Vectors For
Trigonometry
October 8, 2004

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Contents

CHAPTER I

Basic Vector Facts October 8, 2004

1. First Steps

A **vector** is an object completely characterized by two quantities, which we call **magnitude** and **direction**. The physical meaning of these quantities in an application of vectors comes from experience and varies from application to application.

Vectors can be represented as **arrows** with the direction given by the "tail-to-tip" direction of the arrow and the magnitude given by its length.

We take the point of view that two arrows located anywhere are merely **instances** of the "same" vector so long as each has the same length and direction. So the arrow to the left represents the "same" vector as the one on the right above, even though it is located at a different place.

You have plenty of experience with "fuzzing out" the distinctions among things which are manifestly different but which exhibit similarities upon which we wish to focus. For example, the fractions $3/4$ and $6/8$ represent different ideas. In the first, you break the "whole" into 4 equal pieces, and you have 3 of them. In the second, you break the "whole" into 8 equal pieces, and you have 6 of them. With these differences, there is something important that is similar about these two fractions: namely, I am just as full if I eat $3/4$ of a pizza or if I eat $6/8$ of a pizza. We choose to focus on that and we say $3/4 = 6/8$. We gather together all the fractions "equal" to $3/4$ and refer to the entire pile of them by picking any convenient representative, such as that in lowest terms or with some specified denominator.

Similarly, the arrows above are different on the page, but by picking one of them we refer to both, and any other arrow with the same magnitude and direction as well!
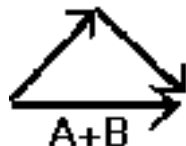
Two vectors $A =$ and $B =$

are added by finding the copy of B which has its tail on the nose of a copy of A. The sum $A + B$ is the arrow that starts with its tail at the tail of this copy of A and ends with its tip at the tip of this copy of B.
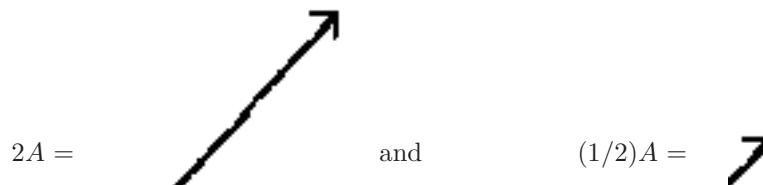
It is important to note the direction of $A + B$, in this case from left to right.

$-B$ is the vector that looks just like B but with tip and tail switched:

$$B = \qquad\qquad\qquad\qquad -B =$$

A positive constant k times a vector A is a new vector pointing in the **same direction as** A but with length stretched (if $k > 1$) or shrunk (if $k < 1$) by the factor k. Negative multiples of A are said to have **direction opposite to** A.

$$2A = \qquad\qquad\qquad \text{and} \qquad\qquad (1/2)A =$$

$A - B$ is defined to be $A + (-B)$. So $A - B$ is the vector on the left of the picture:

The process of adding two vectors is called **vector addition**. Multiplying a vector by a constant is called **scalar multiplication**.

The vector with zero magnitude is hard to represent as an arrow: it is called the **zero vector**, denoted 0. It doesn't really have a direction—or perhaps it has any direction. You pick. Context distinguishes it from the number 0.


1.1. ***Exercise***. *You should satisfy yourself that:*

$$A + B = B + A, \ A - A = 0 \ and \ that \ 2A = A + A.$$
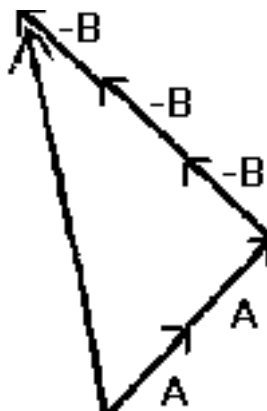
On the far left is a picture of the vector sum $2A - 3B$. A vector such as this, formed as a sum of multiples of vectors, is often referred to as a **resultant vector**. It is also called a **linear combination** of the vectors involved, in this case A and B.

1.2. **_Exercise_**. *Draw a picture of $3C - 2D$ and $C + \frac{1}{2}D$ where C and D are given by:*
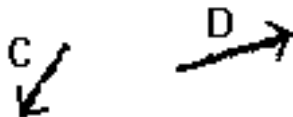
There are a number of things in the world with which you are no doubt familiar that are commonly represented as vectors, and you should think a bit about the meaning of "magnitude" and "direction" in each specific case.

- **Displacement**—a representation of a movement from a starting place to an ending place, with emphasis on the completed movement rather than how it occurred or the specific starting spot.
- **Velocity**—a description of motion, whose magnitude is the **speed** and whose direction "points the way." The velocity would be the displacement vector over one time unit, if the motion continued unchanged for the whole time unit.
- **Forces**—these describe "pushes" by one thing against another. A force is the cause of acceleration. If you see changes in the motion of something, it is because there is a force acting on that thing. No such changes require that the resultant of all forces have zero magnitude.
- A representation of a uniform **Wind** or **Current**—in the air or water. This example is tied to velocity. It can be interpreted as the velocity of a dust particle swept up and carried along by an unvarying wind or current.

Why should vectors describe faithfully these categories of real-world experiences? I don't know. It is a puzzle. They just do. It is ONLY through experience, conjecture bolstered by many experiments, that we (i.e. physicists, engineers, you, me) decide that vectors are a reasonable tool to try to describe something in the world. Mathematicians can tell you how vectors behave. Only you can decide if vectors mimic well some aspect of the world.

There is, obviously, overlap and relationships among the items listed above. Each will be useful, alone and in combinations.

There are a couple of points I would like to make before getting down to business.

First, each instance of a vector in the world actually occurs at some specific spot, and whenever an arrow is drawn it is drawn somewhere specifically. When we think of something in the world as a vector, we take the point of view that any specific representative refers not only to itself but to all others with the same magnitude and direction too. When you refer to 7/3 you are often making a statement about 21/9 at the same time even without specifically mentioning that second fraction.

Second, a given push (a force) is a real thing that exists however we decide to describe it. The wind is just whatever it is and doesn't need us to tell it that it is 30 miles per hour from the North. A displacement across a room is a real

thing, in itself. But in physics and other classes we try to describe things, often using mathematics and numbers. This association always involves a huge pile of assumptions including, for example, a choice of a distance unit, a time unit, an "origin," directions for coordinate axes and methods for measuring lengths and angles and the passage of time and some way of gauging the magnitude of a "push" and on and on. There is also a conceptual framework, frequently generated by the esthetic sensibilities of the creators of the model, which help us think about the measurements. It is not always clear which among the conceptual underpinnings are necessary, or even if they are consistent. Our description depends not only on the real thing, but on all these choices involved in a representation too.

When we go through this process of assembling a model we must never forget that the map is not the territory. A nickname for a thing is not the thing itself. The universe **names itself**, and whatever shorthand we use to describe part of it leaves out almost everything. In applications we must always be looking "out the window" to make sure the world is still answering to our nickname for it. It is astounding how often, over the last couple of centuries, it comes when we call. We must be doing something right.

1.3. **Exercise**. *This is an exercise designed to connect experiences people have had with the more abstract ideas we will be thinking about. It is best done in a group with the discussion guided by an instructor. Alternatively, it could be done by a single student accompanied by a good imagination. In each item below an activity will be described. After each activity there should be a discussion about what happened and possible alternative responses. The exercise will be repeated at the end of Section 4.*

A(i) The instructor picks a volunteer toward the back of the room. The instructor then moves about ten feet across the front of the room. The instructor then asks the volunteer to mimic that movement.
A(ii) The instructor moves ten feet in a different direction. Does this seem like "the same movement" as the first one?
A(iii) The instructor moves two feet in the same direction as the first movement. Does this seem like "the same movement" as the first one?

B(i) The instructor picks two volunteer who move toward an open area where everyone can see. The instructor gives each volunteer a heavy smooth ball, such as a billiard ball, and asks the first volunteer to roll it across the floor. The instructor asks the second volunteer to stand ready and, while the first ball is still rolling, mimic the motion of the first ball using his or her ball.
B(ii) The instructor rolls a ball with about the same speed but in a different direction. Does this seem like "the same motion" as the first one?
B(iii) The instructor rolls a ball in the same direction as the first motion but much faster. Does this seem like "the same motion" as the first one?

C(i) The instructor picks four volunteers, at least two of which claim to be physically durable. The instructor asks a volunteer to give "a medium sized push" to one of the durable volunteers. The instructor then asks another volunteer to mimic that push on a different durable volunteer.
C(ii) The instructor walks up to the original durable volunteer and gives a "medium

push" in a direction different from the first push. Does this seem like "the same push" as the first one?

C(iii) The instructor walks up to the original durable volunteer and gives a "very light push" in the same direction as the first push. Does this seem like "the same push" as the first one?

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2. Position Vectors and Constant Velocity Motion: Part 1

In this section we will use vectors to describe specific locations and then the path and motion of a "particle" traveling in a straight line with constant speed.

Vectors which are used to describe specific locations are called **position vectors**. Since vectors do not **have** specific locations, we need to create a "convention" for how we should interpret a vector thought of in this way.

The only thing we need to create the interpretation beyond the ideas of the first section of this chapter, is to agree on a "center of the universe." This will be a known and agreed upon place, usually called the **origin**. All other locations will be described by the displacement vector needed to get to the location from this origin.

To reiterate: a vector which is to be used to describe a particular spot is called a **position vector**. To use a vector this way, you must first decide upon a "center" point called the origin. The copy of the vector with its tail at this origin is said to be in **standard position**. When in standard position, the nose will "point" to the spot in the plane we are trying to identify.


With this idea in hand, we will use position vectors to describe **constant velocity motion** of an object—that is, an object moving in such a way that displacements during any two equal time intervals are the same.

You can think of a moving object as a "moving point of light" that leaves a trace (a burn mark or a smoke trail perhaps) after it has passed over a point. If that doesn't suit your fancy, you can think of it as a "moving **slug**" that leaves a **slime** trail after it has passed over a point. We will label points on the track by the time or times when the light (or slug) passes over the point. That is what people mean when they say the motion is **parameterized**, and the time in this case is called the **parameter**.

First, we remind the reader of the distinction between speed and velocity. Velocity is a vector that "points the way" of the motion. It is the displacement during one time unit. Its length is the speed.

We have thrown a new element into the mix here: to talk about velocity we must agree on an idea of **time** and a **means of measuring intervals of time.**

For specificity, let's think of our motion as describing the position of a constant velocity slug, with time measured in minutes from a moment we all agree is "time 0" and that the slug has constant velocity V. Each minute the slug moves the length of the velocity vector (the speed) in the direction of the velocity vector. If we find the copy of the velocity vector with tail at the slug starting place, the tip will be at the slug position after one minute of motion. If we only go for a fraction of a minute, or for many minutes, the slug position will be the tip of the appropriate multiple of the velocity vector when the tail of this vector is at the starting place. The displacement of the slug position over t minutes from **any** starting place will be the vector tV.
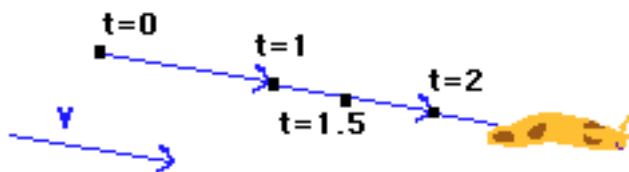

