UNCONVENTIONAL SERIES CONVERGENCE

LARRY SUSANKA

> "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."
>
> Niels Henrik Abel (1828)[1]

Abel may not have been entirely correct, though it is unclear which portion of his statement is problematic. We consider the matter in this note.

As you may be aware, convergence of series is defined in terms of the limit of the sequence of partial sums. It is, of course, *not* the sum of an infinite number of terms. Addition is a binary operation. You can only add things two at a time. There is no way to perform an infinite number of operations in arithmetic.

Sometimes people assert that the "value" of such a limit is ∞ or $-\infty$ but this is really just additional information about how the limit fails to converge.

In an absolutely convergent series, we learn that the terms can be rearranged any way we like and the new series will yield the same limit.

In a conditionally convergent series this is not true: rearrangement of such series can lead to new series combining the same terms in different order and which converge to any number at all, or fail to converge in various ways if you prefer.

Some students find this to be surprising or even alarming. This "infinite sum" yields a different "answer" *depending on how the exact same summands are listed and combined* and people are warned, thereby, that intuition based on finite sums can lead one astray.

But other ways of combining an infinite number of terms are possible.

For instance we can take averages of the list of partial sums. Or we can use power series methods and meditate upon convergence on the boundary of the disk of convergence of such series.

Or we can take the individual terms and *split them into pieces and combine these with pieces from other terms* to form new series arranged so, ultimately, all the bits are accounted for; no part of any term will be "left uncounted" somewhere. *We get to choose the order in which we will add these pieces to the rest.*

By doing this in specified ways we can create limits for what were formerly series that failed to converge such as

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots .$$

Date: May 1, 2017.

[1]This comment by Abel and several other facts mentioned here are extracted from G.H. Hardy's delightful "Divergent Series."

In one of these other contexts the sum, written as above, is misleading, inviting the observer to add the numbers in order as one would with ordinary series, and of course that procedure fails to converge. *But that is not the procedure used to form this "sum."*

Before embarking on our main discussion, I would like to discuss the two other approaches to dealing with some series of this kind, mentioned above.

First is Cesáro summation, a process that takes the average of the first n partial sums of a sequence and then takes the limit of this average as $n \to \infty$.

In the case of the divergent series listed above the sequence of partial sums is

$$(1, 0, 1, 0, 1, \dots)$$

and the Cesáro means of these sums converge to $1/2$. The original sequence is said to be Cesáro summable by virtue of that fact.

Another approach was used by Euler himself, who had no qualms about dealing with divergent series. Examine

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

For real r near -1 but slightly smaller in magnitude the left side looks very much like our series, and the right side is (nearly) $1/2$.

We have to be careful with power series here. For $|r| < 1$ we have

$$\frac{1+r}{1+r+r^2} = 1 - r^2 + r^3 - r^5 + r^6 - r^8 + r^9 - \cdots .$$

which at first blush *seems* to imply that

$$\frac{2}{3} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots .$$

The *actual* series given by that power series (with missing powers included with zero coefficients) is

$$\frac{2}{3} = 1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + 1 - \cdots$$

which corresponds to the "sum" of an entirely distinct sequence of values.

Substitute $2t$ for r in $1 + r + r^2 + r^3 + \cdots$ and you have

$$1 + 2t + 4t^2 + 8t^3 + \cdots = \frac{1}{1-2t} \quad \text{and so}$$

$$1 + 2 + 4 + 8 + \cdots = \frac{1}{1-2} = -1 \quad \text{and} \quad 1 - 2 + 4 - 8 + \cdots = \frac{1}{1+2} = \frac{1}{3}.$$

If you have trouble believing that, try (for example) letting

$$\alpha = 1 - 2 + 4 - 8 + \cdots = 1 - 2(1 - 2 + 4 - 8 + \cdots) = 1 - 2\alpha$$

from which $\alpha = 1/3$. Or

$$\alpha = 1 + 2 + 4 + 8 + \cdots = 1 + 2(1 + 2 + 4 + 8 + \cdots) = 1 + 2\alpha$$

from which, again, we find $\alpha = -1$. Nota bene: *These terms are all positive.* For that reason we may prefer to regard this as implying that $\alpha = \infty$.

