RELATIVITY

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1. Spacetime

In this section we outline the basic ideas of this work and definitions of observer. This will lead up to, but not include, the creation of Physical Law. We emphasize limitations and the essential arbitrariness, and consequent flexibility, of aspects of the Scientific Enterprise described thereby.

Our purpose is to very carefully identify the relationship between aspects of the real world that we experience and various mathematical structures which have been created to provide a framework for building models of this "reality."

One goal is to describe the location occupied by physical objects in space and time, and we must presume we (think we) know what that means. Physicists want to imagine something physical happening at a place and time and moving, at later times, to different places. We are not dealing with those physical things here, *only the collective of possible locations these things might occupy.* This is sufficient to develop Physical Law for simple situations, like the motion of "idealized particles." If successful, the program outlined herein could be expanded to include other measurable features of the world.

The "we" of the previous paragraph is a literal plural: a collective of **observers** who agree on a common way of describing the world and who, though they may

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disagree about many aspects, have the same fundamental understanding of its nature and who use a shared vocabulary to describe it. The observers who agree on these things will be called **allowable** for their list of assumptions.

(i) Each allowable observer must agree, first of all, that there actually *is* a world and that other observers are experiencing the *same* world, each from his or her own point of view. Allowable observers agree that some of the physical entities that inhabit this world possess a quality called **location**. The collective of possible locations will be represented by $\boldsymbol{\mathcal{W}}$, a set whose elements may also be called **events**. $\boldsymbol{\mathcal{W}}$ may be referred to as **spacetime**.

(ii) Each allowable observer must agree that there is something called **time**[1] associated with each event. Allowable observers agree that the real numbers $\mathbb{R}$ are an appropriate tool to describe times, and to use these real numbers to impose **temporal order** on different events with bigger numbers corresponding to later times. Other than that, the nature of this association is unspecified. However, given the agreement of all observers about the essential nature of the world we have the following restriction about locations that could describe **actual physical objects** in the world.

> **If one physical object is found by any observer to be at two different events then that observer must perceive these two events to be at distinct times and every observer must perceive these two events to be *in the same temporal order.***

So, for instance, if an explosion takes place at one event and a light particle emanating therefrom traverses space to create a chemical reaction in a silver iodide crystal, no allowable observer could perceive the world in a way in which the chemical reaction at the crystal *precedes* the explosion.

(iii) Each allowable observer must agree that there is a property of events called **place**. Each allowable observer must agree that the collection of all possible places, the collection of potential values of this quality, does not change with time. Each observer believes he or she knows what it means to be at the same place for different times, and that any choice of place and time will characterize exactly one event.

A **worldline** is a choice of events for an interval of times. Thus, each worldline is dependent on the perception of time of its observer. Later assumptions will restrict the options for time perception available to allowable observers.

We imagine the entities studied by Physicists, "particles" in particular, to be imbued with various qualities and to be at locations. It is the business of Physicists to identify and predict the worldlines associated with these entities. So Physicists try to make simple statements about *perceptions*, specifically the perceptions of allowable observers, things they can measure such as place and time.

This note will have little to say about how these particles behave, beyond the most primitive generalities, though many of the assumptions anticipate the necessities of Mechanics. For instance the assumptions about time allow for differentiability of worldlines with respect to time, necessary in the formulation of Newton's Laws, a central topic in first year Physics courses.

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[1] The Clepsydra (water clock), the hourglass, the candle-clock and the sundial represent ancient attempts to quantify this elusive . . . something. This effort extends back *at least* 4500 years.

We are interested, instead, in events and worldlines in general, and necessary relationships among ***perception*** *of these by observers who share certain assumptions about how the world, and all it contains, must behave.*

If these assumptions change, so we have a different collection of observers, we will see that the relationships among these perceptions will change too. When these assumptions change statements made about these perceptions by Physicists to explain (for instance) particle motion might go from simple to complex, or conversely.

We will later focus on two classes of observers, the allowable observers of Newtonian Mechanics and the allowable observers of (Special) Relativistic Mechanics.

But in this section we continue with the most basic assumptions, shared by all these observers.

(iv) Allowable observers will have decided to identify events in our world $\mathcal{W}$ with $\mathbb{R}^4$, the 4-tuples of numbers, where the first coordinate represents time and, at each time, the last three prescribe a place.

Each allowable observer is required, somehow, to make this association explicit, and is said to "preside" over an **invertible** function called a **coordinate map**

$$\boldsymbol{x} \colon \mathcal{W} \to \mathbb{R}^{\mathbf{4}}.$$

If it should be convenient to name an observer apart from coordinates we will use a locution such as "**the observer $\mathcal{O}$ who presides over coordinates $\boldsymbol{x}$.**"

If $q \in \mathcal{W}$ we denote the time coordinate of q in coordinates x by $\boldsymbol{x_q^0}$ and the space part of q in these coordinates by $\boldsymbol{\overline{x}_q = (x_q^1, x_q^2, x_q^3)} \in \mathbb{R}^{\mathbf{3}}$.

$$x_q = (x_q^0, \overline{x}_q) = (x_q^0, x_q^1, x_q^2, x_q^3) \in \mathbb{R}^4.$$

A worldline can be identified with a function

$$\boldsymbol{f} \colon \mathbb{R} \to \mathcal{W}$$

where $f(t)$ is the event corresponding to time t according to some observer and **it is an ultimate goal of mechanics** (Newtonian or SR or any other) **to predict and explain worldlines in a way that is as simple as possible, whose parameterizations are by definition based on the time concept of a specific allowable observer.**

There is one type of worldline each allowable observer can identify. Given any event in the world, specified by a combination of time and place, there is the constant worldline: the worldline that assigns that place to all times.