The starting time might have been chosen for convenience rather than the actual time the slug started moving. So negative times would merely refer to times before then. So to get to the location of the slug one minute before time 0 you would use a displacement vector $-V$ from its position at time 0.
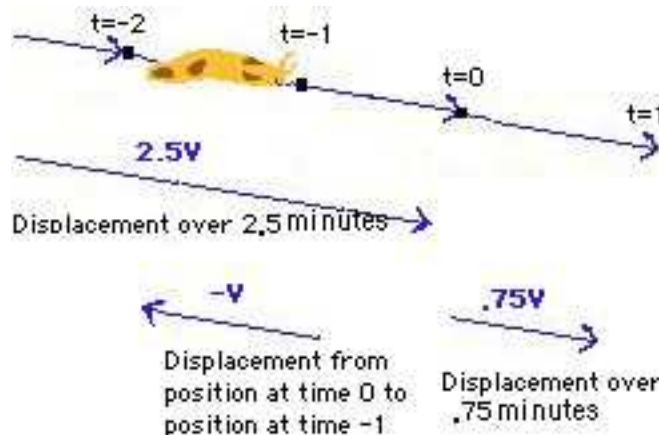
Finally, we get to a position vector description of the journey of the slug. If an origin is chosen and the position vector of the slug at time zero is P with respect to this origin then the position vector of the slug at time t is given by

$$Q(t) = P + tV.$$

This is called a **parametric vector equation** for the position of the slug.
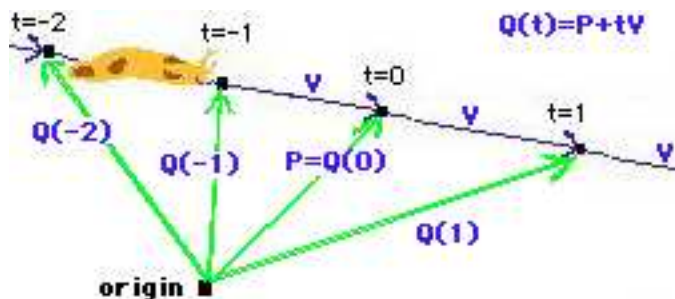
2.1. **Exercise**. *In the picture of constant velocity slug motion you see below, the origin and slug positions at times 4 and 8 are identified. From this picture, create a vector V which represents the velocity of the motion. Identify the position vector P which points to the position of the slug at time 0. Finally, use the formula $Q(t) = P + tV$ to help you find the position on this picture at times $-1, 1$ and 2.*

## 3. Decomposition of Vectors: Part 1

We know how to add vectors, combine them into a single resultant vector. A natural next step might be to see how we can break them into pieces in various ways. In this section we will think about how to break a vector into the sum of two others. One of these will be a multiple of a specified vector and the other perpendicular to that specified vector. This is a very important process in applications. The process is called **decomposition**. When we finally get to precise calculations, decompositions will be easy to find from an arithmetical standpoint.

In order to create the picture of a decomposition you need to know only one thing not shown in the first section of this chapter. You must have a concept of **perpendicularity**, and be able to tell when two arrows are perpendicular to each other by some method. In this section, the old standby "eyeball" method will suffice.

A common and very important usage of vector decomposition occurs when we consider force vectors. A classic example would be that of a box sliding down a slanted board. The most obvious force here is the weight of the box. But that force is directed straight down and the surface of the board prevents movement in that direction. The right way to handle this is to decompose the force caused by gravity into two perpendicular pieces: the part that is straight into the surface of the board (the source of friction) and the other pointing along the line of the board. It is only this last part which makes the box slide.

Whatever the source of the vectors involved may be, we will draw some pictures here to see "how to do it." We want to learn how to decompose a vector V into the sum of a vector P which is a multiple of some vector W and another vector $V - P$ which is perpendicular to W. We call the second vector $V - P$ because whenever $V = P + A$ it must be that $A = V - P$, so there is no point in introducing an independent name for the perpendicular part of the decomposition. Find below three different decompositions of this type in picture form.
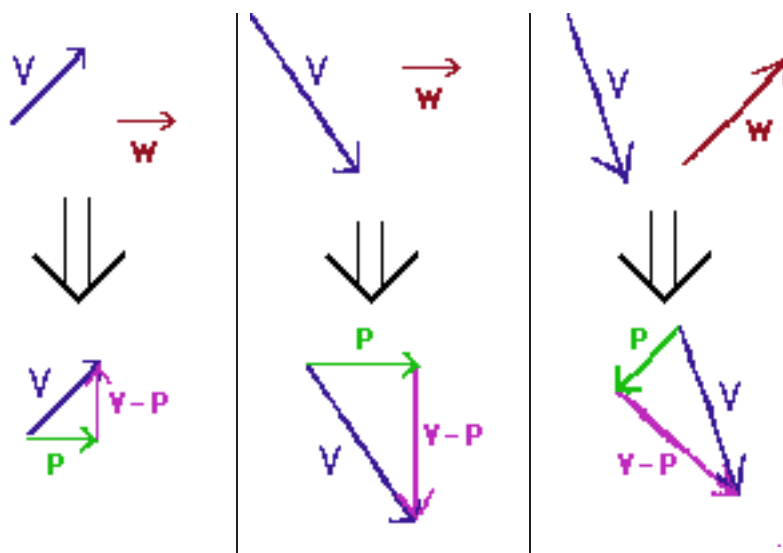
Notice two things about the pictures above: First, in each case P and $V - P$ are perpendicular to each other. Second, P is a multiple of W.

In constructing the decomposition, the length of W is irrelevant. The only thing that is important about W is its direction.

To create the decomposition, draw a fresh picture of a V and W pair from one of the pictures above on a sheet of paper.

V should be somewhere in the middle and W off to the side for reference. Draw a dotted line through the tail of the copy of V. The dotted line must go along the same direction as W. Extend this dotted line a good bit on either side of V, across the whole paper.

Next lay your pencil down on the paper. Put the eraser on this dotted line with the point on the same side as V is on. Make the shaft of the pencil perpendicular to this dotted line. This is where you need to know about perpendicularity.

Slide the pencil up or down the dotted line, keeping it perpendicular to the dotted line, till the pencil tip points at the tip of V or the shaft crosses the tip of V. Stop. This gives the decomposition.

3.1. **Exercise**.
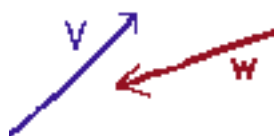
(i) Draw a picture showing the decomposition as described above for the indicated V and W on the left.
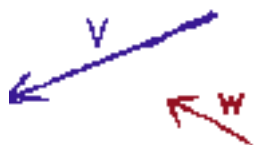
(ii) Draw a picture showing the decomposition as described above for the indicated V and W on the left.

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4. Vectors in the Plane

We have spent considerable time thinking about arrows and drawing pictures and introducing vocabulary. Those pictures will guide you in the later work, and help you to understand what the calculations are telling you. But if we are to produce exact answers rather than qualitative approximations we must do things differently. Our first precise description of vectors will be as arrows in the ordinary XY plane, **freighted with all of its preliminary choices of axes at right angles to each other, origin, choice of units and so on.**

To describe each arrow we must identify a tail followed by a tip. In the XY plane this requires a number of pieces of information—the coordinates of each and also which is tail and which is tip. There is, evidently, some redundancy here and we can cut this down by choosing that representative of a vector which has its tail at the origin: the **standard position** version of the vector. With this convention the vector can be completely described by referring to its tip alone.

The following arrows all represent the SAME vector (draw pictures and convince yourself of this!)

- tail at $(0,0)$ and tip at $(2,2)$

- tail at $(3,-7)$ and tip at $(5,-5)$

- tail at $(-2,-2)$ and tip at $(0,0)$

The vector corresponding to ALL these arrows, or any one of them, will be denoted $\langle 2,2 \rangle$. The difference between $(2,2)$ and $\langle 2,2 \rangle$ is the following: $(2,2)$ is the location of a spot in the plane; $\langle 2,2 \rangle$ is a vector which, when represented as an arrow with its tail at the origin, points at $(2,2)$.

An additional benefit of the standard position description of a vector is that it is ready to use as a position vector whenever we want. $\langle 2,2 \rangle$ is the position vector for the point located at $(2,2)$.

You should satisfy yourself by drawing pictures that multiplication by constants and vector addition can be handled using this standard representation according to the following pattern: If $A = \langle 2,2 \rangle$ and $B = \langle 2,-2 \rangle$ then

$$2A - 3B = 2\langle 2,2 \rangle - 3\langle 2,-2 \rangle = \langle 4,4 \rangle - \langle 6,-6 \rangle = \langle -2,10 \rangle .$$

To multiply vectors by a constant, multiply the coordinates of the standard representation by the constant. To add or subtract vectors, add or subtract corresponding coordinates.

Usually we will denote vectors by capital Latin letters, such as V, W, A or B and the coordinates of the tip of a vector when in standard position will be represented as lower case letters with subscripts. So we might write $V = \langle v_1, v_2 \rangle$. The individual coordinates of the tip in standard position are sometimes called the **components** of the vector.

The **magnitude** of a vector $V = \langle v_1, v_2 \rangle$ is denoted $|V|$ and is defined to be the length of any arrow representing V.

By the Pythagorean distance formula, this length is: $|V| = \sqrt{v_1^2 + v_2^2}$.

4.1. ***Exercise***. *Verify directly that, unless $V = 0$, the vector $\frac{V}{|V|} = \left\langle \frac{v_1}{|V|}, \frac{v_2}{|V|} \right\rangle$ has length 1. Why does it point in the same direction as V?*

Note: if θ is the counterclockwise angle from the positive X axis to V we have

$$V = |V| \left\langle \frac{v_1}{|V|}, \frac{v_2}{|V|} \right\rangle = |V| \langle cos(\theta), sin(\theta) \rangle .$$
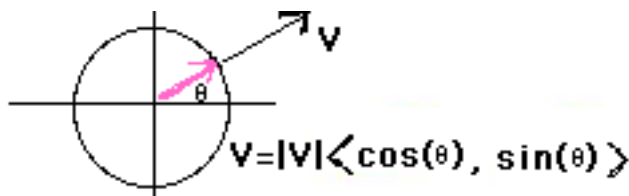
The angle θ is related to $arctan\left(\frac{v_2}{v_1}\right)$.

This is a very important representation for vectors in the plane. It separates cleanly the two things that make up a vector (direction and magnitude) in a way that the XY coordinates of the tip do not. The direction is indicated by an arrow $\langle cos(\theta), sin(\theta)\rangle$ of unit length—these are called **unit vectors**—with its nose resting on the **unit circle** and "pointing the way." Unit vectors are also sometimes called **direction vectors**. The magnitude $|V|$ is the "stretch or shrink factor" which modifies the length of the direction vector.

So we have two ways of representing vectors in the plane. The first gives the XY coordinates of the tip when the vector is in standard position and is better for doing most kinds of vector algebra. The second is more intuitive and emphasizes the two defining properties of a vector. The ability to translate from one form to the other is a key skill.

There is an alternative notation for the XY representation of a vector that you will see from time to time.
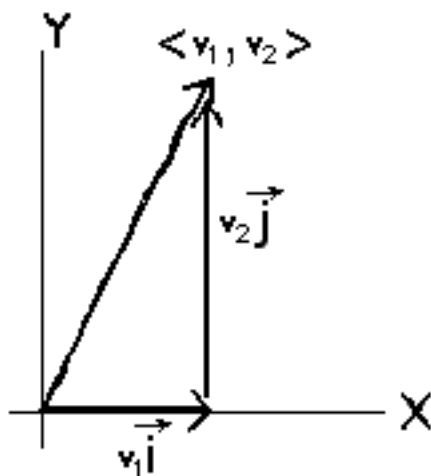
We let $\vec{i}$ stand for the unit vector in the positive X direction, and let $\vec{j}$ stand for the unit vector in the positive Y direction.

$$\vec{i} = \langle 1, 0\rangle \text{ and } \vec{j} = \langle 0, 1\rangle.$$

Do not confuse this vector $\vec{i}$ with the complex number i. With this convention, any vector can be written as a vector sum involving $\vec{i}$ and $\vec{j}$.

$$\begin{aligned} V &= \langle v_1, v_2\rangle \\ &= v_1 \langle 1, 0\rangle + v_2 \langle 0, 1\rangle \\ &= v_1\vec{i} + v_2\vec{j}. \end{aligned}$$

There is nothing terribly significant about this alternative notation (other than that you will see it often) but it does introduce the idea of breaking a vector up into a sum of two perpendicular vectors.