Whether or not sense can be made out of this, there are some things that cannot be done. For instance

$$\alpha = 1 + 0 + 1 + 0 + 1 + \cdots = 1 + 0 + (1 + 0 + 1 + 0 + \cdots) = 1 + \alpha$$

is contradictory or, if you prefer, requires $\alpha = \infty$. It is evident that any sequence composed of a repeating segment whose sum is nonzero will produce the same contradiction/unboundedness conundrum. Thank heavens there are *some* boundaries to all this!

By similar argument, it follows that if we can create any consistent means of defining "sums" of sequences, and if the real number $\beta = 1 - 1 + 1 - 1 + \cdots$ and *if that method is linear in the defining sequences and agrees with ordinary summation for finite sequences* then $\beta = 1 - \beta$ so the value of β *must* be $1/2$.

And then for that method if

$$\gamma = 1 - 2 + 3 - 4 + 5 - \cdots$$

we have

$$\gamma = (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - 4 + \cdots) = \beta - \gamma.$$

So $2\gamma = 1/2$ and then $\gamma = 1/4$.

I am not asserting at this point in the note that all, or even *any*, of these things can be done.

I am simply claiming that *if* there is a means of assigning a number to a sequence such as $(1, 2, 4, 8, \ldots)$ and if that procedure is linear on a vector space of sequences and *if* that method agrees with summation on finite sequences *then the number assigned cannot be anything but* -1. And no real number can be assigned to $1 + 0 + 1 + 0 + \cdots$ by this method at all.

And if I had some ham I could have ham and eggs if I had some eggs.

There are thriving communities of Mathematicians who are charmed and intrigued by this sort of thing.

The type of manipulation mentioned above, yielding the only possible value for one of these sums (or ruling out the possibility of a numerical assignment) is important for us because the methods we discuss below *are* linear in the defining sequence and *do* agree with ordinary summation for finite sums, so we can play this game among sequences for which our definition applies.

If it was good enough for Euler ... who are we to argue?

But let's organize things more precisely and see where it gets us.

Let $\mathbb{R}^{\mathbb{N}}$ denote the real-valued sequences, the set of all functions of the form

$$x\colon \mathbb{N} \to \mathbb{R} \quad \text{sometimes denoted} \quad (x_0, x_1, x_2, \ldots).$$

$\mathbb{R}^{\mathbb{N}}$ is a vector space with the obvious operations.

The first thing you do with such a sequence in Calculus class is talk about its limit, if it has one. We let $\mathcal{A}$ denote the set of sequences with (finite) limits and define

$$\mathcal{L}\colon \mathcal{A} \to \mathbb{R} \quad \text{by} \quad \mathcal{L}(x) = \lim_{n\to\infty} x_n.$$

This limit, when it exists, is the height of the "horizontal asymptote" of the graph of x.

$\mathcal{L}$ is linear and has other useful properties. Perhaps the most useful is that

$$\mathcal{L}(z) = f(\mathcal{L}(x))$$

when sequence z is defined by $z_i = f(x_i)$ and f is a real-valued function which is continuous at $\mathcal{L}(x)$.

We now want to bring series into the mix.

Define

$$\mathcal{S} \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$$

to be the sequence formed from partial sums

$$\left(x_0,\ x_0 + x_1,\ x_0 + x_1 + x_2,\ \dots,\ \sum_{i=0}^{n} x_i,\ \dots \right).$$

$\mathcal{S}$ is linear from the vector space $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$ and it is not hard to show that $\mathcal{S}$ is both one-to-one and onto $\mathbb{R}^{\mathbb{N}}$.

The nth term $\mathcal{S}(x)_n$ of $\mathcal{S}(x)$ is the number $\sum_{i=0}^{n} x_i$.