Typically, different observers will not agree on which worldlines are constant.

We explicitly do not insist that our allowable observers assign coordinates that reflect this constancy. If $f \colon \mathbb{R} \to \mathcal{W}$ is a constant worldline according to the observer with coordinates x then we cannot conclude (yet) that $x \circ f$ has constant space coordinates.

The procedures which allow us to differentiate among observers and define classes of observers apparently require at least some Physics: a bit of unaesthetic circular reasoning that can't be avoided. Allowable observers will identify themselves by agreeing to measure things in the world in specified ways, and the act of measurement must involve physical principle.

Physicists see if they can build a convenient and simple Physics ***for these observers.*** We can then refine our perception of allowable observer, and repeat as our power to describe and measure and theorize improves. This is the evolutionary approach described below.

Willard Van Orman Quine quotes Otto Neurath:[2]

> "We are like sailors who on the open sea must reconstruct their ship but are never able to start afresh from the bottom. Where a beam is taken away a new one must at once be put there, and for this the rest of the ship is used as support. In this way, by using the old beams and driftwood the ship can be shaped entirely anew, but only by gradual reconstruction."

But at a certain point we may discover that there is simply no way to make the natural evolution of our Physical Laws both simple and "true." In order to accommodate observer's perceptions Physical Law might need to add "bells and whistles" (or epicycles) of debilitating complexity, and looking at the world with greater precision only increases this complexity.

We might then throw out the Laws and try for new Laws using these same observers. Or we can try to make a whole new definition of allowable observer. Perhaps these newly "privileged" observers can create, once again, simple Physical Law which allows them to accurately predict worldlines.

This step would not be undertaken lightly: it would precipitate an instance of ***paradigm shift*** as discussed by Thomas Kuhn[3] in his influential *"The Structure of Scientific Revolutions."*

The "ship-under-sail" analogy might be slightly misplaced, in that our ship might possess less structural integrity than is implied there. People who are uncomfortable with this lack of "inevitability" in these theories are following in a long tradition of those who want more. George Berkeley's scathing critique of Newton's methods and Calculus in particular were answered by the followers of Newton with the plain fact that these methods *worked* and provided verifiable answers to problem-after-problem that had been around for thousands of years. So these answers provided justification for the methods, which had abandoned the logical rigidity of that gold standard of Mathematics, Euclidean geometry.

To carry on with our business, we understand that perfect observation is impossible and do not require that of our allowable observers. ***We do require procedures that we assume can be carried out with all the precision available to the current "generation" of observer/Physicists.***

(v) We assume that we can describe a clock that everyone would agree measures time accurately. The period associated with a vibration mode of certain atoms comes to mind as a good choice. Or the duration of a day.

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[2]Quine (1908-2000) was a Logician and wrote extensively on Philosophy of Science. Neurath (1882-1945) was influential in the development of Logical Empiricism and a founding member of the Vienna Circle.

[3]Thomas Kuhn (1922-1996) was a Physicist, Historian and an influential (and popular) Philosopher of Science. He worked, in temporal order, at Harvard, Berkeley, Princeton and finally MIT.

All allowable observers agree to use the same type of clock to measure time displacements between two events in the world. This allows us to create a time unit, the **second**, whose definition could be agreed-upon by each allowable observer. Our allowable observers is required to use a unit proportionate to this one to measure time displacements.

This does not imply that observers would agree on specific time displacements, even if they decide to use the same units. The agreement is on the method of calculating time displacement, not the values themselves.

(vi) Each allowable observer $\mathcal{O}$ selects an origin $\mathcal{O}_0$ in the world and uses the constant worldline through $\mathcal{O}_0$ to provide an origin at each time. Each allowable observer will choose coordinates for which the constant worldline through this location has space coordinates which are the natural origin in $\mathbb{R}^3$ for each time. This *does not imply that more general constant worldlines have coordinates which are independent of time. At this point this only must be true for this observer for the constant worldline through the origin.*

And different observers may disagree on this choice of origin and its unmoving nature.

(vii) To some observers, the collective of displacements in the world at each particular time may seem to be a three dimensional real vector space. Our allowable observer is one of these, and agrees to use a coordinate basis at each time to reflect this structure. Because of choice of origin in (vi), this allows the observer to use vectors, which by their nature only refer to displacements, to identify the actual places associated with events.

(viii) We presume that there are objects in the world accessible to all observers which exhibit a fixed size. A standard based on the diameter of a type of atom in the observer's lab might be appropriate. Or the length of a king's foot.

We will call this unit the **meter** and our allowable observers agree to use a unit of distance proportionate to this one to describe the magnitude of displacement.

