solution y.

4. Laplace Transforms

Finally, we come to the third topic. This is a means of producing exact (not approximate) solutions to a considerable variety of initial value problems. In this situation, you transform a differential equation into an equation of entirely different type: an algebraic equation. The hope is that this one will be easier to understand. Then one must transform the solution back into the original framework. This is a part of a powerful and beautiful branch of mathematics, and we only touch the surface here.

First, if y is a piecewise continuous (that is, continuous except for a finite number of simple "jump" discontinuities) real valued function defined on $[0, \infty)$ we define for real numbers s

$$\mathcal{L}[y](s) = \int_0^\infty y(t)\, e^{-st} dt = \lim_{M\to\infty} \int_0^M y(t)\, e^{-st} dt$$

when the integral exists. It is possible that $\mathcal{L}[y](s)$ might fail to exist for certain, or possibly all, values of s. **However, if the integral exists for some $\mathbf{s_0}$ then the integral exists for all s in the interval** $[\mathbf{s_0}, \infty)$. (Can you show this?)

If y is given as a formula $f(t)$ involving t it is convenient to write $\mathcal{L}[y] = \mathcal{L}[f(t)]$ even though the dummy variable t is not actually involved in the final transform. It is also common to use the notation $\widehat{y}$ (read this "y hat") in place of $\mathcal{L}[y]$.

A function f defined on the interval $[0, \infty)$ is said to be of **exponential order** if there is a positive number s for which $\lim_{t\to\infty} f(t)e^{-st} = 0$. Polynomials, simple exponentials, rational functions and linear combinations of sines and cosines are all of exponential order. Sums and products of functions of exponential order are of exponential order. An example of a function that grows too fast to be of exponential order is $g(t) = e^{t^2}$.

The reason we are interested in these functions is because $\mathcal{L}[y](s)$ **exists if y is piecewise continuous functions of exponential order, for sufficiently large s**.

$\mathcal{L}$ is a function that operates on a certain family of functions containing, at least, all piecewise continuous functions of exponential order on $[0, \infty)$ and gives as output another function. $\mathcal{L}$ is called the **Laplace transform** and $\mathcal{L}[y]$ is called **the Laplace transform of the function y**. There are many other transforms too (some very simple, some more complicated) but the Laplace transform is customarily the first one people think about in this way.

$\mathcal{L}$ **is linear on these functions.** In this context that means

$$\mathcal{L}[cy + x](s) = c\mathcal{L}[y](s) + \mathcal{L}[x](s)$$

for functions x and y and constants c, at least for those s at which all integrals involved are defined.

It is also true (though it is plausible, we do not prove this) that if y is piecewise continuous and $\mathcal{L}[y](s) = 0$ for all $s \in [s_0, \infty)$ for any real s_0 then y itself must have been the zero function except at a finite number of points.

Coupled with linearity from above we then find that $\mathcal{L}$ **is "essentially" one-to-one on piecewise continuous continuous functions of exponential order**: $\mathcal{L}[x] = \mathcal{L}[y]$ implies $x(t) = y(t)$ except, possibly, at a finite number of values of t.

Here are examples of Laplace transforms, and it is a good exercise to verify the integration steps to prove it.

3.1 Let $y(t) = e^{at}$. Then $\int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt$. If $s \le a$ then this integral does not exist. But if $s > a$ then this integral is $\dfrac{1}{s-a}$.

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \text{ for } s > a.$$

3.2 $\mathcal{L}[\sin(\omega t)](s) = \dfrac{\omega}{s^2+\omega^2}$.

3.3 $\mathcal{L}[\cos(\omega t)](s) = \dfrac{s}{s^2+\omega^2}$.