4.2. ***Exercise***. *You should verify (and be able to translate from one form to the other yourself if needed) the following equations:*

$$\langle 2, 2\rangle = \sqrt{8}\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \sqrt{8}\left\langle cos\left(\frac{\pi}{4}\right), sin\left(\frac{\pi}{4}\right)\right\rangle.$$

$$\langle -2, -2\rangle = \sqrt{8}\left\langle \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right\rangle = \sqrt{8}\left\langle cos\left(\frac{5\pi}{4}\right), sin\left(\frac{5\pi}{4}\right)\right\rangle.$$

$$\langle 0, -7\rangle = 7\langle 0, -1\rangle = 7\left\langle cos\left(\frac{3\pi}{2}\right), sin\left(\frac{3\pi}{2}\right)\right\rangle.$$

$$\langle -2 \ , 3 \rangle = \sqrt{13} \ \langle \ cos\left(\pi + arctan\left(-1.5\right)\right), sin\left(\pi + arctan\left(-1.5\right)\right) \ \rangle$$
$$\approx 3.61 \ \langle \ cos\left(2.16\right), sin\left(2.16\right) \ \rangle \approx \ 3.61 \ \langle -.556, .832 \rangle .$$

4.3. **Exercise**. *Repeat Exercise 1.3. Then discuss and compare the three different ways of thinking about vectors we have seen:*

- *vectors as "experiences:" pushes on your shoulder, actual completed movement of your body from one spot to a second spot and as a description of a motion which you witness as it happens*

- *vectors as arrows (where arrows are "the same" if they have the same direction and length)*

- *vectors given by a coordinate pair such as* $\langle 3, 8 \rangle .$

5. Dot Products

We will now define a way of multiplying two vectors called **dot product**.

If $V = \langle v_1, v_2 \rangle$ and $W = \langle w_1, w_2 \rangle$ we define $V \cdot W = v_1 w_1 + v_2 w_2.$

This product takes two vectors and produces a number obtained by simple arithmetic involving the coordinates of each.

5.1. **Exercise**. *(i) Show that* $V \cdot W = W \cdot V.$

(ii) Show that $(7V) \cdot W = V \cdot (7W) = 7(V \cdot W).$

(iii) Satisfy yourself that, in general, constants can be moved around in a dot product like the 7 above.

(iv) Show that $(V + P) \cdot W = V \cdot W + P \cdot W$ *and* $W \cdot (V + P) = W \cdot V + W \cdot P.$

(v) $V \cdot V = |V|^2.$

Properties (iii) and (iv) of the exercise together make the dot product an example of a **tensor**. Since there are two vectors involved, it is called a 2-tensor. Property (i) in this context is called **symmetry**. The dot product is an example of a symmetric 2-tensor. If you stay in this business awhile you will run into tensors of various kinds.

Dot products are incredibly useful.

If $\quad V \ = |V| \langle cos(\alpha), \ sin(\alpha) \rangle \quad$ and $\quad W \ = |W| \langle cos(\beta), \ sin(\beta) \rangle$

then $\quad V \cdot W \ = |V||W| \langle cos(\alpha), \ sin(\alpha) \rangle \cdot \langle cos(\beta), \ sin(\beta) \rangle$

$= |V||W|(cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)) = |V||W|cos(\alpha - \beta).$

This means that you can use elementary arithmetic to get information about the angle $\alpha - \beta$ between two vectors!

We have just show that if θ is the **angle between vectors** V and W then

$$V \cdot W \ = \ |V||W| cos(\theta).$$

For example, the angle θ between $\langle 3, 7 \rangle$ and $\langle 5, -3 \rangle$ satisfies

$$\langle 3, 7 \rangle \cdot \langle 5, -3 \rangle = 15 - 21 = \sqrt{9 + 49} \ \sqrt{25 + 9} \ cos(\theta)$$

and so $\qquad \dfrac{-6}{\sqrt{9+49} \ \sqrt{25+9}} \ = \ cos(\theta) \quad$ which means $\quad \theta \ \approx \ 98^o$.

5.2. ***Exercise***. *(i) Verify the angle found above by drawing a careful picture and doing trigonometry.*

Also, satisfy yourself that the following statements are true:

(ii) $V \cdot W > 0$ implies that V and W are less than 90° apart.

*(iii) $V \cdot W = 0$ implies that V and W are **perpendicular**—that is, are at right angles to each other. Vectors that are perpendicular to something (a line, a plane, another vector) are often called **normal** or **orthogonal** to that other object. This vocabulary is most useful when neither V nor W is 0.*

(iv) $V \cdot W < 0$ implies that V and W are more than 90° apart.

*(v) $V \cdot W = |V||W|$ implies that V and W point in the same direction.*

(vi) $V \cdot W = -|V||W|$ implies that V and W point in the opposite direction.

(vii) $\langle 5, 3 \rangle$ is normal to $\langle -3, 5 \rangle$. Find a vector normal to $\langle a, b \rangle$.

5.3. ***Exercise***. *If $V = \langle v_1, v_2 \rangle$ and $W = \langle w_1, w_2 \rangle$ we define $\det(\mathbf{V}, \mathbf{W})$ to be the number $v_1 w_2 - v_2 w_1$. The notation comes from the fact that, for those among you who know about **determinants**, it is the determinant, usually denoted $\det A$, of the matrix A where*

$$A \ = \ \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}.$$

Here is an application for this number:
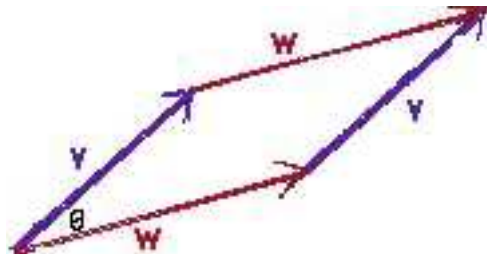
V and W can be used to form a parallelogram. Show that the area of this parallelogram is

$$|W||V|sin(\theta) = |W||V|\sqrt{1 - cos^2(\theta)}.$$

Now square this last number and verify that it is $(v_1 w_2 - v_2 w_1)^2$. Conclude that the area of the parallelogram is $|\det(V, W)|$.

### 6. Problems in the Plane: Displacements, Forces and Velocity

The reader should work through all the statements made in this section.

Problem 1: Displacements

Bearings are commonly used in navigation problems to indicate direction. A bearing looks like this:

$$N10^oW \quad or \quad S38^oW \quad or \quad S18^oE.$$

The meaning of a bearing is as follows: You first point your nose straight north or south, whichever is indicated by the first letter. Then you rotate the given angle toward the compass heading suggested by the second letter, either east or west. That is the direction.
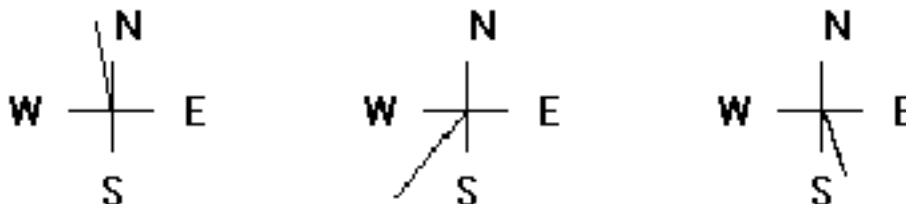
The problem is as follows:

Suppose a cross country runner jogs 5 miles $N10^oW$ and 3 miles $S38^oW$ and then 4 miles $S18^oE$. How far is the runner from home, and what bearing should the runner take to get there?

The Solution:

The resultant displacement vector is:

$$5 \langle cos(100^o), sin(100^o) \rangle$$
$$+3 \langle cos(-128^o), sin(-128^o) \rangle$$
$$+4 \langle cos(-72^o), sin(-72^o) \rangle$$

This vector is, approximately, $\langle -1.479, -1.244 \rangle$, or about

$$1.93 \langle cos(220^o), sin(220^o) \rangle.$$

It is about 1.93 miles to home and the runner should head 180° away from the angle of the resultant vector. The bearing will be $N50^oE$.

Problem 2: Forces

Three people are pushing on different sides of a huge ball but the ball is not moving. The pushes of first and second of these people are represented by force

vectors $\langle 30, 20 \rangle$ and $\langle 15, -10 \rangle$, respectively. Which force vector represents the push of the third?

The Solution:

Since the ball is not moving, the sum of the three forces must add to the zero vector. So if F is the force of the third person, $F + \langle 30, 20 \rangle + \langle 15, -10 \rangle = \langle 0, 0 \rangle$. So $F = -\langle 30, 20 \rangle - \langle 15, -10 \rangle = \langle -45, 10 \rangle$.

Problem 3: Forces
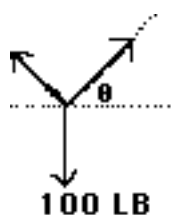
We suppose that there is a 100 pound weight hooked (not moving but free to slide) over a length of rope. The rope is at a given angle (on both sides) from the horizontal as indicated. The rope is very light in comparison to 100 pounds, so the weight tightens it to (nearly) a straight line. The **tension** is a measure of how hard the rope is pulling, and is the magnitude of a force vector whose direction lies along the rope. What is the tension, T, on this rope?

The Solution:

Since nothing is moving, all the forces must counteract each other at each point along the rope and, in particular, at the point where the weight is attached. The resultant of the three forces there, indicated by arrows, must be zero. The sum is

$$\langle 0, 0 \rangle = 100 \langle 0, -1 \rangle + T \langle \cos(\theta), \ \sin(\theta) \rangle + T \langle \cos(\pi - \theta), \ \sin(\pi - \theta) \rangle.$$
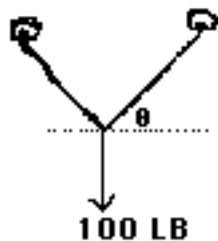
After fiddling around with some algebra, this gives

$$\langle 0, 0 \rangle = \langle 0, -100 + 2T \ \sin(\theta) \rangle \quad \text{and so} \quad T = \frac{50}{\sin(\theta)}.$$

Note: the minimum tension occurs when the angle is 90°. When the angle gets small enough, the weight WILL break the rope.

Problem 4: Forces

Let's embellish the last problem a bit and think about it from the standpoint of a bolt anchored to a wall and to which the rope is attached.

In one situation, the rope is tied off at each anchor point. In the other situation, the rope slides through rings at the same height and is allowed to run freely in a big loop. The issue here is to find the forces acting on the anchor bolt in each case.
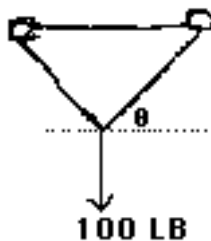
The Solution:

Experience shows that a bolt like this will refuse to move until it breaks or pulls out of the wall. So it musters up whatever force is required to keep it from moving (to counteract the other forces on it) until that catastrophic event. Unbalanced forces always generate a change in motion. No change in motion means that the

resultant of all forces is the zero vector in each case. In the diagram on the left, the situation is clear. The force which the upper right bolt must counteract is

$$T \langle -cos(\theta), -sin(\theta) \rangle \quad \text{which has magnitude} \quad T = \frac{50}{sin(\theta)}.$$

But in the second case, there are TWO forces tugging on the upper right bolt, one along each segment of the rope leading away from that bolt. The resultant force to which it must respond is

$$T \langle -cos(\theta), -sin(\theta) \rangle + T \langle -1, 0 \rangle = T \langle -1 - cos(\theta), -sin(\theta) \rangle.$$

This vector has magnitude

$$T \sqrt{1 + 2cos(\theta) + cos^2(\theta) + sin^2(\theta)} = T\sqrt{2}\sqrt{1 + cos(\theta)}.$$

By merely looping the rope around the pair of bolts rather than tying it off at each one you increase the magnitude of the force on the bolt by at least 40% and up to 100%. The worst case also happens for smaller angles, exacerbating an already bad situation.

If you made the apparatus using real bolts and a rope, friction each time the rope changes direction at an anchor would reduce this effect somewhat. Calculating friction can be quite complicated, involving the diameters of rope and bolt shaft, angle of contact and so on. But friction must be considered to make sense of more complicated situations or to make accurate predictions. But that is for a physics class!

Problem 5: Combining Velocities

Suppose that a plane is pointing $N10^o W$ and its throttle and altitude are such that if it were flying on a windless day it would be moving at 400 kilometers per hour. However there is a wind, coming from $N40^o W$ at 50 KPH. How fast is the plane moving with respect to the ground and with what bearing?