Conditions (vii) and (viii) refer to displacements. These displacements are from one event to another with the same time component. *It is left to each observer to devise means to deduce facts about linear combinations, magnitudes and relative directions involving these displacements at fixed time, perhaps by some kind of approximation method.*

If it turns out that these requirements cannot be assumed consistently, that observers are wrong about this, there may be contradictions that emanate from theories that assume them. This will "tip off" these observers that there are problems which need to be addressed.

(ix) Some observers may agree that the distance specified in (viii) is Euclidean for each time, that distances in space can be calculated using a Euclidean inner product. Angle can be defined for these observers who can therefore find an orthonormal basis in the world for each time and associate that choice of basis with the standard basis in $\mathbb{R}^3$ for each time. We do not (yet) insist that these basis vectors are related to each other at different times.

We henceforth assume that allowable observers agree with this and make a specific choice of orthonormal basis at each time to create the coordinates over which that observer presides.

This does not imply that allowable observers would agree on the magnitude of any specific space displacement in the world, even if they decide to use the same units, or whether two particular displacements in the world are at right angles. Again, the agreement is on the method of calculation, not the results of that calculation.

In this note an allowable observer must agree on all nine assumptions, and the collection of assumptions embodied thereby in coordinate map $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ for that observer is called an **allowable framework.**

2. The Relativity Principle

So we have identified a collection of observers[4] each of whom presides over an allowable framework and who may begin to create, together, Physical Law.

Suppose we have observer $\mathcal{O}$ who presides over coordinates $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ and observer $\mathcal{O}'$ who presides over coordinates $y\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$.

Since $\mathcal{O}$ and $\mathcal{O}'$ are working with the same world, though they may perceive it differently, there is a translator, an invertible function

$$C = x \circ y^{-1} \colon \mathbb{R}^4 \to \mathbb{R}^4$$

from the coordinates preferred by $\mathcal{O}'$ to those preferred by $\mathcal{O}$.

We now can state the basic consistency principle required for any successful Physical Law, and which is intended to be a useful guide in its creation for any collection of observers.

The Relativity Principle

All allowable frames of reference are to be completely equivalent for our formulation of Physical Law.

In this context, this has the following meaning.

If observer $\mathcal{O}'$ comes up with a solution to some physical question using a Physical Law that has been developed for a group of observers (i.e. a prediction about a worldline) and transforms the answer in an appropriate way corresponding to some C then observer $\mathcal{O}$, who is related to $\mathcal{O}'$ via translator C, should produce that transformed answer when he or she "solves the same problem" directly.

It should be pointed out that each observer is free to develop Physical Law independently. At the least, each observer calculates solution-worldlines using a Physical Law based on his or her *perceptions.* No other observer is involved in this.

[4] Any observer class might, a priori, be empty. We assume it is not and see where that goes.

The point of the Relativity Principle is to recognize conflict, not create law. There is no possibility of dispute among observers who cannot communicate with each other, so in that case the principle is inapplicable.

But a collection of allowable observers must agree on many things to create their allowable frameworks. That means they must communicate to do this and the foundation of these agreements is, presumably, more certain and more obvious than any Physical Law.

So contradictions provided by violations of the Relativity Principle would be extremely important.

One possibility is that the observer class is actually empty: observers were deluding themselves about the nature of the world and possibility of doing the things they thought they could.

Another possibility is that the Physical Law they built together gives wrong predictions about worldlines: it is "false."

Sometimes Law and Observer(s) must both restructured, as in the abandonment of the Ptolomaic model of the solar system in favor of Galileo and Newton.

But Einstein only had to abandon Galileo to save Maxwell.

3. Galileo

Galileo created in "Dialogue Concerning the Two Chief World System," his 1632 defense of his scientific ideas, a thought experiment which leads to a Relativity Principle in the formulation of scientific law.

The statements contained therein directly contradict the assumptions about the world believed to be true from antiquity and, by implication, support heliocentrism.

Defending the physical truth of Copernican heliocentrism[5] was already deemed heretical in 1616. The Dialogues were found to be a defence of this idea[6], in spite of Galileo's rather unconvincing denial, in 1633.

In this work Galileo creates a debate between the foil, Simplicio, defender of Aristotle and Ptolemy, and Salviati, who attempts to enlighten him with modern ideas about nature. In particular, Salviati defends the idea that one should actually look at the world to see how it works, rather than simply accept ancient beliefs, unquestioningly. A third participant, Segredo, acts as an—initially—neutral party, an "intelligent layman."

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[5]Nicolaus Copernicus (1473-1543) in his work *De Revolutionibus Orbium Coelestium* placed the Sun as stationary, rather than the Earh, at the center of the universe. Though "closer" to the truth, this theory suffered from many of the flaws of the theory proposed by Ptolemy (100-170) in *The Almagest* in that it, too, presumed the motion of the planets to be confined to circles. This flaw in both theories required a plague of epicycles (though fewer and smaller in the Copernican theory) to deal with the increasing accuracy of celestial observation.

[6]Galileo was convicted through a plea bargain to a lesser crime than heresy by 7 of the 10 cardinals on the panel who judged him. This was very unusual, as almost all verdicts in trials for heresy were unanimous, and reflects the complex church politics of the day.