3.4 There is a fact that is sometimes useful in calculating the transform of a periodic function. Suppose y is of exponential order and periodic on $[0,\infty)$ with period T, and also piecewise continuous on the interval $[0,T)$. Then

$$\begin{aligned}
\mathcal{L}[y](s) &= \int_0^\infty y(t)\, e^{-st}dt = \sum_{n=0}^{\infty} \int_{nT}^{nT+T} y(t)\, e^{-st}dt \\
&= \sum_{n=0}^{\infty} \int_0^T y(u+nT)\, e^{-s(u+nT)}du \qquad \text{(use the substitution } u = t - nT\text{)} \\
&= \sum_{n=0}^{\infty} e^{-snT} \int_0^T y(u)\, e^{-su}du \qquad \text{(periodicity of } y\text{)} \\
&= \frac{1}{1-e^{-sT}} \int_0^T y(u)\, e^{-su}du.
\end{aligned}$$

So you only need to calculate this integral over a single period (and include the exponential factor) to find the transform of a periodic function.

3.5 For $a \geq 0$ let u_a denote the discontinuous function

$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Let $T_a(t) = t + a$. Show (use a change of variable in the integral) that if f is a function which has a transform (so the line below makes sense) then

$$\mathcal{L}[f \circ T_a](s) = e^{sa}\mathcal{L}[u_a f](s).$$

In other words, multiplying $\mathcal{L}[u_a f]$ by e^{sa} on the transform side corresponds to a left-shift by a on the domain side. Turning this around (let $g = f \circ T_a$) this can be rephrased as:

$$e^{-sa}\mathcal{L}[g](s) = \mathcal{L}[u_a\, g \circ T_{-a}](s).$$

Formerly we had no easy way of working with DEs that involved discontinuous functions. An example might be a forced harmonic motion with discontinuous forcing term, corresponding to a switch turned on to drive the system and then, after a while, flipped off again. This last observation will be the key to handling these situations efficiently.

There are many more transforms of functions worked out and assembled in books of tables (or in computer mathematics programs such as Maple or Mathematica) which are commonly used when faced with an atypical function whose transform you require.

3.6 $\mathcal{L}[ty(t)] = -\frac{d}{ds}\widehat{y}$. The derivative of the Laplace transform of y is the negative of the transform of the function given by the formula $ty(t)$.

To see this examine

$$\frac{d}{ds}\widehat{y}(s) = \frac{d}{ds}\int_0^\infty y(t)e^{-st}dt = \int_0^\infty \frac{d}{ds}y(t)e^{-st}dt = \int_0^\infty -ty(t)e^{-st}dt.$$

Can you justify the second equality in the line above?

3.7 We now consider the fact that makes this transform particularly useful for us. Integration by parts gives

$$\begin{aligned}\mathcal{L}\left[\frac{dy}{dt}\right](s) &= \lim_{M\to\infty}\int_0^M \frac{dy}{dt}e^{-st}dt \\ &= \lim_{M\to\infty} e^{-st}y(t)\Big|_{t=0}^{t=M} + s\int_0^M ye^{-st}dt \\ &= -\,y(0) + s\,\mathcal{L}[y](s)\end{aligned}$$

provided the transforms $\mathcal{L}\left[\frac{dy}{dt}\right](s)$ and $\mathcal{L}[y](s)$ exist.

More concisely (but be careful with the derivative: it is with respect to t!),

$$\widehat{y'}(s) \;=\; -y(0) + s\,\widehat{y}(s).$$

3.8 There is an integral form of this last result. If y has Laplace transform $\widehat{y}$ and antiderivative $Y(t) = \int_0^t y(u)\,du$ then $\widehat{Y}(s) \;=\; \frac{1}{s}\widehat{y}(s)$.

So here is the important point: **Laplace transform converts differentiation of functions on its domain side into multiplication by the variable on the transform side.** A *differential* equation is converted into an *algebraic* equation involving the transform of the function we want.

3.9 Suppose both x and y are piecewise continuous and of exponential order. Define $x\star y$ by $(x\star y)(t) = \int_0^t x(u)y(t-u)\,du$. This function is called **the convolution** of x and y. The convolution operation is commutative: that is, $x\star y = y\star x$. It is also true that

$$\widehat{x\star y} = \widehat{x}\,\widehat{y}.$$

To prove this, write the right side as a multiple integral, justified by the exponential order and continuity properties of x and y.