The Solution:

$$400 \langle cos(100^o), sin(100^o) \rangle - 50 \langle cos(130^o), sin(130^o) \rangle$$
$$\approx \langle -37.3, \ 355.6 \rangle \approx 357.6 \langle cos(96^o), sin(96^o) \rangle.$$

So the plane is flying at a speed of about 358 KPH at a bearing of $N6^o W$.

7. Position Vectors and Constant Velocity Motion: Part 2

In this section we will take another look at the material of Section 2, this time with coordinates. For specificity, we will measure the parameter t in seconds.

Suppose we have an object moving in the plane with velocity that never changes and starting at some place at time $t = 0$, such as $(1, 2)$. In this context, $\langle 1, 2 \rangle$ in standard position **points at** the spot and we will say that the object **is at** $P = \langle 1, 2 \rangle$. Let us say that the velocity vector is found to be $V = \langle -3, 4 \rangle$. The velocity vector is the displacement vector after one second of motion, so you could get these two pieces of information by watching the motion for one second. We will say that the velocity vector **lies in** the line of the motion. The vocabulary means

that if you choose the copy of V with its tail on the line of the motion, the tip (and so the whole arrow) is in the line too.

$$\text{Note that} \quad V \ = \ \langle -3, 4 \rangle = 5 \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle \approx 5 \left\langle \cos(126.9^o), \ \sin(126.9^o) \right\rangle.$$

After 1 second the object will have moved 5 meters in the direction of V and so will be at $P + V$. After 2 seconds it will have moved another 5 meters in the direction $\langle -3, 4 \rangle$ and so will be at $P + V + V = P + 2V$.

At time $t = -1$, that is one second before it was at P, it must have been at $P - V$ (if it is to arrive at P one second later.) In general, for any time t, the position at time t, which we will denote Q or $Q(t)$, will be given by

$$Q(t) = P + tV.$$
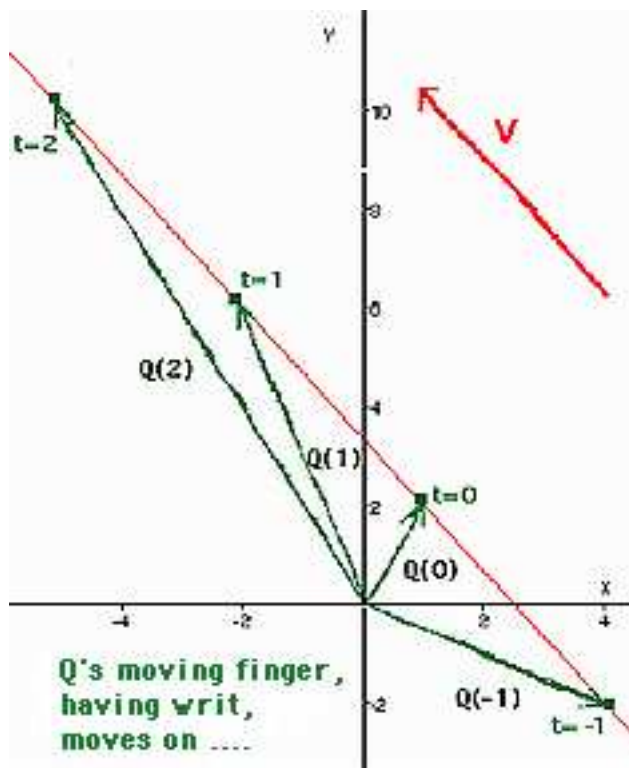
As you will recall from Section 2, this is called a **parametric vector equation** for the motion. The tip of the arrow $Q(t) = P + tV$ sweeps out the motion as time passes. You might think of this as like a "slope-intercept form" for the linear motion. P is the location at time 0, and V carries the direction and speed information about the motion.

Note that:

$$Q(t) = P + tV = \langle 1 - 3t, 2 + 4t \rangle \ \text{ so } X = 1 - 3t \text{ and } Y = 2 + 4t,$$

where X and Y represent the X and Y coordinates of the tip of Q as functions of time. We can **eliminate** t from this whole business as follows:

$$t \;=\; \frac{1-X}{3} \qquad \text{so} \qquad Y = 2 + 4\left(\frac{1-X}{3}\right) \qquad \text{or} \qquad Y = \frac{-4}{3}\,X + \frac{10}{3}.$$

Another (easier) method of eliminating t is to use the fact that, for a generic point $Q = \langle X, Y \rangle$ on the line, $Q - P = tV$: that is, $Q - P$ is a multiple of the velocity vector V. So if N is any vector perpendicular to V then $(Q - P) \cdot N = 0$. This is called the **normal form** for the line in the XY plane..

In our situation we can pick $N = \langle 4, 3 \rangle$ so $(\langle X, Y \rangle - \langle 1, 2 \rangle) \cdot \langle 4, 3 \rangle = 0$ which yields $(X - 1)4 + (Y - 2)3 = 0$ or $4X + 3Y = 10$.

In any case, we can see the motion is along a line in the plane, and we have a formula for this **geometrical track** of the parametric motion. Note how the slope is related to the components of the velocity vector $\langle -3, 4 \rangle$. This is not chance.

There is something important missing when we eliminate the parameter. In this form we lose all knowledge of **when** we are anywhere on this line, which in applications is often the whole point! Knowing the geometrical track can be an aid, however, in drawing the graph or for other reasons.

In applications you will not always be given the velocity and position at time 0, just as in the old line exercises from elementary algebra you were not always given the slope and Y axis intercept. Still, we should be able to cook up an equation for the motion given two independent facts about it.

In the problems below we assume constant velocity motion in the plane. If you get stuck - DRAW PICTURES FOR VARIOUS t VALUES!

Problem 1: A Parametric Vector Equation—Where Will the Object Be?

Write a parametric vector equation for the position of a moving object located at $\langle 3, 5 \rangle$ at time 0 and $\langle 4, 7 \rangle$ at time 1. Where will it be at $t = 2$?

Movement during 1 second: $\langle 4, 7 \rangle - \langle 3, 5 \rangle = \langle 1, 2 \rangle = $ velocity.

$Q(t) = \langle 3, 5 \rangle + t \langle 1, 2 \rangle$. So it will be at $Q(2) = \langle 3, 5 \rangle + 2 \langle 1, 2 \rangle = \langle 5, 9 \rangle$ at time 2. You can also get this by adding a copy of the velocity vector to the position at time 1.

Problem 2: A Parametric Vector Equation Plus Elimination

Write a parametric vector equation for the position of a moving object with velocity vector $\langle 1, -5 \rangle$ and which is located at $\langle 5, 9 \rangle$ at time 4. What is the XY formula for the geometrical track upon which the motion takes place?

At time 0 it was at $\langle 5, 9 \rangle - 4 \langle 1, -5 \rangle = \langle 1, 29 \rangle$.

So $Q(t) = \langle 1, 29 \rangle + t \langle 1, -5 \rangle = \langle 1 + t, 29 - 5t \rangle$. So $X = 1 + t$ and $Y = 29 - 5t$. So $Y = 29 - 5(X - 1)$ or, in slope-intercept form, $Y = -5X + 34$.

Problem 3: A Parametric Vector Equation Plus a "Wall"

Write a parametric vector equation for the position of a moving object located at $\langle -3, 0 \rangle$ at time 5 and $\langle 12, 6 \rangle$ at time 8. When will the object pass through a barrier set up on the line $Y = -X + 100$?

Movement during the 3 seconds from $t = 5$ to $t = 8$: $\langle 12, 6 \rangle - \langle -3, 0 \rangle = \langle 15, 6 \rangle$. So the movement in 1 second is one third as much: $(1/3) \langle 15, 6 \rangle = \langle 5, 2 \rangle$ = velocity. At time 0 it was at $\langle 12, 6 \rangle - 8 \langle 5, 2 \rangle = \langle -28, -10 \rangle$.

So $Q(t) = \langle -28, -10 \rangle + t \langle 5, 2 \rangle$. This will be at the barrier at $Y = -X + 100$ when $-10 + 2t = -(-28 + 5t) + 100$. That means $t = 138/7$ seconds.

7.1. **Exercise**. (i) *Write a parametric vector equation for the position of a moving object located at $\langle 0, 5 \rangle$ at time 0 and with velocity vector $\langle -1, 7 \rangle$. Where will it be at time 10? Where was it at time -5? When will it hit a wall set up on the line $Y = X + 100$?*

(ii) *Write a parametric vector equation for the position of a moving object located at $\langle -2, 3 \rangle$ at time 0 and $\langle -4, 1 \rangle$ at time 1. Draw a picture of this motion.*

(iii) *Write a parametric vector equation for the position of a moving object with velocity vector $\langle -1, 8 \rangle$ and which is located at $\langle 1, -9 \rangle$ at time 6. Draw a picture of this motion. What is the XY formula for the geometrical track upon which the motion takes place?*

(iv) *Write a parametric vector equation for the position of a moving object located at $\langle -2, 9 \rangle$ at time 5 and $\langle 12, 16 \rangle$ at time 12. When will the object hit a wall set up on the line $Y = -X + 100$?*

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8. Decomposition of Vectors: Part 2

In this section we revisit the ideas of Section 3 only this time we use coordinates and can obtain more precise information. You will recall that the goal was to decompose a vector into the sum of two other vectors one of which was along a given direction while the other was perpendicular to that direction.

Being able to draw the picture is important and will allow you to estimate the decomposition fairly accurately in the plane, so you might want to go back and take another look at Section 3 at this point.

The fact we need for our calculation is:

$V \cdot W = |V||W|cos(\theta)$ where θ is the angle between nonzero vectors V and W.

We will use the fact that $\frac{V \cdot W}{|W|} = |V|cos(\theta)$ in the calculation below.

Notice in the following picture that, with W as shown, the length of P is $|V|cos(\theta)$.
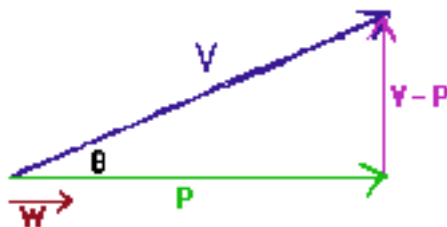
So $P = |V|cos(\theta)\frac{W}{|W|} = \left(\frac{V \cdot W}{|W|}\right)\frac{W}{|W|} = \left(\frac{V \cdot W}{W \cdot W}\right)W.$

The vector $V - P$ should be perpendicular to W if the picture is correct.

Let's see:

$$W \cdot (V - P) = W \cdot V - W \cdot P = W \cdot V - W \cdot \left(\frac{V \cdot W}{W \cdot W}\right)W$$

$$= W \cdot V - \left(\frac{V \cdot W}{W \cdot W}\right)W \cdot W$$

$$= W \cdot V - W \cdot V = 0.$$

So $V - P$ and W are perpendicular as advertised.

The complete decomposition calculation is in three steps:

- Calculate $P = \left(\frac{V \cdot W}{W \cdot W}\right)W.$
- Then calculate $V - P.$
- Then verify that $P \cdot (V - P) = 0.$ (This catches most arithmetic "issues.")

Problem: A Decomposition

Decompose $\langle 4, 5 \rangle$ into the sum of a vector which is a multiple of $\langle 1, 2 \rangle$ and another which is perpendicular to $\langle 1, 2 \rangle$.

$$P = \left(\frac{\langle 4, 5 \rangle \cdot \langle 1, 2 \rangle}{\langle 1, 2 \rangle \cdot \langle 1, 2 \rangle}\right)\langle 1, 2 \rangle$$

$$= \left(\frac{4 + 10}{1 + 4}\right)\langle 1, 2 \rangle = \left\langle \frac{14}{5}, \frac{28}{5} \right\rangle.$$

$$V - P = \left\langle \frac{20}{5}, \frac{25}{5} \right\rangle - \left\langle \frac{14}{5}, \frac{28}{5} \right\rangle$$

$$= \left\langle \frac{6}{5}, \frac{-3}{5} \right\rangle.$$

Verify:

$$P \cdot (V - P) = \left\langle \frac{14}{5}, \frac{28}{5} \right\rangle \cdot \left\langle \frac{6}{5}, \frac{-3}{5} \right\rangle$$

$$= \frac{84}{25} - \frac{84}{25} = 0.$$

8.1. **_Exercise_**. *(i) Draw a picture to verify that the decomposition in the problem, given algebraically, is "about right."*

(ii) Decompose $\langle -4, 1 \rangle$ into the sum of a vector which is a multiple of $\langle -1, 8 \rangle$ and another which is perpendicular to $\langle -1, 8 \rangle$. Using a picture, verify that your decomposition is "about right."