Segredo and Salviati are the names of two of Galileo's friends. Galileo claimed that Simplicio was named after Simplicius of Cilicia, a sixth-century commentator on Aristotle, but the dialog targets two of Galileo's critics Lodovico delle Colombe and the Padovan Cesare Cremonini, who refused Galileo's offer to view his astronomical discoveries through the new telescope.

After his conviction Galileo was confined to house arrest until his death in 1642. His case was not helped by the fact that the foolish-seeming Simplicio often argued as did the Pope, Pius VIII, in *his* writings.

In Stillman Drake's 1967 translation (page 186-7, The Second Day) we find Salviati holding forth as follows:

> *Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.*
>
> *When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.*

Salviati argues that descriptions of objects moving in the world are simple when the ship is "standing still" and remain simple when it is not, "so long as the motion is uniform and not fluctuating this way and that."

(a) We presume that at least *some* observers have access to telescopes and other basic tools and so observers who preside over an allowable framework can pick a **non-rotating orthonormal frame—an ordered basis in the world—whose vectors seem to point to fixed locations against the background of nearby stars**, and which will be used, for each time, to define the "space part" of that observer's coordinate map $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$.

At this point, together with the presumed constancy of the space coordinates of the origin worldline, we *can* conclude that the space coordinates of any worldline that an observer perceives as a constant worldline must be constant *for that observer.*

Different observers may use different bases to create their coordinates.

As we develop our Physics there may be improvements or modifications that some or all observers may use to certify "allowability." For instance a non-rotating frame may be redefined as one in which two small objects of equal mass connected by a string can never be oriented in such a way that they rotate about the center of the string yet the string is under no tension.

Obviously, this is a modification to Nuerath's ship that could be undertaken only well into the voyage: the words mass and tension must be carefully unpacked for this substitution to succeed.

(b) **At least some observers must be able and willing to choose an origin whose worldline coordinates have constant velocity with respect to the worldline coordinates of some presumed and agreed-upon non-accelerating standard object in the world.**

Possibly, the average velocity of nearby stars will be good enough. Or your lab tabletop.

Alternatively, we could try define a "good" origin by declaring that a massive object subject to no forces should, when caused to move in three independent directions, have worldline coordinates which are straight lines. This redefinition also requires much preliminary work involving the construction of Physical Law.

(c) **Allowable observers are required to use the second for time displacement, and if one allowable observer perceives two events to be separated by a specific numerical time increment then all do.**

The effect of this is to divide potential allowable observers into admissibility classes based on perception of simultaneity and for whom the time unit, the second, for one allowable observer coincides with the second for all.

With this assumption the only disagreement about worldline parameterizations for *allowable* observers is an adjustment of "time zero."

So if we are careful to talk about "time differences" corresponding to motion from one event to another, all allowable observers would agree on the chronology of movement from event to event on a worldline.

(d) **Each allowable observer must use the meter for space displacement magnitude and each allowable observer must measure the same distance between any pair of simultaneous events as any other allowable observer would measure.**

We have by (d) again subdivided potential allowable observers into admissibility classes, this time based on isometry at each time. This is a very powerful assumption.

For each (corresponding) time the space-part of the function $C = x \circ y^{-1}$ that translates between coordinates x for allowable observer $\mathcal{O}$ and coordinates y for allowable observer $\mathcal{O}'$ is an isometry from $\mathbb{R}^3$ to itself.

Given our 3-dimensional vector space assumptions about time slices (the set of fixed-time events in the view of each observer) in the world, and our insistence that time increments match and space coordinates be coordinates with respect to a choice of basis in the world, the translator function C must correspond, at each time, to translation of origin together with left multiplication by a matrix $\widetilde{M}$, when

$\mathbb{R}^4$ is represented by columns.

$$\widetilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} \\ 0 & m_{21} & m_{22} & m_{23} \\ 0 & m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

Let M be the submatrix in the lower right of $\widetilde{M}$.

Because distances are preserved M is orthogonal and (by the Cartan-Dieudonné Theorem) corresponds therefore to a reflection, a rotation (two reflections) or a rotation and a reflection (three reflections) for each time.

Because both observers agree that the direction of their respective axes are non-rotating with respect to the "firmament" of background stars, the matrix M does not depend on time.

There is quite a bit of "rigidity" in the list of assumptions we have assembled. If we perform a translation at each particular time to account for a (possibly) moving-but-constant-velocity origin with respect to one observer or the other, we have at each instant an isometry onto $\mathbb{R}^3$ that sends the origin to the origin.

The Mazur-Ulam theorem tells us that any such isometry must be a linear isomorphism and so the translator C from $\mathcal{O}'$ space coordinates to $\mathcal{O}$ space coordinates corresponds to matrix multiplication by some invertible matrix M for each time. This is an alternative route to this conclusion, which we found directly above.

We point out this alternative because this result follows by the isometry condition (d), and *doesn't* appeal to other features, or the source, of the translator function C. Given assumption (d) it might, therefore, be possible to "tweak" the earlier assumptions while retaining this conclusion. We won't go further in this direction here.

A collection of observers who agree on (i)-(ix) and (a)-(d) are called **Galilean Observers** and each is said to preside over a **Galilean frame** or **Galilean coordinates**.

Observers "tests themselves" in items (i)-(ix) and (a) and (b) for admissibility status, and are responsible for carrying out the instructions codified in these assumptions independently. But they must communicate while their ships are sailing to see if they agree with each other in assumptions (c) and (d), at least.