So if we have a function of s and recognize it as the product of the transforms of known functions, then this product is the transform of the convolution of the two known functions.

$$\frac{s}{(s^2+4)^2} = \frac{1}{2}\,\frac{s}{(s^2+4)}\,\frac{2}{(s^2+4)}$$

so we recognize the first fraction as one half of the transform of the convolution of $\sin(2t)$ with $\cos(2t)$.

Here is an example of how we use Laplace transforms in this class, once again taken from our text.

Consider the IVP

$$\frac{dy}{dt} = y - 4e^{-t} \quad \text{and} \quad y(0) = 1.$$

Taking Laplace transforms of both sides of the IVP produces

$$s\widehat{y} - 1 = \widehat{y} - 4\mathcal{L}[e^{-t}] = \widehat{y} - \frac{4}{s+1}.$$

Isolating terms involving $\widehat{y}$ yields

$$(s-1)\widehat{y} = 1 - \frac{4}{s+1} = \frac{s-3}{s+1}$$

which yields (partial fraction decomposition)

$$\widehat{y} = \frac{s-3}{(s+1)(s-1)} = \frac{2}{s+1} - \frac{1}{s-1}.$$

Using the inverse transform, we find

$$y = 2e^{-t} - e^{t}.$$

To reiterate, there is a wide variety of functions whose Laplace transforms are known, including the transforms of discontinuous functions, such as "jump functions" defined piecewise. If an IVP generates, on the transform side, an algebraic solution which is a linear combination of known transforms we have found an exact solution of the original problem.

Here is an example of a simple spring IVP with a discontinuous forcing function.

$$y'' + y = u_5, \qquad y(0) = 0, \qquad y'(0) = 2.$$

Common sense suggests that before time 5 we will have simple harmonic motion, and after time 5 there will be some kind of oscillation centered to the right of the origin by some amount.

$$\widehat{y''} = s\widehat{y'} - y'(0) = s\widehat{y'} - 2.$$

Since $\widehat{y'} = s\widehat{y} - y(0) = s\widehat{y}$ this gives a transformed IVP of

$$s^2\widehat{y} - 2 + \widehat{y} = \frac{e^{-5s}}{s}$$

Isolating $\widehat{y}$ and using partial fraction decomposition we find that

$$\widehat{y} = \frac{2}{s^2+1} + e^{-5s}\frac{1}{s} - e^{-5s}\frac{s}{s^2+1}.$$

Looking at the little table we have assembled we see that y is given by

$$\begin{aligned} y(t) &= 2\sin(t) + u_5(t) - u_5(t)\cos(t-5) \\ &= \begin{cases} 2\sin(t), & \text{if } t < 5; \\ 2\sin(t) + 1 - \cos(t-5), & \text{if } t \geq 5. \end{cases} \end{aligned}$$

Now we will find a particular solution to an undamped oscillator with resonant driver, a situation whose solution required the "guess and check" technique described in the text. Our IVP is

$$y'' + 2\,y = \cos(\sqrt{2}t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

The Laplace transform of this equation with these initial conditions is

$$s^2\widehat{y} + 2\widehat{y} = \frac{s}{s^2+2} \quad \Longrightarrow \quad \widehat{y} = \frac{s}{(s^2+2)^2} = \frac{d}{ds}\,\frac{-1}{2(s^2+2)} = \frac{1}{2\sqrt{2}}\,\frac{d}{ds}\,\frac{-\sqrt{2}}{s^2+2}.$$

Using our rules from above we have the particular solution $y(t) = \frac{1}{2\sqrt{2}}\, t\, \sin(\sqrt{2}\, t)$.

As you can see, there is considerable technique involved in solving problems this way, but it allows us to solve certain types of problems routinely that are awkward to handle by other means. We can *find* the solutions, and are not dependent on “guess-and-check” as used earlier in the text. And the basic idea, that of transforming an equation (a *differential* equation) whose solution is required in one function space to a different type of equation (an *algebraic* equation) in a different function space, is both conceptually beautiful and has widespread applicability.

Have Fun In Your Later Math Classes!