(iii) Decompose $\langle 0, 1 \rangle$ into the sum of a vector which is a multiple of $\langle -1, 1 \rangle$ and another which is perpendicular to $\langle -1, 1 \rangle$.

We will make two final points here:

First, the vector P found above is also called the **vector projection of** V **in the direction of** W and often denoted $Proj_W(V)$. The specificity provided by this notation is needed when there are decompositions of different vectors or along various directions within one problem. It is apparent from the definition that $Proj_W(V) = Proj_U(V)$ when U is a nonzero multiple of W.

Second, sometimes the decomposition vectors P and $V - P$ themselves are not needed but only their magnitudes. The number $\dfrac{V \cdot W}{|W|}$ is called the **scalar projection of** V **in the direction of** W. It is a positive number if the angle θ between V and W is less than 90°. Its absolute value is the magnitude of P.

The various magnitudes are are related by:

$$|P| = |V||cos(\theta)| = \frac{|V \cdot W|}{|W|} \quad \text{and} \quad |V - P| = |V||sin(\theta)|.$$

Also note that:

$$|V|^2 = |P|^2 + |V - P|^2.$$

Using this last equation, you can get the magnitude of the third from the magnitudes of any two of the terms involved.

9. Problems in the Plane: Work, the Inclined Plane and a Robot Arm

In this section we are going to use dot products to perform some calculations whose meaning will be explored in much greater depth in physics and engineering classes.

Work

The concept of "work" in physics is not the usual idea of work that pops to mind when we say "I worked all day weeding the garden." In the colloquial sense, work means "I accomplished something today." or "I expended a lot of effort today." Neither of these fuzzy notions corresponds to the idea of work as defined in physics, which is part of the energy "accounting system."

Work done by a constant force, in physics, involves the magnitude of the part of the force that lies in the direction of displacement times the magnitude of that displacement. The speed of the motion is not relevant: only the displacement.

So if I am standing in a room holding a 200 pound sack of flour, beads of sweat popping out on my forehead—but not moving—then I am doing no work. If there is no movement there can be no work.

Even if I move, so long as I move horizontally and not up or down, no work has been done by or against gravity. At least part the force must lie in the direction of motion.

When I have found the component of the force along the line of the displacement, I calculate its length and multiply by the length of the displacement vector. If the angle between the force and the displacement is less than 90° this number is

the work. If the angle between the force and the displacement is more than 90° I multiply this number by -1. This number is the work.

When the work is positive the displacement took place aided by the force. When negative, the force was hindering the displacement.
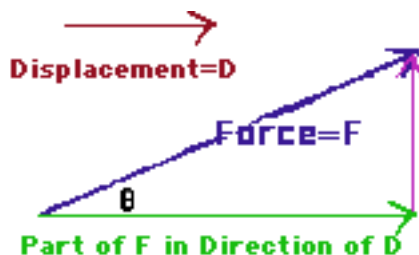
From a calculation standpoint, if F is the force vector and D is the displacement vector we have, from the picture above,

$$\text{Work} = |D||F|cos(\theta) = |D||F|\left(\frac{D \cdot F}{|D||F|}\right) = D \cdot F.$$

So to calculate the work, dot the displacement vector against the force vector.

Why should this odd number be useful in physics? Good question. The story is pretty long and is the object of much thought in physics or engineering classes. However now, if anyone should ask, you can calculate it.

We should make several points here about the picture shown above. The force vector shown is less than 90° away from the displacement vector, so the dot product is positive. If the force acts to "slow down" the displacement instead of "helping it along" the angle exceeds 90° and the dot product will be negative. Second, this calculation is for constant force and straight line motion situations. Other situations are considered after you have had calculus. Finally, notice that the part of the force perpendicular to the motion is discarded—it serves no purpose in the work calculation.

Problem 1: Work

Suppose I am walking due east and a very hard wind is pushing on me 40 LB from $N20^oE$. How much work is done if I move 200 feet?

The Solution:

$D = (200 \; feet) \langle 1, 0 \rangle$ and $F = (40 \; pounds) \langle -cos(70^o), -sin(70^o) \rangle$.

So the work is $D{\cdot}F = -8000 \; cos(70^o) \; foot \, pounds$ or roughly $2736 \; foot \, pounds$.

The Inclined Plane

If an object is sitting on an **inclined plane** the force of gravity is pulling straight down. But "straight down" is not a direction that the object can move. Any push directly into the face of the plane is countered by a corresponding push by the molecules of the surface to prevent the object from sinking in. Because of this, it is only the part of the force of gravity pointing down **along the hill** that can influence the motion of the object. The rest is "wasted" against the **constraint**—the hard surface of the hill.
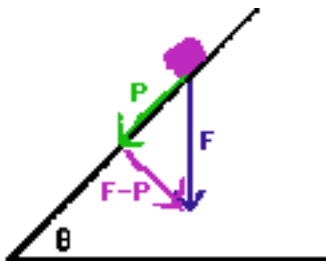
It is the vector P (shown above) that acts to accelerate the object down the hill and you learn how to calculate the effect of the magnitude of P on the motion in basic physics classes. Sometimes in these classes the part $F - P$ of F that is perpendicular to the hill is discarded. We say the hill is "frictionless." Usually we say this because we don't want to think about **friction**, which is very complicated to understand physically, involving molecular forces and tiny rugosities binding on each other. But in real inclined planes, the push directly into the surface creates friction. $F - P$ causes an "effective" force up the hill and this will completely counteract the downhill force if the downhill force is too small. In many cases, the magnitude of the maximum "up the hill" force that could be generated by friction is $k|F - P|$ where k is a positive constant called the **coefficient of static friction**. k depends on how the surfaces are prepared, the material of which they are composed and a myriad of other complicated factors, and is measured for a specific situation, not calculated.

Problem 2: The Inclined Plain

Find P and $F - P$ in the picture above if the object weighs 100 pounds and the plane is inclined 37° from horizontal.

The Solution:

To calculate the decomposition we can use ANY vector that points along the hill. The easiest one is $\langle cos(37^o), sin(37^o)\rangle$. So

$$
\begin{aligned}
P &= \frac{\langle cos(37^o),\ sin(37^o)\rangle \cdot \langle 0, -100\rangle}{\langle cos(37^o),\ sin(37^o)\rangle \cdot \langle cos(37^o), sin(37^o)\rangle}\ \langle cos(37^o), sin(37^o)\rangle \\
&= -\ 100\ sin(37^o)\ \langle cos(37^o), sin(37^o)\rangle \approx 60.18\ \langle cos(37^o), sin(37^o)\rangle \\
&\approx \langle -48.06\ , -36.22\rangle \\
F - P &= \langle 0, -100\rangle - 100 sin(37^o)\ \langle cos(37^o), sin(37^o)\rangle \\
&\approx \langle 0, -100\rangle - \langle -48.06, -36.22\rangle = \langle 48.06, -63.78\rangle \\
&\approx 79.86\ \langle cos(-53^o), sin(-53^o)\rangle
\end{aligned}
$$

If all you needed was the magnitude of these two vectors, that is easier:

$$|P| = |F|sin(\theta) \qquad \text{and} \qquad |F - P| = |F|cos(\theta).$$

In our case that means $|P|$ is about 60.18 pounds and $|F - P|$ is approximately 79.86 pounds.

Problem 3: The Inclined Plain

Consider the situation from Problem 2. Suppose the coefficient of static friction is $k = .65$. Will the object slide down the hill?

The Solution:

The maximum force that could be generated by friction has magnitude $(.65)(79.86) \approx 51.9$ pounds up the hill.

This is less than the magnitude of P so there will be a net force down the hill.

If there is a net force there is always a change in the motion, so the object will move.

A Tool on a Robot Arm

If you are trying to **control** the activity of a tool at the end of a robot arm, the first step is to have a means of sensing the location of the tool. One way of doing this is to have a device that measures the angle of each joint relative to the previous arm segment or an anchor point. The picture below indicates a robot arm confined to move in a vertical plane with two arm segments, together with the angles α and β which can be measured by the sensors at each joint: one sensor at the shoulder measures the angle between the first arm segment and the floor and one at the elbow which measures the angle between the upper arm and the forearm. These measurements are called **feedback**. The robot arm segments have fixed lengths, L_A and L_B respectively. Let us also suppose that the mechanism of the joint allows you to control the angles, but that the angles must be positive and cannot exceed 180°.
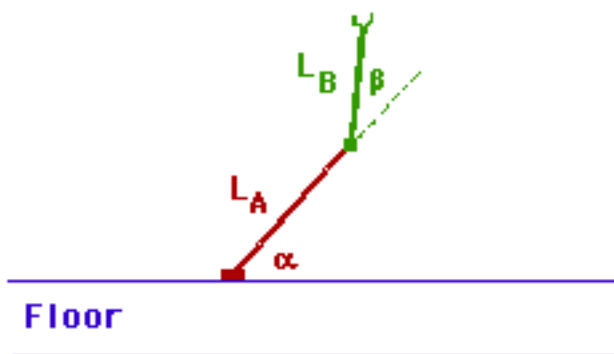
If a specific angle feedback arrives at the controller computer, where is the tool? Using vectors, the tool is at the tip of the resultant vector

$$L_A \langle cos(\alpha), sin(\alpha) \rangle + L_B \langle cos(\alpha + \beta), sin(\alpha + \beta) \rangle .$$

9.1. ***Exercise***. *If the shoulder-to-elbow part of the arm has length* 2 *and the elbow-to-tool part has length* 1:

(i) Describe the part of the plane which is accessible to the tool.

(ii) Will there be more than one way to get to any accessible point?

*(iii) * Suppose that you want to get to a specific point with the tool. How should you control the arm to get there? (I am asking for* α *and* β. *hint: Represent the point as* $a \langle cos(\gamma), sin(\gamma) \rangle$ *where* $a > 0$ *and* $0 \le \gamma \le 180^o$.)

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10. Vectors in Three Dimensions

Locations in the plane require two numbers (X and Y coordinates, for instance) to describe. Sometimes people say that the plane has **two dimensions**, or is **2D** because of that. Locations in space require three, commonly called X, Y and Z. Space is said to have **three dimensions**, or to be **3D**. When we describe locations by the X and Y coordinates in $2D$ or X, Y and Z coordinates in $3D$ we are said to be using **rectangular coordinates.** Vectors in space require three components too: the X, Y and Z components of the tip of the arrow when the tail is at the origin. When we make this identification for a particular application of vectors we assume concepts of origin, perpendicularity and distance to be agreed upon and reflected in the identification.