As an alternative viewpoint we could posit a "master" allowable observer, the "keeper of the standard meter." This special observer examines the universe and decides if the origin is accelerating or if the orthonormal basis is rotating. Other observers are then allowable if their origins are seen as constant velocity with respect to the master, and their representations of the world are isometric to this one for each time in such a way that the rotation matrix is constant with time.

However you conceive of this, we have codified rules for a class of observers, the Galilean observers.

Enter Newton and his Newtonian Mechanics for this class, an incredibly successful theory whose consequences are still being explored and profitably mined for applications.

4. The Galilean Group

We will see that the translators from one Galilean coordinate system to another form a group with composition of functions, called the **Galilean group**[7].

All allowable observers agree on the passage of time and the unit of distance. The origins may move with respect to each other with constant velocity. The axis differences may involve reflection and rotation.

To take advantage of matrix operations we identify $\mathbb{R}^4$ with column matrices.

Suppose Galilean observer $\mathcal{O}$ presides over coordinates $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ and Galilean observer $\mathcal{O}'$ presides over coordinates $y\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ with translator

$$C = x \circ y^{-1} \colon \mathbb{R}^4 \to \mathbb{R}^4$$

from y-coordinates to x-coordinates.

Suppose p is an event in the world and $y_p = a$ and $b = x\,(\mathcal{O}'_0)$, the x-coordinates of the $\mathcal{O}'$ origin. Then coordinates x_p are given by $C(a)$ as

$$\begin{aligned} C\begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} &= \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} + a^0 \begin{pmatrix} 0 \\ v_0^1 \\ v_0^2 \\ v_0^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1^1 & m_2^1 & m_3^1 \\ 0 & m_1^2 & m_2^2 & m_3^2 \\ 0 & m_1^3 & m_2^3 & m_3^3 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \\ &= \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_0^1 & m_1^1 & m_2^1 & m_3^1 \\ v_0^2 & m_1^2 & m_2^2 & m_3^2 \\ v_0^3 & m_1^3 & m_2^3 & m_3^3 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}. \end{aligned}$$

There are 7 free variables corresponding to translation and velocity of the $\mathcal{O}'$ origin and the collection of 3×3 rotation/reflection matrices have just three free variables.

So there are 10 free variables here: technically, this representation of the Galilean group is a ten dimensional Lie group, one dimension for each free variable. Any choice of these 10 variables (subject to the orthogonality condition) is a translator between any one Galilean observer and *some other* Galilean observer related to the first by origin translation and velocity and orthonormal basis choice specified in the representation.

Suppose $f\colon \mathbb{R} \to \boldsymbol{\mathcal{W}}$ is the worldline according to observer $\mathcal{O}'$, of the $\mathcal{O}'$-origin.

In coordinates of $\mathcal{O}'$ this is a constant worldline whose space coordinate is the zero vector in $\mathbb{R}^3$ but to $\mathcal{O}$ it is not a constant worldline. To $\mathcal{O}$ that worldline has coordinates $C \circ f(t)$.

According to $\mathcal{O}$, the vector $v = (1, v_0^1, v_0^2, v_0^3)^T$ is the 4-velocity at any point along the worldline of the $\mathcal{O}'$ origin and for all times $\overline{v} = (v_0^1, v_0^2, v_0^3)^T$ is the space velocity of that worldline.

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[7]A group is a mathematical structure. It is a nonempty set with an associative binary operation that has an identity and for which each element has an inverse with respect to the operation.

The m^i_j form the orthogonal 3×3 rotation/reflection matrix M. It modifies space coordinates to match the orthonormal basis used by observer $\mathcal{O}$.

So if $\mathcal{O}'$ assigns coordinates a to event p then $\mathcal{O}$ assigns to p the time coordinate $b^0 + a^0$ and location coordinates

$$\overline{b} + a^0\overline{v} + M\overline{a}.$$

Defining the 4×4 matrix $\widetilde{M}$ as

$$\widetilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} \\ 0 & m_{21} & m_{22} & m_{23} \\ 0 & m_{31} & m_{32} & m_{33} \end{pmatrix}$$

we have

$$\begin{aligned} C\begin{pmatrix} a^0 \\ \overline{a} \end{pmatrix} &= b + a^0\begin{pmatrix} 0 \\ \overline{v} \end{pmatrix} + \widetilde{M}\begin{pmatrix} a^0 \\ \overline{a} \end{pmatrix} \\ &= \begin{pmatrix} b^0 \\ \overline{b} \end{pmatrix} + \begin{pmatrix} 0 \\ a^0\overline{v} \end{pmatrix} + \begin{pmatrix} a^0 \\ M\,\overline{a} \end{pmatrix} = \begin{pmatrix} b^0 + a^0 \\ \overline{b} + a^0\overline{v} + M\,\overline{a} \end{pmatrix}. \end{aligned}$$