We say that the series formed from x converges provided $\mathcal{S}(x) \in \mathcal{A}$.

And when

$$\lim_{n\to\infty} \mathcal{S}(x)_n = \lim_{n\to\infty} \sum_{i=0}^{n} x_i$$

exists we often indicate the limit

$$\mathcal{S}er(x) = \mathcal{L}(\mathcal{S}(x)) = \mathcal{L} \circ \mathcal{S}(x) \quad \text{by} \quad \sum_{i=0}^{\infty} x_i$$

and refer to it, colloquially, as a "sum."

We emphasize the obvious: unless sequence x has only finitely many nonzero terms $\mathcal{S}er(x)$ *is not actually a sum.* It is a limit of sums, and if you insist on thinking of it as a sum you should expect to see behavior, from time to time, that runs counter to normal intuition about sums.

For notational convenience, when $j > 0$ a series limit such as $\sum_{i=j}^{\infty} x_i$ is intended to denote the limit of $\mathcal{S}$ applied to the sequence $(0, \dots, 0, x_j, x_{j+1}, \dots)$ for an initial segment of j zeroes.

And when $x = (x_0, x_1, \dots, x_n, 0, 0, \dots)$ has only finitely many nonzero entries we say that x is a finite sequence.

Let $\mathcal{A}_{\mathcal{S}er} = \mathcal{S}^{-1}(\mathcal{A})$.

$$\mathcal{S}er : \mathcal{A}_{\mathcal{S}er} \to \mathbb{R}$$

is the map that takes x to the limit of its series: that is,

$$\mathcal{S}er(x) = \lim_{n\to\infty} \mathcal{S}(x)_n = \sum_{i=0}^{\infty} x_i.$$

It is easy to show that $\mathcal{A}_{\mathcal{S}er} \subset \mathcal{A}$ since if $x \in \mathcal{A}_{\mathcal{S}er}$ then $\mathcal{L}(x) = 0$.

$\mathcal{S}er$ has many nice properties such as linearity, and one proves all the standard theorems about $\mathcal{S}er$ on members of $\mathcal{A}_{\mathcal{S}er}$ and explores types of sequences in $\mathcal{A}_{\mathcal{S}er}$, such as those leading to conditionally or absolutely convergent series and so on.

And $\mathcal{S}er(x)$ is the sum of the entries of x when x is a finite sequence.

Here is a specific example. We seek to evaluate for positive integer j

$$\sum_{n=j}^{\infty} \frac{1}{(n+1)\cdot n} = \lim_{k\to\infty} \sum_{n=j}^{k} \frac{1}{(n+1)\cdot n} = \lim_{k\to\infty} \sum_{n=j}^{k} \frac{1}{n} - \frac{1}{n+1}$$
$$\text{which telescopes to} \quad \lim_{k\to\infty} \frac{1}{j} - \frac{1}{k+1} = \frac{1}{j}.$$

There are many members of $\mathbb{R}^{\mathbb{N}}$ not in $\mathcal{A}_{\mathcal{S}er}$ and we would like to create a way of "summing" these sequences too, or at least some of them, using a method that will agree with $\mathcal{S}er$ for members of $\mathcal{A}_{\mathcal{S}er}$ if we can.

In other words we are hoping to extend the linear map $\mathcal{S}er$ to a larger domain.

Let's define the "mean map"

$$\mathcal{M}\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$$

by sending $x = (x_0, x_1, x_2, \dots)$ to

$$\mathcal{M}(x) = \left(x_0,\ \frac{x_0+x_1}{2},\ \frac{x_0+x_1+x_2}{3}, \dots,\ \frac{x_0+x_1+\cdots+x_n}{n+1}\ \cdots \right).$$

The map $\mathcal{M}$ is one-to-one and onto $\mathbb{R}^{\mathbb{N}}$ and also linear.