Vectors $V = \langle v_1, v_2, v_3 \rangle$ still have a length, found by using Pythagoras twice in the picture found below—first to find the length of L. You will find that the **magnitude** of a vector V, denoted $|V|$, is $\sqrt{v_1^2 + v_2^2 + v_3^2}$.
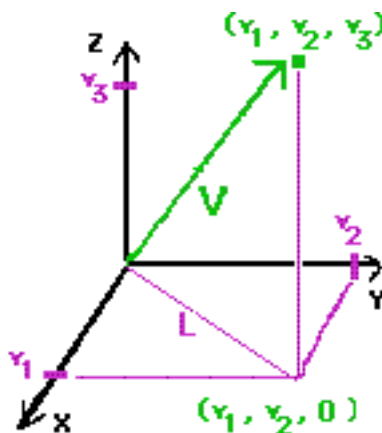

There are three special planes, called the **coordinate planes**, to which we refer in order to get "oriented" in a $3D$ picture. These are called the XY, YZ and XZ planes, and correspond to all the points with Z, X and Y coordinates, respectively, equal to 0. They divide space into eight **octants**.
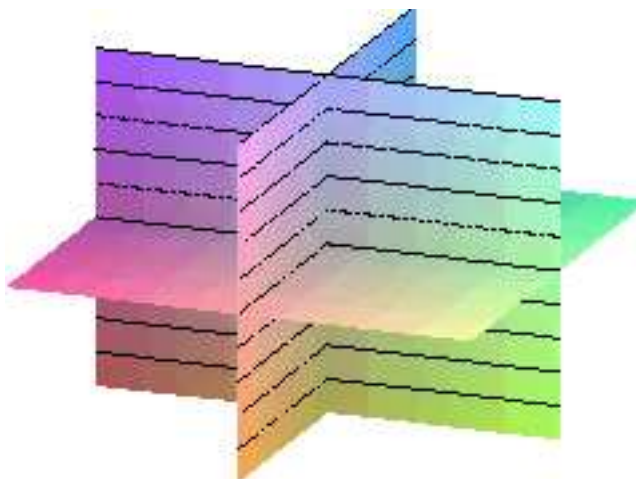
There are three special vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ (not just $\vec{i}$ and $\vec{j}$ anymore) pointing along the three coordinate axes:

$$\vec{i} = \langle 1, 0, 0 \rangle, \qquad \vec{j} = \langle 0, 1, 0 \rangle \qquad \text{and} \qquad \vec{k} = \langle 0, 0, 1 \rangle.$$

This second definition for $\vec{i}$ and $\vec{j}$ invites confusion. However, with use you will come to consider that a "feature" rather than a "bug."
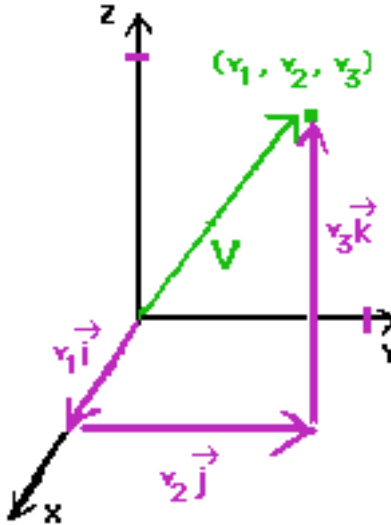
Any vector $V = \langle v_1, v_2, v_3 \rangle$ in space can be written also as

$$V = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

and both notations for vectors in space are in common use. One advantage of this notation is facilitate the transition from $2D$ to $3D$ which is frequently useful in applications. A vector $V = \langle v_1, v_2 \rangle$ in $2D$ is different from the vector $\langle v_1, v_2, 0 \rangle$ in space, but sums and multiples of vectors in the XY plane in space all correspond to the same operations on the related vectors in $2D$. You have merely changed your description of the arrows, not the arrows themselves. By using the notation $V = v_1 \vec{i} + v_2 \vec{j}$ you can make the move from $2D$ to the XY plane in $3D$ by merely redefining which $\vec{i}$ and $\vec{j}$ you mean in a formula. Mathematicians don't like this kind of thing very much, but Engineers seem to like it a lot.

When V is any nonzero $3D$ vector it is still true that $\frac{V}{|V|}$ is a unit vector. $\frac{V}{|V|}$ is called the **direction vector for** V.

So any nonzero vector V can be written as $V = |V| \frac{V}{|V|}$. The vector part has its nose on the **unit sphere** which consists of all points (X, Y) with $X^2 + Y^2 + Z^2 = 1$. This part indicates direction of V, while $|V|$ carries the magnitude information about V.

We define the **dot product** of two vectors $V = \langle v_1, v_2, v_3 \rangle$ and $W = \langle w_1, w_2, w_3 \rangle$ to be

$$V \cdot W = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

It is still true but a bit harder to show that

$$V \cdot W \ = \ |V||W|cos(\theta)$$

where θ is the angle between V and W as measured in a plane containing both vectors. There is a discussion of this in the next section if you are interested.

If you accept that formula as fact, then there is a simple meaning for the entries in

$$\frac{V}{|V|} = \left\langle \frac{v_1}{|V|}, \frac{v_2}{|V|}, \frac{v_3}{|V|} \right\rangle.$$

They are $\langle cos(\theta_1), cos(\theta_2), cos(\theta_3) \rangle$ where the angles θ_1, θ_2 and θ_3 are the angles between V and the coordinate axes! To show this, dot $\frac{V}{|V|}$ against $\vec{i}, \vec{j}$ and $\vec{k}$ one at a time. The numbers $cos(\theta_1)$, $cos(\theta_2)$ and $cos(\theta_3)$ are called **direction cosines** for the vector.

It is just as easy in $3D$ as in $2D$ to come up with **vectors perpendicular to a given one:** simply select them to have dot product 0 with the first. For example $\langle 3, 6, -1 \rangle$ is normal to both $\langle -6, 3, 0 \rangle$ and $\langle 0, 1, 6 \rangle$.

The formulas for decomposition (and the proof that the decomposition has the expected properties) and the formulas involved in parameterized constant velocity motion are unchanged in three dimensions!

To **decompose** V into the sum of a vector (call it P) which is a multiple of W and another perpendicular to W we proceed as before:

Let $P \ = \ \left(\frac{V \cdot W}{W \cdot W} \right) W$. Then $V \ = \ P + (V - P)$ is the decomposition. Dotting P against $V - P$ to yield 0 provides the check for your arithmetic.

To describe **constant velocity motion** the formula is still:

$$Q(t) = P + tV$$

where P is the position at time $t = 0$ and V is the velocity. The length of V is the speed.

Eliminating the parameter can be done in $3D$ too, although this has less utility than in $2D$. Essentially, you describe the geometrical track as the solution of a system of two linear equations in the three variables, X, Y and Z.

A parametric equation in $3D$:

$$Q(t) = \langle X, Y, Z \rangle = P + tV$$

yields the 3 equations:

$$X = p_1 + tv_1$$
$$Y = p_2 + tv_2$$
$$Z = p_3 + tv_3$$

Solving for t in one of these equations and **eliminating this parameter** from the other two gives the system of two linear equations in three unknowns: a line in space.

If you can find two vectors N_1 and N_2 which are not multiples of each other and which are both normal to V you can get formulas for a system of two equations (whose solution is the line) with less effort.

$$(Q - P) \cdot N_1 = 0$$
$$(Q - P) \cdot N_2 = 0.$$

This system is called the **normal form** for the line.

Problem 1: A Decomposition

Decompose $\langle 4, 5, 1 \rangle$ into the sum of a vector which is a multiple of $\langle 1, 2, 3 \rangle$ and another vector perpendicular to $\langle 1, 2, 3 \rangle$.

The Solution:

$$\begin{aligned}
P &= \left(\frac{\langle 4, 5, 1 \rangle \cdot \langle 1, 2, 3 \rangle}{\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle} \right) \langle 1, 2, 3 \rangle \\
&= \left(\frac{4 + 10 + 3}{1 + 4 + 9} \right) \langle 1, 2, 3 \rangle = \left\langle \frac{17}{14}, \frac{34}{14}, \frac{51}{14} \right\rangle. \\
V - P &= \left\langle \frac{56}{14}, \frac{70}{14}, \frac{14}{14} \right\rangle - \left\langle \frac{17}{14}, \frac{34}{14}, \frac{51}{14} \right\rangle = \left\langle \frac{39}{14}, \frac{36}{14}, \frac{-37}{14} \right\rangle.
\end{aligned}$$

Verification:

$$P \cdot (V - P) = \left\langle \frac{17}{14}, \frac{34}{14}, \frac{51}{14} \right\rangle \cdot \left\langle \frac{39}{14}, \frac{36}{14}, \frac{-37}{14} \right\rangle = \frac{663}{196} + \frac{1224}{196} - \frac{1887}{196} = 0.$$

Problem 2: Parametric Constant Velocity Motion

(i) Write a parametric vector equation for the position of an object moving with constant velocity and with position vector $\langle -3, 0, 6 \rangle$ at time 5 and $\langle 12, 6, 18 \rangle$ at time 8.

(ii) When will it run into the plane $X + Y + Z = 100$?

(iii) Eliminate the parameter to give a system of equations for this line.

The Solution:

The movement in the 3 seconds from $t = 5$ to $t = 8$ is:

$$\langle 12, 6, 18 \rangle - \langle -3, 0, 6 \rangle = \langle 15, 6, 12 \rangle.$$

The movement each second is one third of this:

$$(1/3) \langle 15, 6, 12 \rangle = \langle 5, 2, 4 \rangle.$$

The position at time 0 is:

$$\langle -3, 0, 6 \rangle - 5 \langle 5, 2, 4 \rangle = \langle -28, -10, -14 \rangle.$$

So the equation for parametric motion can be written in three ways:

$$Q(t) = \langle -28, -10, -14 \rangle + t \langle 5, 2, 4 \rangle$$

or

$$Q(t) = \langle -28 + 5t, -10 + 2t, -14 + 4t \rangle$$

or

$$X = -28 + 5t, \ Y = -10 + 2t \text{ and } Z = -14 + 4t$$

whichever you prefer.

It remains to solve the second part of the question: When will it hit "The Wall?"

$$X + Y + Z = (-28 + 5t) + (-10 + 2t) + (-14 + 4t) = 100$$

which implies $t = 152/11$ seconds is the time when it "hits the wall."

Finally, we eliminate t: The vectors $\langle -2, 5, 0 \rangle$ and $\langle 0, 4, -2 \rangle$ are both perpendicular to the velocity vector $\langle 5, 2, 4 \rangle$. So the equations

$$(\langle X, Y, Z \rangle - \langle -28, -10, -14 \rangle) \cdot \langle -2, 5, 0 \rangle = 0$$
$$(\langle X, Y, Z \rangle - \langle -28, -10, -14 \rangle) \cdot \langle 0, 4, -2 \rangle = 0$$

give a non-parametric system for this line, which is the solution to the third part of the problem. These equations reduce to the system:

$$-2X + 5Y = 6 \text{ and } 4Y - 2Z = -12.$$

Problem 3: Draw a Picture of Parametric Constant Velocity Motion in $3D$.

Draw a picture to describe the constant velocity motion that has position vector $\langle 0, 1, -1 \rangle$ at time $t = 0$ and $\langle 1, 2, 1 \rangle$ at time $t = 1$.

The Solution:

We simply plot several points in a representation of a $3D$ coordinate system and connect the dots in the obvious way. There are a couple of issues to be emphasized. First, you should pick a scale on the axes so that the points you are plotting are well separated. You might want to adjust the "point of view" from which you look at the axes when you draw your picture to make this easier. Second, you should label your points with times to indicate the direction of motion. Third, don't expect too much. Unless you can rotate the picture around and see your picture from various angles the fact that you are attempting to represent three dimensions on a two dimensional piece of paper means you will lose a lot of information. Computer algebra systems are a HUGE benefit when graphing $3D$ pictures for that reason.

In our problem, the velocity vector is $\langle 1, 1, 2 \rangle$ so as time passes we add on increasing multiples of that vector.
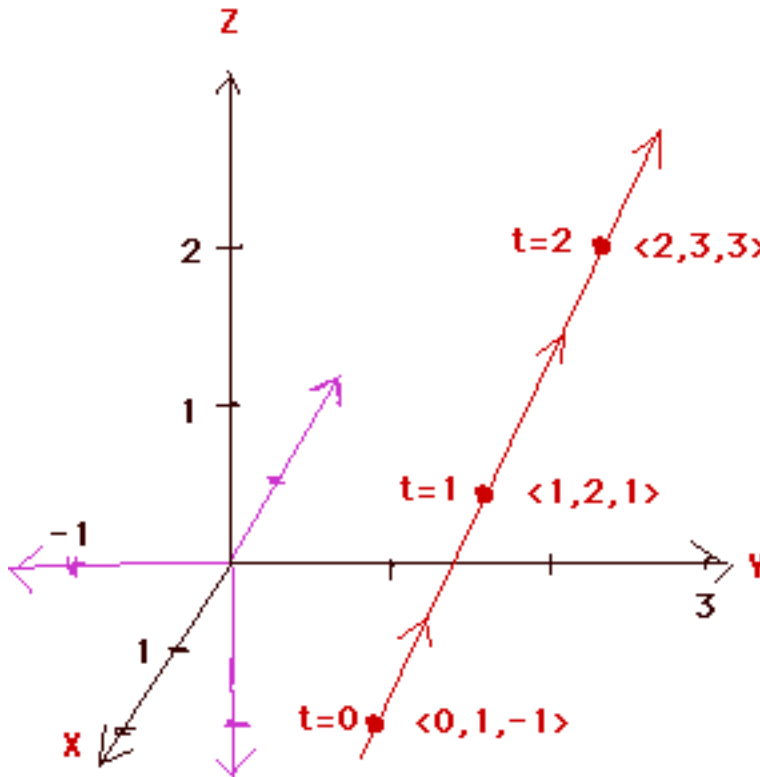
10.1. ***Exercise***. *(i) Decompose $\langle -4, 1, 6 \rangle$ into the sum of a vector which is a multiple of $\langle -1, 8, 1 \rangle$ and another perpendicular to $\langle -1, 8, 1 \rangle$.*

(ii) Decompose $\langle 0, 1, -1 \rangle$ into the sum of a vector which is a multiple of $\langle -1, 1, 2 \rangle$ and another perpendicular to $\langle -1, 1, 2 \rangle$.