If D is another Galilean transformation

$$D(a) = g + a^0\begin{pmatrix} 0 \\ \overline{w} \end{pmatrix} + \widetilde{N}a = \begin{pmatrix} g^0 \\ \overline{g} \end{pmatrix} + a^0\begin{pmatrix} 0 \\ \overline{w} \end{pmatrix} + \begin{pmatrix} a^0 \\ N\,\overline{a} \end{pmatrix} = \begin{pmatrix} g^0 + a^0 \\ \overline{g} + a^0\overline{w} + N\,\overline{a} \end{pmatrix}.$$

then $D \circ C$ is given by

$$\begin{aligned} D \circ C(a) &= \left[\, g + b^0\begin{pmatrix} 0 \\ \overline{w} \end{pmatrix} + \widetilde{N}\,b \,\right] + a^0\begin{pmatrix} 0 \\ \overline{w} + N\overline{v} \end{pmatrix} + \widetilde{NM}\,a \\ &= \begin{pmatrix} g^0 + b^0 + a^0 \\ \overline{g} + b^0\overline{w} + N\overline{b} + a^0\left(\overline{w} + N\overline{v}\right) + NM\overline{a} \end{pmatrix}. \end{aligned}$$

So the composition of two Galilean transformations is given by this specific formula which is itself a transformation of the type that is qualified to be a Galilean transformation.

C is one-to-one and hence invertible. We can produce its inverse by looking at this composition.

Guessing $N = M^{-1}$ and setting $D \circ C(a) = a$ we find in turn that $g^0 = -b^0$ and $\overline{w} = -M^{-1}\overline{v}$ and finally $\overline{g} = b^0M^{-1}\overline{v} - M^{-1}\overline{b}$.

So the set of Galilean transformations does indeed form a group with composition, as stated earlier.

Any coordinate transformation C can be expressed as

$$\begin{aligned} C(a) =& x + a^0\begin{pmatrix} 0 \\ v \end{pmatrix} + \widetilde{M}a = x + \widetilde{M}\left[\, a + a^0\widetilde{M}^{-1}\begin{pmatrix} 0 \\ v \end{pmatrix} \right] \\ =& x + \widetilde{M}\left[\, a + a^0\begin{pmatrix} 0 \\ M^{-1}v \end{pmatrix} \right] \end{aligned}$$

So C can be thought of as composed of three consecutive operations.

First, a movement in space with velocity $\begin{pmatrix} 0 \\ M^{-1}v \end{pmatrix}$.

Second, a rotation/reflection in space.

Then a translation of all four coordinates.

The inverse of C can, of course, be calculated by "doing" the opposite of these three operations in the opposite order.

Looking at a generic member of the Galilean group

$$C\begin{pmatrix}a^0\\a^1\\a^2\\a^3\end{pmatrix}=\begin{pmatrix}b^0\\b^1\\b^2\\bb^3\end{pmatrix}+a^0\begin{pmatrix}0\\v^1\\v^2\\v^3\end{pmatrix}+\begin{pmatrix}1&0&0&0\\0&m_{11}&m_{12}&m_{13}\\0&m_{21}&m_{22}&m_{23}\\0&m_{31}&m_{32}&m_{33}\end{pmatrix}\begin{pmatrix}a^0\\a^1\\a^2\\a^3\end{pmatrix}$$

we can see lots of subgroups of this 10-dimensional Lie group, and identify some of these below.

Subgroups are useful in testing the observer/Physical Law combination for validity. Rather than testing against a generic change of coordinates we can see if there is a problem with one of these simpler subgroups, and these might give hints at the simplest cure if a problem is found.

- The simple translation subgroup, of dimension 4
- The translation-moving-origin subgroup, of dimension 7
- The reflection/rotation only subgroup, of dimension 3 with two components, corresponding to an even number of reflections (none or two, so $\det(M) = 1$) or an odd number (i.e. one, so $\det(M) = -1$.) This is called the Euclidean group.
- The rotation-only subgroup. This is also of dimension 3 but has one component only, corresponding to $\det(M) = 1$. These are called the direct Euclidean isometries.
- The rotation-around-fixed-axis subgroups. There are three types of these, depending whether translation or relative motion of origin are allowed. These types have dimensions 1, 5 or 8.
- The translation-moving-origin-rotation subgroup. This only restricts $\det(M)$ to be 1, allowing freedom otherwise. Like the full Galilean group, it too has dimension 10, but with one component.

These groups of "translators" between representations of events by different observers can be used to test Physical Law for its adherence to the Relativity Principle for this class of observers.

5. The SR Observer

We define here a new class of observers, the **Special Relativistic (SR)** observers. There are assumptions for these shared with Galilean observers but there are also rather dramatically different assumptions too.

Our admissible observers are taken, just as before, from observers who agree to assumptions (i)-(ix).

Our universe seems to these observers, at each time, to be a three dimensional vector space. This collection of observers agrees on a method of establishing units of distance and time. They *could* agree on the second and the meter, as Galilean observers did, but we do not assume that here. However whatever units they do use will be proportionate to these units. With these units space seems to be Euclidean at each time.

The instructions leading to these units must be carried out by, and from the viewpoint, of each observer.

As before, each observer $\mathcal{O}$ selects an orthonormal basis at each time and origin $\mathcal{O}_0$ and agrees to use the constant worldline through that origin to provide an origin for each constant-time slice of $\boldsymbol{\mathcal{W}}$.

Each observer presides over coordinates $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ that represent the world in that basis, with observer-measured time as the first coordinate and origin having the natural origin of $\mathbb{R}^3$ as its spacial coordinate.