If x has a limit (i.e. $x \in \mathcal{A}$) then certainly $\mathcal{M}(x)$ has a limit too, in fact the same limit as has x. But it is far easier for $\mathcal{M}(x)$ to have a limit than for x itself to have a limit. The *averages* of the sequence values must converge, and averaging smooths out variations in the terms being averaged.

It follows that it is far easier for the sequence of Cesáro means

$$\mathcal{M}(\mathcal{S}(x)) = \left(\mathcal{S}(x)_0,\ \frac{\mathcal{S}(x)_0 + \mathcal{S}(x)_1}{2},\ \frac{\mathcal{S}(x)_0 + \mathcal{S}(x)_1 + \mathcal{S}(x)_2}{3}, \dots\dots \right)$$

to converge than for $\mathcal{S}(x)$ itself to converge. But definitely if $x \in \mathcal{A}_{\mathcal{S}er}$ then $\mathcal{M}\circ\mathcal{S}(x)$ will converge, and to the same limit as does $\mathcal{S}(x)$ itself.

As a reminder, we saw above that $(1, -1, 1, -1, \dots)$ is Cesáro summable to $1/2$.

So $\mathcal{C}es = \mathcal{L} \circ \mathcal{M} \circ \mathcal{S}$ is a genuine linear extension of $\mathcal{S}er = \mathcal{L} \circ \mathcal{S}$ to the Cesáro summable sequences, the vector subspace $\mathcal{A}_{\mathcal{C}es}$ of $\mathbb{R}^{\mathbb{N}}$.

$$\mathcal{C}es\colon \mathcal{A}_{\mathcal{C}es} \to \mathbb{R} \quad \text{and} \quad \mathcal{A}_{\mathcal{S}er} \subset \mathcal{A}_{\mathcal{C}es} \quad \text{and} \quad \mathcal{C}es(x) = \mathcal{S}er(x)\ \forall x \in \mathcal{A}_{\mathcal{S}er}.$$

Now let's try a different approach.

Given $x \in \mathbb{R}^{\mathbb{N}}$ we define $\mathcal{T}(x)$ to be the sequence y given by

$$\begin{array}{rcccccccc}
y_0 & = & x_0 & & & & & & \\
y_1 & = & \frac{1}{2}x_1 & & & & & & \\
y_2 & = & \frac{1}{3\cdot 2}x_1 & + & \frac{2}{3\cdot 2}x_2 & & & & \\
y_3 & = & \frac{1}{4\cdot 3}x_1 & + & \frac{2}{4\cdot 3}x_2 & + & \frac{3}{4\cdot 3}x_3 & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
y_n & = & \frac{1}{(n+1)\cdot n}x_1 & + & \frac{2}{(n+1)\cdot n}x_2 & + & \frac{3}{(n+1)\cdot n}x_3 & +\cdots+ & \frac{n}{(n+1)\cdot n}x_n \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The series formed from any column of this triangle "sums" to one of the x_j for each j. That is simply another way of saying that all the "bits" of each x_j are present (exactly once!) somewhere in this big triangle.

But what happens if we form the series from the rows? Each row sum is finite so you might have to go a *long* way down the sequence y before any particular piece of x_2 is included, not to mention a particular piece of x_3 or the other x_i.

Still, all the pieces of all the x_i are *eventually included somewhere* and one has the feeling that $\mathcal{L}\circ\mathcal{S}\circ\mathcal{T}(x)$ *should* equal $\mathcal{S}er(x) = \mathcal{L}\circ\mathcal{S}(x)$. But does it?