(iii) Write a parametric vector equation for the position of a moving object located at $\langle 0, 5, 1 \rangle$ at time 0 and with velocity vector $\langle -1, 7, 7 \rangle$. Where will it be at time 10? Where was it at time -5? Draw a picture of this motion.

(iv) The vectors $\langle 5, 3, 0 \rangle$ and $\langle 0, 7, -5 \rangle$ are both perpendicular to $\langle -3, 5, 7 \rangle$. Following that pattern, find two vectors perpendicular to $\langle a, b, c \rangle$.

(v) Write a parametric vector equation for the position of a moving object whose position vector is $\langle -2, 9, 1 \rangle$ at time 5 and $\langle 12, 16, 8 \rangle$ at time 12. Draw a picture of this motion. When will it "hit a roof" at $Z = 120$? Finally, eliminate the parameter to form a system of two equations for this line.

10.2. ***Exercise***. * *(i) We suppose $Q(t) = P + tV$ to be a parametric vector equation with geometrical track which we denote L. Show that $\overline{Q} = \overline{P} + t\overline{V}$ has the*

same geometrical track as Q, where

$$\overline{P} = P - \frac{P \cdot V}{V \cdot V} \, V \quad and \quad \overline{V} = \frac{V}{|V|} \, .$$

Note also that $\overline{P} \cdot \overline{V} = 0$ and $\overline{V} \cdot \overline{V} = 1$.

(ii) The point on L nearest to the origin has position vector $\overline{Q}$ at which $\overline{Q} \cdot \overline{Q}$ is a minimum. But $\overline{Q} \cdot \overline{Q} = \overline{P} \cdot \overline{P} + t^2$ which obviously has a unique minimum for $t = 0$. So the near spot has position vector

$$\overline{Q}(0) = \overline{P} = P - \frac{P \cdot V}{V \cdot V} \, V.$$

This is the part of P perpendicular to V and is the one and only position vector of a point on L which is perpendicular to the line.

(iii) Show that the point on L closest to the point with position vector A has position vector

$$B = P - \frac{(P - A) \cdot V}{V \cdot V} \, V.$$

B is the only position vector of a point on L for which $B - A$ is perpendicular to the line. Draw pictures to convince yourself of why this should be true.

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11. Angles in Higher Dimensions

We are going to do some calculations here to make it seem reasonable to use the dot product to determine the angles between vectors in higher dimensions and not just the 2 dimensional XY plane.

We will presume V and W are two vectors in some higher dimensional setting, such as 3 dimensional space. We presume that neither is a multiple of the other. The case where one is a multiple of the other is left as an exercise. We presume that distances in this setting are defined by an extension of the old Pythagorean distance formula. So the distance between the points $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ is given by

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots}.$$

The three dots signify that you should keep on going in the same pattern till you run out of coordinates.

So the dot product of the vectors V and W is $V \cdot W = v_1 w_1 + v_2 w_2 + \dots$ and the magnitude of a vector such as V is

$$|V| \; = \; \sqrt{v_1^2 + v_2^2 + \dots} \; = \; \sqrt{V \cdot V}.$$

Our goal is to show that with these assumptions it makes sense to write

$$V \cdot W \; = \; |V||W| \cos(\theta)$$

where θ is the angle between V and W.

Let's define the three vectors:

$$P = \left(\frac{V \cdot W}{W \cdot W}\right) W \quad \text{and}$$

$$Q = \frac{V - P}{|V - P|} \quad \text{and}$$

$$T = \frac{W}{|W|}$$

It is an interesting exercise to show that $Q \cdot T = 0$.

It is obvious that $T \cdot T = 1$ and $Q \cdot Q = 1$.

We now define a function F that takes ordered pairs in the plane to points in the space where the vectors V and W live.

We define $F(r, s) = (rq_1 + st_1, rq_2 + st_2, \ldots)$, where the q_i and t_i come from the coordinates of Q and T respectively.

The following messy calculation is the key result.

The square of the distance between two points $F(a, b)$ and $F(r, s)$ is

$$(aq_1 + bt_1 - rq_1 - st_1)^2 + (aq_2 + bt_2 - rq_2 - st_2)^2 + \ldots$$
$$= ([a - r] q_1 + [b - s] t_1)^2 + ([a - r] q_2 + [b - s] t_2)^2 + \ldots$$
$$= (a - r)^2 \left(q_1^2 + q_2^2 + \ldots\right) + 2(a - r)(b - s)(q_1 t_1 + q_2 t_2 + \ldots)$$
$$+ (b - s)^2 \left(t_1^2 + t_2^2 + \ldots\right)$$
$$= (a - r)^2 + (b - s)^2$$
$$(\text{because } Q \cdot T = 0, \ Q \cdot Q = 1 \text{ and } T \cdot T = 1.)$$

This is just the square of the distance between (a, b) and (r, s)! This means that the distances between all pairs of points in the XY plane are unchanged when transported by F.

So any triangle in the XY plane, determined by three points in the XY plane, is taken by F to three points the same distance apart from each other. So the triangle formed from these has the same interior angles as the triangle formed from the original three points in the XY plane.

Let $A = \langle |V - P|, V \cdot W/|W| \rangle$ and $B = \langle 0, |W| \rangle$.

The tips (when in standard position) of A and B are taken by F to the standard-position tips of V and W respectively (check that out!)

So the angle between A and B is the same as the angle between V and W.

So $A \cdot B = |A||B|\cos(\theta)$ where θ is the angle between V and W.

The result now follows from a little algebra:

$$V \cdot W \;=\; A \cdot B \;=\; |A||B|cos\,(\theta)$$

$$=\; \sqrt{(V-P)^2 + \frac{(V \cdot W)^2}{W \cdot W}}\ \ |W|\ cos\,(\theta)$$

$$=\; \sqrt{V \cdot V - 2V \cdot W\frac{V \cdot W}{W \cdot W} + \frac{(\,V \cdot W\,)^2}{(\,W \cdot W\,)^2}\,W \cdot W + \frac{(\,V \cdot W\,)^2}{W \cdot W}}\ \ |W|\ cos\,(\theta)$$

$$=\; \sqrt{V \cdot V}\ \ |W|\ cos\,(\theta)\quad=\quad |V|\,|W|\ cos\,(\theta)\,.$$

11.1. *Exercise*. *(i) Under what conditions, exactly, will the equation*

$$|V + W|^2 = |V|^2 + |W|^2$$

be true? (hint: $|V + W|^2 = (V + W) \cdot (V + W)$.)

(ii) If the angle between V and W is θ, what is the angle between $-V$ and W?

(iii) * Show that $|\,|V|\;-\;|W|\,| \le |V + W| \;\le\; |V| + |W|$. This is called the **Triangle Inequality**. *What does this inequality have to do with triangles? When, exactly, will you have equality on one side or the other?*

11.2. *Exercise*. *Show that F takes any point on a line segment between (a, b) and (r, s) to a point on the line segment between $F(a, b)$ and $F(r, s)$.*

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12. The Cross Product

The **cross product** is another way of multiplying vectors, like the dot product. In this case, however, the "answer" is another vector rather than a number. We define this product for vectors in $3D$ only—if you have $2D$ vectors involved you must think of them as $3D$ vectors with zero third component to use the cross product.

If $V = \langle v_1, v_2, v_3 \rangle$ and $W = \langle w_1, w_2, w_3 \rangle$ we define

$$V \times W = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}.$$

Notice that the first entry in the product does NOT involve the first entry of either vector. The second entry in the product does NOT involve the second entry of either vector. The third entry in the product does NOT involve the third entry of either vector. Also, the positive term in the first and third entry in the product has the V and W entries "in order" while the middle term has them backwards.

If V and W are in the XY plane note that $V \times W$ is a multiple of $\vec{k}$.

Another interesting and useful product is called the **triple scalar product**. It is defined for triples of $3D$ vectors P, V and W by:

$$\det(\mathbf{P}, \mathbf{V}, \mathbf{W}) = P \cdot (\,V \times W\,)\,.$$

The triple scalar product is a number and can be calculated by dotting the first vector against the cross product of the last two vectors. The det notation comes from the fact that, for those among you who know about **determinants**, it is the determinant, usually denoted $\det A$, of the matrix A where

$$A \;=\; \begin{pmatrix} p_1 & p_2 & p_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

12.1. **Exercise**. *(i) Show that $V \times W = -\,W \times V$.*

(ii) Show that $(kV) \times W = k\,(V \times W) = V \times (kW)$ for any real number k.

(iii) Show that $(\,P + V\,) \times W \;=\; P \times W + V \times W$ and $W \times (\,P + V\,) = W \times P + W \times V$.

(iv) Show that $V \times W$ is perpendicular to both V and W.

(v) Show that if you switch any two of the vectors in a triple product, the sign of its value switches. For example $\det(P, V, W) = -\det(V, P, W)$.

(vi) * Suppose $V \times W \neq 0$ and N is a vector with $N \cdot V = 0$ and $N \cdot W = 0$. Show that N must be a multiple of $V \times W$.

This last fact is an algebraically messy exercise involving coordinates, but it is quite important. Here is a hint.

Write out the components of $C = V \times W$. Now consider the equations $N \cdot V = 0$ and $N \cdot W = 0$. They give you two equations involving n_1, n_2 and n_3. Use the "elimination method" three times on these two equations to eliminate n_1, n_2 and n_3, one at a time. Recognize the messy coefficients as c_1, c_2 and c_3. This will yield three equations:

$$c_2 n_3 = c_3 n_2 \quad and \quad c_3 n_1 = c_1 n_3 \quad and \quad c_1 n_2 = c_2 n_1.$$

Since $C \neq 0$ at least one of the components of C is nonzero. If $c_i \neq 0$ then $N = \frac{n_i}{c_i} C$.

The facts $(i) - (iii)$ shown in the exercise above combined with the properties of dot products make the triple scalar product another example of a **tensor**. Since there are three vectors involved it is called a 3-tensor. The property from part (v) in this context is called **antisymmetry**. The triple scalar product is an example of an antisymmetric 3-tensor.

12.2. **Exercise**. *Recall Exercise 5.3. If $V = \langle v_1, v_2 \rangle$ and $W = \langle w_1, w_2 \rangle$ we defined $\det(V, W)$ to be the number $v_1 w_2 - v_2 w_1$. Show that for any 2D vectors V, P and W and any constant a*

$$\det(aV + P, W) = a\ \det(V, W) + \det(P, W) \quad and \quad \det(V, W) = -\det(W, V).$$

These properties imply that $\det(V, W)$, defined for pairs of 2D vectors, is an antisymmetric 2-tensor.

You can show by doing a lot of multiplying and gathering of like terms that
$$(V \times W) \cdot (V \times W) + (V \cdot W)(V \cdot W) = (V \cdot V)(W \cdot W).$$

We observe that the second and third terms above can rewritten as:
$$(V \times W) \cdot (V \times W) \ + |V|^2 |W|^2 \ cos^2(\theta) = |V|^2 |W|^2 \ ,$$

where θ is the angle between V and W . It follows that:
$$|V \times W| = |V||W| \ sin(\theta).$$

This is reminiscent of the similar fact about dot products and is just as useful, though in different contexts.