However SR-allowable observers don't agree, as did Galilean ones, that measured distances at each time, or time intervals between events, be in agreement even if they do decide to use the same units for these. There is no equivalent of assumptions (c) or (d) here.

Instead we have Assumption **(SR1):** To some observers, the worldline of any light particle, whatever its source and however the observer measures it, appears to pass through coordinates of events on **straight lines and always seems to these observers to move at the same speed. All allowable observers agree on the direction of past-to-future along this worldline and that this speed is consistent with $c = 2.99792458 \times 10^8$ meters per second. Any straight line path at that constant speed is a potential path of a light particle.**

This assumption replaces the two assumptions in the Galilean set-up of the existence of a non-accelerating-origin in a non-rotating frame of reference.

All SR-allowable observers must agree on assumptions (i)-(ix) and (SR1).

Later we will find it convenient to add a second SR assumption regarding scaling and units, but we put that off for now.

Assumption (SR1) seems preposterous on its face.

• *Is (SR1) even possible?* Is there even *one* of these observers?

This is contingent on "reality." If we see it, there is one. And apparently there is one, at least approximately.

• If there is more than one *how are they related?*

The translators between pairs of SR observers satisfy certain properties and we will see that they too form a group, as did the Galilean transformations. This group will be called the **Poincaré group.**

Accepting for now the existence of at least one SR-observer, the Poincaré group will contain, at least, the identity map and space translations. In fact it also contains the (time invariant) rotations and coordinate transformations called **boosts**, connecting coordinates of observers in constant velocity motion with respect to each other.

The subgroup containing just the rotations and boosts is called the **Lorentz group**.

• Given that we have established the existence of such observers, what kind of physical law can we build in emulation of the very successful Newtonian approximation?

We will not even gesture at this aspect in these notes. That is one primary topic in any standard Modern Physics course.

However we will develop properties of the Poincaré group, and see that these properties alone have surprising, even shocking, physical consequences.

Observers who agree on (i)-(ix) and (SR1) could also agree to measure time in units of the amount of time it takes for light to move 1 meter. With this choice, the numerical value c of the speed of light is 1. Alternatively, such an observer could measure distance in units corresponding to the space displacement of a particle of light in one second. In this case too c has numerical value 1.

As a matter of convenience, we assume that units have been chosen so that in these units the numerical value of c is 1. This is assumption **(SR2).**

An SR observer is, henceforth, one who **adheres to assumptions (i)-(ix) and (SR1) and (SR2).**

6. Light Cones

We assume $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ is the coordinate map for an SR observer $\mathcal{O}$.

An event p in the world has coordinates x_p where

$$x_p = (x_p^0 \; x_p^1 \; x_p^2 \; x_p^3)^T \in \mathbb{R}^4.$$

x_p^0 is the time coordinate of p for this observer and the other three coordinates are space coordinates relative to a choice of orthonormal basis.

If a light ray passes from point with coordinates x_p to point with coordinates x_q we have, in units adhering to (SR2), the numerical equality

$$(x_q^0 - x_p^0)^2 = (x_q^1 - x_p^1)^2 + (x_q^2 - x_p^2)^2 + (x_q^3 - x_p^3)^2.$$

The coordinates x for which

$$(x^0 - x_p^0)^2 - (x^1 - x_p^1)^2 - (x^2 - x_p^2)^2 - (x^3 - x_p^3)^2 = 0$$

all lie on a cone in $\mathbb{R}^4$ called the **light cone** (also called the **null cone**) at x_p for the SR observer $\mathcal{O}$, who presides over these coordinates.

The collection of all such points will be denoted $\boldsymbol{\mathcal{N}_{\mathcal{O}}(p)} \subset \mathbb{R}^4$.

Any generic $x \in \mathbb{R}^4$ satisfying this equation corresponds to a unique event q in the world (so $x = x_q$) *and it would be possible for light to traverse the line segment between p and this point q under our assumptions.* In other words, any such x is on the worldline of a possible light particle "emanating" from p.

The future cone $\mathcal{N}^+_{\mathcal{O}}(p)$ and past cone $\mathcal{N}^-_{\mathcal{O}}(p)$ are the cone points with greater or, respectively, lesser time coordinate than has x_p.

The (unbounded) path $R_{\mathcal{O},p,q}$ of the coordinates of a particular light "photon" that moves from event p and, at a later time, arrives at event q is parameterized in $\mathbb{R}^4$ by

$$F_{\mathcal{O},p,q}(t) = x_p + \left(\frac{t - x^0_p}{x^0_q - x^0_p} \right) (x_q - x_p)$$

where t is the clock time as measured by this observer.

$R^+_{\mathcal{O},p,q}$ is a ray of points corresponding to those 4-vectors of the form $F_{\mathcal{O},p,q}(t)$ for $t > x^0_p$, the "future ray" for this observer of the worldline emanating from p and arriving later at q.

$R^-_{\mathcal{O},p,q}$ is the corresponding "past ray" for this observer.

The future and past light cones at x_p are formed as the union of coordinates of all possible "world-rays" emanating from p that correspond to the various possible photons passing through p in all directions.