The row series created from $y = \mathcal{S}\circ\mathcal{T}(x)$ may be represented generally as

$$\begin{aligned}
x_0 + &\frac{1}{2}\,x_1 + \frac{1}{(3)(2)}(x_1 + 2\,x_2) \\
&+ \frac{1}{(4)(3)}(x_1 + 2\,x_2 + 3\,x_3) + \frac{1}{(5)(4)}(x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4) \\
&+ \cdots + \frac{1}{(n+1)\,n}(x_1 + 2\,x_2 + \cdots + n\,x_n) + \cdots
\end{aligned}$$

We want to understand the relationship between the two series

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty}\frac{i}{j(j+1)}\,x_i\right) \quad \text{and} \quad \sum_{i=1}^{\infty} y_i = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{j}\frac{i}{j(j+1)}\,x_i\right).$$

Fubini's Theorem with counting measure on $\mathbb{N}$ gives equality when the series formed from x is absolutely convergent.

There is a pattern to the sequence of partial sums of $\mathcal{T}(x)$, the entries of $\mathcal{S}\circ\mathcal{T}(x)$. For convenience we suppose $x_0 = 0$.

First sum: $\frac{1}{2}\,x_1$. The second sum is $\frac{1}{2}\,x_1 + \frac{1}{(3)(2)}(x_1 + 2\,x_2) = \frac{1}{3}(2\,x_1 + x_2)$.

Suppose we have shown that the sum of the first n terms has the form

$$\frac{1}{n+1}(n\,x_1 + (n-1)\,x_2 + \cdots + (n+1-k)\,x_k + \cdots + x_n).$$

In that case the sum of the first $n+1$ terms is

$$
\begin{aligned}
&\frac{1}{n+1}(n\,x_1 + (n-1)\,x_2 + \cdots + (n+1-k)\,x_k + \cdots + x_n) \\
&\quad + \frac{1}{(n+2)(n+1)}(x_1 + 2\,x_2 + \cdots k\,x_k + \cdots + n\,x_n + (n+1)\,x_{n+1}) \\
&\quad = \frac{1}{(n+2)(n+1)}([(n+2)n+1]\,x_1 + [(n+2)(n-1)+2]\,x_2 \\
&\qquad\qquad + \cdots + [(n+2)(n+1-k)+k]\,x_k \\
&\qquad\qquad\quad + \cdots + [(n+2)+n]\,x_n + (n+1)\,x_{n+1}) \\
&\quad = \frac{1}{(n+2)(n+1)}((n+1)^2\,x_1 + (n+1)n\,x_2 \\
&\qquad\qquad + \cdots + (n+1)(n+2-k)\,x_k \\
&\qquad\qquad\quad + \cdots + (n+1)2\,x_n + (n+1)\,x_{n+1}) \\
&\quad = \frac{1}{(n+2)}((n+1)\,x_1 + n\,x_2 + \cdots + (n+1+1-k)\,x_k \\
&\qquad\qquad\quad + \cdots + 2\,x_n + x_{n+1})
\end{aligned}
$$

Therefore the formula presumed for the nth partial sum of $\mathcal{T}(x)$ does hold for all n:

$$
\begin{aligned}
\mathcal{S} \circ \mathcal{T}(x)_n &= \frac{1}{n+1}(n\,x_1 + (n-1)\,x_2 + \cdots + (n+1-k)\,x_k + \cdots + x_n) \\
&= \frac{1}{n+1}(\mathcal{S}(x)_1 + \mathcal{S}(x)_2 + \cdots + \mathcal{S}(x)_n).
\end{aligned}
$$

So (possibly surprisingly) convergence[2] of the series formed from y is equivalent to convergence of the Cesáro means of x, and these limits coincide.

So in the end we have discovered one way (in two different forms) of creating a linear map on sequences that agrees with ordinary partial sum limits when those limits exist and that is a proper extension of the ordinary summation.

With this definition the sequence $x_i = (-1)^i$ for all i sums to $1/2$.

We also conclude that there are some sequences such as $x_i = 1$ for all i that cannot be assigned a real number by any such process.

We have not ruled out the possibility that there could be a process by which the sequence such as $x_i = 2^i$ for all i could be assigned a real number, but we have concluded that *if* it is possible then that number must be -1, which certainly calls into question our intuition about how an extension of summability should behave.

[2]Thanks to John Baxter for pointing out that second equality.