Here we will examine four related geometrical uses for cross product.

Area of a Parallelogram

First we find the **area of a parallelogram**. By dividing the picture below into two triangles we see that the area is just twice "one half the base times the height" which is
$$|V \times W| = |V||W| \ sin(\theta).$$

Cross product can be used to find the area of this region.
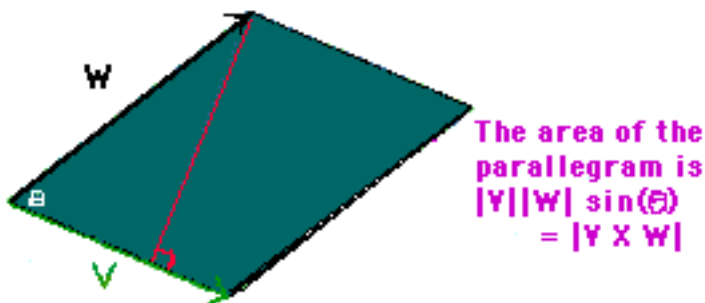
12.3. ***Exercise***. *In the calculation above for the area of the parallelogram, we drew a picture with the angle between V and W less than 90°. Is the formula still valid if the angle exceeds 90°?*

Volume of a Parallelepiped

Second, we consider the **parallelepiped** below. In general, parallelepipeds are $3D$ objects bounded by three pairs of parallel parallelogram faces.

Our parallelepiped is a (possibly) bent box determined by the three vectors T, V and W. You can think of it as a deck of parallelogram cards that has been pushed so that the edge of the stack runs at an angle—along the vector T.

If you pushed the stack straight again, the shape of the cards is unchanged and so is the volume of the stack.
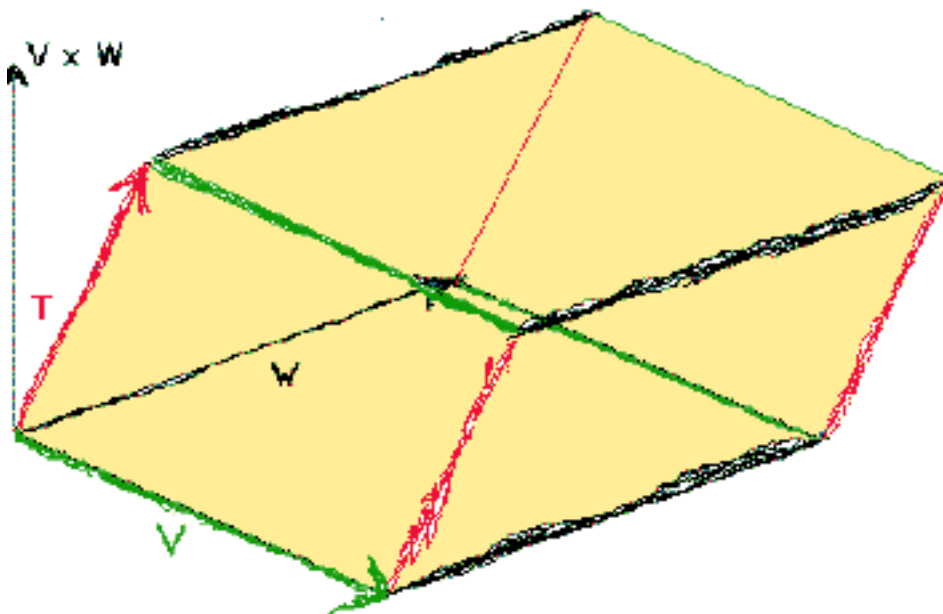
If we want the volume of the stack we need to know the height of the stack. Then the volume will be the height times the area of a card. We just found that the area of a card is $|V \times W|$.

The height of the stack is the length of the part of T that is perpendicular to the cards. The vector $\frac{V \times W}{|V \times W|}$ is a unit vector perpendicular to the cards.

So the height of the stack is $\left| T \cdot \frac{V \times W}{|V \times W|} \right|$.

Finally, we have the **volume of the parallelepiped** as

$$\left| T \cdot \frac{V \times W}{|V \times W|} \right| |V \times W| = |\, T \cdot (\, V \times W \,)\,| = |\det(T, V, W)|$$

This gives us an interpretation for the triple scalar product as the volume of the shape generated by the three vectors. This turns out to be more useful than you might at first imagine.

The discussion from above might (or might not) constitute a compelling argument for you as to why we call this number the volume of a parallelepiped. In any case, that is *the definition* of volume. It is our dog. We can call it **Fido** if we want to.

At this point we digress to contemplate the **meaning** of procedures such as this. We all have an intuitive concept of the word "volume" as having to do with the extent of a "fat" object, and in the last paragraphs we aligned that concept for

a certain type of fat object with a number. This number is calculated by vector operations on edges. It is (at least) twice removed from our intuition about the extent of any real physical object.

In the first place, we model the edges of some real object with which we have experience by vectors. Then we perform operations to obtain a number.

At risk of boring you with an obvious point, this number is not the "extent" of any object in the world. It is just a number. Its value may be relevant in comparison to other numbers calculated with other vectors with their own imaginary link to still *other* real objects. It is useful to a scientist or engineer only to the extent it gives explanatory or predictive power over events in the world. Does the number correspond to intuition built on other cases already understood? Does it invite extension to cases not yet understood? Is it simple, or simpler than competing alternatives? If the answers are affirmative, the scientist will carry on, thinking of these numbers as a measure of "fat extent." If not, the model is discarded, at least in some circumstances. Every now and then a model violates a long-held, even cherished, intuition that must be abandoned based on the surprising success of a model yielding peculiar predictions. The creators of the model may win Nobel Prizes.

Mathematicians, on the other hand, might be intrigued by the structure of a model itself, and might not be bothered excessively if it didn't match intuition obtained from "looking out the window" at something real. Surprising or downright weird behavior is not rare in these models. As a means of assigning numbers to vectors, for example, our volume idea from above stands by itself. Trying to puzzle out the details of this structure and the relationships among this and other structures— such as "flat extent" and "straight extent"—could occupy a mathematician for a very long time.

You might regard this as a "division of labor" or even "symbiosis." The mathematicians stock the shelves with well understood and beautiful models, packaged and ready-to-go. They seek to understand the patterns they see as they create these ideas. They build new models, by analogy and lifetimes of effort and inspiration, hunting for unifying ideas and a "big picture" that will clarify that which was cloudy. The scientists root around on the shelves looking for models that match what they see out the window guided by their esthetic sensibilities and previously acquired intuitions about the world. Experimenters create ever more subtle ways of stealing a glimpse at some previously hidden part of the world. Every now and then someone makes a great leap of intuition and builds a brand new model, of a type never thought of by anyone, to try to understand what has been seen. And in this way they inspire and repay their mathematical brothers and sisters.

12.4. **Exercise**. *A **tetrahedron** is a four-sided three dimensional object with equilateral triangular faces.*

Show that the vectors

$$\langle 1, 0, 0 \rangle \ , \ \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle \ , \ \left\langle \frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3} \right\rangle$$

lie on three edges of unit length of a tetrahedron. Conclude that a tetrahedron of edge length d has volume $\dfrac{d^3\sqrt{2}}{4}$.

<div align="center">

Area of a Shadow

</div>

As a third example we consider the area of a tilted parallelogram and how that area is related to the **area of a shadow** of this parallelogram on a plane.

Let us presume that P, Q, A and B are vectors and $P \times Q = N \neq 0$ and $A = P + aN$ and $B = Q + bN$ for certain numbers a and b. Let $M = A \times B$. The picture below describes the situation. A and B form a tilted parallelogram with area $|M|$ while P and Q form the shadow parallelogram with smaller area $|N|$.
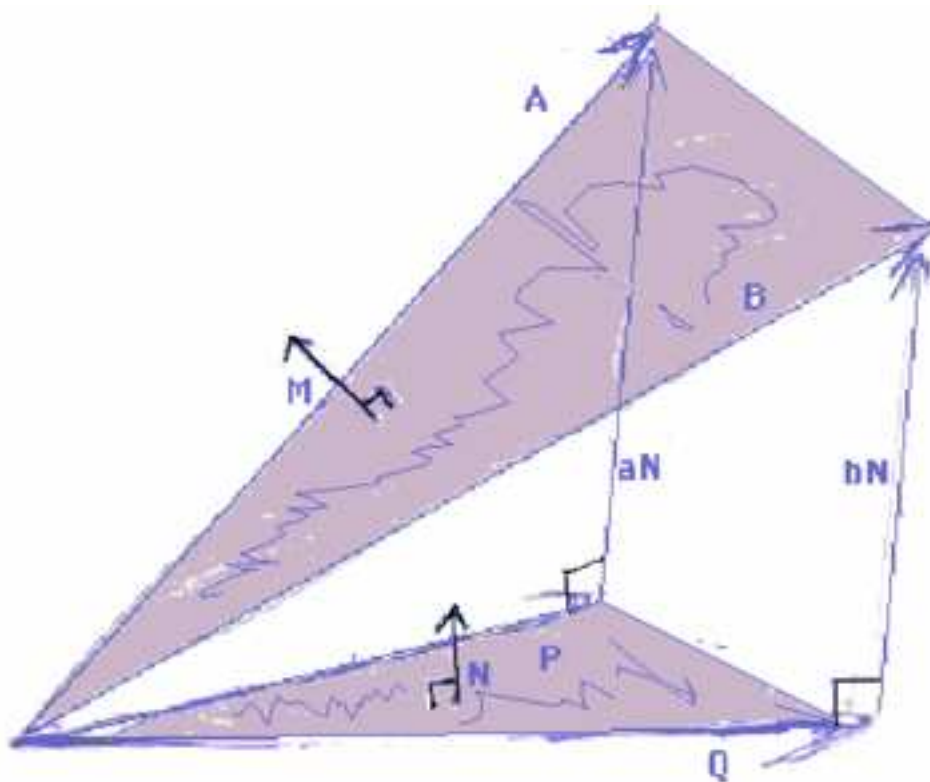
The **angle between two flat surfaces in 3D** is defined to be the angle between their normal vectors. In our case, the angle between the two parallelograms (or the triangles in the picture) is the angle between M and N. If θ is this angle

we have

$$
\begin{aligned}
|M||N|cos\,(\theta) \;=\; M \cdot N \;&=\; (A \times B) \cdot N \\
&=\; [(P + aN) \times (Q + bN)] \cdot N \\
&=\; (P \times Q) \cdot N + (P \times bN) \cdot N + (aN \times Q) \cdot N + (aN \times bN) \cdot N \\
&=\; N \cdot N \quad \text{because the last three terms are 0} \\
&=\; |N|^2 .
\end{aligned}
$$

We have just shown that

$$
|M|cos\,(\theta) \;=\; |N| \quad \text{or} \quad |A \times B|cos\,(\theta) \;=\; |P \times Q|
$$

where θ is the angle between vectors M and N, which are normal to the tilted and shadow parallelograms. In words this is:

(The Area of a Tilted Parallelogram) $\cos(\theta)$ $=$ Area of the Shadow.

A Consequence of the Pythagorean Theorem

There is an interesting consequence of the Pythagorean Theorem applied to this last calculation. Recall that any vector, M included, can be written as

$$
M = |M| \langle cos(\theta_1), cos(\theta_2), cos(\theta_3) \rangle
$$

where the angles are those between M and the coordinate vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$. These coordinate vectors are each normal vectors to one of the coordinate planes.

The equation

$$
|M|^2 = |M|^2 cos^2(\theta_1) + |M|^2 cos^2(\theta_2) + |M|^2 cos^2(\theta_3)
$$

is nothing more than the Pythagorean Theorem in $3D$ applied to M and hardly a surprise. But we have just shown that the area of the shadow of the tilted parallelogram onto the relevant coordinate plane is $|M|cos(\theta_i)$. So the last equation has the following intriguing interpretation.

The Square of the Area of the Tilted Parallelogram
is the Sum of the Squares of the Areas
of the Shadow Parallelograms on the Coordinate Planes.

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