With our unit requirement the coordinates of each such ray (past and future) make angle 45^o relative[8] to the hyperplane in $\mathbb{R}^4$ of all points with time coordinate x^0_p.

We will show that $R_{\mathcal{O},p,q} = \mathcal{N}_{\mathcal{O}}(p) \cap \mathcal{N}_{\mathcal{O}\,q}$.

Certainly $R_{\mathcal{O},p,q} \subset \mathcal{N}_{\mathcal{O}}(p) \cap \mathcal{N}_{\mathcal{O}\,q}$.

If there is a third event r whose coordinates are in the intersection of the two light cones assume, for convenience, that the time coordinates of the list of points x_p, x_q, x_r are arranged to be non-decreasing.

A "light-sphere" emanates from p and passes q. At that instant a light-sphere emanates from q. At a later time the bigger sphere and the smaller sphere both touch r. The space part of $x_r - x_p$ has magnitude $x^0_r - x^0_p = x^0_r - x^0_q + x^0_q - x^0_p$ which is the sum of the magnitudes of the space-parts of $x_r - x_q$ and $x_q - x_p$. By the triangle inequality these space parts must be positive multiples of each other which implies the three points x_p, x_q, x_r are collinear.

It follows then that $\mathcal{N}_{\mathcal{O}}(p) \cap \mathcal{N}_{\mathcal{O}}(q) \subset R_{\mathcal{O},p,q}$.

Define the segment $S_{\mathcal{O},p,q}$ which consists of the $\mathcal{O}$-coordinates of a particular light "photon" that moves from event p and, at a later time, arrives at event q to be the points in $\mathbb{R}^4$ given by

$$F_{\mathcal{O},p,q}(t) = x_p + \left(\frac{t - x^0_p}{x^0_q - x^0_p} \right) (x_q - x_p)$$

where t is the clock time as measured by this observer and

$$x^0_p \le t \le x^0_q.$$

A point x in the future light cone $\mathcal{N}^+_{\mathcal{O}}(p)$ for event p corresponds to a real place q in the world. You can visualize the cone by imagining a flash of light at the time and place corresponding to p. A spherical "light surface" moves away from

[8]This is the usual angle in $\mathbb{R}^4$ thought of as Euclidean space.

p, illuminating the points whose coordinates form the cone, one spherical "layer" after another eventually arriving at $x = x_q$.

Every observer must agree that a particular place in the world has already been illuminated or has not yet seen the light.

"Before illumination" and "after illumination" are well defined concepts for each event in the world and *therefore well defined for the coordinates that describe them.*

The specific *time* at which that illumination takes place could be the subject of several kinds of disagreement among allowable observers, and *the space coordinates will vary*, but not whether the illumination has occurred or not.

So if $x\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ are coordinates for allowable observer $\mathcal{O}$ and $y\colon \boldsymbol{\mathcal{W}} \to \mathbb{R}^4$ are coordinates for allowable observer $\mathcal{O}'$ we must have

$$\begin{aligned} \mathcal{N}^+_{\mathcal{O}}(p) &= x \circ y^{-1}\left(\mathcal{N}^+_{\mathcal{O}'}(p)\right) \\ \text{and}\quad R^+_{\mathcal{O},p,q} &= x \circ y^{-1}\left(R^+_{\mathcal{O}',p,q}\right) \\ \text{and}\quad \mathcal{N}^-_{\mathcal{O}}(p) &= x \circ y^{-1}\left(\mathcal{N}^-_{\mathcal{O}'}(p)\right) \\ \text{and}\quad R^-_{\mathcal{O},p,q} &= x \circ y^{-1}\left(R^-_{\mathcal{O}',p,q}\right) \\ \text{and}\quad S_{\mathcal{O},p,q} &= x \circ y^{-1}\left(S_{\mathcal{O}',p,q}\right). \end{aligned}$$

The time coordinate has a distinct function in this theory and is *not* on the same footing as the other three coordinates.

Our observer—any SR observer—can detect displacements in the world which lie along the 4-dimensional track of light rays.

If $x_q - x_p$ represents a specific light displacement from event p to event q then $g(x_q - x_p, x_q - x_p) = 0$ where g is the indefinite inner product, the Lorentz metric, which for observer $\mathcal{O}$ in SR coordinates $x\colon \mathcal{M} \to \mathbb{R}^4$ is

$$g = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3.$$

Minkowski space is a 4-dimensional vector space, in our case $\mathcal{W}$, together with a metric that can be represented this way:

$$\boldsymbol{g = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3.}$$

Every SR observer produces such a metric, naturally, along with an orthonormal basis, en route to qualifying as an SR observer.

Here is the critical thing—though different SR observers might produce metrics built from different bases, all SR observers would agree when a displacement is or is not a light displacement, and which direction along the displacement is future-directed.

These coordinate displacements Δx are exactly those for which the Lorentz metric evaluates to

$$g(\,\Delta x,\, \Delta x\,) = 0.$$

Coordinates may change among different observers. A different SR observer, for instance, may decide that the vectors in the world used to create your x space coordinates are *not* orthogonal. There is no obvious reason, for example, to think—based solely on the listed assumptions for SR obserers—that a change of coordinates from one set of SR coordinates to another is continuous.

But SR observers *cannot* disagree about light cones and the zero space-time "length" of light displacements.